

# Lecture 9: Linear Quadratic Regulator

## 1 LQR

### Motivation:

Last week, we introduced the concept of state feedback, with the idea of using the states of the system and its dynamics to synthesize a controller using desired pole-placement. This week we introduce Linear Quadratic control (LQ) and the special case of the Linear Quadratic Regulator (LQR). The concept behind this control strategy has a key role for control theory and is worth a detailed explanation.

Moreover, pole placement has some drawbacks:

- Does not work well with model uncertainty.
- Does not allow specific tuning of desired trade-offs (e.g. cost vs. performance).

### 1.1 Problem Definition

Given the dynamics of a system

$$\frac{d}{dt}x(t) = A \cdot x(t) + B \cdot u(t), \quad A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \quad (1.1)$$

find a controller

$$u(t) = f(x(t), t), \quad t \in [0, \infty] \quad (1.2)$$

that brings  $x(t)$  asymptotically to zero (with  $x(0) \neq 0$ ). In other words it should hold:

$$\lim_{t \rightarrow \infty} x(t) = 0, \quad (1.3)$$

i.e. the cost

$$J_{\text{LQR}}(x(t), u(t)) = \int_0^{\infty} \|z(t)\|_2^2 + \rho \|u(t)\|_2^2 dt, \quad \rho \in \mathbb{R}^+ \quad (1.4)$$

is minimized, where  $\rho$  allows trade-off between energy of the input and energy of the controlled signal and  $z(t) = Ex(t) + Fu(t)$  can be chosen to contain the state variables of interest. The LQR standard control loop is reported in Figure 1.

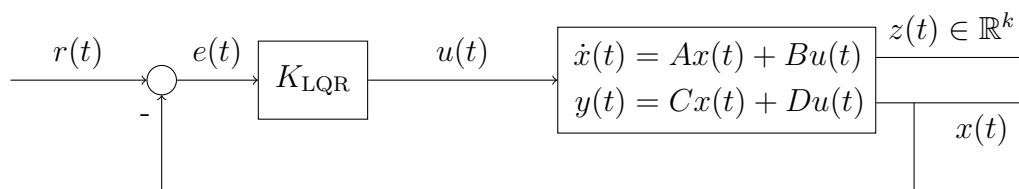


Figure 1: LQR Problem: Closed loop system.

## 1.2 General Form

### 1.2.1 Preliminary Definitions

Since the cost function introduced in Equation 1.4 contains the euclidean norm, one recalls its definition:

$$\begin{aligned}\|u(t)\|_2^2 &= u(t)^\top u(t) \\ &= \sum_i u_i(t)^2.\end{aligned}\tag{1.5}$$

The *weighted* euclidean norm reads

$$\|u(t)\|_{R,2}^2 = u(t)^\top R u(t).\tag{1.6}$$

**Definition 1.** A pair  $(A, C)$  is observable if and only if  $\text{rank}(\mathcal{O}) = n = \dim(A)$ , where

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}.\tag{1.7}$$

**Definition 2.** A pair  $(A, C)$  is detectable if all the unobservable modes are stable.

**Definition 3.** A real matrix  $M$  is said to be **positive definite** (denoted as  $M > 0$ ) when the associated *quadratic form*  $V(z)$  is non-negative, i.e.

$$V(z) = \sum_{i,j} m_{i,j} z_i z_j = z^\top M z \geq 0, \quad \forall z \neq 0.\tag{1.8}$$

**Definition 4.** A real matrix  $M$  is said to be **positive semi-definite** (denoted as  $M \geq 0$ ) when the associated *quadratic form*  $V(z)$  is positive, i.e.

$$V(z) = \sum_{i,j} m_{i,j} z_i z_j = z^\top M z > 0, \quad \forall z \neq 0.\tag{1.9}$$

How can we check if a matrix is positive definite?

### Eigenvalue Test

A real, symmetric matrix is (semi)-positive definite if and only if it has all (non-negative) positive eigenvalues.

### Sylvester's Criterion

An symmetric matrix  $M \in \mathbb{R}^{m \times m}$  is positive definite if and only if all the *upper-left*  $i \times i$  *submatrices* (*principal minors*),  $i \in 1, \dots, m$  have positive determinant. In the case of

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix},\tag{1.10}$$

the conditions are

1.  $a > 0$ .
2.  $ad - b^2 > 0$ .

### 1.2.2 Weighted LQR

With these definitions, one can write (by dropping the time dependency for simplicity) the weighted LQR problem as

$$\begin{aligned}
J_{\text{LQR}}(x(t), u(t)) &= \int_0^\infty \|z(t)\|_{\bar{Q},2}^2 + \rho \|u(t)\|_{\bar{R},2}^2 dt \\
&= \int_0^\infty z^\top \bar{Q} z + \rho u^\top \bar{R} u dt \\
&= \int_0^\infty (Ex + Fu)^\top \bar{Q} (Ex + Fu) + \rho u^\top \bar{R} u dt \\
&= \int_0^\infty (x^\top E^\top + u^\top F^\top) (\bar{Q} Ex + \bar{Q} Fu) + \rho u^\top \bar{R} u dt \\
&= \int_0^\infty x^\top E^\top \bar{Q} Ex + x^\top E^\top \bar{Q} Fu + u^\top F^\top \bar{Q} Ex + u^\top F^\top \bar{Q} Fu + \rho u^\top \bar{R} u dt \\
&= \int_0^\infty x^\top (E^\top \bar{Q} E) x + u^\top (F^\top \bar{Q} F + \rho \bar{R}) u + 2x^\top (E^\top \bar{Q} F) u dt \\
&= \int_0^\infty x^\top Q x + u^\top R u + 2x^\top N u dt,
\end{aligned} \tag{1.11}$$

where  $\bar{Q}, \bar{R}$  are symmetric and positive definite,  $\rho \in \mathbb{R}^+$ ,  $u(t) \in \mathbb{R}^{m \times 1}$ ,  $z(t) \in \mathbb{R}^{k \times 1}$ ,  $x \in \mathbb{R}^{n \times 1}$  and

$$\begin{aligned}
R &= F^\top \bar{Q} F + \rho \bar{R}, \quad R \in \mathbb{R}^{m \times m} \\
Q &= E^\top \bar{Q} E, \quad Q \in \mathbb{R}^{n \times n} \\
N &= E^\top \bar{Q} F.
\end{aligned} \tag{1.12}$$

**Example 1.** You are given the criterion

$$J(x, u) = \int_0^\infty (x_1^2 + 6 \cdot x_1 \cdot x_2 + 100 \cdot x_2^2 + 6 \cdot u_1^2 + 10 \cdot u_2^2) dt. \tag{1.13}$$

Matrix  $N$  is the zero matrix and matrices  $Q$  and  $R$  are

$$Q = \begin{pmatrix} 1 & 3 \\ 3 & 100 \end{pmatrix} \tag{1.14}$$

and

$$R = \begin{pmatrix} 6 & 0 \\ 0 & 10 \end{pmatrix}. \tag{1.15}$$

## 1.3 Solution

If

- The system  $(A, B)$  is stabilizable (all unstable modes are reachable). **Intuition:** this is necessary for state feedback to work (stabilizable means that the unstable modes must be controllable). Controllability is the same as reachability for continuous time systems).

- the pair  $(\tilde{A}, \tilde{Q}) = (A - BR^{-1}N^\top, Q - NR^{-1}N^\top)$  is detectable. **Intuition:** this is necessary to ensure internal stability, i.e. that the closed loop is asymptotically stable. If the system input were known a priori to stabilize the closed loop, then this condition would not be necessary. For now, just remember that internal stability corresponds to input-output stability when no unstable zero/pole cancellations occur.

then

$$u_{\text{LQR}}(t) = - \underbrace{R^{-1}(N + PB)^\top}_{K_{\text{LQR}}} x(t), \quad (1.16)$$

where  $P$  is the **real, symmetric, positive definite** solution of the algebraic Riccati equation

$$(N + P \cdot B) \cdot R^{-1} \cdot (N^\top + B^\top \cdot P) - P \cdot A - A^\top \cdot P - Q = 0. \quad (1.17)$$

### 1.3.1 Solving the ARE

#### Hamiltonian Method

Starting from Equation 1.17, one can rearrange as

$$\begin{aligned} (N + P \cdot B) \cdot R^{-1} \cdot (N^\top + B^\top \cdot P) - P \cdot A - A^\top \cdot P - Q &= 0 \\ (NR^{-1} + PBR^{-1})(N^\top + B^\top P) - P \cdot A - A^\top \cdot P - Q &= 0 \\ NR^{-1}N^\top + NR^{-1}B^\top P + PBR^{-1}N^\top + PBR^{-1}B^\top P - P \cdot A - A^\top \cdot P - Q &= 0 \\ - \underbrace{(A - BR^{-1}N^\top)^\top P}_{\tilde{A}^\top} - P \underbrace{(A - BR^{-1}N^\top)}_{\tilde{A}} + P \underbrace{(BR^{-1}B^\top)}_{\tilde{R}} P - \underbrace{(Q - NR^{-1}N^\top)}_{\tilde{Q}} &= 0. \end{aligned} \quad (1.18)$$

Hence, one gets the Riccati equation

$$\tilde{A}^* P + P \tilde{A}^\top + P \tilde{R} P + \tilde{Q} = 0, \quad (1.19)$$

with the unknown quadratic matrix  $P$ . Note that this equation can be rewritten as

$$(P \quad -\mathbb{I}) \underbrace{\begin{pmatrix} \tilde{A} & \tilde{R} \\ -\tilde{Q} & -\tilde{A}^* \end{pmatrix}}_{H \in \mathbb{R}^{2n \times 2n}} \begin{pmatrix} \mathbb{I} \\ P \end{pmatrix} = 0, \quad (1.20)$$

where  $H$  is the hamiltonian matrix. In order to find the solution of the ARE, we assume two things:

1.  $H$  has no eigenvalues on the imaginary axis, i.e. it will have  $n$  eigenvalues in the LHP and  $n$  in the RHP. Let the subspace spanned by the eigenvectors associated to the stable eigenvalues (i.e. in the LHP) be

$$X_H = \text{Im} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad (1.21)$$

where  $X_1, X_2 \in \mathbb{C}^{n \times n}$ .

2.  $X_1$  is invertible.

If the two assumptions are met, one says that *the Hamiltonian belongs to the domain of the Riccati operator*, i.e.  $H \in \text{dom}(\text{Ric})$ . With these two assumptions, the solution of the ARE can be computed as

$$P = X_2 X_1^{-1}. \quad (1.22)$$

*Remark.* The ARE has in general more than one solution, but only one is stabilizing, i.e. it makes the closed loop asymptotically stable (see assumption before).

**Theorem 1.**  $H \in \text{dom}(\text{Ric})$  if there exists symmetric matrices  $P, H \in \mathbb{R}^{n \times n}$  with stable  $H$  such that

$$H \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} H, \quad (1.23)$$

where

- a)  $P = X_2 X_1^{-1}$  is real and symmetric.
- b)  $P$  satisfies the ARE.
- c) The matrix  $\tilde{A} + \tilde{R}P$  is stable (all eigenvalues are in the open LHP).

### 1.3.2 Direct Method

The direct method consists in introducing a matrix  $P$  with the correct dimensions and unknowns, apply the Riccati equation and solve the system of equations.

## 1.4 Examples

**Example 2.** You have to design a LQ regulator for a plant with 2 inputs, 3 outputs and 6 state variables.

- (a) What are the dimensions of  $A$ ,  $B$ ,  $C$  and  $D$ ?
- (b) What is the dimension of the transfer function  $u \rightarrow y$ ?
- (c) What is the dimension of the matrix  $Q$  of  $J_{LQR}$ ?
- (d) What is the dimension of the matrix  $R$  of  $J_{LQR}$ ?
- (e) What is the dimension of the matrix  $K$ ?

**Solution.**

- (a) One can find the solution by analyzing the meaning of the matrices:
  - Since we are given 6 states variables, the matrix  $A$  should have 6 rows and 6 columns, i.e.  $A \in \mathbb{R}^{6 \times 6}$ .
  - Since we are given 2 inputs, the matrix  $B$  should have 2 columns and 6 rows, i.e.  $B \in \mathbb{R}^{6 \times 2}$ .
  - Since we are given 3 outputs, the matrix  $C$  should have 6 columns and 3 rows, i.e.  $C \in \mathbb{R}^{3 \times 6}$ .
  - Since we are given 2 inputs and 3 outputs, the matrix  $D$  should have 2 columns and 3 rows, i.e.  $D \in \mathbb{R}^{3 \times 2}$ .
- (b) Since we are dealing with a system with 2 inputs and 3 outputs,  $P(s) \in \mathbb{R}^{3 \times 2}$ . Moreover,  $P(s)$  should have the same dimensions of  $D$  because of its formula.
- (c) From the formulation of  $Q$  one can easily see that its dimensions are the same of the dimensions of  $A$ , i.e.  $Q \in \mathbb{R}^{6 \times 6}$ .
- (d) From

$$u(t) = -K \cdot x(t).$$

we can see that  $K$  should have 6 columns and 2 rows, i.e.  $K \in \mathbb{R}^{2 \times 6}$ .

**Example 3.** A system is given as

$$\begin{aligned}\dot{x}_1(t) &= 3 \cdot x_2(t) \\ \dot{x}_2(t) &= 3 \cdot x_1(t) - 2 \cdot x_2(t) + \frac{1}{2} \cdot u(t) \\ y(t) &= 4 \cdot x_1(t) + \frac{7}{3} \cdot x_2(t).\end{aligned}\tag{1.24}$$

a) Solve the LQR problem for the criterion

$$J(x(t), u(t)) = \int_0^\infty 7 \cdot x_1(t)^2 + 3 \cdot x_2(t)^2 + \frac{1}{4} \cdot u(t)^2 dt\tag{1.25}$$

and find the state feedback controller  $K$  using the direct method.

b) Solve a) using the Hamiltonian method.

c) Find the eigenvalues of the closed-loop system with the LQ regulator  $K$ .

d) Does the new criterion

$$J_{\text{new}} = \int_0^\infty 70 \cdot x_1^2 + 30 \cdot x_2^2 + \frac{10}{4} \cdot u^2 dt\tag{1.26}$$

affect the solution for  $K$ ?

**Solution.**

a) Using quadratic forms, one can identify  $\tilde{Q}$  and  $\tilde{R}$  to be (using the null matrix for  $N$ )

$$Q = \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix}\tag{1.27}$$

and

$$R = \frac{1}{4}.\tag{1.28}$$

The state-space description of the system can be re-written in standard form as

$$\begin{aligned}\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} &= \underbrace{\begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix}}_A \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}}_B u(t) \\ y(t) &= \underbrace{\begin{pmatrix} 4 & \frac{7}{3} \end{pmatrix}}_C \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \underbrace{0}_D u(t).\end{aligned}\tag{1.29}$$

In order to find the controller  $K$ , one has to compute the symmetric, positive definite solution of the Riccati equation related to this problem. First, one has to look at the form that this solution should have. Here  $B \in \mathbb{R}^{2 \times 1}$ . This means that since  $\Phi = \Phi^\top$  we are dealing with  $\Phi \in \mathbb{R}^{2 \times 2}$  of the form

$$\Phi = \begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_2 & \varphi_3 \end{pmatrix}.\tag{1.30}$$

With the Riccati equation, it holds

$$\begin{aligned}
\Phi \cdot (B \cdot R^{-1} \cdot B^\top) \cdot \Phi - \Phi \cdot (A - BR^{-1}N^\top) - (A - BR^{-1}N^\top)^\top \Phi - Q &= 0 \\
\Phi \cdot B \cdot R^{-1} \cdot B^\top \cdot \Phi - \Phi \cdot A - A^\top \cdot \Phi - Q &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\Phi \cdot \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \cdot 4 \cdot \begin{pmatrix} 0 & \frac{1}{2} \end{pmatrix} \cdot \Phi - \Phi \cdot \begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix} - \begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix} \cdot \Phi - \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 2\varphi_2 \\ 2\varphi_3 \end{pmatrix} \cdot \begin{pmatrix} \varphi_2 & \varphi_3 \\ \varphi_2 & \varphi_3 \end{pmatrix} - \begin{pmatrix} 3\varphi_2 & 3\varphi_1 - 2\varphi_2 \\ 3\varphi_3 & 3\varphi_2 - 2\varphi_3 \end{pmatrix} - \begin{pmatrix} 3\varphi_2 & 3\varphi_3 \\ 3\varphi_1 - 2\varphi_2 & 3\varphi_2 - 2\varphi_3 \end{pmatrix} - \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\begin{pmatrix} \varphi_2^2 & \varphi_2\varphi_3 \\ \varphi_2\varphi_3 & \varphi_3^2 \end{pmatrix} - \begin{pmatrix} 6\varphi_2 + 7 & 3\varphi_1 - 2\varphi_2 + 3\varphi_3 \\ 3\varphi_1 - 2\varphi_2 + 3\varphi_3 & 6\varphi_2 - 4\varphi_3 + 3 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{1.31}$$

Hence, one gets 3 equations (two elements are equal because of symmetry):

$$\begin{aligned}
\varphi_2^2 - 6 \cdot \varphi_2 - 7 &= 0 & (I) \\
\varphi_2 \cdot \varphi_3 - 3 \cdot \varphi_1 + 2 \cdot \varphi_2 - 3 \cdot \varphi_3 &= 0 & (II) \\
\varphi_3^2 - 6 \cdot \varphi_2 + 4 \cdot \varphi_3 - 3 &= 0 & (III).
\end{aligned} \tag{1.32}$$

*Sylvester's Criterion:* An Hermitian (here symmetric) matrix  $M \in \mathbb{C}^{m \times m}$  is positive definite if and only if all the *upper-left  $i \times i$  submatrices (leading minors)*,  $i \in 1, \dots, m$  has positive determinant. Applying this to  $\Phi$  one gets the conditions:

- (a)  $\varphi_1 > 0$ .
- (b)  $\varphi_1\varphi_3 - \varphi_2^2 > 0$ .

From the Equation (I), one gets

$$\begin{aligned}
\varphi_2 &= \frac{6 \pm \sqrt{64}}{2} \\
&= \{-1, 7\}.
\end{aligned} \tag{1.33}$$

Since we cannot discard a specific value, we pursue with  $\varphi_{2,1} = -1$  and  $\varphi_{2,2} = 7$ :

**Case  $\varphi_{2,1} = -1$ :**

Plugging this into the Equation (III), one gets

$$\begin{aligned}
\varphi_3^2 + 6 + 4\varphi_3 - 3 &= 0 \\
\varphi_3^2 + 4\varphi_3 + 3 &= 0,
\end{aligned} \tag{1.34}$$

and

$$\begin{aligned}
\varphi_3 &= \frac{-4 \pm \sqrt{4}}{2} \\
&= \{-3, -1\}.
\end{aligned} \tag{1.35}$$

In order for these two values to fulfill the second Sylvester condition, it should hold  $\varphi_1 < 0$ , which violates the first condition. For this reason this is not a possible choice.



**Case**  $\varphi_{2,1} = 7$ :

Plugging this into the Equation (III), one gets

$$\begin{aligned}\varphi_3^2 - 42 + 4\varphi_3 - 3 &= 0 \\ \varphi_3^2 + 4\varphi_3 - 45 &= 0,\end{aligned}\tag{1.36}$$

and

$$\begin{aligned}\varphi_3 &= \frac{-4 \pm \sqrt{196}}{2} \\ &= \{-9, 5\}.\end{aligned}\tag{1.37}$$

$\varphi_3 = 5$  is the only value which does not violate the two Sylvester's conditions. Plugging the values into Equation (II) one gets

$$\begin{aligned}35 - 3\varphi_1 + 14 - 15 &= 0 \\ \varphi_1 &= \frac{34}{3}.\end{aligned}\tag{1.38}$$

The solution of the Riccati equation hence is

$$\Phi = \begin{pmatrix} \frac{34}{3} & 7 \\ 7 & 5 \end{pmatrix}.\tag{1.39}$$

The controller  $K$  can be computed as

$$\begin{aligned}K &= R^{-1}B^T\Phi \\ &= 4 \cdot \begin{pmatrix} 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{34}{3} & 7 \\ 7 & 5 \end{pmatrix} \\ &= (14 \quad 10).\end{aligned}\tag{1.40}$$

b) Before applying the Hamiltonian method, one need to check

- $(A, B)$  stabilizable (all unstable modes are reachable). The reachability matrix for this pair

$$\begin{aligned}\mathcal{R} &= (B \quad AB) \\ &= \begin{pmatrix} 0 & \frac{3}{2} \\ \frac{1}{2} & -1 \end{pmatrix}\end{aligned}\tag{1.41}$$

has full rank, hence the system is reachable.

- $(A, Q)$  detectable. The observability matrix for the pair

$$\begin{aligned}\mathcal{O} &= \begin{pmatrix} Q \\ QA \end{pmatrix} \\ &= \begin{pmatrix} 7 & 0 \\ 0 & 3 \\ 0 & 21 \\ 9 & -6 \end{pmatrix}\end{aligned}\tag{1.42}$$

has full column rank, hence the system is observable.

In order to use the Hamiltonian method, one needs to build the Hamiltonian matrix

$$H = \begin{pmatrix} \tilde{A} & \tilde{R} \\ -\tilde{Q} & -\tilde{A}^\top \end{pmatrix}, \quad (1.43)$$

where

$$\begin{aligned} \tilde{A} &= A - BR^{-1}N^\top \\ \tilde{R} &= -BR^{-1}B^\top \\ \tilde{Q} &= Q - NR^{-1}N^\top. \end{aligned} \quad (1.44)$$

For this specific example ( $N = 0$ ), one has

$$\begin{aligned} \tilde{A} &= A. \\ \tilde{R} &= -\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \frac{1}{\frac{1}{4}} \begin{pmatrix} 0 & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}. \\ \tilde{Q} &= Q. \end{aligned} \quad (1.45)$$

Therefore, the Hamiltonian is

$$H = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 3 & -2 & 0 & -1 \\ -7 & 0 & 0 & -3 \\ 0 & -3 & -3 & 2 \end{pmatrix}. \quad (1.46)$$

In order to have the eigenvalues, one computes

$$\begin{aligned} \det(H - \lambda\mathbb{I}) &= \det \begin{pmatrix} -\lambda & 3 & 0 & 0 \\ 3 & -2 - \lambda & 0 & -1 \\ -7 & 0 & -\lambda & -3 \\ 0 & -3 & -3 & 2 - \lambda \end{pmatrix} \\ &= -\lambda \det \begin{pmatrix} -2 - \lambda & 0 & -1 \\ 0 & -\lambda & -3 \\ -3 & -3 & 2 - \lambda \end{pmatrix} - 3 \det \begin{pmatrix} 3 & 0 & -1 \\ -7 & -\lambda & -3 \\ 0 & -3 & 2 - \lambda \end{pmatrix} \\ &= \lambda \left[ (2 + \lambda) \det \begin{pmatrix} -\lambda & -3 \\ -3 & 2 - \lambda \end{pmatrix} + \det \begin{pmatrix} 0 & -\lambda \\ -3 & -3 \end{pmatrix} \right] \\ &\quad - \left[ 9 \det \begin{pmatrix} -\lambda & -3 \\ -3 & 2 - \lambda \end{pmatrix} - 3 \det \begin{pmatrix} -7 & -\lambda \\ 0 & -3 \end{pmatrix} \right] \\ &= (\lambda^2 + 2\lambda - 9) \det \begin{pmatrix} -\lambda & -3 \\ -3 & 2 - \lambda \end{pmatrix} - 3\lambda^2 + 63 \\ &= (\lambda^2 - 9 - 2\lambda)(\lambda^2 - 9 + 2\lambda) - 3\lambda^2 + 63 \\ &= \lambda^4 - 25\lambda^2 + 144. \end{aligned} \quad (1.47)$$

Therefore, the eigenvalues are

$$\lambda_{1,2} = \pm 3, \quad \lambda_{3,4} = \pm 4. \quad (1.48)$$

Since we only care about stable eigenvalues (in LHP), we compute the eigenvectors for  $\lambda_2 = -3$  and  $\lambda_4 = -4$ .

It holds:

- $E_{\lambda_2} = E_{-3}$ : from  $(H - \lambda_2 \mathbb{I}) \cdot x = 0$  one gets the linear system of equations

$$\left( \begin{array}{cccc|c} 3 & 3 & 0 & 0 & 0 \\ 3 & 1 & 0 & -1 & 0 \\ -7 & 0 & 3 & -3 & 0 \\ 0 & -3 & -3 & 5 & 0 \end{array} \right).$$

Using the first row as reference and subtracting the correct multiples of it from the other rows, one gets the form

$$\left( \begin{array}{cccc|c} 3 & 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & 7 & 3 & -3 & 0 \\ 0 & -3 & -3 & 5 & 0 \end{array} \right).$$

Using the second row as reference and subtracting the correct multiples of it from the other rows, one gets the form

$$\left( \begin{array}{cccc|c} 3 & 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & 3 & -\frac{13}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

Since one has one zero row, one can introduce a free parameter. Let  $x_4 = s$ , then  $x_3 = \frac{13}{6}s$ ,  $x_2 = -\frac{s}{2}$ ,  $x_1 = \frac{s}{2}$ ,  $s \in \mathbb{R}$ . This defines the first eigenspace, which is (multiplying everything by 6)

$$E_{-3} = \left\{ \begin{pmatrix} 3 \\ -3 \\ 13 \\ 6 \end{pmatrix} \right\}. \quad (1.49)$$

- $E_{\lambda_4} = E_{-4}$ : from  $(H - \lambda_4 \mathbb{I}) \cdot x = 0$  one gets the linear system of equations

$$\left( \begin{array}{cccc|c} 4 & 3 & 0 & 0 & 0 \\ 3 & 2 & 0 & -1 & 0 \\ -7 & 0 & 4 & -3 & 0 \\ 0 & -3 & -3 & 6 & 0 \end{array} \right).$$

Using the first row as reference and subtracting the correct multiples of it from the other rows, one gets the form

$$\left( \begin{array}{cccc|c} 4 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -4 & 0 \\ 0 & 21 & 16 & -12 & 0 \\ 0 & -3 & -3 & 6 & 0 \end{array} \right).$$

Using the second row as reference and subtracting the correct multiples of it from the other rows, one gets the form

$$\left( \begin{array}{cccc|c} 4 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -4 & 0 \\ 0 & 0 & 16 & -96 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

Since one has one zero row, one can introduce a free parameter. Let  $x_4 = s$ , then  $x_3 = 6s$ ,  $x_2 = -4s$ ,  $x_1 = 3s$ ,  $s \in \mathbb{R}$ . This defines the second eigenspace, which is (multiplying everything by 6)

$$E_{-4} = \left\{ \begin{pmatrix} 3 \\ -4 \\ 6 \\ 1 \end{pmatrix} \right\}. \quad (1.50)$$

Stacking the eigenvectors one gets

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ -4 & -3 \\ 6 & 13 \\ 1 & 6 \end{pmatrix}. \quad (1.51)$$

It holds

$$\begin{aligned} \Phi &= X_2 X_1^{-1} \\ &= \begin{pmatrix} 6 & 13 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ -4 & -3 \end{pmatrix}^{-1} \\ &= \frac{1}{3} \begin{pmatrix} 6 & 13 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} -3 & -3 \\ 4 & 3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{34}{3} & 7 \\ 7 & 5 \end{pmatrix}, \end{aligned} \quad (1.52)$$

which confirms the result of a).

c) The closed-loop matrix to analyse is

$$\begin{aligned} A - B \cdot K &= \begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \cdot (14 \quad 10) \\ &= \begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 7 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ -4 & -7 \end{pmatrix}. \end{aligned} \quad (1.53)$$

The eigenvalues of the closed loop system are given by

$$\begin{aligned} \det((A - B \cdot K) - \lambda \cdot \mathbb{I}) &= 0 \\ \lambda^2 + 7\lambda + 12 &= 0 \end{aligned}$$

from which it follows:  $\lambda_1 = -3$  and  $\lambda_2 = -4$ .

d) No. Since it holds  $J_{\text{new}} = 10 \cdot J$ ,  $K$  remains the same.

**Example 4.** You design with MATLAB a LQ Regulator:

```
1 A = [1 0 0 0; 1 1 0 0; 1 1 1 0; 0 0 1 1];
2 B = [1 1 1 1; 0 1 0 2];
3 C = [0 0 0 1; 0 0 1 1; 0 1 1 1];
4
5 nx = size(A,1); Number of state variables of the plant, in Script: n
6 nu = size(B,2); Number of input variables of the plant, in Script: m
7 ny = size(C,1); Number of output variables of the plant, in Script: p
8
9 q = 1;
10 r = 1;
11 Q = q*eye(###);
12 R = r*eye(###);
13
14 K = lqr(A,B,Q,R);
```

Fill the following rows:

```
11 : ### =
12 : ### =
```

**Solution.** The matrix  $Q$  is a weight for the states and the matrix  $R$  is a weight for the inputs. The correct filling is

```
11 : ### = nx
12 : ### = nu
```

## References

- [1] Essentials of Robust Control, Kemin Zhou.
- [2] Karl Johan Amstroem, Richard M. Murray *Feedback Systems for Scientists and Engineers*. Princeton University Press, Princeton and Oxford, 2009.
- [3] Sigurd Skogestad, *Multivariate Feedback Control*. John Wiley and Sons, New York, 2001.