# Lecture 4: Analysis of MIMO Systems

## 1 Norms

The concept of norm will be extremely useful for evaluating signals and systems quantitatively during this course. In the following, we will present vector norms and matrix norms.

### 1.1 Vector Norms

**Definition 1.** A **norm** on a linear space $(V, F)$ is a function $\| \cdot \| : V \to \mathbb{R}_+$ such that

  a) $\forall v_1, v_2 \in V, \|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$ (triangle inequality).

  b) $\forall v \in V, \forall \alpha \in F, \|\alpha v\| = |\alpha| \cdot \|v\|$.

  c) $\|v\| = 0 \Leftrightarrow v = 0$.

*Remark.* Note that norms are always non-negative. This can be noticed by seeing that a norm always maps to $\mathbb{R}_+$.

Considering $x \in \mathbb{C}^n$, i.e. $V = \mathbb{C}^n$, one can define the $p-$**norm** as

$$\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}, \quad p = 1, 2, \dots \tag{1.1}$$

The easiest example of such a norm is the case where $p = 2$, i.e. the **euclidean norm** (shortest distance between two points):

$$\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}. \tag{1.2}$$

Another important example of such a norm is the **infinity norm** (largest element in the vector):

$$\|x\|_\infty = \max_i |x_i|. \tag{1.3}$$

**Example 1.** You are given the vector

$$x = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}. \tag{1.4}$$

Compute the $\|x\|_1, \|x\|_2$ and $\|x\|_\infty$ norms of $x$.

**Solution.** It holds

$$\begin{aligned}
\|x\|_1 &= |-1| + |2| + |3| \\
&= 6. \\
\|x\|_2 &= \sqrt{1 + 4 + 9} \\
&= \sqrt{14}. \\
\|x\|_\infty &= \max\{1, 2, 3\} \\
&= 3.
\end{aligned} \tag{1.5}$$

## 1.2   Matrix Norms

In addition to the defined axioms for norms, matrix norms fulfill

$$\|A \cdot B\| \leq \|A\| \cdot \|B\|. \tag{1.6}$$

Considering the linear space $(V, F)$, with $V = \mathbb{C}^{m \times n}$, and assuming a matrix $A \in \mathbb{C}^{m \times n}$ is given, one can define the **Frobenius norm** (euclidean matrix norm) as:

$$\|A\|_F = \|A\|_2 = \left( \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}^2 \right)^{\frac{1}{2}}, \tag{1.7}$$

where $a_{ij}$ are the elements of $A$. This can also be written as

$$\|A\|_F = \sqrt{\operatorname{tr}(A^*A)}, \tag{1.8}$$

where tr is the trace (i.e. sum of eigenvalues or diagonal elements) of the matrix and $A^*$ is the Hermitian transpose of $A$ (complex transpose), i.e.

$$A^* = (conj(A))^T. \tag{1.9}$$

**Example 2.** Let

$$A = \begin{pmatrix} 1 & -2 - i \\ 1 + i & i \end{pmatrix}. \tag{1.10}$$

Then it holds

$$\begin{aligned}
A^* &= (conj(A))^T \\
&= \begin{pmatrix} 1 & -2 + i \\ 1 - i & -i \end{pmatrix}^T \\
&= \begin{pmatrix} 1 & 1 - i \\ -2 + i & -i \end{pmatrix}.
\end{aligned} \tag{1.11}$$

The **maximum matrix norm** is the largest element of the matrix and is defined as

$$\|A\|_\infty = \max_{i=1,\dots,m} \max_{j=1,\dots,n} |a_{ij}|. \tag{1.12}$$

### 1.2.1   Matrix Norms as Induced Norms

Matrix norms can always be defined as induced norms.

**Definition 2.** Let $A \in \mathbb{C}^{m \times n}$. Then, the **induced norm** of matrix $A$ can be written as

$$\begin{aligned}
\|A\|_p &= \sup_{x \in \mathbb{C}^n,\ x \neq 0} \left( \frac{\|Ax\|_p}{\|x\|_p} \right) \\
&= \sup_{x \in \mathbb{C}^n,\ x = 1} (\|Ax\|_p)
\end{aligned} \tag{1.13}$$

*Remark.* At this point, one would ask what is the difference between sup and max. A maximum is the largest number **within** a set. A sup is a number that bounds a set. A sup may or may not be part of the set itself (0 is not part of the set of negative numbers, but it is a sup because it is the least upper bound). If the sup **is** part of the set, it is also the max.

Figure 1: Interpretation of induced norm.

The definition of induced norm is interesting because one can interpret this as in Figure 1. In fact, using the induced norm

$$\|G\|_p = \sup_{y \neq 0} \left( \frac{\|Gu\|}{\|u\|} \right), \tag{1.14}$$

one can quantify the **maimum gain** (amplification) of an output vector for any possible input direction at a given frequency. This turns out to be extremely useful for evaluating system interconnections. Referring to Figure 2 and using the multiplication property for norms, it holds

$$\|y\|_p = \|G_2 G_1 u\|_p \leq \|G_2\|_p \cdot \|G_1\|_p \cdot \|u\|_p.$$
$$\Rightarrow \frac{\|y\|_p}{\|u\|_p} \leq \|G_2\|_p \cdot \|G_1\|_p. \tag{1.15}$$

In words, the input-output gain of a system series is upper bounded by the product of the induced matrix norms.
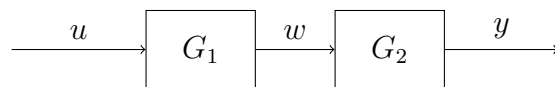


Figure 2: Interpretation of induced norm.

### 1.2.2   Properties of the Euclidean Norm

We can list a few useful properties for the Euclidean norm, intended as induced norm:

(i) If $A$ is squared (i.e. $m = n$), the norm is defined as

$$\|A\|_2 = \sqrt{\mu_{max}}$$
$$= \sqrt{\text{maximal eigenvalue of } A^* \cdot A}. \tag{1.16}$$

(ii) If $A$ is **orthogonal**:
$$\|A\|_2 = 1. \tag{1.17}$$

Note that this is the case because orthogonal matrices always have eigenvalues with magnitude 1.

(iii) If $A$ is **symmetric** (i.e. $A^\intercal = A$):

$$\|A\|_2 = \max_i(|\lambda_i|), \tag{1.18}$$

where $\lambda_i$ are the eigenvalues of $A$.

(iv) If $A$ is **invertible**:

$$\|A^{-1}\|_2 = \frac{1}{\sqrt{\mu_{min}}}$$

$$= \frac{1}{\sqrt{\text{minimal eigenvalue of } A^* \cdot A}}. \tag{1.19}$$

(v) If $A$ is **invertible and symmetric**:

$$\|A^{-1}\|_2 = \frac{1}{\min_i(|\lambda_i|)}. \tag{1.20}$$

*Remark.* Remember: the matrix $A^*A$ is always a square matrix.

## 1.3   Signal Norms

The norms we have seen so far are **space** measures. **Temporal** norms (signal norms), take into account the variability of signals in time/frequency. Let

$$e(t) = \begin{pmatrix} e_1(t) & \dots e_n(t) \end{pmatrix}^{\mathsf{T}}, \quad e_i(t) \in \mathbb{C}, \quad i = 1, \dots, n. \tag{1.21}$$

The *p*-**norm** is defined as

$$\|e(t)\|_p = \left( \int_{-\infty}^{\infty} \sum_{i=1}^{n} |e_i(\tau)|^p \mathrm{d}\tau \right)^{\frac{1}{p}}. \tag{1.22}$$

A special case of this is the two-norm, also called euclidean norm, integral square error, energy of a signal:

$$\|e(t)\|_2 = \sqrt{\left( \int_{-\infty}^{\infty} \sum_{i=1}^{n} |e_i(\tau)|^2 \mathrm{d}\tau \right)}. \tag{1.23}$$

The **infinity norm** is defined as

$$\|e(t)\|_\infty = \sup_\tau \left( \max_i |e_i(\tau)| \right). \tag{1.24}$$

## 1.4   System Norms

Considering linear, time-invariant, causal systems of the form depicted in Figure 1, one can write the relation

$$y(t) = G * u(t)$$

$$= \int_{-\infty}^{\infty} G(t - \tau) u(\tau) \mathrm{d}\tau. \tag{1.25}$$

The **two-norm** for the transfer function $\hat{G}$ reads

$$\|\hat{G}(s)\|_2 = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 \mathrm{d}\omega \right)^{\frac{1}{2}}$$

$$= \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} \left( \hat{G}^*(j\omega) \hat{G}(j\omega) \right) \mathrm{d}\omega \right)^{\frac{1}{2}} \tag{1.26}$$

$$= \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{i,j=1}^{n} |g_{ij}|^2 \mathrm{d}\omega \right)^{\frac{1}{2}}.$$

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*Remark.* Note that this norm is a measure of the combination of system gains in all directions, over all frequency. This is not an induced norm, as it does not respect the multiplicative property.

The **infinity norm** is

$$\|\hat{G}(s)\|_\infty = \sup_\omega \|\hat{G}(j\omega)\|_2 \tag{1.27}$$

*Remark.* This norm is a measure of the peak of the maximum singular value, i.e. the biggest amplification the system may bring at any frequency, for any input direction (worst case scenario). This is an induced norm and respects the multiplicative property.

# 2   Singular Value Decomposition (SVD)

The Singular Value Decomposition plays a central role in MIMO frequency response analysis. Let's recall some concepts from the course *Lineare Algebra I/II*:

## 2.1   Preliminary Definitions

The **induced** norm $\|A\|$ of a matrix that describes a linear function like

$$y = A \cdot u \tag{2.1}$$

is defined as

$$\begin{aligned}
\|A\| &= \max_{u \neq 0} \frac{\|y\|}{\|u\|} \\
&= \max_{\|u\|=1} \|y\|.
\end{aligned} \tag{2.2}$$

Let's recall Equation 1.16, and let's notice that if $A \in \mathbb{R}^{n \times m}$ it holds

$$A^* = A^T. \tag{2.3}$$

In order to define the SVD we have to go a step further. Let's consider a Matrix $A$ and the linear function given in Equation 2.1. It holds

$$\begin{aligned}
\|A\|_2^2 &= \max_{\|u\|=1} y^* \cdot y \\
&= \max_{\|u\|=1} (A \cdot u)^* \cdot (A \cdot u) \\
&= \max_{\|u\|=1} u^* \cdot A^* \cdot A \cdot u \\
&= \max_i \mu(A^* \cdot A) \\
&= \max_i \sigma_i^2.
\end{aligned} \tag{2.4}$$

where $\sigma_i$ are the **singular values** of matrix $A$. They are defined as

$$\sigma_i = \sqrt{\mu_i} \tag{2.5}$$

where $\mu_i$ are the eigenvalues of $A^* \cdot A$.
Combining Equations 2.2 and 2.5 one gets

$$\sigma_{min}(A) \leq \frac{\|y\|}{\|u\|} \leq \sigma_{max}(A). \tag{2.6}$$
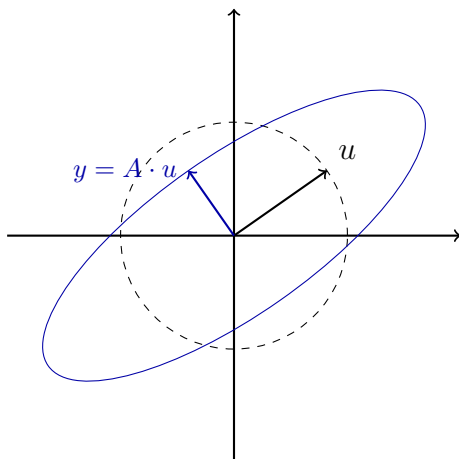
Figure 3: Illustration of the singular values.

## 2.2 Singular Value Decomposition

Our goal is to write a general matrix $A \in \mathbb{C}^{p \times m}$ as product of three matrices: $U$, $\Sigma$ and $V^*$. It holds

$$A = U \cdot \Sigma \cdot V^* \text{ with } U \in \mathbb{C}^{p \times p}, \ \Sigma \in \mathbb{R}^{p \times m}, \ V \in \mathbb{C}^{m \times m}. \tag{2.7}$$

*Remark.* $U$ and $V$ are orthogonal, $\Sigma$ is a diagonal matrix.

### 2.2.1 Kochrezept:

Let $A \in \mathbb{C}^{p \times m}$ be given:

(I) Compute all the eigenvalues and eigenvectors of the matrix

$$A^* \cdot A \in \mathbb{C}^{m \times m}.$$

and sort them as

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_r > \mu_{r+1} = \ldots = \mu_m = 0 \tag{2.8}$$

(II) Compute an orthogonal basis from the eigenvectors $v_i$ and write it in a matrix as

$$V = \begin{pmatrix} v_1 & \ldots & v_m \end{pmatrix} \in \mathbb{C}^{m \times m}. \tag{2.9}$$

(III) We have already found the singular values: they are defined as

$$\sigma_i = \sqrt{\mu_i} \text{ for } i = 1, \ldots, \min\{p, m\}. \tag{2.10}$$

By ordering them from the biggest to the smallest, we can then write $\Sigma$ as

$$\Sigma = \begin{pmatrix} \sigma_1 & & 0 & \ldots & 0 \\ & \ddots & & \vdots & \vdots \\ & & \sigma_m & 0 & \ldots & 0 \end{pmatrix} \in \mathbb{R}^{p \times m}, \ p < m \tag{2.11}$$

$$
\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{p \times m}, \ p > m. \tag{2.12}
$$

(IV) One finds $u_1, \dots, u_r$ from

$$
u_i = \frac{1}{\sigma_i} \cdot A \cdot v_i \text{ for all } i = 1, \dots, r \text{ (for } \sigma_i \neq 0) \tag{2.13}
$$

(V) If $r < p$ one has to complete the basis $u_1, \dots, u_r$ (with ONB from Gram-Schmid) to obtain an orthogonal basis, with $U$ orthogonal.

(VI) If you followed the previous steps, you can write

$$
A = U \cdot \Sigma \cdot V^*.
$$

**Motivation for the computation of $\Sigma$, $U$ und $V$.**

$$
\begin{aligned}
A^* \cdot A &= (U \cdot \Sigma \cdot V^*)^* \cdot (U \cdot \Sigma \cdot V^*) \\
&= V \cdot \Sigma^* \cdot U^* \cdot U \cdot \Sigma \cdot V^* \\
&= V \cdot \Sigma^* \cdot \Sigma \cdot V^* \\
&= V \cdot \Sigma^2 \cdot V^*.
\end{aligned} \tag{2.14}
$$

This is nothing else than the **diagonalization** of the matrix $A^* \cdot A$. The columns of $V$ are the eigenvectors of $A^* \cdot A$ and the $\sigma_i^2$ the eigenvalues.
For $U$:

$$
\begin{aligned}
A \cdot A^* &= (U \cdot \Sigma \cdot V^*) \cdot (U \cdot \Sigma \cdot V^*)^* \\
&= U \cdot \Sigma \cdot V^* \cdot V \cdot \Sigma \cdot U^* \\
&= U \cdot \Sigma^* \cdot \Sigma \cdot U^* \\
&= U \cdot \Sigma^2 \cdot U^*.
\end{aligned} \tag{2.15}
$$

This is nothing else than the **diagonalization** of the matrix $A \cdot A^*$. The columns of $U$ are the eigenvectors of $A \cdot A^*$ and the $\sigma_i^2$ the eigenvalues.

*Remark.* In order to derive the previous two equations I used that:

- The matrix $A^* \cdot A$ is symmetric, i.e.

$$
\begin{aligned}
(A^* \cdot A)^* &= A^* \cdot (A^*)^* \\
&= A^* \cdot A.
\end{aligned}
$$

- $U^{-1} = U^*$ (because $U$ is unitary).

- $V^{-1} = V^*$ (because $V$ is unitary).

*Remark.* Since the matrix $A^* \cdot A$ is always symmetric and positive semidefinite, the singular values are always real numbers.

*Remark.* The MATLAB command for the singular value decomposition is

`[U,S,V]=svd`

One can write $A^T$ as `A.'=transpose(A)` and $A^*$ as `A'=conj(transpose(A))`. Those two are equivalent for real numbers.

## 2.3   Intepretation

Considering the system depicted in Figure 1, one rewrite the system $G$ as $G = U\Sigma V^*$. The matrix $V$ is **orthogonal** and contains the **input directions** of the system. The matrix $U$ is **orthogonal** as well and contains the **output directions** of the system (unfortunate notation). It holds

$$
\begin{aligned}
G &= U\Sigma V^* \\
GV &= U\Sigma \\
Gv_i &= \sigma_i u_i, \quad \forall i,
\end{aligned}
\tag{2.16}
$$

which is similar to an eigenvalue equation. This can be rewritten as

$$
\sigma_i = \frac{\|Gv_i\|}{\|u_i\|}.
\tag{2.17}
$$

For a unitary input, i.e. $\|u\|_2 = 1$, one has

$$
\begin{aligned}
y_1 &= \sigma_1 u_1, \\
y_m &= \sigma_m u_m,
\end{aligned}
\tag{2.18}
$$

where $\sigma_m$ is the last (and hence the smallest) singular value. This can be interpreted using Figure 3 and interpreting the circle as a unit circle.

**Example 3.** Let $u$ be

$$
u = \begin{pmatrix} \cos(x) \\ \sin(x) \end{pmatrix}
$$

with $||u|| = 1$. The matrix $M$ is given as

$$
M = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.
$$

We know that the product of $M$ and $u$ defines a linear function

$$
\begin{aligned}
y &= M \cdot u \\
&= \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \cos(x) \\ \sin(x) \end{pmatrix} \\
&= \begin{pmatrix} 2 \cdot \cos(x) \\ \frac{1}{2} \cdot \sin(x) \end{pmatrix}.
\end{aligned}
$$

We need the maximum of $||y||$. In order to avoid square roots, one can use that the $x$ that maximizes $||y||$ should also maximize $||y||^2$.

$$||y||^2 = 4 \cdot \cos^2(x) + \frac{1}{4} \cdot \sin^2(x)$$

has maximum

$$\frac{\mathrm{d}||y||^2}{\mathrm{d}x} = -8 \cdot \cos(x) \cdot \sin(x) + \frac{1}{2} \cdot \sin(x) \cdot \cos(x) \overset{!}{=} 0$$

$$\Rightarrow \quad x_{\max} = \left\{ 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \right\}.$$

Inserting back for the maximal $||y||$ one gets:

$$||y||_{\max} = 2, \qquad ||y||_{\max} = \frac{1}{2}.$$

The singular values can be calculated with $M^* \cdot M$:

$$M^* \cdot M = M^\mathsf{T} \cdot M$$

$$= \begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \quad \Rightarrow \quad \lambda_i = \left\{ 4, \frac{1}{4} \right\} \quad \Rightarrow \quad \sigma_i = \left\{ 2, \frac{1}{2} \right\}.$$

As stated before, one can see that $||y|| \in [\sigma_{\min}, \sigma_{\max}]$. The matrix $U$ has eigenvectors of $M \cdot M^\mathsf{T}$ as coulmns and the matrix $V$ has eigenvectors of $M^\mathsf{T} \cdot M$ as columns.
In this case

$$M \cdot M^\mathsf{T} = M^\mathsf{T} \cdot M,$$

hence the two matrices are equal. Since their product is a diagonal matrix one should recall from the theory that the eigenvectors are easy to determine: they are nothing else than the standard basis vectors. This means

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

**Interpretation:**

Referring to Figure 4, let's interprete these calculations. One can see that the maximal *amplification* occurs at $v = V(:,1)$ and has direction $u = U(:,1)$, i.e. the vector $u$ is doubled ($\sigma_{max}$). The minimal *amplification* occurs at $v = V(:,2)$ and has direction $u = U(:,2)$, i.e. the vector $u$ is halved ($\sigma_{min}$)
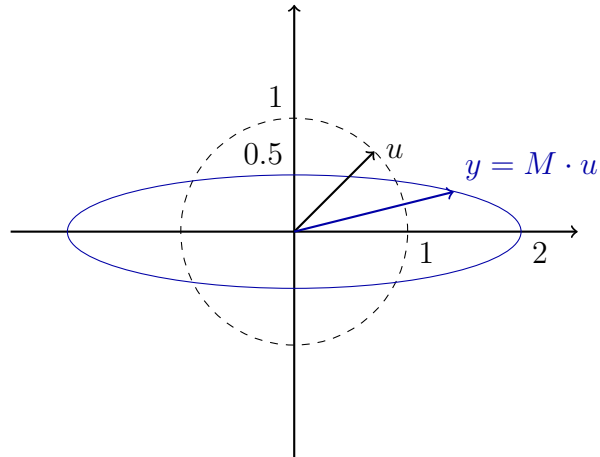
Figure 4: Illustration of the singular value decomposition.

**Example 4.** Let

$$A = \begin{pmatrix} -3 & 0 \\ 0 & 3 \\ \sqrt{3} & 2 \end{pmatrix}$$

be given.
**Question:** Find the singular values of $A$ and write down the matrix $\Sigma$.

**Solution.** Let's compute $A^T A$:

$$A^T A = \begin{pmatrix} -3 & 0 & \sqrt{3} \\ 0 & 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} -3 & 0 \\ 0 & 3 \\ \sqrt{3} & 2 \end{pmatrix} = \begin{pmatrix} 12 & 2\sqrt{3} \\ 2\sqrt{3} & 13 \end{pmatrix}$$

One can see easily that the eigenvalues are

$$\lambda_1 = 16, \ \lambda_2 = 9.$$

The singular values are

$$\sigma_1 = 4, \ \sigma_2 = 3.$$

One writes in this case

$$\Sigma = \begin{pmatrix} 4 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix}.$$

**Example 5.** A transfer function $G(s)$ is given as

$$
\begin{pmatrix}
\frac{1}{s+3} & \frac{s+1}{s+3} \\
\frac{s+1}{s+3} & \frac{1}{s+3}
\end{pmatrix}
$$

Find the singular values of $G(s)$ at $\omega = 1\frac{\text{rad}}{s}$.

**Solution.** The transfer function $G(s)$ evaluated at $\omega = 1\frac{\text{rad}}{s}$ has the form

$$
G(j) =
\begin{pmatrix}
\frac{1}{j+3} & \frac{j+1}{j+3} \\
\frac{j+1}{j+3} & \frac{1}{j+3}
\end{pmatrix}
$$

In order to calculate the singular values, we have to compute the eigenvalues of $H = G^* \cdot G$:

$$
\begin{aligned}
H &= G^* \cdot G \\
&=
\begin{pmatrix}
\frac{1}{-j+3} & \frac{-j+1}{-j+3} \\
\frac{-j+1}{-j+3} & \frac{1}{-j+3}
\end{pmatrix}
\cdot
\begin{pmatrix}
\frac{1}{j+3} & \frac{j+1}{j+3} \\
\frac{j+1}{j+3} & \frac{1}{j+3}
\end{pmatrix} \\
&=
\begin{pmatrix}
\frac{3}{10} & \frac{2}{10} \\
\frac{2}{10} & \frac{3}{10}
\end{pmatrix} \\
&= \frac{1}{10} \cdot
\begin{pmatrix}
3 & 2 \\
2 & 3
\end{pmatrix}.
\end{aligned}
$$

For the eigenvalues it holds

$$
\begin{aligned}
\det(H - \lambda \cdot \mathbb{I}) &= \det
\begin{pmatrix}
\frac{3}{10} - \lambda & \frac{2}{10} \\
\frac{2}{10} & \frac{3}{10} - \lambda
\end{pmatrix} \\
&= \left(\frac{3}{10} - \lambda\right)^2 - \left(-\frac{2}{10}\right)^2 \\
&= \lambda^2 - \frac{6}{10}\lambda + \frac{5}{100} \\
&= \left(\lambda - \frac{1}{10}\right) \cdot \left(\lambda - \frac{5}{10}\right).
\end{aligned}
$$

It follows

$$
\begin{aligned}
\lambda_1 &= \frac{1}{10} \\
\lambda_2 &= \frac{1}{2}
\end{aligned}
$$

and so

$$
\begin{aligned}
\sigma_1 &= \sqrt{\frac{1}{10}} \\
&\approx 0.3162. \\
\sigma_2 &= \sqrt{\frac{1}{2}} \\
&\approx 0.7071.
\end{aligned}
$$

**Example 6.** Let be

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} j & 1 \end{pmatrix}.$$

Find the singular values of the two matrices.

**Solution.**

- Let's begin with matrix $A$. It holds

$$H = A^* \cdot A$$

$$= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

In order to find the eigenvalues of $H$ we compute

$$\det(H - \lambda \cdot \mathbb{I}) = \det \begin{pmatrix} 1 - \lambda & 2 \\ 2 & 5 - \lambda \end{pmatrix}$$

$$= (1 - \lambda) \cdot (5 - \lambda) - 4$$

$$= \lambda^2 - 6\lambda + 1.$$

This means that the eigenvalues are

$$\lambda_1 = 3 + 2\sqrt{2}$$

$$\lambda_2 = 3 - 2\sqrt{2}.$$

The singular values are then

$$\sigma_1 \approx 2.4142$$

$$\sigma_2 \approx 0.4142.$$

- Let's look at matrix $B$. It holds

$$F = B^* \cdot B$$

$$= \begin{pmatrix} -j \\ 1 \end{pmatrix} \cdot \begin{pmatrix} j & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix}.$$

In order to find the eigenvalues of $F$ we compute

$$\det(F - \lambda \cdot \mathbb{I}) = \det \begin{pmatrix} 1 - \lambda & -j \\ j & 1 - \lambda \end{pmatrix}$$

$$= (1 - \lambda)^2 - 1$$

$$= \lambda^2 - 2\lambda$$

$$= \lambda \cdot (\lambda - 2).$$

This means that the eigenvalues are

$$\lambda_1 = 0$$
$$\lambda_2 = 2.$$

The singular values are then

$$\sigma_1 = 0$$
$$\sigma_2 = \sqrt{2}.$$

## 2.4 Directions of poles and zeros

### 2.4.1 Directions of zeros

Assume a system $G(s)$ has a zero at $s = z$. Then, it must holds

$$G(z)u_z = 0, \qquad y_z^* G(z) = 0, \tag{2.19}$$

where $u_z$ is the **input zero direction** and $y_z$ is the **output zero direction**. Furthermore, it holds

$$\|u_z\|_2 = 1, \qquad \|y_z\|_2 = 1. \tag{2.20}$$

### 2.4.2 Directions of poles

Assume a system $G(s)$ has a pole at $s = p$. Then, it must holds

$$G(p)u_p \to \infty, \qquad y_p^* G(p) \to \infty, \tag{2.21}$$

where $u_p$ is the **input pole direction** and $y_p$ is the **output pole direction**. Furthermore, it holds

$$\|u_p\|_2 = 1, \qquad \|y_p\|_2 = 1. \tag{2.22}$$

*Remark.* In both cases, if the zero/pole causes an unfeasible calculation, one consider feasible variations, i.e. $z + \varepsilon$, $p + \varepsilon$.

## 2.5 Frequency Responses

As we learned for SISO systems, if one excites a system with an *harmonic* signal

$$u(t) = h(t) \cdot \cos(\omega \cdot t), \tag{2.23}$$

the answer after a big amount of time is still an harmonic function with equal frequency $\omega$:

$$y_\infty(t) = |P(j \cdot \omega)| \cos(\omega \cdot t + \angle(P(j \cdot \omega))). \tag{2.24}$$

One can generalize this and apply it to MIMO systems. With the assumption of $p = m$, i.e. equal number of inputs and outputs, one excite a system with

$$u(t) = \begin{pmatrix} \mu_1 \cdot \cos(\omega \cdot t + \phi_1) \\ \vdots \\ \mu_m \cdot \cos(\omega \cdot t + \phi_m) \end{pmatrix} \cdot h(t) \tag{2.25}$$

and get

$$y_\infty(t) = \begin{pmatrix} \nu_1 \cdot \cos(\omega \cdot t + \psi_1) \\ \vdots \\ \nu_m \cdot \cos(\omega \cdot t + \psi_m) \end{pmatrix}. \tag{2.26}$$

Let's define two diagonal matrices

$$\begin{aligned} \Phi &= \mathrm{diag}(\phi_1, \ldots, \phi_m) \in \mathbb{R}^{m \times m}, \\ \Psi &= \mathrm{diag}(\psi_1, \ldots, \psi_m) \in \mathbb{R}^{m \times m} \end{aligned} \tag{2.27}$$

and two vectors

$$\begin{aligned} \mu &= \begin{pmatrix} \mu_1 & \ldots & \mu_m \end{pmatrix}^T, \\ \nu &= \begin{pmatrix} \nu_1 & \ldots & \nu_m \end{pmatrix}^T. \end{aligned} \tag{2.28}$$

With these one can compute the Laplace Transform of the two signals as:

$$U(s) = e^{\frac{\Phi \cdot s}{\omega}} \cdot \mu \cdot \frac{s}{s^2 + \omega^2}. \tag{2.29}$$

and

$$Y(s) = e^{\frac{\Psi \cdot s}{\omega}} \cdot \nu \cdot \frac{s}{s^2 + \omega^2}. \tag{2.30}$$

With the general equation for a systems one gets

$$\begin{aligned} Y(s) &= P(s) \cdot U(s) \\ e^{\frac{\Psi \cdot s}{\omega}} \cdot \nu \cdot \frac{s}{s^2 + \omega^2} &= P(s) \cdot e^{\frac{\Phi \cdot s}{\omega}} \cdot \mu \cdot \frac{s}{s^2 + \omega^2} \\ e^{\frac{\Psi \cdot j \cdot \omega}{\omega}} \cdot \nu &= P(s) \cdot e^{\frac{\Phi \cdot j \cdot \omega}{\omega}} \cdot \mu \\ e^{\Psi \cdot j} \cdot \nu &= P(s) \cdot e^{\Phi \cdot j} \cdot \mu. \end{aligned} \tag{2.31}$$

We then recall that the **induced norm** for the matrix of a linear transformation $y = A \cdot u$ from 2.2. Here it holds

$$\begin{aligned} ||P(j \cdot \omega)|| &= \max_{e^{\Phi \cdot j} \cdot \mu \neq 0} \frac{||e^{\Psi \cdot j} \cdot \nu||}{||e^{\Phi \cdot j} \cdot \mu||} \\ &= \max_{||e^{\Phi \cdot j} \cdot \mu|| = 1} ||e^{\Psi \cdot j} \cdot \nu||. \end{aligned} \tag{2.32}$$

Since

$$||e^{\Phi \cdot j} \cdot \mu|| = ||\mu|| \tag{2.33}$$

and

$$||e^{\Psi \cdot j} \cdot \nu|| = ||\nu||. \tag{2.34}$$

One gets

$$\begin{aligned} ||P(j \cdot \omega)|| &= \max_{\mu \neq 0} \frac{||\nu||}{||\mu||} \\ &= \max_{||\mu|| = 1} ||\nu||. \end{aligned} \tag{2.35}$$

Here one should get the feeling of why we introduced the singular value decomposition. From the theory we've learned, it is clear that

$$\sigma_{min}(P(j \cdot \omega)) \leq ||\nu|| \leq \sigma_{max}(P(j \cdot \omega)). \tag{2.36}$$

and if $||\mu|| \neq 1$

$$\sigma_{min}(P(j \cdot \omega)) \leq \frac{||\nu||}{||\mu||} \leq \sigma_{max}(P(j \cdot \omega)). \tag{2.37}$$

with $\sigma_i$ singular values of $P(j \cdot \omega)$. These two are worst case ranges and is important to notice that there is no exact formula for $\nu = f(\mu)$.

### 2.5.1   Maximal and minimal Gain

You are given a singular value decomposition

$$P(j \cdot \omega) = U \cdot \Sigma \cdot V^*. \tag{2.38}$$

One can read out from this decomposition several informations: the maximal/minial gain will be reached with an excitation in the direction of the column vectors of $V$. The response of the system will then be in the direction of the coulmn vectors of $U$.

Let's look at an example and try to understand how to use these informations:

**Example 7.** We consider a system with $m = 2$ inputs and $p = 3$ outputs. We are given its singular value decomposition at $\omega = 5\frac{\text{rad}}{\text{s}}$:

$$\Sigma = \begin{pmatrix} 0.4167 & 0 \\ 0 & 0.2631 \\ 0 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} 0.2908 & 0.9568 \\ 0.9443 - 0.1542 \cdot j & -0.2870 + 0.0469 \cdot j \end{pmatrix},$$

$$U = \begin{pmatrix} -0.0496 - 0.1680 \cdot j & 0.1767 - 0.6831 \cdot j & -0.6621 - 0.1820 \cdot j \\ 0.0146 - 0.9159 \cdot j & -0.1059 + 0.3510 \cdot j & -0.1624 + 0.0122 \cdot j \\ 0.0349 - 0.3593 \cdot j & 0.1360 - 0.5910 \cdot j & 0.6782 + 0.2048 \cdot j \end{pmatrix}.$$

For the singular value $\sigma_{\max} = 0.4167$ the eigenvectors are $V(:,1)$ and $U(:,1)$:

$$V_1 = \begin{pmatrix} 0.2908 \\ 0.9443 - 0.1542 \cdot j \end{pmatrix}, \qquad |V_1| = \begin{pmatrix} 0.2908 \\ 0.9568 \end{pmatrix}, \qquad \angle(V_1) = \begin{pmatrix} 0 \\ -0.1618 \end{pmatrix},$$

$$U_1 = \begin{pmatrix} -0.0496 - 0.1680 \cdot j \\ 0.0146 - 0.9159 \cdot j \\ 0.0349 - 0.3593 \cdot j \end{pmatrix}, \qquad |U_1| = \begin{pmatrix} 0.1752 \\ 0.9160 \\ 0.3609 \end{pmatrix}, \qquad \angle(U_1) = \begin{pmatrix} -1.8581 \\ -1.5548 \\ -1.4741 \end{pmatrix}.$$

The maximal gain is then reached with

$$u(t) = \begin{pmatrix} 0.2908 \cdot \cos(5 \cdot t) \\ 0.9568 \cdot \cos(5 \cdot t - 0.1618) \end{pmatrix}.$$

The response of the system is then

$$y(t) = \sigma_{\max} \cdot \begin{pmatrix} 0.1752 \cdot \cos(5 \cdot t - 1.8581) \\ 0.9160 \cdot \cos(5 \cdot t - 1.5548) \\ 0.3609 \cdot \cos(5 \cdot t - 1.4741) \end{pmatrix} = 0.4167 \cdot \begin{pmatrix} 0.1752 \cdot \cos(5 \cdot t - 1.8581) \\ 0.9160 \cdot \cos(5 \cdot t - 1.5548) \\ 0.3609 \cdot \cos(5 \cdot t - 1.4741) \end{pmatrix}.$$

Since the three signals $y_1(t)$, $y_2(t)$ and $y_3(t)$ are not in phase, the maximal gain will never be reachen. One can show that

$$\max_t ||y(t)|| \approx 0.4160 < 0.4167 = \sigma_{\max}$$

The reason for this difference stays in the phase deviation between $y_1(t)$, $y_2(t)$ and $y_3(t)$. The same analysis can be computed for $\sigma_{\min}$.

**Example 8.** Given the MIMO system

$$P(s) = \begin{pmatrix} \frac{1}{s+3} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{3}{s+1} \end{pmatrix}.$$

Starting at $t = 0$, the system is excited with the following input signal:

$$u(t) = \begin{pmatrix} \cos(t) \\ \mu_2 \cos(t + \varphi_2) \end{pmatrix}.$$

Find the parameters $\varphi_2$ and $\mu_2$ such that for steady-state conditions the output signal

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

has $y_1(t)$ equal to zero.

**Solution.** For a system excited using a harmonic input signal

$$u(t) = \begin{pmatrix} \mu_1 \cos(\omega t + \varphi_1) \\ \mu_2 \cos(\omega t + \varphi_2) \end{pmatrix}$$

the output signal $y(t)$, after a transient phase, will also be a harmonic signal and hence have the form

$$y(t) = \begin{pmatrix} \nu_1 \cos(\omega t + \psi_1) \\ \nu_2 \cos(\omega t + \psi_2) \end{pmatrix}.$$

As we have learned, it holds

$$e^{\Psi \cdot j} \cdot \nu = P(j\omega) \cdot e^{\Phi \cdot j} \cdot \mu.$$

One gets

$$\begin{pmatrix} e^{\psi_1 \cdot j} & 0 \\ 0 & e^{\psi_2 \cdot j} \end{pmatrix} \cdot \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} P_{11}(j\omega) & P_{12}(j\omega) \\ P_{11}(j\omega) & P_{21}(j\omega) \end{pmatrix} \cdot \begin{pmatrix} e^{\varphi_1 \cdot j} & 0 \\ 0 & e^{\varphi_2 \cdot j} \end{pmatrix} \cdot \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}.$$

For the first component one gets

$$e^{\psi_1 \cdot j} \cdot \nu_1 = P_{11}(j\omega) \cdot e^{\varphi_1 \cdot j} \cdot \mu_1 + P_{12}(j\omega) \cdot e^{\varphi_2 \cdot j} \cdot \mu_2.$$

For $y_1(t) = 0$ to hold we must have $\nu_1 = 0$. In the given case, some parameters can be easily copied from the signals:

$$\mu_1 = 1$$
$$\varphi_1 = 0$$
$$\omega = 1.$$

With the given transfer functions, one gets

$$0 = \frac{1}{j+3} + \mu_2 \cdot \frac{1}{j+1} \cdot e^{\varphi_2 \cdot j}$$

$$0 = \frac{3-j}{10} + \mu_2 \cdot \frac{1-j}{2} \cdot e^{\varphi_2 \cdot j}$$

$$0 = \frac{3-j}{10} + \mu_2 \cdot \frac{1-j}{2} \cdot (\cos(\varphi_2) + j \sin(\varphi_2))$$

$$0 = \frac{3}{10} + \mu_2 \cdot \frac{1}{2} \cdot (\cos(\varphi_2) + \sin(\varphi_2)) + j \cdot \left( \mu_2 \cdot \frac{1}{2} \cdot (\sin(\varphi_2) - \cos(\varphi_2)) - \frac{1}{10} \right).$$

Splitting the real to the imaginary part, one can get two equations that are easily solvable:

$$\mu_2 \cdot \frac{1}{2} \cdot (\cos(\varphi_2) + \sin(\varphi_2)) + \frac{3}{10} = 0$$

$$\mu_2 \cdot \frac{1}{2} \cdot (\sin(\varphi_2) - \cos(\varphi_2)) - \frac{1}{10} = 0.$$

Adding and subtracting the two equations one can reach two better equations:

$$\mu_2 \cdot \sin(\varphi_2) + \frac{1}{5} = 0$$

$$\mu_2 \cdot \cos(\varphi_2) + \frac{2}{5} = 0.$$

One of the solutions (periodicity) reads

$$\mu_2 = \frac{1}{\sqrt{5}}$$

$$\varphi_2 = \arctan\left(\frac{1}{2}\right) + \pi.$$

**Example 9.** A $2 \times 2$ linear time invariant MIMO system with transfer function

$$P(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{2}{s+1} \\ \frac{s^2+1}{s+10} & \frac{1}{s^2+2} \end{pmatrix}$$

is excited with the signal

$$u(t) = \begin{pmatrix} \mu_1 \cdot \cos(\omega \cdot t + \varphi_1) \\ \mu_2 \cdot \cos(\omega \cdot t + \varphi_2) \end{pmatrix}.$$

Because we bought a cheap signal generator, we cannot know exactly the constants $\mu_{1,2}$ and $\varphi_{1,2}$. A friend of you just found out with some measurements, that the excitation frequency is $\omega = 1\frac{\text{rad}}{\text{s}}$. The cheap generator, cannot produce signals with magnitude of $\mu$ bigger than 10, i.e. $\sqrt{\mu_1^2 + \mu_2^2} \leq 10$. This works always at maximal power, i.e. at 10. Choose all possible responses of the system after infinite time.

☐ $y_\infty(t) = \begin{pmatrix} 5 \cdot \sin(t + 0.114) \\ \cos(t) \end{pmatrix}.$

☐ $y_\infty(t) = \begin{pmatrix} 5 \cdot \sin(t + 0.114) \\ \cos(2 \cdot t) \end{pmatrix}.$

☐ $y_\infty(t) = \begin{pmatrix} \sin(t + 0.542) \\ \sin(t + 0.459) \end{pmatrix}.$

☐ $y_\infty(t) = \begin{pmatrix} 19 \cdot \cos(t + 0.114) \\ \cos(t + 1.124) \end{pmatrix}.$

☐ $y_\infty(t) = \begin{pmatrix} 5 \cdot \cos(t + 0.114) \\ 5 \cdot \cos(t) \end{pmatrix}.$

☐ $y_\infty(t) = \begin{pmatrix} 10 \cdot \sin(t + 2.114) \\ 11 \cdot \sin(t + 1.234) \end{pmatrix}.$

**Solution.**

☑ $y_\infty(t) = \begin{pmatrix} 5 \cdot \sin(t + 0.114) \\ \cos(t) \end{pmatrix}.$

☐ $y_\infty(t) = \begin{pmatrix} 5 \cdot \sin(t + 0.114) \\ \cos(2 \cdot t) \end{pmatrix}.$

☐ $y_\infty(t) = \begin{pmatrix} \sin(t + 0.542) \\ \sin(t + 0.459) \end{pmatrix}.$

☐ $y_\infty(t) = \begin{pmatrix} 19 \cdot \cos(t + 0.114) \\ \cos(t + 1.124) \end{pmatrix}.$

☑ $y_\infty(t) = \begin{pmatrix} 5 \cdot \cos(t + 0.114) \\ 5 \cdot \cos(t) \end{pmatrix}.$

☑ $y_\infty(t) = \begin{pmatrix} 10 \cdot \sin(t + 2.114) \\ 11 \cdot \sin(t + 1.234) \end{pmatrix}.$

**Explanation**

We have to compute the singular values of the matrix $P(j \cdot 1)$. These are

$$\sigma_{max} = 1.8305$$
$$\sigma_{min} = 0.3863.$$

With what we have learned it follows

$$10 \cdot \sigma_{min} = 3.863 \leq ||\nu|| \leq 18.305 = 10 \cdot \sigma_{max}.$$

The first response has $||\nu|| = \sqrt{26}$ that is in this range. The second response also has $||\nu|| = \sqrt{26}$ but the frequency in its second element changes and that isn't possible for linear systems. The third response has $||\nu|| = \sqrt{2}$ that is too small to be in the range. The fourth response has $||\nu|| = \sqrt{362}$ that is too big to be in the range. The fifth response has $||\nu|| = \sqrt{50}$ that is in the range. The sixth response has $||\nu|| = \sqrt{221}$ that is in the range.

**Example 10.** A $3 \times 2$ linear time invariant MIMO system is excited with the input

$$u(t) = \begin{pmatrix} 3 \cdot \sin(30 \cdot t) \\ 4 \cdot \cos(30 \cdot t) \end{pmatrix}.$$

You have forgot your PC and you don't know the transfer function of the system. Before coming to school, however, you have saved the MATLAB plot of the singular values of the system on your phone (see Figure 5. Choose all the possible responses of the system.
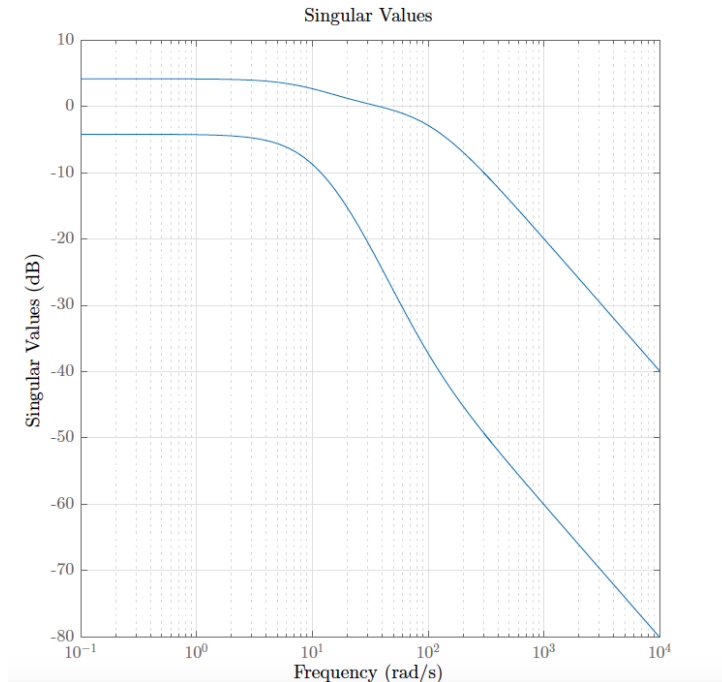


Figure 5: Singular values behaviour.

$$\square \ y_\infty(t) = \begin{pmatrix} 0.5 \cdot \sin(30 \cdot t + 0.314) \\ 0.5 \cdot \cos(30 \cdot t) \\ 0.5 \cdot \cos(30 \cdot t + 1) \end{pmatrix}.$$

$$\square \ y_\infty(t) = \begin{pmatrix} 4 \cdot \sin(30 \cdot t + 0.314) \\ 3 \cdot \cos(30 \cdot t) \\ 2 \cdot \cos(30 \cdot t + 1) \end{pmatrix}.$$

$$\square \ y_\infty(t) = \begin{pmatrix} 0.1 \cdot \sin(30 \cdot t + 0.314) \\ 0.1 \cdot \cos(30 \cdot t) \\ 0.1 \cdot \cos(30 \cdot t + 1) \end{pmatrix}.$$

$$\square \ y_\infty(t) = \begin{pmatrix} 0 \\ 4 \cdot \cos(30 \cdot t) \\ 2 \cdot \cos(30 \cdot t + 1) \end{pmatrix}.$$

$$\square \ y_\infty(t) = \begin{pmatrix} 2 \cdot \cos(30 \cdot t + 0.243) \\ 2 \cdot \cos(30 \cdot t + 0.142) \\ 2 \cdot \cos(30 \cdot t + 0.252) \end{pmatrix}.$$

**Solution.**

$$\blacktriangleright\ y_\infty(t) = \begin{pmatrix} 0.5 \cdot \sin(30 \cdot t + 0.314) \\ 0.5 \cdot \cos(30 \cdot t) \\ 0.5 \cdot \cos(30 \cdot t + 1) \end{pmatrix}.$$

$$\square\ y_\infty(t) = \begin{pmatrix} 4 \cdot \sin(30 \cdot t + 0.314) \\ 3 \cdot \cos(30 \cdot t) \\ 2 \cdot \cos(30 \cdot t + 1) \end{pmatrix}.$$

$$\square\ y_\infty(t) = \begin{pmatrix} 0.1 \cdot \sin(30 \cdot t + 0.314) \\ 0.1 \cdot \cos(30 \cdot t) \\ 0.1 \cdot \cos(30 \cdot t + 1) \end{pmatrix}.$$

$$\blacktriangleright\ y_\infty(t) = \begin{pmatrix} 0 \\ 4 \cdot \cos(30 \cdot t) \\ 2 \cdot \cos(30 \cdot t + 1) \end{pmatrix}.$$

$$\blacktriangleright\ y_\infty(t) = \begin{pmatrix} 2 \cdot \cos(30 \cdot t + 0.243) \\ 2 \cdot \cos(30 \cdot t + 0.142) \\ 2 \cdot \cos(30 \cdot t + 0.252) \end{pmatrix}.$$

**Explanation**

From the given input one can read

$$||\mu|| = \sqrt{3^2 + 4^2} = 5.$$

From the plot one can read at $\omega = 30\frac{\text{rad}}{\text{s}}$ $\sigma_{min} = 0.1$ and $\sigma_{max} = 1$. It follows

$$5 \cdot \sigma_{min} = 0.5 \leq ||\nu|| \leq 5 = 5 \cdot \sigma_{max}.$$

The first response has $||\nu|| = \sqrt{0.75}$ that is in the range. The second response has $||\nu|| = \sqrt{29}$ that is too big to be in the range. The third response has $||\nu|| = \sqrt{0.03}$ that is to small to be in the range. The fourth response has $||\nu|| = \sqrt{20}$ that is in the range. The fifth response has $||\nu|| = \sqrt{12}$ that is in the range.

# References

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