Lecture 6: MIMO Robust Stability, Performance

1 MIMO Robust Stability

Definition 1. Robust stability: Given a controller C, one determines whether the system remains stable for all possible plants P in the uncertaint set.

1.1 SISO Case

In order to understand what we will address in this section, let's have a look at the SISO case. Let's assume *multiplicative uncertainty*, i.e.

$$P(s) = P_0(s)\left(1 + W(s)\Delta(s)\right), \qquad \text{with } |\Delta(j\omega)| \leq 1 \forall\omega, \tag{1.1}$$

where $P(s)$ represents the perturbed plant and $P_0(s)$ the nominal plant. Assuming a controller which *stabilizes* the nominal plant, one has

$$|1 + L_0(j\omega)| > 0. \tag{1.2}$$

If one looks at the perturbed plant, instead, one has

$$\begin{aligned} 1 + L(j\omega) &= 1 + P(j\omega)C(j\omega) \\ &= 1 + \underbrace{P_0(j\omega)C(j\omega)}_{L_0(j\omega)} + W(j\omega)\Delta(j\omega)P_0(j\omega)C(j\omega). \end{aligned} \tag{1.3}$$

In order to ensure stability even in the worst case scenario, it should hold

$$\begin{aligned} |1 + L(j\omega)| &> |1 + L_0(j\omega)| - |L_0(j\omega)W(j\omega)\Delta(j\omega)| \\ \overset{|\Delta| \leq 1}{} \quad &> |1 + L_0(j\omega)| - |L_0(j\omega)W(j\omega)| \\ &> 0. \end{aligned} \tag{1.4}$$

From this it follows

$$\left|\frac{W(j\omega)L_0(j\omega)}{1 + L_0(j\omega)}\right| < 1. \tag{1.5}$$

What we want to do, is to be able to write such relations for MIMO systems.

1.2 Linear Fractional Transform (LFT)

In order to analyze robust stability, it is worth first to separate the nominal plant from the uncertainty which affects it. Assuming a nominal plant $P_0(s)$ and a feedback controller $C(s)$ that stabilizes $P_0(s)$, one can write the problem as in Figure 1. Note that the generalization for an uncertainty block $W_1(s)\Delta(s)W_2(s)$ instead of $\Delta(s)$ is trivial and can be used as well. Note that $w(t) = (r(t), d(t), n(t))$ represents the exogenous inputs and $z(t) = (y(t), u(t), e(t))$ represents the regulated variables. One can write

$$\begin{pmatrix} U_\Delta(s) \\ Z(s) \end{pmatrix} = \begin{pmatrix} M(s) & N(s) \\ J(s) & L(s) \end{pmatrix} \cdot \begin{pmatrix} Y_\Delta(s) \\ W(s) \end{pmatrix}. \tag{1.6}$$

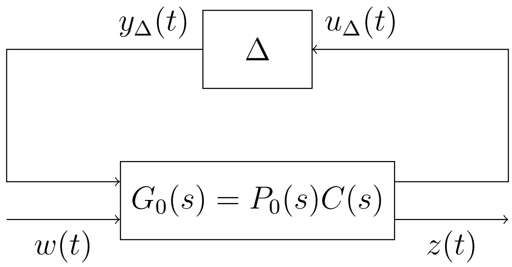

Figure 1: Standard feedback control system structure.

Note that $\begin{pmatrix} U_\Delta(s) \\ Z(s) \end{pmatrix}$ represents the plant outputs and $\begin{pmatrix} Y_\Delta(s) \\ W(s) \end{pmatrix}$ the plant inputs, where $Y_\Delta(s) = \Delta(s) \cdot U_\Delta(s)$. In order for the system to be **internally stable**, each element of the matrix

$$\begin{pmatrix} M(s) & N(s) \\ J(s) & L(s) \end{pmatrix} \tag{1.7}$$

must be stable itself. By looking at the transfer function which relates $z(t)$ to $w(t)$ one has

$$\begin{aligned} Z(s) &= J(s)Y_\Delta(s) + L(s)W(s) \\ &= J(s)\Delta(s)U_\Delta(s) + L(s)W(s) \end{aligned} \tag{1.8}$$

Furthermore, the first equation of the system reads

$$\begin{aligned} U_\Delta(s) &= M(s)Y_\Delta(s) + N(s)W(s) \\ W(s) &= N^{-1}\left[\left(\mathbb{I} - M(s)\Delta(s)\right)U_\Delta(s)\right]. \end{aligned} \tag{1.9}$$

By pluggin Equation 1.9 in Equation 1.8 one gets

$$\begin{aligned} Z(s) &= J(s)\Delta(s)U_\Delta(s) + L(s)W(s) \\ &= \left[J(s)\Delta(s)U_\Delta(s)W^{-1}(s) + L(s)\right]W(s) \\ &= \left[J(s)\Delta(s)U_\Delta(s)U_\Delta(s)^{-1}\left(\mathbb{I} - M(s)\Delta(s)\right)^{-1}N(s) + L(s)\right]W(s) \\ &= \left[J(s)\Delta(s)\left(\mathbb{I} - M(s)\Delta(s)\right)^{-1}N(s) + L(s)\right]W(s). \end{aligned} \tag{1.10}$$

This means that the transfer function from $w(t)$ to $z(t)$ is

$$G_{zw}(s) = J(s)\Delta(s)\left(\mathbb{I} - M(s)\Delta(s)\right)^{-1}N(s) + L(s). \tag{1.11}$$

The **internal stability** of the perturbed closed-loop system requires this transfer function to be stable for all possible perturbations $\Delta(s)$. Since from above $M(s), N(s), J(s), L(s)$ are stable, $G_{zw}(s)$ is stable for all stable $\left(\mathbb{I} - M(s)\Delta(s)\right)^{-1}$.

1.3 Unstructured Small Gain Theorem

Theorem 1. Let the set of allowable model uncertainties be

$$\tilde{\Delta} = \{\Delta : \|\Delta\|_\infty \leq 1\} \tag{1.12}$$

and let M be stable. Then, $\left(\mathbb{I} - M(s)\Delta(s)\right)^{-1}$ and $\Delta\left(\mathbb{I} - M(s)\Delta(s)\right)^{-1}$ are stable, for all $\Delta \in \tilde{\Delta}$, **if and only if** $\|M\|_\infty < 1$.

Proof. We first prove **sufficiency** and then **necessity**.

(I) **Sufficiency:** we show that $(\mathbb{I} - M(s)\Delta(s))$ has no zeros ζ in the right-half plane. In particular, we show

$$\|M\|_\infty < 1 \Rightarrow (\mathbb{I} - M(s)\Delta(s))^{-1} \text{ stable.} \tag{1.13}$$

It holds

$$\begin{aligned}
& \|(\mathbb{I} - M(\zeta)\Delta(\zeta))x\|_2 > 0, \qquad x \neq 0, \forall \Delta \in \tilde{\Delta} \\
\textit{triangle inequality} \quad & \|(\mathbb{I} - M(\zeta)\Delta(\zeta))x\|_2 \geq \|x\|_2 - \|M(\zeta)\Delta(\zeta)x\|_2 \\
\textit{induced matrix norm} \quad & \geq \|x\|_2 - \bar{\sigma}\left(M(\zeta)\Delta(\zeta)\right)\|x\|_2 \\
& \geq \|x\|_2 - \underbrace{\|M(\zeta)\|_\infty \|\Delta(\zeta)\|_\infty}_{\leq 1} \|x\|_2 \\
& > 0,
\end{aligned} \tag{1.14}$$

where in the last step we used the fact that $\bar{\sigma}(H(s)) \leq \|H(s)\|_\infty$ for stable and causal $H(s)$.

(II) **Necessity**: we show by construction, that if $\bar{\sigma}(M(j\omega_0)) > 1$, there exists a $\Delta \in \tilde{\Delta}$ such that $(\mathbb{I} - M\Delta)^{-1}$ is unstable, i.e.

$$\det(\mathbb{I} - M(j\omega_0)\Delta(j\omega_0) = 0. \tag{1.15}$$

In particular, we show

$$\neg\|M\|_\infty < 1 \Rightarrow \neg\,(\mathbb{I} - M(s)\Delta(s))^{-1} \text{ stable.} \tag{1.16}$$

Let's write the singular value decomposition of M as

$$M(j\omega_0) = U \begin{pmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_p \end{pmatrix} V^*, \ \sigma_1 > 1. \tag{1.17}$$

We choose a Δ such that

$$\Delta(j\omega_0) = V \begin{pmatrix} \sigma_1^{-1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} U^*, \ \|\Delta\|_\infty < 1. \tag{1.18}$$

It holds then

$$
\begin{aligned}
(\mathbb{I} - M(j\omega_0)\Delta(j\omega_0)) &= \mathbb{I} - U \begin{pmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_p \end{pmatrix} V^* V \begin{pmatrix} \sigma_1^{-1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} U^* \\
&= U \left(\mathbb{I} - \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \right) U^* \\
&= U \underbrace{\begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}}_{I} U^*,
\end{aligned}
\tag{1.19}
$$

where I is clearly **not invertible**.

□

1.4 From the Block-Diagram to the LFT

One follows usually this procedure

1. Define the input and the output of each perturbation block Δ_i as $(u_{\Delta,i}, y_{\Delta,i})$ and let

$$
u_\Delta = \begin{pmatrix} u_{\Delta,1} & \dots & u_{\Delta,q} \end{pmatrix}^\mathsf{T}, \qquad y_\Delta = \begin{pmatrix} y_{\Delta,1} & \dots & y_{\Delta,q} \end{pmatrix}^\mathsf{T}, \tag{1.20}
$$

where q is the number of uncertainties in the loop.

2. Compute each component of the transfer matrix M as the map between the (i, j)-th inputs and outputs to each uncertainty block, assuming $\Delta_i = \mathbb{I}\ \forall i = 1, \dots, q$, i.e.

$$
M(s) = \begin{pmatrix} M_{1,1}(s) & M_{1,2}(s) & \dots & M_{1,q}(s) \\ M_{2,1}(s) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ M_{q,1}(s) & \dots & \dots & M_{q,q}(s) \end{pmatrix}, \qquad M_{i,j} = \frac{U_{\Delta,i}}{Y_{\Delta,j}}. \tag{1.21}
$$

3. The uncertainty block will be block diagonal in the MIMO case and diagonal in the SISO one:

$$
\Delta = \operatorname{diag}(\Delta_1, \dots, \Delta_q), \quad \|\Delta_i\|_\infty < 1. \tag{1.22}
$$

Example 1. (**Additive Uncertainty**) You are given the system depicted in Figure 5 and the input output behaviour depicted in Figure 3, where

$$
\eta(s) = P_0(s)U(s) + W_1(s)\Delta(s)W_2(s)U(s). \tag{1.23}
$$

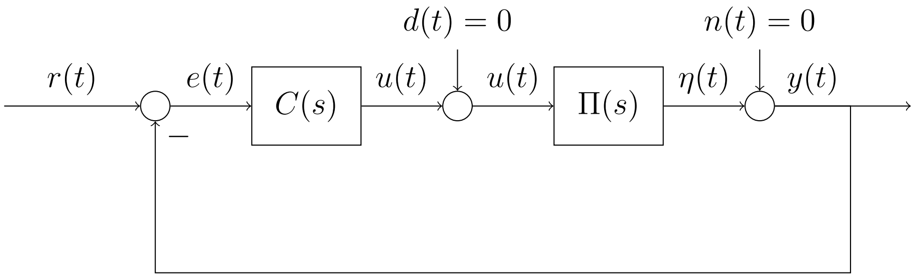

Figure 2: Additive Uncertainty Control System Loop

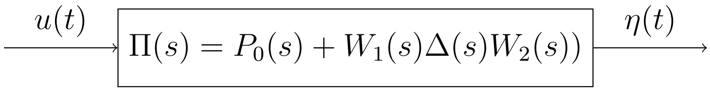

Figure 3: Input/Output Behaviour

In order to find the transfer function M, one rewrites the problem as depicted in Figure 4. It holds

$$U_\Delta(s) = W_2(s)U(s) \tag{1.24}$$

and

$$\begin{aligned}
U(s) &= C(s)(R(s) - Y(s)) \\
&= -C(s)\left(P_0(s)U(s) + W_1(s)Y_\Delta(s)\right) \\
&= -C(s)W_1(s)Y_\Delta(s) - C(s)P_0(s)U(s) \\
(\mathbb{I} + C(s)P_0(s))U(s) &= -C(s)W_1(s)Y_\Delta(s) \\
U(s) &= -(\mathbb{I} + C(s)P_0(s))^{-1}C(s)W_1(s)Y_\Delta(s),
\end{aligned} \tag{1.25}$$

from which it follows

$$\begin{aligned}
U_\Delta(s) &= W_2(s)U(s) \\
&= -\underbrace{W_2(s)(\mathbb{I} + C(s)P_0(s))^{-1}C(s)W_1(s)}_{M(s)}Y_\Delta(s).
\end{aligned} \tag{1.26}$$

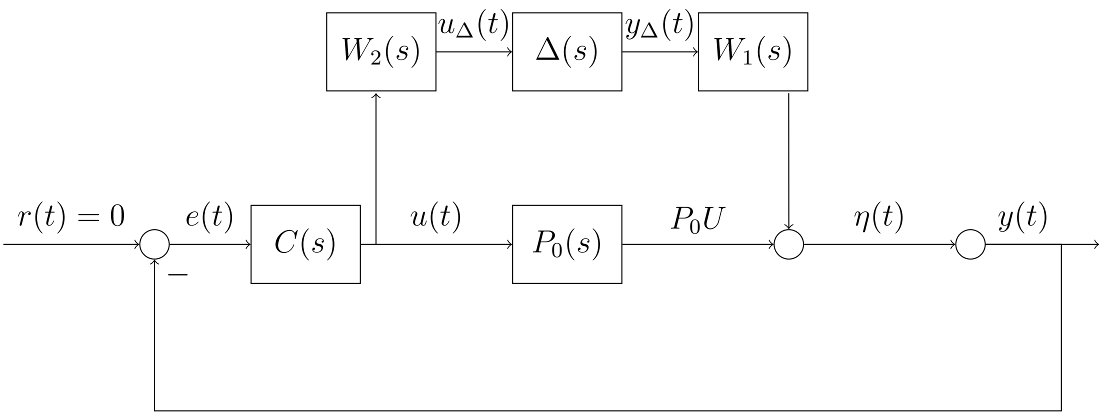

Figure 4: Additive Uncertainty Control System Loop

Example 2. (**Multiplicative Uncertainty**) You are given the system depicted in Figure 5 and the input output behaviour depicted in Figure 6, where

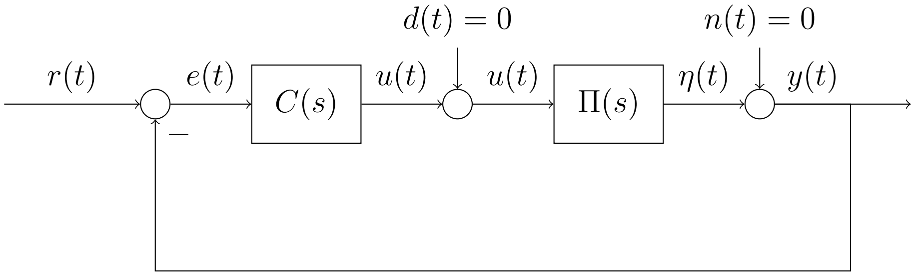

Figure 5: Multiplicative Uncertainty Control System Loop

$$\eta(s) = P_0(s)U(s) + P_0(s)W_1(s)\Delta(s)W_2(s)U(s). \tag{1.27}$$

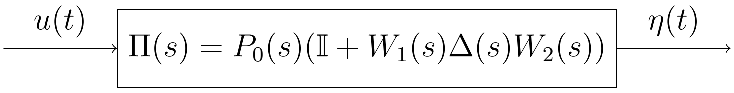

Figure 6: Input/Output Behaviour

In order to find the transfer function M, one rewrites the problem as depicted in Figure 7. It holds

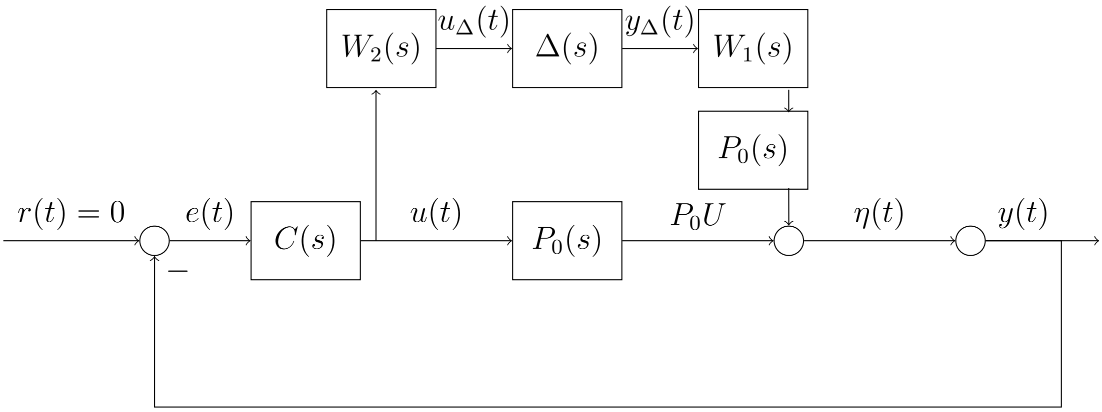

Figure 7: Multiplicative Uncertainty Control System Loop

$$U_\Delta(s) = W_2(s)U(s) \tag{1.28}$$

and

$$\begin{aligned}
U(s) &= C(s)(R(s) - Y(s)) \\
&= -C(s)\left(P_0(s)U(s) + P_0(s)W_1(s)Y_\Delta(s)\right) \\
&= -C(s)P_0(s)W_1(s)Y_\Delta(s) - C(s)P_0(s)U(s) \qquad (1.29) \\
(\mathbb{I} + C(s)P_0(s))U(s) &= -C(s)P_0(s)W_1(s)Y_\Delta(s) \\
U(s) &= -(\mathbb{I} + C(s)P_0(s))^{-1}C(s)P_0(s)W_1(s)Y_\Delta(s),
\end{aligned}$$

from which it follows

$$U_\Delta(s) = W_2(s)U(s) \\ = -\underbrace{W_2(s)(\mathbb{I}+C(s)P_0(s))^{-1}C(s)P_0(s)W_1(s)}_{M(s)}Y_\Delta(s). \quad (1.30)$$

1.5 Recasting Performance in a Robust Stability Problem

One can summarize robust stability conditions in bounding the infinity norm of selected functions. Assuming we want to attenuate noise on the output. Let $\|n\|_2$ and $W_n(s)$ a weighting function to rescale and shape the frequency content of the signal. Using the nominal performance approach, one uses the diagram depicted in Figure 8 and writes

$$Y(s) = (\mathbb{I}+P(s)C(s))^{-1}W_n(s)N(s) + \dots \qquad \Rightarrow \|S_0(s)W_n(s)\|_\infty \ll 1 \quad (1.31)$$

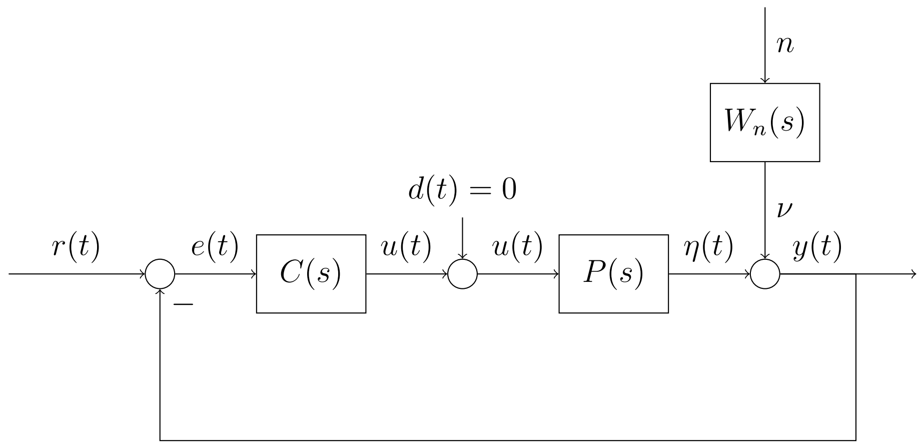

Figure 8: Robust Performance Problem

The same result is obtained considering the following loop and treating it as a robust stability problem, as in Figure 9. One can then identify the transfer function

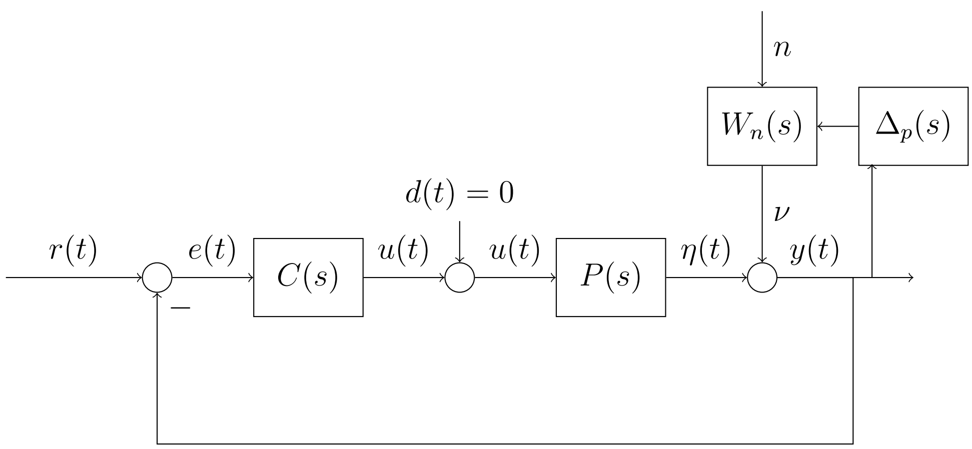

Figure 9: Robust Stability Problem

$$M(s) = \frac{Y}{N} = S_O(s)W_n(s) \Rightarrow \|M\|_\infty. \quad (1.32)$$

2 MIMO Robust Performance

Definition 2. Robust Performance: The effect of exogenous signals in presence of plant uncertainty can degrade performance to unacceptable levels before the system goes unstable. We need a *robust performance* test to evaluate the worst case effect of performance, given uncertainty.

Before having a closer look to the problem, let's recall what we have seen so far:

- **Nominal Stability (NS)**: The controller internally stabilizes the (nominal) plant.
- **Robust Stability (RS)** The controller internally stabilizes all plants parametrized through model uncertainty.
- **Nominal Performance (NP)**: is guaranteed by imposing constraints on the infinity norm of some sensitivity function, given nominal stability.
- **Robust Performance (RP)**: like NP, but for all plants within a given model set.

2.1 Problem Definition

Given a nominal plant $P_0(s)$ and a model uncertainty parametrization $\Delta(s)$, find conditions on the nominal closed loop system, such that

1. The controller $C(s)$ stabilizes the closed loop system for all $P \in \Pi$ with

$$\Pi = \{(\mathbb{I} + W_1(s)\Delta(s)W_2(s))P_0(s) : \; W_1(s), W_2(s), \Delta, \; \text{ rational, proper, stable}\} \tag{2.1}$$

2. A performance metric on some relevant transfer function is satisfied for all $P \in \Pi$.

2.2 M-Delta Approach: from RP to RS

A robust performance as the one depicted in Figure 10 (with $\|\Delta_r\|_\infty < 1$ and $\|\Delta_p\|_\infty < 1$), can be transformed in a robust stability problem as the one depicted in Figure 11.

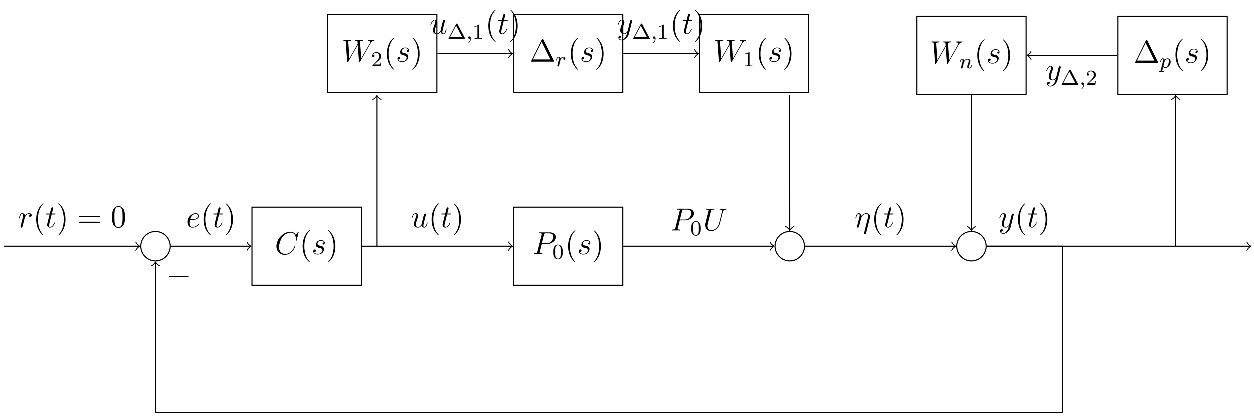

Figure 10: M-Delta Approach

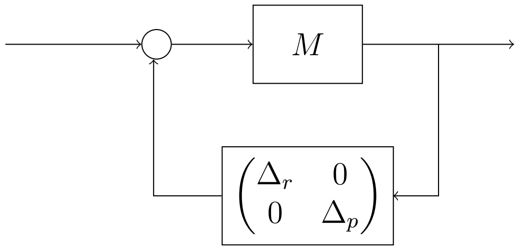

Figure 11: M-Delta Approach

In particular, one can show

$$\begin{pmatrix} U_{\Delta,1} \\ U_{\Delta,2} \end{pmatrix} = \underbrace{\begin{pmatrix} -W_2(s)T_O(s)W_1(s) & -W_2(s)T_O(s)W_n(s) \\ S_O(s)W_1(s) & S_O(s)W_n(s) \end{pmatrix}}_{M} \begin{pmatrix} Y_{\Delta,1} \\ Y_{\Delta,2}. \end{pmatrix} \tag{2.2}$$

Using the small gain theorem, a sufficient condition for robust performance is

$$\|M\|_\infty < 1. \tag{2.3}$$

2.3 Structured Singular Value

2.3.1 Definition

The approach we have seen in the previous section applies to a diagonal uncertainty. How can we handle any uncertainty $\Delta \in \tilde{\Delta}$?

Intuition: The Structured Singular Value is a generalization of the maximum singular value and the spectral radius. Through SSV, a *generalized small gain theorem* is obtained. This accounts for the structure of uncertainty.

Definition 3. Mu: Given Δ, find the smallest (in terms of $\bar{\sigma}(\Delta)$) Δ which makes

$$\det(\mathbb{I} - M(s)\Delta(s)) = 0. \tag{2.4}$$

Then:

$$\mu(M) = \frac{1}{\bar{\sigma}(\Delta)}. \tag{2.5}$$

If $\det(\mathbb{I} - M(s)\Delta(s)) \neq 0 \ \forall \Delta \in \tilde{\Delta}$, then $\mu(M) = 0$.

Theorem 2. (**SSV Robust Stability**) The $M - \Delta$ system is stable for all $\Delta \in \tilde{\Delta}$ with $\|\Delta\|_\infty < 1$ **if and only if**

$$\sup_\omega \mu(M(j\omega)) < 1. \tag{2.6}$$

Remark. Mu is a measure of the smallest perturbation that sends the system unstable.

2.3.2 Properties

(I)

$$\mu(M) \geq 0. \tag{2.7}$$

(II) It holds

$$\tilde{\Delta} = \{\Delta | \Delta \in \mathbb{C}^{p\times q},\ \text{full matrix}\} \Rightarrow \mu(M) = \bar{\sigma}(M). \tag{2.8}$$

(III) It holds

$$\tilde{\Delta} = \{\lambda \mathbb{I} | \lambda \in \mathbb{C}\} \Rightarrow \mu(M) = \rho(M) = |\lambda_{max}(M)|. \tag{2.9}$$

because $\inf \lambda^{-1}(M) = \rho(M)$.

(IV) It holds

$$\tilde{\Delta} = \{\text{diag}(\Delta_1, \ldots, \Delta_q) | \Delta_i \text{ is complex}\} \Rightarrow \rho(M) \leq \mu(M) \leq \bar{\sigma}(M) \tag{2.10}$$

(V) It holds

$$\tilde{\Delta} = \{\text{diag}(\Delta_1, \ldots, \Delta_q) | \Delta_i \text{ is complex}\} \Rightarrow \mu(M) = \mu(D^{-1}MD),\ \forall D \in \mathcal{D}, \tag{2.11}$$

where $\mathcal{D} = \{D = \text{diag}(d_1, \ldots, d_n) | d_i > 0\}$, $D\Delta = \Delta D$.

Remark.

- SSV provides a necessary and sufficient condition for RS (and thus RP), provided mu. This leads to a less conservative bound than the infinity norm condition.
- Computing mu is very tricky. There exist numerical approaches to refine upper and lower bounds for mu.
- The bounds are defined as

$$\mu(M) = \mu(D^{-1}MD) \leq \inf_{D \in \mathcal{D}} \bar{\sigma}(D^{-1}MD) \tag{2.12}$$

2.3.3 Robust Performance Noise Rejection: SISO Case

One can recover the structure defined in the previous chapters, but the SISO case offers some simplifications:

$$W_1(s) = 1, \quad S_0, T_0 \to S, T. \tag{2.13}$$

It follows

$$M = \begin{pmatrix} -W_2(s)T(s) & -W_2(s)T(s)W_n(s) \\ S(s) & S(s)W_n(s) \end{pmatrix}. \tag{2.14}$$

Let $D = \text{diag}(d_1, d_2)$ and $\alpha = \frac{d_2}{d_1}$ with $|d_1|, |d_2| < 1$. It must hold

$$\begin{aligned} \mu(M(j\omega)) &= \mu(\underbrace{D^{-1}M(j\omega)D}_{A(\alpha)}) \\ &\leq \inf_{|\alpha|>0} \lambda_{\max}^{\frac{1}{2}}(A^*(\alpha)A(\alpha)) \\ &< 1. \end{aligned} \tag{2.15}$$

We perform the analysis following specific steps:

1. We fix ω and find $A(\alpha)$ and $A^*(\alpha)A(\alpha)$.
   In the SISO case, the matrix Δ is diagonal and we define

$$\Delta = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \qquad d_1, d_2 < 1, \qquad \alpha = \frac{d_2}{d_1}. \tag{2.16}$$

   Since $\mu(M) = \mu(D^{-1}MD)$, let's set $D = \Delta$ and write

$$\begin{aligned} A(\alpha) = D^{-1}MD &= \begin{pmatrix} \frac{1}{d_1} & 0 \\ 0 & \frac{1}{d_2} \end{pmatrix} \begin{pmatrix} -W_2(s)T(s) & -W_2(s)T(s)W_n(s) \\ S(s) & S(s)W_n(s) \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{W_2(s)T(s)}{d_1} & -\frac{W_2(s)T(s)W_n(s)}{d_1} \\ \frac{S(s)}{d_2} & \frac{S(s)W_n(s)}{d_2} \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \\ &= \begin{pmatrix} -W_2(s)T(s) & -\alpha W_2(s)T(s)W_n(s) \\ \frac{1}{\alpha}S(s) & S(s)W_n(s) \end{pmatrix}. \end{aligned} \tag{2.17}$$

   Furthermore, it holds (by dropping the s in the notation for simplicity)

$$\begin{aligned} A^*(\alpha)A(\alpha) &= \begin{pmatrix} -\bar{W}_2\bar{T}(s) & \frac{1}{\alpha}\bar{S} \\ -\alpha\bar{W}_2\bar{T}\bar{W}_n & \bar{S}\bar{W}_n \end{pmatrix} \begin{pmatrix} -W_2T & -\alpha W_2TW_n \\ \frac{1}{\alpha}S & SW_n \end{pmatrix} \\ &= \begin{pmatrix} \|W_2\|_2^2\|T\|_2^2 + \frac{\|S\|_2^2}{\alpha^2} & \alpha\|W_2\|_2^2\|T\|_2^2W_n + \frac{1}{\alpha}\|S\|_2^2W_n \\ \alpha\|W_2\|_2^2\|T\|_2^2\bar{W}_n + \frac{1}{\alpha}\|S\|_2^2\bar{W}_n & \alpha^2\|W_2\|_2^2\|T\|_2^2\|W_n\|_2^2 + \|S\|_2^2\|W_n\|_2^2 \end{pmatrix}. \end{aligned} \tag{2.18}$$

2. We find $\lambda_{\max}(\alpha)$, i.e. the biggest λ from $\det(\underbrace{A^*(\alpha)A(\alpha) - \lambda\mathbb{I}}_{I}) = 0$. It holds

$$\begin{aligned} \det(I) &= \det\begin{pmatrix} \|W_2\|_2^2\|T\|_2^2 + \frac{\|S\|_2^2}{\alpha^2} - \lambda & \alpha\|W_2\|_2^2\|T\|_2^2W_n + \frac{1}{\alpha}\|S\|_2^2W_n \\ \alpha\|W_2\|_2^2\|T\|_2^2\bar{W}_n + \frac{1}{\alpha}\|S\|_2^2\bar{W}_n & \alpha^2\|W_2\|_2^2\|T\|_2^2\|W_n\|_2^2 + \|S\|_2^2\|W_n\|_2^2 - \lambda \end{pmatrix} \\ &= \alpha^2\|W_2\|_2^4\|T\|_2^4\|W_n\|_2^2 + \|W_2\|_2^2\|W_2\|_2^2\|T\|_2^2\|S\|_2^2\|W_n\|_2^2 - \lambda\|W_2\|_2^2\|T\|_2^2 \\ &+ \|W_2\|_2^2\|T\|_2^2\|S\|_2^2\|W_n\|_2^2 + \frac{\|S\|_2^4\|W_n\|_2^2}{\alpha^2} - \frac{\lambda}{\alpha^2}\|S\|_2^2 - \lambda\alpha^2\|W_2\|_2^2\|T\|_2^2\|W_n\|_2^2 \\ &- \lambda\|S\|_2^2\|W_n\|_2^2 + \lambda^2 - \alpha^2\|W_2\|_2^4\|T\|_2^4\|W_n\|_2^2 - 2\|W_2\|_2^2\|W_2\|_2^2\|T\|_2^2\|S\|_2^2\|W_n\|_2^2 \\ &- \frac{1}{\alpha^2}\|S\|_2^4\|W_n\|_2^2 \\ &= \lambda^2 - \lambda\left(\|W_2\|_2^2\|T\|_2^2 + \alpha^2\|W_2\|_2^2\|T\|_2^2\|W_n\|_2^2 + \|S\|_2^2\|W_n\|_2^2 + \frac{1}{\alpha^2}\|S\|_2^2\right), \end{aligned} \tag{2.19}$$

   from which it follows

$$\lambda_{\max}(A^*(\alpha)A(\alpha)) = \|W_2\|_2^2\|T\|_2^2 + \alpha^2\|W_2\|_2^2\|T\|_2^2\|W_n\|_2^2 + \|S\|_2^2\|W_n\|_2^2 + \frac{1}{\alpha^2}\|S\|_2^2. \tag{2.20}$$

3. We now want to minimize this with respect to α. It holds

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\lambda_{\max}\right) &= 0 \\ 2\alpha\|W_2\|_2^2\|T\|_2^2\|W_n\|_2^2 - 2\frac{1}{\alpha^3}\|S\|_2^2 &= 0 \\ 2\alpha^4\|W_2\|_2^2\|T\|_2^2\|W_n\|_2^2 - 2\|S\|_2^2 &= 0 \\ \alpha^2 &= \frac{\|S\|_2}{\|W_2\|_2\|T\|_2\|W_n\|_2}. \end{aligned} \tag{2.21}$$

4. By plugging this into the original equation one gets

$$\begin{aligned} \mu(M) = \lambda_{\max} &= \|W_2\|_2^2\|T\|_2^2 + \frac{\|S\|_2}{\|W_2\|_2\|T\|_2\|W_n\|_2}\|W_2\|_2^2\|T\|_2^2\|W_n\|_2^2 + \|S\|_2^2\|W_n\|_2^2 \\ &+ \frac{\|W_2\|_2\|T\|_2\|W_n\|_2}{\|S\|_2}\|S\|_2^2 \\ &= \|W_2\|_2^2\|T\|_2^2 + 2\|S\|_2\|W_2\|_2\|T\|_2\|W_n\|_2 + \|S\|_2^2\|W_n\|_2^2 \\ &= \left(\|S\|_2\|W_n\|_2 + \|W_2\|_2\|T\|_2\right)^2 \end{aligned} \tag{2.22}$$

5. The condition on μ implies

$$\|S\|_2\|W_n\|_2 + \|W_2\|_2\|T\|_2 < 1. \tag{2.23}$$

References

[1] Essentials of Robust Control, Kemin Zhou.

[2] Karl Johan Amstroem, Richard M. Murray *Feedback Systems for Scientists and Engineers*. Princeton University Press, Princeton and Oxford, 2009.

[3] Sigurd Skogestad, *Multivariate Feedback Control*. John Wiley and Sons, New York, 2001.