Lecture 5: MIMO Stability, Performance and Robustness

1 MIMO Stability, Controllability, Observability

1.1 External Stability

The **input-output** stability (also known as external stability) describes the stability properties of a system with respect to its input-output behaviour. Let's consider the system interaction depicted in Figure 1.

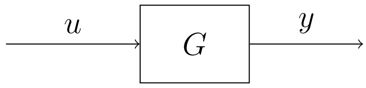

Figure 1: Interpretation of induced norm.

Definition 1. A MIMO system $y = Gu$ is said to be **BIBO stable** (i.e. bounded input bounded output) if there exists a finite constant $k \in \mathbb{R}$ such that

$$\|y\|_\infty \leq k\|u\|_\infty. \tag{1.1}$$

Remark. A **necessary and sufficient** condition for BIBO stability is: the closed loop transfer function

$$P(s) = C(s\mathbb{I} - A)^{-1} + D \tag{1.2}$$

has all poles in the open left-half of the complex plane (all poles have real part strictly smaller than 0).

1.2 Internal Stability

Consider the linear time invariant system

$$\begin{aligned} \dot{x}(t) &= A \cdot x(t) + B \cdot u(t), & x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \\ y(t) &= C \cdot x(t) + D \cdot u(t), & y(t) \in \mathbb{R}^p \end{aligned} \tag{1.3}$$

where

$$x(t) \in \mathbb{R}^{n\times 1},\ u(t) \in \mathbb{R}^{m\times 1},\ y(t) \in \mathbb{R}^{p\times 1},\ A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}. \tag{1.4}$$

Such a system is internally stable if for all initial conditions, and all bounded signals injected at any place in the system, all states remain bounded for all future time.

Definition 2. The MIMO linear time invariant system described in Equation 1.3 is BIBO stable if and only if $C(s\mathbb{I} - A)^{-1} + D$ has all poles on the open left-half of the complex plane (all poles have real part strictly smaller than 0).

Remark.

- Internal stability **implies** BIBO stability. The converse is not true.
- BIBO stability with controllability and observability **imply** internal stability.

This is a crucial concept: it is not sufficient for the input-output transfer function of the system to be stable. In fact, internal transfer functions, related to the sensitivity functions, must be stable as well to prevent pole/zero cancellations, which could hide instabilities.

1.2.1 Internal Stability Check

Assume a MIMO loop as the one depiced in Figure 2. It holds

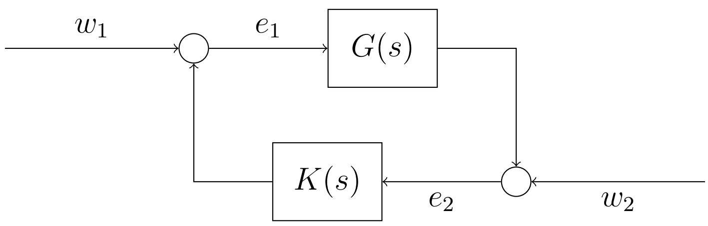

Figure 2: MIMO Loop.

$$\begin{aligned} E_1(s) &= W_1(s) + K(s)E_2(s) \\ &= W_1(s) + K(s)\left[G(s)E_1(s) + W_2(s)\right] \\ &= W_1(s) + K(s)G(s)E_1(s) + K(s)W_2(s), \end{aligned} \tag{1.5}$$

from which it follows

$$\begin{aligned} \left(\mathbb{I} - K(s)G(s)\right)E_1(s) &= W_1(s) + K(s)W_2(s) \\ E_1(s) &= \left(\mathbb{I} - K(s)G(s)\right)^{-1}W_1(s) + \left(\mathbb{I} - K(s)G(s)\right)^{-1}K(s)W_2(s). \end{aligned} \tag{1.6}$$

Similarly, one can write

$$\begin{aligned} E_2(s) &= W_2(s) + G(s)E_1(s) \\ &= W_2(s) + G(s)\left[K(s)E_2(s) + W_1(s)\right] \\ &= W_2(s) + G(s)K(s)E_2(s) + G(s)W_1(s), \end{aligned} \tag{1.7}$$

from which it follows

$$\begin{aligned} \left(\mathbb{I} - G(s)K(s)\right)E_2(s) &= W_2(s) + G(s)W_1(s) \\ E_2(s) &= \left(\mathbb{I} - G(s)K(s)\right)^{-1}W_2(s) + \left(\mathbb{I} - G(s)K(s)\right)^{-1}G(s)W_1(s). \end{aligned} \tag{1.8}$$

Resuming the calculations into matrix form, one gets

$$\begin{pmatrix} E_1(s) \\ E_2(s) \end{pmatrix} = \begin{pmatrix} \left(\mathbb{I} - K(s)G(s)\right)^{-1} & \left(\mathbb{I} - K(s)G(s)\right)^{-1}K(s) \\ \left(\mathbb{I} - G(s)K(s)\right)^{-1}G(s) & \left(\mathbb{I} - G(s)K(s)\right)^{-1} \end{pmatrix} \cdot \begin{pmatrix} W_1(s) \\ W_2(s) \end{pmatrix}. \tag{1.9}$$

The necessary and sufficient condition for **internal stability** is: each of the four transfer functions in relation 1.9 must be stable. (Note: even if three of four are stable, the system is not internally stable).

1.3 Lyapunov Stability

The Lyapunov stability theorem analyses the behaviour of a system near to its equilibrium points when $u(t) = 0$. Because of this, we don't care if the system is MIMO or SISO. The three cases are

- Asymptotically stable: $\lim_{t\to\infty} \|x(t)\| = 0$;
- Stable: $\|x(t)\| < \infty \, \forall \, t \geq 0$;
- Unstable: $\lim_{t\to\infty} \|x(t)\| = \infty$.

As it was done for the SISO case, one can show by using $x(t) = e^{A\cdot t} \cdot x_0$ that the stability can be related to the eigenvalues of A through:

- Asymptotcally stable: $\text{Re}(\lambda_i) < 0 \, \forall \, i$;
- (Marginally) Stable: $\text{Re}(\lambda_i) \leq 0 \, \forall \, i$;
- Unstable: $\text{Re}(\lambda_i) > 0$ for at least one i.

1.4 Controllability and Observability

1.4.1 Controllability

Controllable: is it possible to control all the states of a system with an input $u(t)$?
Mathematically, a linear time invariant system is controllable if, for every state $x^*(t)$ and every finite time $T > 0$, there exists an input function $u(t)$, $0 < t \leq T$ such that the system can be driven from the initial state $x(0) = x_0$ to $x(T) = x^*(t)$.
A system of the form of the one represented in Equation 1.3 is said to be **completely controllable**, if the **controllability Matrix**

$$\mathcal{R} = \begin{pmatrix} B & A\cdot B & A^2\cdot B & \dots & A^{n-1}\cdot B \end{pmatrix} \in \mathbb{R}^{n\times(n\cdot m)}. \tag{1.10}$$

has full rank n (easy by checking row rank).

1.4.2 Observability

Observable: is it possible to reconstruct the initial conditions of all the states of a system from the output $y(t)$?
A system is said to be **completely observable**, if the **observability Matrix**

$$O = \begin{pmatrix} C \\ C\cdot A \\ C\cdot A^2 \\ \vdots \\ C\cdot A^{n-1} \end{pmatrix} \in \mathbb{R}^{(n\cdot p)\times n}. \tag{1.11}$$

has full rank n (easy by checking column rank).

Example 1. The dynamics of a system are given as

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot x(t) + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot u(t) \\ y(t) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot x(t). \end{aligned} \tag{1.12}$$

Moreover the transfer function of the system is given as

$$P(s) = \begin{pmatrix} \frac{(s-2)}{s^2-6s+9} & 0 \\ \frac{-1}{s^2-6s+9} & \frac{1}{s-2} \end{pmatrix}. \tag{1.13}$$

(a) Is the system Lyapunov stable, asymptotically stable or unstable?

(b) Is the system completely controllable?

(c) Is the system completely observable?

(d) The poles of the system are $\pi_1 = 2$ and $\pi_{2,3} = 3$. The zero of the system is $\zeta_1 = 2$. Are there any zero-pole cancellations?

Solution.

(a) First of all, one identifies the matrices as:

$$\begin{aligned} \dot{x}(t) &= \underbrace{\begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{A} \cdot x(t) + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}_{B} \cdot u(t) \\ y(t) &= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}}_{C} \cdot x(t) + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{D} \cdot u(t). \end{aligned} \tag{1.14}$$

We have to compute the eingevalues of A. It holds

$$\begin{aligned} det(A - \lambda \cdot \mathbb{1}) &= \left| \begin{pmatrix} 4-\lambda & 1 & 0 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} \right| \\ &= (2-\lambda) \cdot \left| \begin{pmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{pmatrix} \right| \\ &= (2-\lambda) \cdot ((4-\lambda) \cdot (2-\lambda) + 1) \\ &= (2-\lambda) \cdot (\lambda^2 - 6\lambda + 9) \\ &= (2-\lambda) \cdot (\lambda - 3)^2. \end{aligned} \tag{1.15}$$

Since all the three eigenvalues are bigger than zero, the system is Lyapunov **unstable**.

(b) The controllability matrix can be found with the well-known multiplications:

$$\begin{aligned} A \cdot B &= \begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 0 \\ -1 & 0 \\ 0 & 2 \end{pmatrix}, \\ A^2 \cdot B &= \begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ -1 & 0 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 15 & 0 \\ -6 & 0 \\ 0 & 4 \end{pmatrix}. \end{aligned} \tag{1.16}$$

Hence, the controllability matrix reads

$$R = \begin{pmatrix} 1 & 0 & 4 & 0 & 15 & 0 \\ 0 & 0 & -1 & 0 & -6 & 0 \\ 0 & 1 & 0 & 2 & 0 & 4 \end{pmatrix}. \tag{1.17}$$

This has full rank 3: the system ist completely controllable.

(c) The observability matrix can be found with the well-known multiplications:

$$\begin{aligned} C \cdot A &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 2 \end{pmatrix}, \\ C \cdot A^2 &= \begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 15 & 6 & 0 \\ -6 & 3 & 4 \end{pmatrix}. \end{aligned} \tag{1.18}$$

Hence, the observability matrix reads

$$O = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 4 & 1 & 0 \\ -1 & 2 & 2 \\ 15 & 6 & 0 \\ -6 & 3 & 4 \end{pmatrix}. \tag{1.19}$$

This has full rank 3: the system is completely observable.

(d) Although $\zeta_1 = 2$ and $\pi_1 = 2$ have the same magnitude, they don't cancel out. Why? Since the system ist completely controllable and completely observable, we have already the minimal realization of the system. This means that no more cancellation is possible. The reason for that is that the directions of the two don't coincide. We will learn more about this in the next chapter.

2 MIMO Performance

2.1 What is Performance?

A good performance means

- good disturbance rejection,
- good noise attenuation,
- good reference tracking,

at input and output.

2.2 Performance Analysis

Recalling the general MIMO loop depicted in Figure 3, we defined inner and outer loop

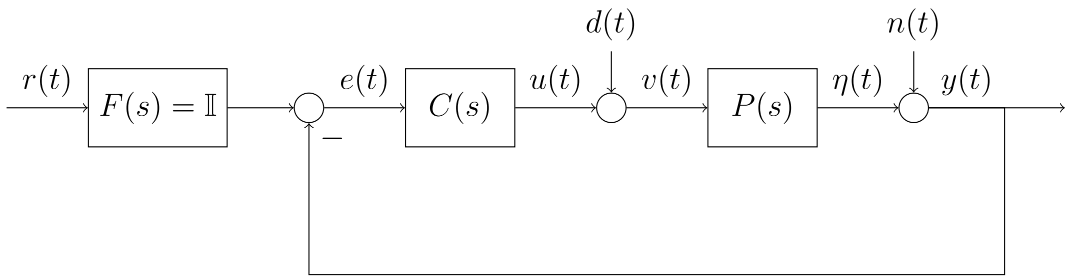

Figure 3: Standard feedback control system structure.

transfer functions

$$L_{\mathrm{O}}(s) = P(s) \cdot C(s) \neq C(s) \cdot P(s) = L_{\mathrm{I}}(s), \tag{2.1}$$

and the input/output sensitivity functions, i.e.

- **Output sensitivity function** $(n \to y)$

$$S_{\mathrm{O}}(s) = (\mathbb{I} + L_{\mathrm{O}}(s))^{-1}. \tag{2.2}$$

- **Output complementary sensitivity function** $(r \to y)$

$$T_{\mathrm{O}}(s) = (\mathbb{I} + L_{\mathrm{O}}(s))^{-1} L_{\mathrm{O}}(s). \tag{2.3}$$

- **Input sensitivity function** $(d \to v)$

$$S_{\mathrm{I}}(s) = (\mathbb{I} + L_{\mathrm{I}}(s))^{-1}. \tag{2.4}$$

- **Input complementary sensitivity function** $(d \to -u)$

$$T_{\mathrm{I}}(s) = (\mathbb{I} + L_{\mathrm{I}}(s))^{-1} L_{\mathrm{I}}(s). \tag{2.5}$$

2.2.1 Output Conditions

Referring to Figure 3, one can write

$$\begin{aligned} Y(s) &= N(s) + \eta(s) \\ &= N(s) + P(s)V(s) \\ &= N(s) + P(s)\left(D(s) + U(s)\right) \\ &= N(s) + P(s)\left(D(s) + C(s)E(s)\right) \\ &= N(s) + P(s)\left(D(s) + C(s)(R(s) - Y(s))\right), \end{aligned} \tag{2.6}$$

from which follows

$$\begin{aligned} (\mathbb{I} + P(s)C(s))Y(s) &= N(s) + P(s)D(s) + P(s)C(s)R(s) \\ Y(s) &= (\mathbb{I} + P(s)C(s))^{-1}\left(N(s) + P(s)D(s) + P(s)C(s)R(s)\right). \end{aligned} \tag{2.7}$$

Using the defined sensitivity functions, one can write

$$Y(s) = S_{\text{O}}(s)N(s) + S_{\text{O}}(s)P(s)D(s) + S_{\text{O}}(s)L_{\text{O}}(s)R(s). \tag{2.8}$$

Disturbance Rejection

Equation 2.8 shows that the effects of the disturbance $D(s)$ on the output can be rejected by making the output sensitivity function $S_{\text{O}}(s)$ small. Since typically disturbances occurr at *low frequencies*, one needs to do that only for this frequency range. How can we relate this to what we have learned about singular values? It must hold

$$\begin{aligned} \bar{\sigma}\left(S_{\text{O}}(j\omega)P(j\omega)\right) &= \bar{\sigma}\left((\mathbb{I} + P(j\omega)C(j\omega))^{-1}P(j\omega)\right) \\ \text{push-through rule} \quad &= \bar{\sigma}\left(P(j\omega)(\mathbb{I} + C(j\omega)P(j\omega))^{-1}\right) \\ &= \bar{\sigma}(P(j\omega)S_{\text{I}}(j\omega)) \\ &\ll 1, \end{aligned} \tag{2.9}$$

where we used the push-through rule

$$G_1(\mathbb{I} - G_2G_1)^{-1} = (\mathbb{I} - G_1G_2)^{-1}G_1, \tag{2.10}$$

and $\bar{\sigma}(H(j\omega)$ refers to the maximum singular value of $H(j\omega)$.

Noise Attenuation

Similarly, Equation 2.8 shows that the effects of the noise $N(s)$ on the output can be attenuated by making the output sensitivity function $S_{\text{O}}(s)$ small. Since typically noise occurs at *high frequencies*, one needs to do that only for this frequency range. It holds

$$\begin{aligned} \bar{\sigma}\left(S_{\text{O}}(j\omega)\right) &= \bar{\sigma}\left((\mathbb{I} + P(j\omega)C(j\omega))^{-1}\right) \\ &\ll 1. \end{aligned} \tag{2.11}$$

2.2.2 Input Conditions

Referring to Figure 3, one can write

$$\begin{aligned} V(s) &= D(s) + U(s) \\ &= D(s) + C(s)E(s) \\ &= D(s) + C(s)\left(R(s) - Y(s)\right) \\ &= D(s) + C(s)\left(R(s) - N(s) - \eta(s)\right) \\ &= D(s) + C(s)\left(R(s) - N(s) - P(s)V(s)\right) \\ &= D(s) + C(s)R(s) - C(s)N(s) - C(s)P(s)V(s), \end{aligned} \tag{2.12}$$

from which follows

$$\begin{aligned} \left(\mathbb{I} + C(s)P(s)\right)V(s) &= D(s) + C(s)R(s) - C(s)N(s) \\ V(s) &= \left(\mathbb{I} + C(s)P(s)\right)^{-1}\left(D(s) + C(s)R(s) - C(s)N(s)\right) \end{aligned} \tag{2.13}$$

Using the defined sensitivity functions, one can write

$$V(s) = S_{\mathrm{I}}(s)D(s) + S_{\mathrm{I}}(s)C(s)R(s) - S_{\mathrm{I}}(s)C(s)N(s). \tag{2.14}$$

Disturbance Rejection

Equation 2.14 shows that the effects of the disturbance $D(s)$ on the input can be rejected by making the input sensitivity function $S_{\mathrm{I}}(s)$ small. Since typically disturbances occurr at *low frequencies*, one needs to do that only for this frequency range. It must hold

$$\begin{aligned} \bar{\sigma}\left(S_{\mathrm{I}}(j\omega)\right) &= \bar{\sigma}\left(\left(\mathbb{I} + C(j\omega)P(j\omega)\right)^{-1}\right) \\ &\ll 1. \end{aligned} \tag{2.15}$$

Noise Attenuation

Similarly, Equation 2.14 shows that the effects of the noise $N(s)$ on the input can be attenuated by making the input sensitivity function $S_{\mathrm{I}}(s)$ small. Since typically noise occurs at *high frequencies*, one needs to do that only for this frequency range. It holds

$$\begin{aligned} \bar{\sigma}\left(S_{\mathrm{I}}(j\omega)C(j\omega)\right) &= \bar{\sigma}\left(\left(\mathbb{I} + C(j\omega)P(j\omega)\right)^{-1}C(j\omega)\right) \\ &\ll 1. \end{aligned} \tag{2.16}$$

2.2.3 Reference Tracking

Referring to Figure 3, one can write

$$\begin{aligned} E(s) &= R(s) - Y(s) \\ &= R(s) - N(s) - \eta(s) \\ &= R(s) - N(s) - P(s)V(s) \\ &= R(s) - N(s) - P(s)\left(D(s) + U(s)\right) \\ &= R(s) - N(s) - P(s)D(s) - P(s)C(s)E(s), \end{aligned} \tag{2.17}$$

from which follows

$$\begin{aligned} \left(\mathbb{I} + P(s)C(s)\right)E(s) &= R(s) - N(s) - P(s)D(s) \\ E(s) &= \left(\mathbb{I} + P(s)C(s)\right)^{-1}\left(R(s) - N(s) - P(s)D(s)\right). \end{aligned} \tag{2.18}$$

Using the defined sensitivity functions, one can write

$$E(s) = S_{\mathrm{O}}(s)(R(s) - N(s)) - S_{\mathrm{O}}(s)P(s)D(s) \tag{2.19}$$

Disturbance Rejection

Equation 2.19 shows that the effects of the disturbance $D(s)$ on the error can be rejected by making the output sensitivity function $S_\text{O}(s)$ small. Since typically disturbances occurr at *low frequencies*, one needs to do that only for this frequency range. It must hold

$$\begin{aligned} \bar{\sigma}\left(S_\text{O}(j\omega)P(j\omega)\right) &= \bar{\sigma}\left((\mathbb{I}+P(j\omega)C(j\omega))^{-1}P(j\omega)\right) \\ \text{push-through rule} \qquad &= \bar{\sigma}\left(P(j\omega)(\mathbb{I}+C(j\omega)P(j\omega))^{-1}\right) \\ &= \bar{\sigma}\left(P(j\omega)S_\text{I}(j\omega)\right) \\ &\ll 1. \end{aligned} \tag{2.20}$$

Noise Attenuation

Similarly, Equation 2.19 shows that the effects of the noise $N(s)$ on the error can be attenuated by making the output sensitivity function $S_\text{O}(s)$ small. Since typically noise occurs at *high frequencies*, one needs to do that for this frequency range **and** for reference relevant frequencies (we have $R(s)$ in the term). It holds

$$\begin{aligned} \bar{\sigma}\left(S_\text{O}(j\omega)\right) &= \bar{\sigma}\left((\mathbb{I}+P(j\omega)C(j\omega))^{-1}\right) \\ &\ll 1. \end{aligned} \tag{2.21}$$

Remark. One can note that the reference tracking case resumes the other two cases.

2.2.4 Useful Properties

Given an invertible matrix A and a matrix B, it holds

(I) **Inverse:**

$$\bar{\sigma}\left(A^{-1}\right) = \frac{1}{\underline{\sigma}\left(A\right)}, \tag{2.22}$$

where $\underline{\sigma}(A)$ represents the smallest singular value of A.

(II) **Sum:**

$$\begin{aligned} \sigma_i(A) - \bar{\sigma}(B) &\leq \sigma_i(A+B) \\ &\leq \sigma_i(A) + \bar{\sigma}(B). \end{aligned} \tag{2.23}$$

In particular, it holds

$$\begin{aligned} \underline{\sigma}(A) - 1 &\leq \underline{\sigma}(\mathbb{I}+A) \\ &\leq \underline{\sigma}(A) + 1. \end{aligned} \tag{2.24}$$

(III) **Product:**

$$\begin{aligned} \bar{\sigma}(AB) &\leq \bar{\sigma}(A)\bar{\sigma}(B) \\ \underline{\sigma}(AB) &\leq \underline{\sigma}(A)\underline{\sigma}(B). \end{aligned} \tag{2.25}$$

2.2.5 Towards Clearer Bounds

Assuming $P(s)$ and $C(s)$ are invertible, one can use the defined properties to write

$$\begin{aligned}\underline{\sigma}(P(j\omega)C(j\omega))-1\leq\underline{\sigma}(\mathbb{I}+P(j\omega)C(j\omega))\leq\underline{\sigma}(P(j\omega)C(j\omega))+1\\ \underline{\sigma}(C(j\omega)P(j\omega))-1\leq\underline{\sigma}(\mathbb{I}+C(j\omega)P(j\omega))\leq\underline{\sigma}(C(j\omega)P(j\omega))+1\end{aligned}\tag{2.26}$$

For disturbance rejection, using Equations 2.9, 2.15, 2.20 one can write

$$\begin{aligned}\bar{\sigma}(P(j\omega)S_{\mathrm{I}}(j\omega))&\ll 1\\ \bar{\sigma}(S_{\mathrm{I}}(j\omega))&\ll 1.\end{aligned}\tag{2.27}$$

For noise attenuation, using Equations 2.11, 2.16, 2.21 one can write

$$\begin{aligned}\bar{\sigma}(S_{\mathrm{O}}(j\omega))&\ll 1\\ \bar{\sigma}(S_{\mathrm{I}}(j\omega)C(j\omega))&\ll 1.\end{aligned}\tag{2.28}$$

With the inverse property of singular values, we know that

$$\begin{aligned}\bar{\sigma}(S_{\mathrm{I}}(j\omega))&=\bar{\sigma}((\mathbb{I}+C(j\omega)P(j\omega))^{-1})\\ &=\frac{1}{\underline{\sigma}(\mathbb{I}+C(j\omega)P(j\omega))},\\ \bar{\sigma}(S_{\mathrm{O}}(j\omega))&=\bar{\sigma}((\mathbb{I}+P(j\omega)C(j\omega))^{-1})\\ &=\frac{1}{\underline{\sigma}(\mathbb{I}+P(j\omega)C(j\omega))}.\end{aligned}\tag{2.29}$$

With Equation 2.26 and $\underline{\sigma}(C(j\omega)P(j\omega))>1$, $\underline{\sigma}(P(j\omega)C(j\omega))>1$, one can write

$$\begin{aligned}\frac{1}{\underline{\sigma}(C(j\omega)P(j\omega))+1}\leq\bar{\sigma}(S_{\mathrm{I}}(j\omega))\leq\frac{1}{\underline{\sigma}(C(j\omega)P(j\omega))-1}\\ \frac{1}{\underline{\sigma}(P(j\omega)C(j\omega))+1}\leq\bar{\sigma}(S_{\mathrm{O}}(j\omega))\leq\frac{1}{\underline{\sigma}(P(j\omega)C(j\omega))-1}.\end{aligned}\tag{2.30}$$

This implies

$$\begin{aligned}\bar{\sigma}(S_{\mathrm{I}}(j\omega))\ll 1\Leftrightarrow\underline{\sigma}(C(j\omega)P(j\omega))\gg 1\\ \bar{\sigma}(S_{\mathrm{O}}(j\omega))\ll 1\Leftrightarrow\underline{\sigma}(P(j\omega)C(j\omega))\gg 1\end{aligned}\tag{2.31}$$

Disturbance Rejection

Suppose that $P(s)$ and $C(s)$ are invertible.

- **Output:** It holds

$$\begin{aligned}\underline{\sigma}(P(j\omega)C(j\omega))\gg 1\Leftrightarrow\bar{\sigma}(S_{\mathrm{O}}(j\omega)P(j\omega))&=\bar{\sigma}((\mathbb{I}+P(j\omega)C(j\omega))^{-1}P(j\omega))\\ &\approx\bar{\sigma}((P(j\omega)C(j\omega))^{-1}P(j\omega))\\ &=\bar{\sigma}(C(j\omega)^{-1})\\ &=\frac{1}{\underline{\sigma}(C(j\omega))}.\end{aligned}\tag{2.32}$$

  This implies:

$$\bar{\sigma}(S_{\mathrm{O}}(j\omega)P(j\omega))\ll 1\Leftrightarrow\underline{\sigma}(C(j\omega))\gg 1,\qquad\forall\omega\in(0,\omega_{\mathrm{low}}).\tag{2.33}$$

- **Input:** Considering Equation 2.30, one can write

$$\bar{\sigma}(S_{\mathrm{I}}(j\omega)) \geq \frac{1}{\underline{\sigma}(C(j\omega)P(j\omega))+1}. \tag{2.34}$$

This implies

$$\bar{\sigma}(S_{\mathrm{I}}(j\omega)) \ll 1 \Leftrightarrow \underline{\sigma}(C(j\omega)P(j\omega)) \gg 1, \qquad \forall\omega \in (0,\omega_{\mathrm{low}}). \tag{2.35}$$

Noise Attenuation

Suppose that $P(s)$ and $C(s)$ are invertible.

- **Output:** Using Equation 2.30, one can write

$$\frac{1}{\underline{\sigma}(P(j\omega)C(j\omega))+1} \leq \bar{\sigma}(S_{\mathrm{O}}(j\omega)) \tag{2.36}$$

This implies

$$\bar{\sigma}(S_{\mathrm{O}}(j\omega)) \ll 1 \Leftrightarrow \underline{\sigma}(P(j\omega)C(j\omega)) \gg 1, \qquad \forall\omega \in (\omega_{\mathrm{high}},\infty). \tag{2.37}$$

- **Input:** It holds

$$\begin{aligned}\underline{\sigma}(C(j\omega)P(j\omega)) \gg 1 \Leftrightarrow \bar{\sigma}(C(j\omega)S_{\mathrm{O}}(j\omega)) &= \bar{\sigma}(C(j\omega)(\mathbb{I}+P(j\omega)C(j\omega))^{-1})\\ &\approx \bar{\sigma}(C(j\omega)(P(j\omega)C(j\omega))^{-1})\\ &= \bar{\sigma}(P(j\omega)^{-1})\\ &= \frac{1}{\underline{\sigma}(P(j\omega))}.\end{aligned} \tag{2.38}$$

This implies

$$\bar{\sigma}(C(j\omega)S_{\mathrm{O}}(j\omega)) \ll 1 \Leftrightarrow \underline{\sigma}(P(j\omega)) \gg 1, \qquad \forall\omega \in (\omega_{\mathrm{high}},\infty). \tag{2.39}$$

2.2.6 Is this the whole Story? Tradeoffs

Robust Stability

One defines **robust stability** to be the stability in the presence of *model uncertainty*. Let Δ be a stable uncertainty matrix, such that

$$P_{\mathrm{real}}(s) = (\mathbb{I}+\Delta)\,P_{\mathrm{nominal}}(s) \tag{2.40}$$

The perturbed closed loop transfer function is then characterized by

$$\det(\mathbb{I}+P(s)C(s)) \to \det(\mathbb{I}+(\mathbb{I}+\Delta)P(s)C(s)) = \det(\mathbb{I}+P(s)C(s))\det(\mathbb{I}+\Delta T_{\mathrm{O}}), \tag{2.41}$$

where we used

$$\det(X+AB) = \det(X)\det(\mathbb{I}+BX^{-1}A), \forall X : \exists X^{-1} \tag{2.42}$$

Since

$$\det(\mathbb{I}+\Delta T_{\mathrm{O}}) \approx 1, \tag{2.43}$$

it holds

$$\|\Delta T_{\mathrm{O}}\| \ll 1. \tag{2.44}$$

This implies

$$\bar{\sigma}(T_{\mathrm{O}}(j\omega)) \ll 1 \Rightarrow \bar{\sigma}(L_{\mathrm{O}}(j\omega)) \ll 1, \qquad \forall\omega \in (\omega_{\mathrm{high}},\infty). \tag{2.45}$$

Remark. Note that typically Δ becomes important at high frequencies.

Actuator Saturation

Using Figure 3, one can derive

$$U(s) = C(s)S_O(s)R(s) - T_I(s)D(s) - C(s)S_O(s)N(s). \tag{2.46}$$

With the defined conditions, it holds

$$U(s) \approx C(s)\left(R(s) - N(s)\right). \tag{2.47}$$

In order to avoid the actuator saturation, the controller gain cannot be chosen too big, i.e.

$$\bar{\sigma}(C(j\omega)) \leq M, \qquad \forall \omega \in (\omega_{\text{high}}, \infty). \tag{2.48}$$

2.2.7 Zämefassig

The specifications we derived are resumed in Figure 4. Mathematically, we have found:

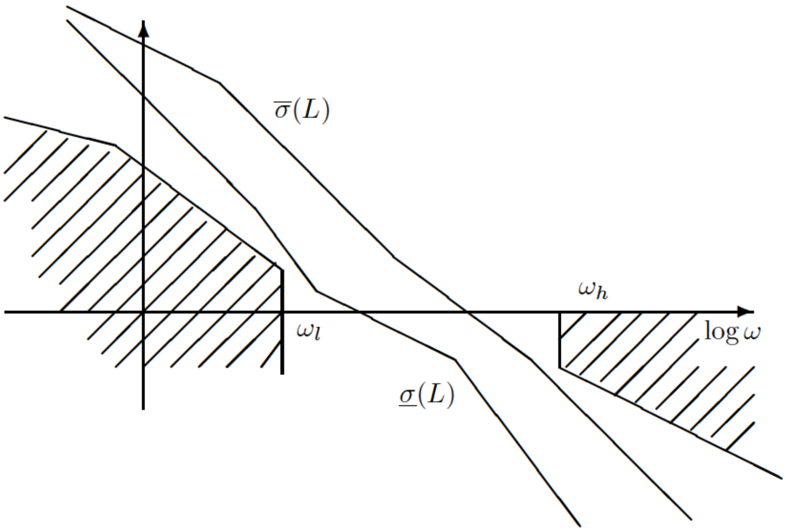

Figure 4: Desired Loop Gain

Disturbance Rejection

At frequency $\omega \in (0, \omega_{\text{low}})$ holds

$$\begin{aligned} \underline{\sigma}(C(j\omega)) &\gg 1, \\ \underline{\sigma}(C(j\omega)P(j\omega) &\gg 1, \\ \underline{\sigma}(P(j\omega)C(j\omega)) &\gg 1. \end{aligned} \tag{2.49}$$

Noise Attenuation

At frequency $\omega \in (\omega_{\text{high}}, \infty)$ holds

$$\begin{aligned} \bar{\sigma}(C(j\omega)) &\ll M, \\ \bar{\sigma}(C(j\omega)P(j\omega) &\ll 1, \\ \bar{\sigma}(P(j\omega)C(j\omega)) &\ll 1. \end{aligned} \tag{2.50}$$

3 MIMO Robustness

$$\textit{All models are wrong, but some are useful.} \tag{3.1}$$

A model maps inputs into outputs and we consider good a model which predicts the outputs accurately. The difference between a model prediction and reality (which is never 0) is referred to as **model uncertainty**.

3.1 Modeling Uncertainty

Let $P(s)$, $C(s)$ be the nominal MIMO plant and an internally stabilizing controller, respectively. Let's define $\Delta(s)$, $W_1(s)$, $W_2(s)$ to be stable, rational and proper transfer matrices. We call $W_1(s)$ and $W_2(s)$ *weighting functions*. $\Delta(s)$ is the *modeling error*. We represent uncertainty as

$$W_1(s)\Delta(s)W_2(s). \tag{3.2}$$

Let's define $\Pi(s)$ to be the set of perturbed plants such that $P(s) \in \Pi(s)$.

3.1.1 Unstructured Uncertainty

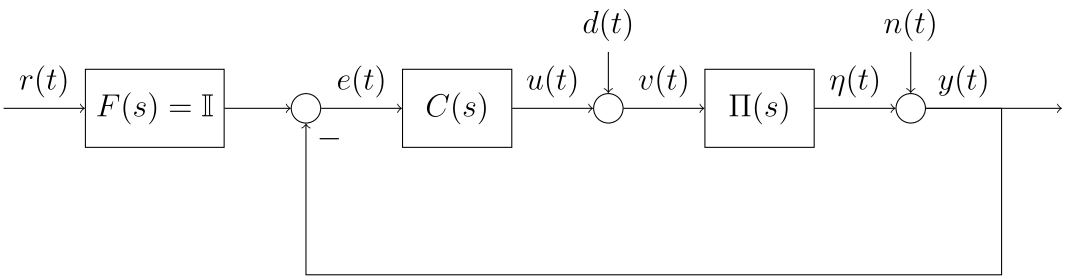

Figure 5: Standard feedback control system structure.

Using Figure 5 and the Equations for internal stability, one can write

$$\begin{pmatrix} E_1(s) \\ E_2(s) \end{pmatrix} = \begin{pmatrix} (\mathbb{I} - C(s)\Pi(s))^{-1} & (\mathbb{I} - C(s)\Pi(s))^{-1} C(s) \\ (\mathbb{I} - \Pi(s)C(s))^{-1}\Pi(s) & (\mathbb{I} - \Pi(s)C(s))^{-1} \end{pmatrix} \cdot \begin{pmatrix} W_1(s) \\ W_2(s) \end{pmatrix}. \tag{3.3}$$

For unstructured uncertainty, nothing more can be said without writing the relation between the uncertainty and the plant.

3.1.2 Additive Uncertainty

Theorem 1. (robust stability under additive uncertainty). Let

$$\Pi(s) = \{P + W_1(s)\Delta(s)W_2(s) : \Delta \text{ rational, proper and stable}\} \tag{3.4}$$

and let $C(s)$ be a stabilizing controller for the nominal plant $P(s)$. Then, the closed loop system is well-posed (i.e., realizable) and internally stable for all $\|\Delta\|_\infty < 1$ if and only if $\|W_2(s)C(s)S_O(s)W_1(s)\|_\infty \leq 1$.

3.1.3 Multiplicative Uncertainty

Theorem 2. (robust stability under multiplicative uncertainty). Let

$$\Pi(s) = \{P + W_1(s)\Delta(s)W_2(s) : \Delta \text{ rational, proper and stable}\} \tag{3.5}$$

and let $C(s)$ be a stabilizing controller for the nominal plant $P(s)$. Then, the closed loop system is well-posed (i.e., realizable) and internally stable for all $\|\Delta\|_\infty < 1$ if and only if $\|W_2(s)T_{\text{O}}W_1(s)\|_\infty \leq 1$.

References

[1] Essentials of Robust Control, Kemin Zhou.

[2] Karl Johan Amstroem, Richard M. Murray *Feedback Systems for Scientists and Engineers*. Princeton University Press, Princeton and Oxford, 2009.

[3] Sigurd Skogestad, *Multivariate Feedback Control*. John Wiley and Sons, New York, 2001.