Lecture 8: IMC and State Feedback

1 Internal Model Control (IMC)

1.1 Recap Q Parametrization

Recalling the standard control Loop repicted in Figure 1, one can write

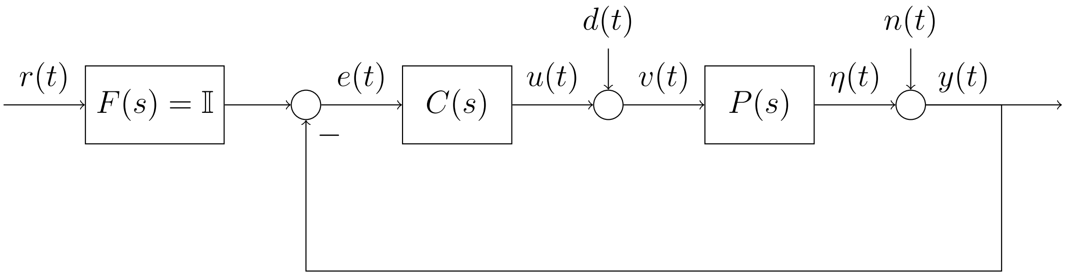

Figure 1: Standard feedback control system structure.

$$
\begin{aligned}
\begin{pmatrix} Y(s) \\ U(s) \end{pmatrix} &= \begin{pmatrix} (\mathbb{I}+P(s)C(s))^{-1}P(s)C(s) & (\mathbb{I}+P(s)C(s))^{-1}P(s) \\ (\mathbb{I}-C(s)P(s))^{-1}C(s) & -(\mathbb{I}+C(s)P(s))^{-1}C(s)P(s) \end{pmatrix} \begin{pmatrix} R(s) \\ D(s) \end{pmatrix} \\
&= \begin{pmatrix} T_O(s) & S_O(s)P(s) \\ S_I(s)C(s) & -T_I(s) \end{pmatrix} \begin{pmatrix} R(s) \\ D(s) \end{pmatrix}.
\end{aligned} \tag{1.1}
$$

In order for the system to be internally stable, $T_O(s), S_O(s)P(s), S_I(s)C(s)$ and $T_I(s)$ must be stable. Ideally, we would like to translate this properties on direct consequences for $C(s)$. However, relations are not linear and it is not obvious how to find direct translations. One defined

$$
Q(s) = C(s)(\mathbb{I}+P(s)C(s))^{-1}. \tag{1.2}
$$

Then, one can write

$$
\begin{aligned}
(\mathbb{I}+P(s)C(s))^{-1} &= (\mathbb{I}+P(s)C(s)-P(s)C(s))(\mathbb{I}+P(s)C(s))^{-1} \\
&= (\mathbb{I}+P(s)C(s))(\mathbb{I}+P(s)C(s))^{-1} - P(s)C(s)(\mathbb{I}+P(s)C(s))^{-1} \\
&= \mathbb{I} - P(s)Q(s),
\end{aligned} \tag{1.3}
$$

and

$$
\begin{aligned}
(\mathbb{I}+C(s)P(s))^{-1} &= (\mathbb{I}+C(s)P(s)-C(s)P(s))(\mathbb{I}+C(s)P(s))^{-1} \\
&= (\mathbb{I}+C(s)P(s))(\mathbb{I}+C(s)P(s))^{-1} - C(s)P(s)(\mathbb{I}+C(s)P(s))^{-1} \\
&= \mathbb{I} - Q(s)P(s).
\end{aligned} \tag{1.4}
$$

It follows

$$
\begin{aligned}
T_O(s) &= (\mathbb{I} + P(s)C(s))^{-1} P(s)C(s) \\
&= P(s)C(s)(\mathbb{I} + P(s)C(s)) \\
&= P(s)Q(s). \\
S_O(s)P(s) &= (\mathbb{I} - P(s)Q(s))P(s) \\
S_I(s)C(s) &= Q(s) \\
T_I(s) &= Q(s)P(s).
\end{aligned}
\tag{1.5}
$$

Theorem 1. Q internal stability: Let $P(s)$ be a stable plant of a negative feedback system, then the closed loop system is internally stable if and only if $Q(s)$ is stable.

This makes the tuning of the controller extremely easier: the sensitivity functions depend **linearly** on Q. Moreover, it holds:

- Supposing that the plant is stable: $Q(s)$ can be any transfer matrix that satisfies the definition.
- If only proper controllers are taken into account, then $Q(s)$ must be proper.
- Finding a $Q(s)$ is equivalent to finding the controller $C(s)$.
- As long as $Q(s)$ is stable, it can vary freely and internal stability will be guaranteed. Even if $Q(s)$ maps to an unstable controller $C(s)$.
- Starting from the formula for $Q(s)$, one can write

$$
C(s) = (\mathbb{I} - Q(s)P(s))^{-1} Q(s) = Q(s)(\mathbb{I} - P(s)Q(s))^{-1}. \tag{1.6}
$$

1.2 Principle

Principle: *Accurate control can be achieved only if the control system encapsulates some representation of the controlled process.*
Approach: We feedback only the mismatch between the model prediction and the actual measured output, i.e. the uncertainty in the control loop.

1.2.1 Connection with Q Parametrization

The control system structure for IMC is depicted in Figure 2. $P(s)$ denotes the plant of the system and $P_0(s)$ the plant model. The measurement $y(t)$ is corrupted by a measurement noise $n(t)$. The signal $y_0(t)$ represents the predicted output. The signal i represents the signal mismatch between the measured and the predicted outputs. The controller $Q(s)$ (please refer to the previous section for its form) produces the input $u(t)$. Relating this structure with the classic one, one can write:

$$
C(s) = Q(s) \left(I(s) - P_0(s)Q(s) \right)^{-1}. \tag{1.7}
$$

Remark. Note that the controller $C(s)$ can be defined with the orange region in Figure 2.

1.2.2 Analysis

By trying to relate the output signal $y(t)$ to the other signals available in the loop, one can write

$$
\begin{aligned}
Y(s) &= N(s) + \eta(s) \\
&= N(s) + P(s)V(s) \\
&= N(s) + P(s)\left(D(s) + U(s)\right) \\
&= N(s) + P(s)D(s) + P(s)Q(s)E(s) \\
&= N(s) + P(s)D(s) + P(s)Q(s)\left(R(s) - I(s)\right) \\
&= N(s) + P(s)D(s) + P(s)Q(s)R(s) - P(s)Q(s)\left(Y(s) - Y_0(s)\right) \\
&= N(s) + P(s)D(s) + P(s)Q(s)R(s) - P(s)Q(s)Y(s) + P(s)Q(s)P_0(s)U(s) \\
&= N(s) + P(s)D(s) + P(s)Q(s)R(s) - P(s)Q(s)\left(N(s) + P(s)V(s)\right) + P(s)Q(s)P_0(s)U(s) \\
&= \underbrace{(\mathbb{I} - PQ)}_{S_O(s)} N + \underbrace{P\,(\mathbb{I} - QP)}_{PS_I(s)} D + \underbrace{PQ}_{T_O(s)} R + PQ\,(P - P_0)\,U,
\end{aligned}
\tag{1.8}
$$

where in the last line we dropped the s dependency for simplicity reasons. In the case where $P(s) = P_0(s)$ and $Q(s) = P^{-1}(s)$, one gets

$$
Y(s) = R(s). \tag{1.9}
$$

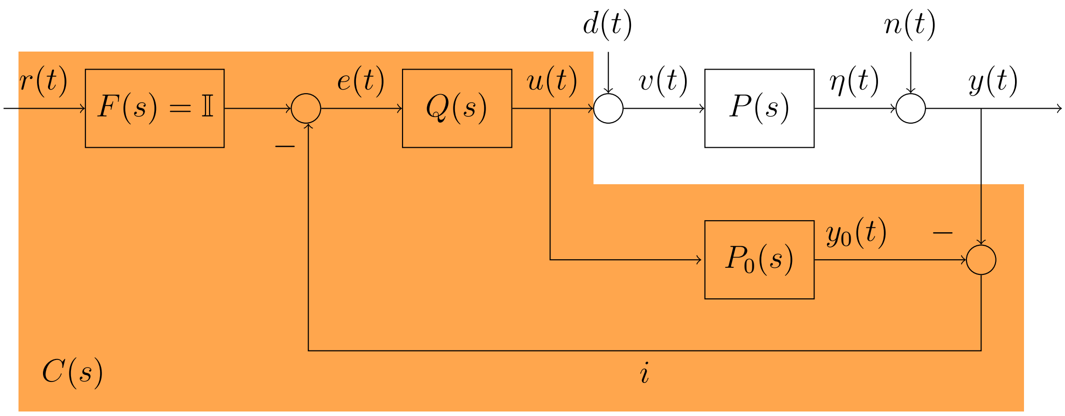

Figure 2: Internal Model Control System Structure.

1.3 Example: Predictive Control

1.3.1 Why predictive control

If a SISO system has *substantial* delays, it is very difficult to control it with a normal PID controller. The I part of the controller causes *impatience*, that is, integrates over time. As a practical example think of taking a shower in the morning: one let the water flow and of course this hasn't the desired temperature. For this reason one chooses warmer water by turning the temperature controller; the water becomes too hot and so one turns it on the other side to have colder water and so on, resulting in a non optimal strategy.

Moreover, the D part of the controller is practically useless[1]. What does the expression *substantial* delays mean? As indicative range one can say that it is worth using predictive control if

$$\frac{T}{T+\tau} > 0.3, \tag{1.10}$$

where T is the delay and τ is the time constant of the system. Other prerequisites are

- The plant must be asymptotically stable.
- A good model of the plant should be available.

1.3.2 The Smith Predictor

One can see the two *equivalent* structures of the Smith Predictor in Figure 3 and Figure 4.

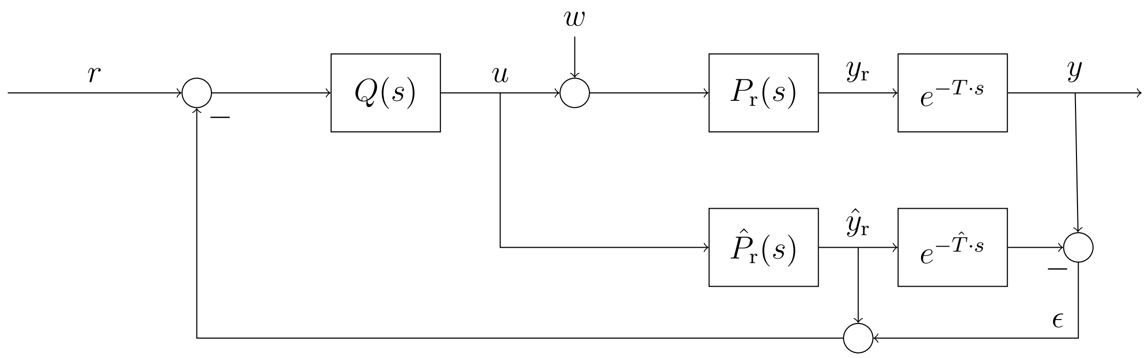

Figure 3: Structure of the Smith predictor.

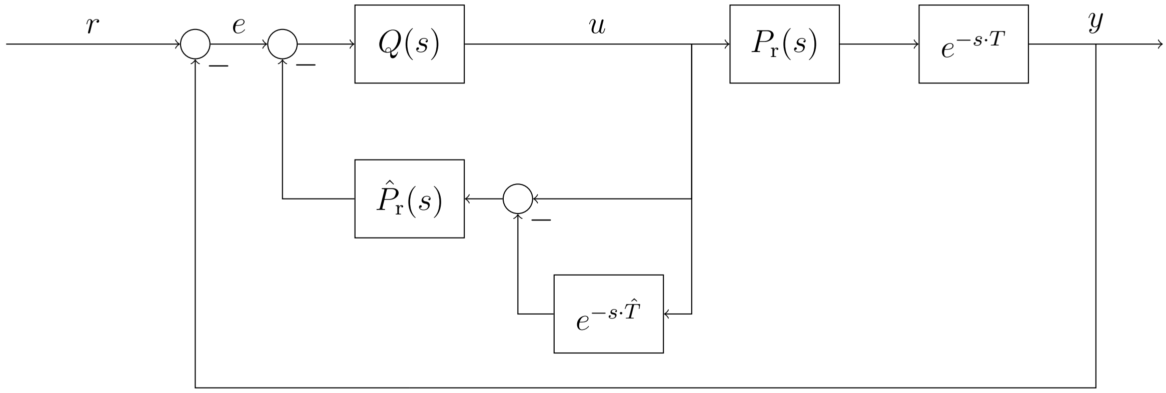

Figure 4: Structure of the Smith predictor.

If the system has big delays, one can assume that it is possible to write the delay element and the nondelayed plant as a product in the frequency domain: that's what is done in the upper right side of Figure 3. This means that the transfer function $u(t) \to y(t)$ can be written as

$$P(s) = P_r(s) \cdot e^{-sT}. \tag{1.11}$$

[1]Taking the derivative of a delay element doesn't help to control it

Main Idea:

As long as we have no disturbance $d(t)$ (i.e. $d(t) = 0$) and our model is good enough (this means $P_r(s) = \hat{P}_r(s),\ T = \hat{T})$[2], we can model a non delayed plant and get the non delayed output $\hat{y}_r(t)$ (one can see this on the lower right side of Figure 3). The feedback signal results from the sum of $\hat{y}_r(t)$ and the correction signal ϵ.

1.3.3 Analysis

The controller of the system is the transfer function $e(t) \to u(t)$, which can be computed as

$$\begin{aligned} U(s) &= Q(s)\left(E(s) - \hat{P}_{\mathrm{r}}(s)\left(U(s) - U(s)e^{-s\hat{T}}\right)\right) \\ &= Q(s)E(s) - Q(s)\hat{P}_{\mathrm{r}}(s)\left(1 - e^{-s\hat{T}}\right)U(s), \end{aligned} \tag{1.12}$$

from which it follows

$$\begin{aligned} C(s) &= \frac{U(s)}{E(s)} \\ &= \frac{Q(s)}{1 + Q(s)\hat{P}_{\mathrm{r}}(s)\left(1 - e^{-s\hat{T}}\right)}. \end{aligned} \tag{1.13}$$

This means that the loop gain transfer function is

$$\begin{aligned} L(s) &= P(s) \cdot C(s) \\ &= \frac{Q(s)P_{\mathrm{r}}(s)e^{-sT}}{1 + Q(s)\hat{P}_{\mathrm{r}}(s)\left(1 - e^{-s\hat{T}}\right)}. \end{aligned} \tag{1.14}$$

If one assumes as stated, that the model is good enough s.t. $P_r(s) = \hat{P}_r(s),\ T = \hat{T}$, one gets

$$\begin{aligned} T(s) &= \frac{L(s)}{1 + L(s)} \\ &= \frac{\frac{Q(s)\cdot P_{\mathrm{r}}(s)\cdot e^{-s\cdot T}}{1+Q(s)\cdot P_{\mathrm{r}}(s)\cdot(1-e^{-s\cdot T})}}{1 + \frac{Q(s)\cdot P_{\mathrm{r}}(s)\cdot e^{-s\cdot T}}{1+Q(s)\cdot P_{\mathrm{r}}(s)\cdot(1-e^{-s\cdot T})}} \\ &= \frac{Q(s)\cdot P_{\mathrm{r}}(s)\cdot e^{-s\cdot T}}{1 + Q(s)\cdot P_{\mathrm{r}}(s)\cdot(1 - e^{-s\cdot T}) + Q(s)\cdot P_{\mathrm{r}}(s)\cdot e^{-s\cdot T}} \\ &= \frac{Q(s)\cdot P_{\mathrm{r}}(s)}{1 + Q(s)\cdot P_{\mathrm{r}}(s)} \cdot e^{-s\cdot T} \\ &= T_{\mathrm{ref}}(s)\cdot e^{-s\cdot T}. \end{aligned} \tag{1.15}$$

[2] We use $.\hat{.}.$ to identify the parameters of the model

Remark.

- This result is very important: we have shown that the delay cannot be completely eliminated and that every tranfer function (here $T(s)$ but also $S(s)$) will have the same delay as the plant $P(s)$.
- Advantages of the prediction are:
  - Very fast.
  - Same Robustness.
- Disadvantages of the prediction are:
  - Very difficult to implement.
  - Very difficult to analyze.
  - Problems if there are model errors.

2 State Feedback

Motivation: Each control strategy we analyzed so far, was based on output feedback. In fact, the main analysis has been based on the fact that system outputs are available through measurements. Imagine now to have the *states* of the system available. Would the control problem benefit from this new information? Intuitively, outputs are nothing else than a linear combination of the states (one can always write this through the dynamics of the system), hence contain less *information*.

2.1 Concept

The big difference to what we have seen so far, is that we are looking at a continuous time control system, which operates in time domain and no more in frequency domain. The basic state feedback control structure is depicted in Figure 5. The basic idea is: we

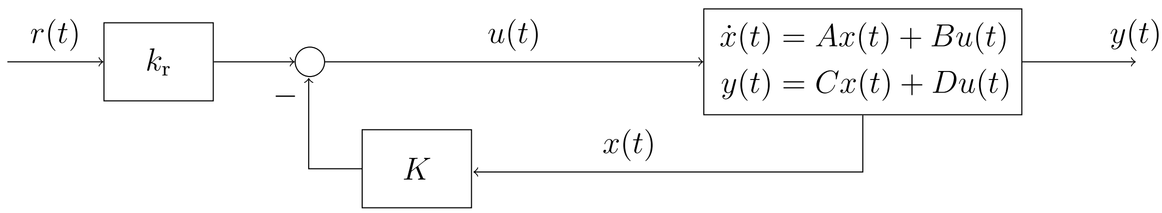

Figure 5: Basic State Feedback Control Structure.

have the dynamics in the loop, with the input $u(t)$ and the output $y(t)$. We negatively feedback the state $x(t)$ with a controller K and add to a reference (or a multiple of it, k_r). In words, we try to keep the state where we want it to be. Assuming for simplicity $D = 0$ (the same analysis can be performed), one gets

$$u(t) = k_r r(t) - Kx(t), \tag{2.1}$$

from which it follows

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ &= Ax(t) + Bk_{\mathrm{r}}r(t) - BKx(t)\\ &= \underbrace{(A - BK)}_{A_{\mathrm{cl}}} x(t) + Bk_{\mathrm{r}}r(t).\end{aligned} \tag{2.2}$$

We get a new closed loop matrix A_{cl}, i.e., state feedback **affects** the poles of the closed loop transfer function.

2.2 Reachability

A key property of a control system is reachability. Which set of points in the state space of the system can be reached through the choice of a specific control input? Reachability plays a central role in deciding if state feedback is a good strategy for the control of a specific dynamic system. Let's assume that a dynamic system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t) + Bu(t). \tag{2.3}$$

is given, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$. We deine the reachable set as the set of all points x_{f} such that there exists an input $u(t)$ with $0 \leq t \leq T$ that steers the system from $x(0) = x_0$ to $x(T) = x_{\mathrm{f}}$.

Definition 1. A linear system is **reachable** if for any $x_0, x_{\mathrm{f}} \in \mathbb{R}^n$, there exists a $T > 0$ and $u(t) : [0, T] \to \mathbb{R}$ such that the corresponding solution satisfies $x(0) = x_0$ and $x(T) = x_{\mathrm{f}} \in \mathbb{R}^n$.

Reachability test: A system (A, B) is reachable if and only if $\mathrm{rank}(\mathcal{R}) = n$, where $x \in \mathbb{R}^n$ and

$$\mathcal{R} = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}. \tag{2.4}$$

2.2.1 Reachable Canonical Form

Given a transfer function matrix

$$\underbrace{P(s)}_{\in\mathbb{R}^{l\times m}} = \frac{B_0 + B_1 s + \ldots + B_{n-1}s^{n-1}}{a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + s^n}, \tag{2.5}$$

one wants to find the system matrices

$$(A, B, C, 0), \tag{2.6}$$

such that

$$P(s) = C(s\mathbb{I} - A)^{-1}B. \tag{2.7}$$

A possible solution is the *reachable canonical form*:

$$\begin{aligned} A &= \begin{pmatrix} 0_m & \mathbb{I}_m & 0_m & \dots & 0_m \\ 0_m & 0_m & \mathbb{I}_m & \dots & 0_m \\ \vdots & \vdots & 0_m & \ddots & \vdots \\ 0_m & 0_m & 0_m & \dots & \mathbb{I}_m \\ -a_0\mathbb{I}_m & -a_1\mathbb{I}_m & -a_2\mathbb{I}_m & \dots & -a_{n-1}\mathbb{I}_m \end{pmatrix}, \quad B = \begin{pmatrix} 0_m \\ 0_m \\ \vdots \\ 0_m \\ \mathbb{I}_m \end{pmatrix} \\ C &= \begin{pmatrix} B_0 & B_1 & \dots & B_{n-2} & B_{n-1} \end{pmatrix}, \end{aligned} \tag{2.8}$$

where 0_m is the $m \times m$ zero matrix and $\mathbb{I}_m$ is the $m \times m$ identity matrix.

Remark. Note that this is the result of a possible change of coordinates. This solution is not unique.

Theorem 2. A system in the reachable canonical form is always reachable.

Theorem 3. Let A and B be the dynamics of a reachable system. Then there exists a transformation $z(t) = Tx(t)$ such that in the transformed coordinates the dynamics matrices are in reachable canonical form and the characteristic polynomial for A is given by

$$\det(s\mathbb{I} - A) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n. \tag{2.9}$$

2.3 Pole Placement

The system dynamics are determined by the poles of the closed loop transfer function. Recalling what we have seen so far, we write the plant transfer function as

$$P(s) = C(s\mathbb{I} - A)^{-1}B, \tag{2.10}$$

the characteristic polynomial as

$$p(s) = \det(s\mathbb{I} - A), \quad p(\pi_i) = 0, \tag{2.11}$$

where π_i are the poles of the system.
Problem definition: We want to find through state feedback a controller K such that the closed loop system has a desired characteristic polynomial

$$p^*_{\text{cl}}(s) = \det(s\mathbb{I} - A_{\text{cl}}) \tag{2.12}$$

But is it always possible to find a solution for the pole placement problem?

Theorem 4. The problem of pole placement has a solution if and only if the system is reachable.

2.3.1 Direct Method

The direct method consists in introducing a matrix K with the correct dimensions and forcing the eigenvalues of the closed loop system matrix $A - BK$ to be the desired ones.

2.3.2 Ackermann Formula

Placing poles by hand is tedious and tricky if the state space dimension grows. The Ackermann's formula provides a one step procedure for calculating the controller K. It holds

$$K = \begin{pmatrix} 0 & \ldots & 0 & 1 \end{pmatrix} \mathcal{R}^{-1} p^*_{\text{cl}}(A) = \gamma p^*_{\text{cl}}(A), \tag{2.13}$$

where

- $\mathcal{R}$ is the reachability matrix. Note that this must be invertible (hence, the system is reachable).
- γ is the last row of the inverse of $\mathcal{R}$.
- $p^*_{\text{cl}}(A)$ is the desired closed loop characteristic polynomial evaluated at $s = A$.

Check in the Problem Set for the derivation of the Ackermann's formula.

Example 1. Your task is to keep a space shuttle in its trajectory . The deviations from the desired trajectory are well described by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + bu(t) \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot u(t),\end{aligned} \tag{2.14}$$

where $x_1(t)$ is the position of the space shuttle and $x_2(t)$ is its velocity. Moreover, $u(t)$ represents the propulsion. The position and the velocity are known for every moment and for this reason we can use a state-feedback regulator. You want to find a state feedback controller using pole placement. The specifications for the system are

- The system should not overshoot.
- The error of the regulator should decrease with $e^{-3 \cdot t}$.

a) Find the poles such that the specifications are met.

b) Find the new state feedback matrix K_2.

c) Use the Ackermann formula to get the same result.

Solution

a) Overshoot or in general oscillations, are due to the complex part of some poles. The real part of these poles is given by the decrease function of the error. Since the system must have two poles (A is 2×2), it holds

$$\pi_1 = \pi_2 = -3. \tag{2.15}$$

b) The closed loop has feedback matrix

$$A - B \cdot K. \tag{2.16}$$

We have to choose K such that the eigenvalues of the state feedback matrix are both -3. The dimensions of K must make the matrix multiplication with B and the subtraction with A feasible. It holds $K \in \mathbb{C}^{1 \times 2}$. It holds

$$\begin{aligned}A - B \cdot K &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} k_1 & k_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ k_1 & k_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -k_1 & -k_2 \end{pmatrix}.\end{aligned} \tag{2.17}$$

The eigenvalues of this matrix are

$$\pi_{1,2} = \frac{-k_2 \pm \sqrt{k_2^2 - 4k_1}}{2}.$$

Since the two eigenvalues should have the same value, we know the the part under the square rooth has to vanish. This means that $-\frac{k_2}{2} = -3 \Rightarrow k_2 = 6$. Moreover:

$$\begin{aligned} k_1 &= \frac{k_2^2}{4} \\ &= 9. \end{aligned} \tag{2.18}$$

The matrix finally reads

$$K_2 = \begin{pmatrix} 9 & 6 \end{pmatrix}. \tag{2.19}$$

c) The Ackermann formula for this problem reads

$$K = \begin{pmatrix} 0 & 1 \end{pmatrix} \mathcal{R}^{-1} p_{\text{cl}}^*(A), \tag{2.20}$$

where $\mathcal{R}$ is the system reachability matrix and $p_{\text{cl}}^*(A)$ is the desired closed loop characteristic polynomial evaluated at $s = A$. For our system it holds

$$\begin{aligned} B &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ AB &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \Rightarrow \mathcal{R} = \mathcal{R}^{-1} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{2.21}$$

With the given poles, it holds

$$\begin{aligned} p_{\text{cl}}^*(s) &= (s+3)^2 \\ p_{\text{cl}}^*(A) &= (A + 3\mathbb{I})^2 \\ &= \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 9 & 6 \\ 0 & 9 \end{pmatrix}. \end{aligned} \tag{2.22}$$

Putting everything together in Equation 2.20, one gets

$$\begin{aligned} K &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 9 & 6 \\ 0 & 9 \end{pmatrix} \\ &= \begin{pmatrix} 9 & 6 \end{pmatrix}, \end{aligned} \tag{2.23}$$

which confirms our previous result.

References

[1] Essentials of Robust Control, Kemin Zhou.

[2] Karl Johan Amstroem, Richard M. Murray *Feedback Systems for Scientists and Engineers*. Princeton University Press, Princeton and Oxford, 2009.

[3] Sigurd Skogestad, *Multivariate Feedback Control*. John Wiley and Sons, New York, 2001.