Lecture 7: MIMO Control Fundamentals

1 Decentralized Control

1.1 Idea and Definitions

As we have introduced in previous lectures, the generalization from SISO to MIMO systems adds crosscouplings and complexities to the control problem. In general, one can divide the control strategies into two philosophies:

1. Avoid the MIMO complexity by trying to use SISO controllers. How?
   - *Decentralized control:* every input signal is determined only by a feedback from **one** output.
   - *Pairing problem:* choose use of input-output paris for feedback.
   - *Decoupled control:* change of variables to facilitate input-output pairing.
2. Centralized multivariable control, optimizing some cost function, e.g.
   - Linear Quadratic Regulator (LQR, next episode).
   - H-infinity control

The first philosophy results in suboptimal solutions and requires less modeling effort. The second philosophy results in optimal results, but the modeling effort increases. Let's address the problem more spefifically:

Definition 1. Decentralized control: when the control systems consists of independent feedback controllers which interconnect a subset of the output measurements with a subset of manipulated inputs. These subsets should not be used by any other controller.

This represents a good strategy if the the MIMO system shows a **low degree of interaction** between inputs and outputs. How can we evaluate this property? Let's have a look at a generic 2×2 MIMO system with full rank and same number of inputs $u_i(t)$ and outputs $u_i(t)$. For the **coupled** system one can write

$$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} P_11(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{pmatrix} \begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix} = \begin{pmatrix} \sum_i P_{1i}(s)U_i(s) \\ \sum_i P_{2i}(s)U_i(s) \end{pmatrix}, \tag{1.1}$$

i.e. each input affects each output. For a **decoupled** system, one can e.g. write

$$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} P_{11}(s) & 0 \\ 0 & P_{22}(s) \end{pmatrix} \begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix} = \begin{pmatrix} P_{11}(s)U_1(s) \\ P_{22}(s)U_2(s) \end{pmatrix}, \tag{1.2}$$

i.e. the system behaves like a union of *non interacting* SISO systems. Furthermore, if one assumes a non-square system, for a general system $P(s) \in \mathbb{R}^{l \times n}$, one can meet the following two cases:

1. **Tall system**($l > m$): we have more outputs than inputs, i.e. not all outputs are affected by an input. Which outputs are best controlled with which inputs?

2. **Fat system**($l < m$): we have more inputs than outputs. How to distribute control action over the inputs?

In the next section, we will introduce a systematic way to address this kind of problems.

1.2 Relative-Gain Array (RGA)

As introduced in the previous section, if a system has a specific decoupled form, one can avoid complex control strategies and use independent SISO controllers. In some cases, this reasoning is actually the good one, but how can one distinguish when to use this approach?
The RGA-matrix tells us how the different subplants of a MIMO plant interact: this matrix is a good indicator of *how SISO* a system is.
This matrix can be generally calculated as

$$\text{RGA}(s) = P(s). \times P(s)^{-T} \tag{1.3}$$

where

$$P(s)^{-T} = (P(s)^T)^{-1}. \tag{1.4}$$

and $A. \times A$ represents the **element-wise, Shur** multiplication (`A.*A` in MATLAB). If $P(s)$ is not invertible (recall tall, fat and non inverbile square systems), one needs to generalize the inverse with the *Moore-Penrose* Inverse. Recalling $P(s) \in \mathbb{R}^{l\times m}$ one can define two cases:

- **Tall system**($l > m$): if $\text{rank}(P(s)) = m$,

$$A^\dagger = (A^*A)^{-1}A^*, \quad A^\dagger A = \mathbb{I}_m. \tag{1.5}$$

- **Fat system**($l < m$): if $\text{rank}(P(s)) = l$,

$$A^\dagger = A^*(AA^*)^{-1}, \quad AA^\dagger = \mathbb{I}_l. \tag{1.6}$$

In general, each element of the matrix gives us a special information:

$$[\text{RGA}]_{ab} = \frac{\text{gain from } u_a \text{ to } y_b \text{ with all other loops open}}{\text{gain from } u_a \text{ to } y_b \text{ with all other loops closed (perfect control)}}. \tag{1.7}$$

Remark. It's intuitive to notice, that if

$$[\text{RGA}]_{ab} \approx 1 \tag{1.8}$$

the numerator and the denominator are equal, i.e. SISO control is enough to bring u_a at y_b.

Remark. The theory behind the relative-gain array goes far beyond the aim of this course and one should be happy with the given examples. If however you are interested in this topic, you can have a look here.

Let's take the example of a 2×2 plant: in order to compute the first element $(1, 1)$ of the $\text{RGA}(s)$ we consider the system depicted in Figure 1. We close with a SISO controller $C_{22}(s)$ the loop from $y_2(t)$ to $u_2(t)$ and try to compute the transfer function from $u_1(t)$ to $y_1(t)$.

Everyone has his special way to decouple a MIMO system. I've always used this procedure: starting from the general equation in frequency domain

$$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{pmatrix} \cdot \begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix}, \tag{1.9}$$

one can read

$$\begin{aligned} Y_1(s) &= P_{11}(s) \cdot U_1(s) + P_{12}(s) \cdot U_2(s) \\ Y_2(s) &= P_{21}(s) \cdot U_1(s) + P_{22}(s) \cdot U_2(s). \end{aligned} \tag{1.10}$$

Since we want to relate $u_1(t)$ and $y_1(t)$ let's express $u_2(t)$ as something we know. Using the controller $C_{22}(s)$ we see

$$\begin{aligned} U_2(s) &= -C_{22}(s) \cdot Y_2(s) \\ &= -C_{22}(s) \cdot P_{21}(s) \cdot U_1(s) - C_{22}(s) \cdot P_{22}(s) \cdot U_2(s) \\ \Rightarrow U_2(s) &= \frac{-C_{22}(s) \cdot P_{21}(s) \cdot U_1(s)}{1 + P_{22}(s) \cdot C_{22}(s)}. \end{aligned} \tag{1.11}$$

With the general equation one can then write

$$\begin{aligned} Y_1(s) &= P_{11}(s) \cdot U_1(s) + P_{12}(s) \cdot U_2(s) \\ &= P_{11}(s) \cdot U_1(s) + P_{12}(s) \cdot \frac{-C_{22}(s) \cdot P_{21}(s) \cdot U_1(s)}{1 + P_{22}(s) \cdot C_{22}(s)} \\ &= \frac{P_{11}(s) \cdot (1 + P_{22}(s) \cdot C_{22}(s)) - P_{12}(s) \cdot C_{22}(s) \cdot P_{21}(s)}{1 + P_{22}(s) \cdot C_{22}(s)} \cdot U_1(s). \end{aligned} \tag{1.12}$$

We have found the general transfer function that relates $u_1(t)$ to $y_1(t)$. We now consider two extreme cases:

- We assume open loop conditions, i.e. *all other loops open*: $C_{22} \approx 0$. One gets

$$Y_1(s) = P_{11}(s) \cdot U_1(s). \tag{1.13}$$

- We assume high controller gains, i.e. *all other loops closed*: $P_{22}(s) \cdot C_{22}(s) \gg 1$. One gets

$$\begin{aligned} &\lim_{C_{22}(s)\to\infty} \frac{P_{11}(s) \cdot (1 + P_{22}(s) \cdot C_{22}(s)) - P_{12}(s) \cdot C_{22}(s) \cdot P_{21}(s)}{1 + P_{22}(s) \cdot C_{22}(s)} \\ &= \frac{P_{11}(s) \cdot P_{22}(s) - P_{12}(s) \cdot P_{21}(s)}{P_{22}(s)}. \end{aligned} \tag{1.14}$$

As stated before, the first element of the RGA is the division of these two. It holds

$$\begin{aligned} [\text{RGA}]_{11} &= \frac{P_{11}(s)}{\frac{P_{11}(s) \cdot P_{22}(s) - P_{12}(s) \cdot P_{21}(s)}{P_{22}(s)}} \\ &= \frac{P_{11}(s) \cdot P_{22}(s)}{P_{11}(s) \cdot P_{22}(s) - P_{12}(s) \cdot P_{21}(s)}. \end{aligned} \tag{1.15}$$

Remark. As you can see, the definition of the element of the RGA matrix does not depend on the chosen controller $C_{22}(s)$. This makes this method extremely powerful.

By repeating the procedure one can try to find $[\text{RGA}]_{22}$. In order to do that one has to close the loop from $y_1(\text{t})$ to $u_1(t)$: the result will be exactly the same:

$$[\text{RGA}]_{11} = [\text{RGA}]_{22}. \tag{1.16}$$

Let's go a step further. In order to compute the element $[\text{RGA}]_{21}$, one has to close the loop from $y_1(t)$ to $u_2(t)$ and find the transfer function from $u_1(t)$ to $y_2(t)$.

Remark. This could be a nice exercise to test your understanding!

With a similar procedure one gets

$$[\text{RGA}]_{21} = \frac{-P_{12}(s) \cdot P_{21}(s)}{P_{22}(s) \cdot P_{11}(s) - P_{21}(s) \cdot P_{12}(s)}. \tag{1.17}$$

and as before

$$[\text{RGA}]_{21} = [\text{RGA}]_{12}. \tag{1.18}$$

How can we now use this matrix, to know if SISO control would be enough? As already stated before, $[\text{RGA}]_{ab} \approx 1$ means SISO control is enough. Moreover, if the diagonal terms differ substantially from 1, the MIMO interactions (also called *cross couplings*) are too important and a SISO control is no more recommended.
If

$$\text{RGA} \approx \mathbb{I} \tag{1.19}$$

evaluated at the relevant frequencies of the system, i.e. at $\omega_c \pm$ **one decade**, one can ignore the cross couplings and can control the system with SISO tools *one loop at time.* If this is not the case, one has to design a MIMO controller. A bunch of observations could be useful by calculations:

1. Rows and columns of the RGA matrix add up to 1. This means one can write the matrix as

$$\begin{pmatrix} [\text{RGA}]_{11} & [\text{RGA}]_{12} \\ [\text{RGA}]_{21} & [\text{RGA}]_{22} \end{pmatrix} = \begin{pmatrix} [\text{RGA}]_{11} & 1 - [\text{RGA}]_{11} \\ 1 - [\text{RGA}]_{11} & [\text{RGA}]_{11} \end{pmatrix}. \tag{1.20}$$

   This allows to calculate just one element of the matrix.

2. If one looks at $\text{RGA}(s = 0)$ and the diagonal entries of the matrix are positive, SISO control is possible.

3. The RGA of a triangular matrix $P(s)$ is the identity matrix.

4. The RGA is invariant to scaling, i.e. for every diagonal matrix D_i it holds

$$[\text{RGA}](P(s)) = [\text{RGA}](D_1 \cdot P(s) \cdot D_2). \tag{1.21}$$

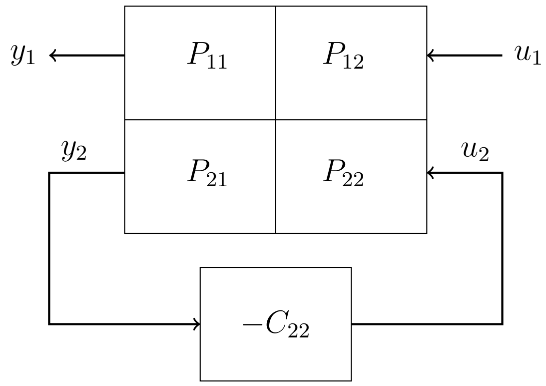

Figure 1: Derivation of the RGA-Matrix for the 2×2 case.

Example 1. For a MIMO system with two inputs and two outputs just the first element of the RGA matrix is given. This is a function of a system parameter p and is given as

$$[\mathrm{RGA}(s)]_{11} = \frac{1}{ps^2 + 2ps + 1}. \tag{1.22}$$

(a) Find the other elements of the RGA matrix.

(b) For which values of p is the system for all frequencies $\omega \in [0, \infty)$ controllable with two independent SISO control loops (*one loop at the time*)?

Now, you are given the following transfer function of another MIMO system:

$$P(s) = \begin{pmatrix} \frac{1}{s} & \frac{s+2}{s+1} \\ 1 & -\frac{1}{s+1} \end{pmatrix}. \tag{1.23}$$

(c) Find the RGA matrix of this MIMO system.

(c) Use the computed matrix to see if for frequencies in the range $\omega \in [3, 10]$ rad/s the system is controllable with two separate SISO controllers.

Solution.

(a) Using the theory we learned, it holds

$$[\mathrm{RGA}(s)]_{11} = [\mathrm{RGA}(s)]_{22} = \frac{1}{ps^2 + 2ps + 1} \tag{1.24}$$

and

$$\begin{aligned} [\mathrm{RGA}(s)]_{12} &= [\mathrm{RGA}(s)]_{21} \\ &= 1 - [\mathrm{RGA}(s)]_{11} \\ &= 1 - \frac{1}{ps^2 + 2ps + 1} \\ &= \frac{ps \cdot (s+2)}{ps^2 + 2ps + 1}. \end{aligned} \tag{1.25}$$

(b) In order to use two independend SISO control loops, the diagonal elements of the RGA matrix should be ≈ 1 and the anti diagonal elements should be ≈ 0. It's easy to see that this is the case for $p = 0$. In fact, if one sets $p = 0$ one gets

$$\mathrm{RGA}(s) = \mathbb{I}.$$

Hence, independently of the frequency one has, i.e. $\omega \in [0, \infty)$, the control problem can be solved with two independent SISO controllers.

(c) Using the learned theory, it holds

$$\begin{aligned} [\mathrm{RGA}(s)]_{11} &= [\mathrm{RGA}(s)]_{22} \\ &= \frac{P_{11}(s) \cdot P_{22}(s)}{P_{11}(s) \cdot P_{22}(s) - P_{12}(s) \cdot P_{21}(s)} \\ &= \frac{-\frac{1}{s\cdot(s+1)}}{-\frac{1}{s\cdot(s+1)} - \frac{s+2}{s+1}} \\ &= \frac{1}{1 + s \cdot (s+2)} \\ &= \frac{1}{s^2 + 2s + 1} \\ &= \frac{1}{(s+1)^2}. \end{aligned} \tag{1.26}$$

and

$$\begin{aligned} [\mathrm{RGA}(s)]_{12} &= [\mathrm{RGA}(s)]_{21} \\ &= 1 - [\mathrm{RGA}(s)]_{11} \\ &= 1 - \frac{1}{(s+1)^2} \\ &= \frac{s \cdot (s+2)}{(s+1)^2}. \end{aligned} \tag{1.27}$$

(d) In order to evaluate the RGA matrix in this range, we have to express it with it's frequency dependence, i.e. $s = j\omega$. For the magnitudes it holds

$$\begin{aligned} |[\text{RGA}(j\omega)]_{11}| &= |[\text{RGA}(j\omega)]_{22}| \\ &= \frac{1}{|j\omega+1|^2} \\ &= \frac{1}{1+\omega^2}. \end{aligned} \tag{1.28}$$

and

$$\begin{aligned} |[\text{RGA}(j\omega)]_{12}| &= |[\text{RGA}(j\omega)]_{21}| \\ &= \frac{1}{|j\omega+1|^2} \cdot |j\omega| \cdot |j\omega+2| \\ &= \frac{\omega \cdot \sqrt{4+\omega^2}}{1+\omega^2}. \end{aligned} \tag{1.29}$$

We can know insert the two limit values of the given range and get

$$\begin{aligned} |[\text{RGA}(j\cdot 3)]_{11}| &= |[\text{RGA}(j\cdot 3)]_{22}| \\ &= \frac{1}{10} \\ &= 0.10. \\ |[\text{RGA}(j\cdot 3)]_{12}| &= |[\text{RGA}(j\cdot 3)]_{21}| \\ &= \frac{3\cdot\sqrt{13}}{10} \\ &\approx 1.08. \end{aligned} \tag{1.30}$$

and

$$\begin{aligned} |[\text{RGA}(j\cdot 10)]_{11}| &= |[\text{RGA}(j\cdot 10)]_{22}| \\ &= \frac{1}{101} \\ &= 0.01. \\ |[\text{RGA}(j\cdot 10)]_{12}| &= |[\text{RGA}(j\cdot 10)]_{21}| \\ &= \frac{10\cdot\sqrt{104}}{101} \\ &\approx 1.01. \end{aligned} \tag{1.31}$$

In both cases the diagonal elements are close to 0 and the antidiagonal elements are close to 1. This means that the system is **diagonal dominant** and SISO control *one loop at time* is permitted. We just need to pay attention to what should be controlled: since the antidiagonal elements are close to 1, we need to use u_1 for y_2 and u_2 for y_1.

Example 2. Figure 2 shows a 2×2 MIMO system. Sadly, we don't know anything about the transfer functions $P_{ij}(s)$ but

$$P_{12}(s) = 0. \tag{1.32}$$

Your boss wants you to use a *one loop at the time* approach as you see in the picture.

(a) Why is your boss' suggestion correct?

(b) Just a reference r_i is affecting both outputs y_i, which one?

(c) Compute the transfer function $r_i \to y_j$ for $i \neq j$?

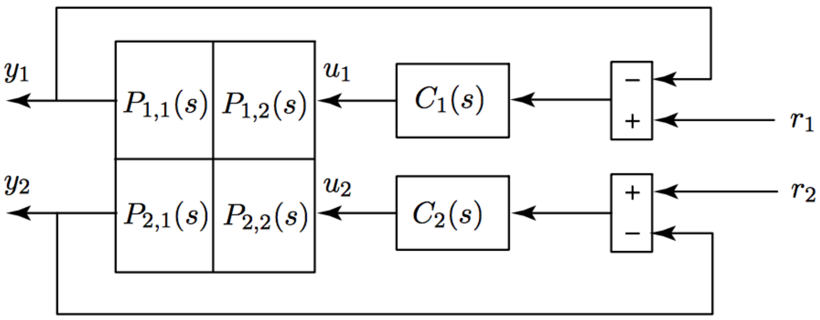

Figure 2: Structure of MIMO system.

Solution.

(a) To check if the suggestion is correct let's have a look at the RGA matrix: it holds

$$\begin{aligned}
[\text{RGA}]_{11} &= [\text{RGA}]_{22} \\
&= \frac{P_{11}(s)\cdot P_{22}(s)}{P_{11}(s)\cdot P_{22}(s) - P_{12}(s)\cdot P_{21}(s)} \\
&= 1. \\
[\text{RGA}]_{12} &= [\text{RGA}]_{21} \\
&= 1 - [\text{RGA}]_{11} \\
&= 0.
\end{aligned} \tag{1.33}$$

since $P_{12}(s) = 0$. This means that the RGA matrix is identical to the identity matrix, resulting in a perfect diagonal dominant system, which can be controlled with the *one loop at the time* approach.

(b) Let's analyze the signals from Figure 2. Since $P_{12}(s) = 0$, the output y_1 is not affected from u_2. Moreover, this means that the reference signal r_2, which influences u_2, cannot affect the output y_1. The only reference that acts on both y_1 and y_2 is r_1: directly through $C_1(s)$ on y_1 and with crosscouplings through $P_{21}(s)$ on y_2.

(c) As usual we set to 0 the reference values we don't analyze: here $r_2 = 0$. Starting from the general equation in frequency domain

$$\begin{aligned}
\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} &= \begin{pmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{pmatrix} \cdot \begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix} \\
&= \begin{pmatrix} P_{11}(s) & 0 \\ P_{21}(s) & P_{22}(s) \end{pmatrix} \cdot \begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix}.
\end{aligned} \tag{1.34}$$

one can read

$$\begin{aligned}
Y_1(s) &= P_{11}(s)\cdot U_1(s) \\
Y_2(s) &= P_{21}(s)\cdot U_1(s) + P_{22}(s)\cdot U_2(s).
\end{aligned} \tag{1.35}$$

Since we want to relate r_1 and y_2 let's express u_1 as something we know.
Using Figure 2 one gets

$$\begin{aligned}
R_1(s)\cdot C_1(s) &= U_1(s) + P_{11}(s)\cdot C_1(s)\cdot U_1(s) \\
U_1 &= \frac{R_1(s)\cdot C_1(s)}{1 + P_{11}(s)\cdot C_1(s)}.
\end{aligned} \tag{1.36}$$

Inserting this into the second equation one gets

$$Y_2(s) = P_{21}(s)\cdot \frac{R_1(s)\cdot C_1(s)}{1 + P_{11}(s)\cdot C_1(s)} + P_{22}(s)\cdot U_2(s). \tag{1.37}$$

One have to find an expression for $U_2(s)$. To do that, we look at the second loop in Figure 2 an see

$$\begin{aligned}
\underbrace{R_2(s)}_{=0}\cdot C_2(s) - Y_2(s)\cdot C_2(s) &= U_2(s) \\
U_2(s) &= -Y_2(s)\cdot C_2(s).
\end{aligned} \tag{1.38}$$

Inserting this into the second equation one gets

$$\begin{aligned}
Y_2(s) &= P_{21}(s) \cdot \frac{R_1(s) \cdot C_1(s)}{1 + P_{11}(s) \cdot C_1(s)} + P_{22}(s) \cdot U_2(s) \\
&= P_{21}(s) \cdot \frac{R_1(s) \cdot C_1(s)}{1 + P_{11}(s) \cdot C_1(s)} + P_{22}(s) \cdot (-Y_2(s) \cdot C_2(s)) \\
Y_2(s) \cdot (1 + P_{22}(s) \cdot C_2(s)) &= P_{21}(s) \cdot \frac{R_1(s) \cdot C_1(s)}{1 + P_{11}(s) \cdot C_1(s)} \\
Y_2(s) &= \underbrace{\frac{P_{21}(s) \cdot C_1(s)}{(1 + P_{11}(s) \cdot C_1(s)) \cdot (1 + P_{22}(s) \cdot C_2(s))}}_{F(s)} \cdot R_1(s).
\end{aligned} \tag{1.39}$$

where $F(s)$ is the transfer function we wanted.

Example 3. Figure 3 shows the structure of a MIMO system, composed of three subsystems $P_1(s), P_2(s)$ and $P_3(s)$. It has inputs u_1 and u_2 and outputs y_1 and y_2. The three

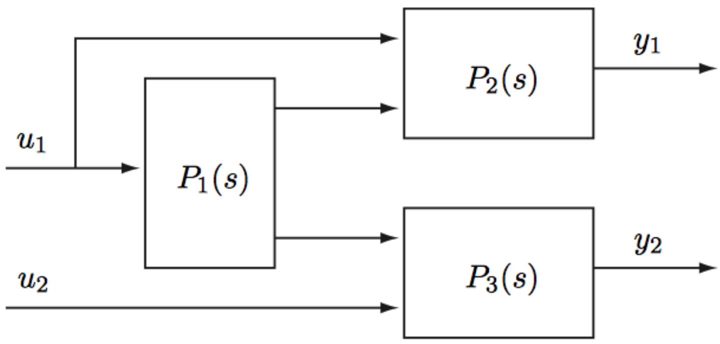

Figure 3: Structure of MIMO system.

subsystems are given as

$$P_1(s) = \begin{pmatrix} \frac{s-5}{s+3} \\ \frac{1}{s+4} \end{pmatrix}, \quad P_2(s) = \begin{pmatrix} \frac{1}{s+3} & \frac{s+4}{s-5} \end{pmatrix}, \quad P_3(s) = \begin{pmatrix} \frac{s+2}{s+5} & \frac{1}{s+1} \end{pmatrix}. \tag{1.40}$$

Compute the transfer function of the whole system.

Solution. One should think with matrix dimensions here. Let's redefine the subsystem's matrices more generally:

$$P_1(s) = \begin{pmatrix} P_1^{11} \\ P_1^{21} \end{pmatrix}, \quad P_2(s) = \begin{pmatrix} P_2^{11} & P_2^{12} \end{pmatrix}, \quad P_3(s) = \begin{pmatrix} P_3^{11} & P_3^{12} \end{pmatrix} \tag{1.41}$$

Together with the structure of the system one gets

$$\begin{aligned} Y_1 &= P_2^{11} \cdot U_1 + P_2^{12} \cdot P_1^{11} \cdot U_1, \\ Y_2 &= P_3^{11} \cdot P_1^{21} \cdot U_1 + P_3^{12} \cdot U_2. \end{aligned} \tag{1.42}$$

This can be written in the general matrix for the transfer function:

$$\begin{aligned} P(s) &= \begin{pmatrix} P_2^{11} + P_2^{12} \cdot P_1^{11} & 0 \\ P_3^{11} \cdot P_1^{21} & P_3^{12} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{s+3} + \frac{s+4}{s-5} \cdot \frac{s-5}{s+3} & 0 \\ \frac{s+2}{s+5} \cdot \frac{1}{s+4} & \frac{1}{s+1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{s+5}{s+3} & 0 \\ \frac{s+2}{(s+5)\cdot(s+4)} & \frac{1}{s+1} \end{pmatrix}. \end{aligned} \tag{1.43}$$

Example 4. Figure 4 shows the structure of a MIMO system, composed of two subsystems $P_1(s), P_2(s)$. It has inputs u_1 and u_2 and outputs y_1 and y_2. The subsystem $P_1(s)$

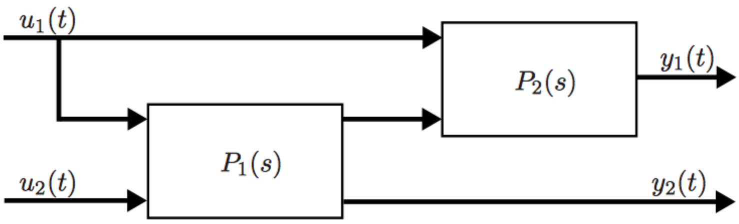

Figure 4: Structure of MIMO system.

is given with its state space description:

$$A_1 = \begin{pmatrix} -3 & 0 \\ 2 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} \quad D_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{1.44}$$

and the subsystem $P_2(s)$ is given as

$$P_2(s) = \begin{pmatrix} \frac{1}{s-2} & \frac{s-1}{(s+4)\cdot(s-2)} \end{pmatrix}. \tag{1.45}$$

Compute the transfer function of the whole system.

Solution. First of all, we compute the transfer function in frequency domain of the first subsystem $P_1(s)$. It holds

$$\begin{aligned}
P_1(s) &= C_1 \cdot (s \cdot \mathbb{I} - A_1)^{-1} \cdot B_1 + D_1 \\
&= \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} s+3 & 0 \\ -2 & s-1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} \cdot \frac{1}{(s+3)\cdot(s-1)} \cdot \begin{pmatrix} s-1 & 0 \\ 2 & s+3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \frac{1}{(s+3)\cdot(s-1)} \cdot \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} s-1 & 0 \\ 2s+8 & s+3 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \frac{1}{(s+3)\cdot(s-1)} \cdot \begin{pmatrix} 4s+16 & 2s+6 \\ 5s+5 & s+3 \end{pmatrix} \\
&= \begin{pmatrix} \frac{4s+16}{(s+3)\cdot(s-1)} & \frac{2}{s-1} \\ \frac{5s+5}{(s+3)\cdot(s-1)} & \frac{1}{s-1} \end{pmatrix}.
\end{aligned} \tag{1.46}$$

One should think with matrix dimensions here. Let's redefine the subsystem's matrices more generally:

$$P_1(s) = \begin{pmatrix} P_1^{11} & P_1^{12} \\ P_1^{21} & P_1^{22} \end{pmatrix}, \quad P_2(s) = \begin{pmatrix} P_2^{11} & P_2^{12} \end{pmatrix}. \tag{1.47}$$

Together with the structure of the system one gets

$$\begin{aligned}
Y_1 &= P_2^{11} \cdot U_1 + P_1^{11} \cdot P_2^{12} \cdot U_1 + P_1^{12} \cdot P_2^{12} \cdot U_2, \\
Y_2 &= P_1^{21} \cdot U_1 + P_1^{22} \cdot U_2.
\end{aligned} \tag{1.48}$$

This can be written in the general matrix for the transfer function:

$$\begin{aligned}
P(s) &= \begin{pmatrix} P_2^{11} + P_1^{11} \cdot P_2^{12} & P_1^{12} \cdot P_2^{12} \\ P_1^{21} & P_1^{22} \end{pmatrix} \\
&= \dots \\
&= \begin{pmatrix} \frac{s+7}{(s+3)\cdot(s-2)} & \frac{s+1}{(s+4)(s-2)} \\ \frac{5s+5}{(s+3)\cdot(s-1)} & \frac{1}{s-1} \end{pmatrix}.
\end{aligned} \tag{1.49}$$

Example 5. The system in Figure 5 can be well controlled with two separate SISO controllers.

- ☐ True.
- ☐ False.

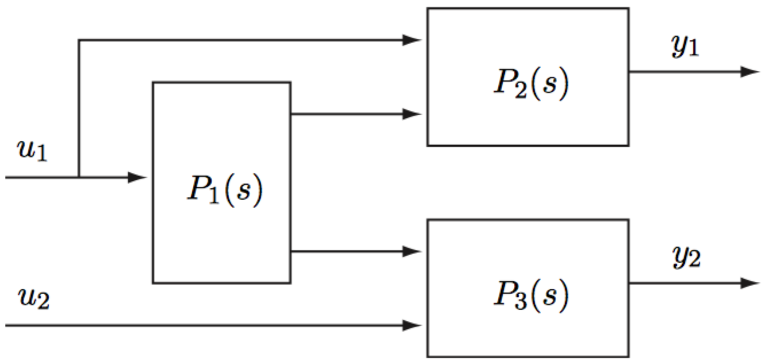

Figure 5: Structure of MIMO system.

Solution.

- ☑ True.
- ☐ False.

Explanation:

One can observe that the input u_2 affects only the output y_2. This means that the transfer function matrix has a **triangular** form and hence, that the RGA matrix is identical to the identity matrix: this means that we can reach good control with two separate SISO controllers.

1.3 Q Parametrization

Recalling the standard control Loop repicted in Figure 6, one can write

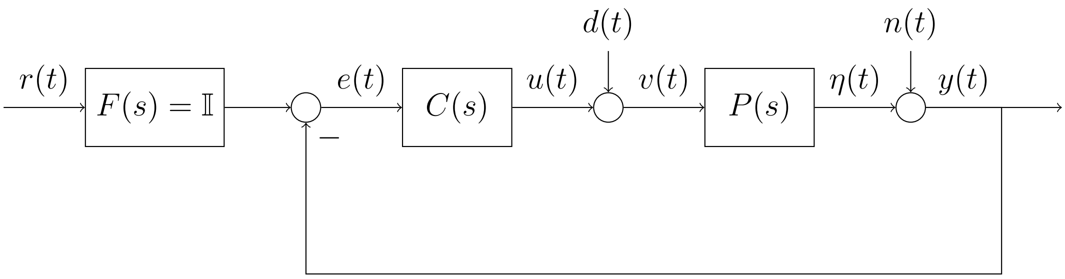

Figure 6: Standard feedback control system structure.

$$\begin{aligned}\begin{pmatrix} Y(s) \\ U(s) \end{pmatrix} &= \begin{pmatrix} (\mathbb{I}+P(s)C(s))^{-1}P(s)C(s) & (\mathbb{I}+P(s)C(s))^{-1}P(s) \\ (\mathbb{I}-C(s)P(s))^{-1}C(s) & -(\mathbb{I}+C(s)P(s))^{-1}C(s)P(s) \end{pmatrix} \begin{pmatrix} R(s) \\ D(s) \end{pmatrix} \\ &= \begin{pmatrix} T_O(s) & S_O(s)P(s) \\ S_I(s)C(s) & -T_I(s) \end{pmatrix} \begin{pmatrix} R(s) \\ D(s) \end{pmatrix}.\end{aligned} \tag{1.50}$$

In order for the system to be internally stable, $T_O(s), S_O(s)P(s), S_I(s)C(s)$ and $T_I(s)$ must be stable. Ideally, we would like to translate this properties on direct consequences for $C(s)$. However, relations are not linear and it is not obvious how to find direct translations. One defined

$$Q(s) = C(s)(\mathbb{I} + P(s)C(s))^{-1}. \tag{1.51}$$

Then, one can write

$$\begin{aligned}(\mathbb{I} + P(s)C(s))^{-1} &= (\mathbb{I} + P(s)C(s) - P(s)C(s))(\mathbb{I} + P(s)C(s))^{-1} \\ &= (\mathbb{I} + P(s)C(s))(\mathbb{I} + P(s)C(s))^{-1} - P(s)C(s)(\mathbb{I} + P(s)C(s))^{-1} \\ &= \mathbb{I} - P(s)Q(s),\end{aligned} \tag{1.52}$$

and

$$\begin{aligned}(\mathbb{I} + C(s)P(s))^{-1} &= (\mathbb{I} + C(s)P(s) - C(s)P(s))(\mathbb{I} + C(s)P(s))^{-1} \\ &= (\mathbb{I} + C(s)P(s))(\mathbb{I} + C(s)P(s))^{-1} - C(s)P(s)(\mathbb{I} + C(s)P(s))^{-1} \\ &= \mathbb{I} - Q(s)P(s).\end{aligned} \tag{1.53}$$

It follows

$$\begin{aligned}T_O(s) &= (\mathbb{I} + P(s)C(s))^{-1}P(s)C(s) \\ &= P(s)C(s)(\mathbb{I} + P(s)C(s)) \\ &= P(s)Q(s). \\ S_O(s)P(s) &= (\mathbb{I} - P(s)Q(s))P(s) \\ S_I(s)C(s) &= Q(s) \\ T_I(s) &= Q(s)P(s).\end{aligned} \tag{1.54}$$

Theorem 1. Q internal stability: Let $P(s)$ be a stable plant of a negative feedback system, then the closed loop system is internally stable if and only if $Q(s)$ is stable.

This makes the tuning of the controller extremely easier: the sensitivity functions depend **linearly** on Q. Moreover, it holds:

- Supposing that the plant is stable: $Q(s)$ can be any transfer matrix that satisfies the definition.
- If only proper controllers are taken into account, then $Q(s)$ must be proper.
- Finding a $Q(s)$ is equivalent to finding the controller $C(s)$.
- As long as $Q(s)$ is stable, it can vary freely and internal stability will be guaranteed. Even if $Q(s)$ maps to an unstable controller $C(s)$.
- Starting from the formula for $Q(s)$, one can write

$$C(s) = (\mathbb{I} - Q(s)P(s))^{-1}Q(s) = Q(s)(\mathbb{I} - P(s)Q(s))^{-1}. \tag{1.55}$$

References

[1] Essentials of Robust Control, Kemin Zhou.

[2] Karl Johan Amstroem, Richard M. Murray *Feedback Systems for Scientists and Engineers*. Princeton University Press, Princeton and Oxford, 2009.

[3] Sigurd Skogestad, *Multivariate Feedback Control*. John Wiley and Sons, New York, 2001.