Control Systems II

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Abstract

This "Skript" is made of my notes from the lecture *Control Systems II* of Dr. Gregor Ochsner (literature of Prof. Dr. Lino Guzzella) and from my lectures as teaching assistant in 2017 for the lecture of Dr. Guillaume Ducard and in 2018 for the lecture of Dr. Jacopo Tani.
This document, should give the chance to repeat one more time the contents of the lecture *Control Systems II* and practice them through many examples and exercises.

The updated version of the Skript is available on *n.ethz.ch/∼gzardini/*.

I cannot guarantee on the correctness of what is included in this Skript: it is possible that small errors occur. For this reason I am very grateful to get feedbacks and corrections, in order to improve the quality of the literature. An *errata* version of these notes will always be available on my homepage
Enjoy your Control Systems II!

Cheers!

Gioele Zardini

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1 Recapitulation from Control Systems I

1.1 Loop Transfer Functions

1.1.1 Standard Feedback Control Loop

The standard feedback control system structure[1] is depicted in Figure 1. This represen-

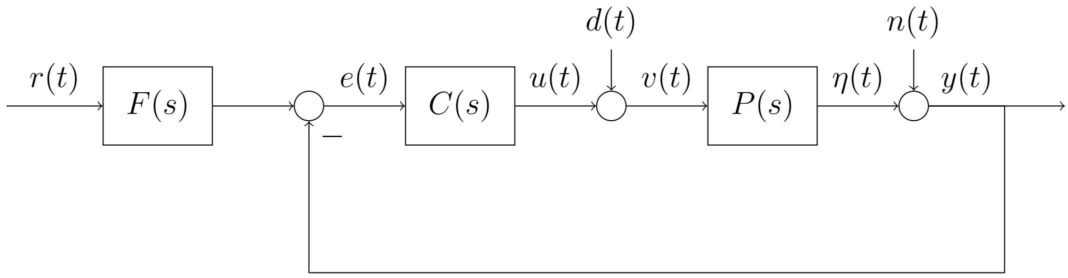

Figure 1: Standard feedback control system structure.

tation will be the key element for your further control systems studies.
The **plant** $P(s)$ represents the system you want to control: let's imagine a Duckiebot. The variable $u(t)$ represents the **real input** that is given to the system and $d(t)$ some **disturbance** that is applyied to it. These two elements toghether, can be resumed into an actuator for the Duckiebot example. The signal $v(t)$ represents the **disturbed input**. The signal $\eta(t)$ describes the **real output** of the system. The variable $y(t)$, instead, describes the **measured output** of the system, which can be eventually measured by a sensor with some **noise** $n(t)$. In the case of the Duckiebot, this can correspond to the position of the vehicle and its orientation (pose). The **feedback controller** $C(s)$ makes sure that the **tracking error** $e(t)$ between the measured output and the **reference** $r(t)$ approaches zero. $F(s)$ represents the **feedforward controller** of the system.

Remark. Note the notation: signals in time domain are written with small letters, such as $n(t)$. Transfer functions in frequency domain (Laplace/Fourier transformed) are written with capital letters, such as $P(s)$.

Why transfer functions? In order to make the analysis of such a system easier, the loop transfer functions are defined. In fact, it is worth transforming a problem from the time domain into the frequency domain, solve it, and back transform it into time domain. The main reason behind this is that *convolutions* (computationally complex operations which relate signals) are multiplications (through Laplace/Fourier transformation) in the frequency domain.

[1]Note that multiple versions of this loop exist

1.1.2 The Gang of Six

The **loop gain** $L(s)$ is the open-loop transfer function defined by

$$L(s) = P(s)C(s). \tag{1.1}$$

The **sensitivity** $S(s)$ is the closed-loop transfer function defined by

$$\begin{aligned} S(s) &= \frac{1}{1+L(s)} \\ &= \frac{1}{1+P(s)C(s)}. \end{aligned} \tag{1.2}$$

Remark. Note that the sensitivity gives measure of the influence of disturbances d on the output y.

The **complmentary sensitivity** $T(s)$ is the closed-loop transfer function defined by

$$\begin{aligned} T(s) &= \frac{L(s)}{1+L(s)} \\ &= \frac{P(s)C(s)}{1+P(s)C(s)}. \end{aligned} \tag{1.3}$$

It can be shown that

$$S(s) + T(s) = 1. \tag{1.4}$$

Recalling that a performant controller minimizes the difference between the reference $R(s)$ and the output $Y(s)$, one can write this difference as an error $E(s)$. This can be computed as

$$\begin{aligned} E(s) &= F(s)R(s) - Y(s) \\ &= F(s)R(s) - (\eta(s) + N(s)) \\ &= F(s)R(s) - (P(s)V(s) + N(s)) \\ &= F(s)R(s) - (P(s)(D(s) + U(s)) + N(s)) \\ &= F(s)R(s) - P(s)D(s) - N(s) - P(s)U(s) \\ &= F(s)R(s) - P(s)D(s) - N(s) - P(s)C(s)E(s). \end{aligned} \tag{1.5}$$

Furthermore, recalling that we started from $E(s)$, one gets the new equation

$$\begin{aligned} E(s) &= F(s)R(s) - P(s)D(s) - N(s) - P(s)C(s)E(s) \\ (1 + P(s)C(s))E(s) &= F(s)R(s) - P(s)D(s) - N(s) \\ E(s) &= \frac{F(s)}{1+P(s)C(s)}R(s) - \frac{P(s)}{1+P(s)C(s)}D(s) - \frac{1}{1+P(s)C(s)}N(s). \end{aligned} \tag{1.6}$$

This procedure can be applied to each pair of signals of the feedback loop depicted in Figure 1. The following equations can be derived:

$$\begin{pmatrix} Y(s) \\ \eta(s) \\ V(s) \\ U(s) \\ E(s) \end{pmatrix} = \frac{1}{1+P(s)C(s)} \begin{pmatrix} P(s)C(s)F(s) & P(s) & 1 \\ P(s)C(s)F(s) & P(s) & -P(s)C(s) \\ C(s)F(s) & 1 & -C(s) \\ C(s)F(s) & -P(s)C(s) & -C(s) \\ F(s) & -P(s) & -1 \end{pmatrix} \cdot \begin{pmatrix} R(s) \\ D(s) \\ N(s) \end{pmatrix}. \tag{1.7}$$

Exercise 1. A good exercise to practice this procedure could be to derive all the other relations reported in Equation (1.7) on your own.

As you can notice, many terms in the relations introduced in Equation (1.7), are repeated. Using the defined sensitivity function $S(s)$ (Equation 1.2) and the complementary sensitivity function $T(s)$ (Equation 1.3), one can define four new important transfer functions. The **load sensitivity** function is defined as

$$P(s)S(s) = \frac{P(s)}{1+P(s)C(s)}, \tag{1.8}$$

and gives us an intuition on how does the disturbance affect the output. The **noise sensitivity** function is defined as

$$C(s)S(s) = \frac{C(s)}{1+P(s)C(s)}, \tag{1.9}$$

and gives us an intuition on how does the noise affect the input. Moreover, one can define two more useful transfer functions:

$$C(s)F(s)S(s) = \frac{C(s)F(s)}{1+P(s)C(s)}, \qquad T(s)F(s) = \frac{P(s)C(s)F(s)}{1+P(s)C(s)}. \tag{1.10}$$

The new introduced four transfer functions together with the sensitivity and the complementary sensitivity functions, describe the so called *gang of six*.

1.1.3 The Gang of Four

The special case where $F(s) = 1$ (i.e., no presence feedforward), leads to the equivalence of some of the defined transfer functions. In particular, we are left with four transfer functions:

$$\begin{aligned}
S(s) &= \frac{1}{1+P(s)C(s)} && \text{sensitivity function,}\\
T(s) &= \frac{P(s)C(s)}{1+P(s)C(s)} && \text{complementary sensitivity function,}\\
P(s)S(s) &= \frac{P(s)}{1+P(s)C(s)} && \text{load sensitivity function,}\\
C(s)S(s) &= \frac{C(s)}{1+P(s)C(s)} && \text{noise sensitivity function.}
\end{aligned} \tag{1.11}$$

At this point one may say: I can define these new transfer functions, but why are they necessary? Let's illustrate this through an easy example.

Example 1. Imagine to deal with a plant $P(s) = \frac{1}{s-1}$ and that you control it through a PID controller of the form $C(s) = k \cdot \frac{(s-1)}{s}$. You can observe that the plant has a pole at $s = 1$, which makes it unstable. If one computes the classic transfer functions learned in Control Systems I (Equations (1.1), (1.2), (1.3)), one gets

$$\begin{aligned}
L(s) &= C(s)P(s) = \frac{1}{s-1} \cdot k \cdot \frac{(s-1)}{s} = \frac{k}{s},\\
S(s) &= \frac{1}{1+L(s)} = \frac{s}{s+k},\\
T(s) &= \frac{L(s)}{1+L(s)} = \frac{k}{s+k}.
\end{aligned} \tag{1.12}$$

You may notice that none of these transfer functions contains the important information about the unstable pole of the plant. However, this information is crucial: if one computes the rest of the gang of four, one gets

$$\begin{aligned} P(s)S(s) &= \frac{\frac{1}{s-1}}{1+\frac{k}{s}} = \frac{1}{(s-1)(s+k)}, \\ C(s)S(s) &= \frac{k \cdot \frac{(s-1)}{s}}{1+\frac{k}{s}} = \frac{k(s-1)}{s+k}. \end{aligned} \tag{1.13}$$

These two transfer functions still contain the problematic term and are extremely useful to determine the influence of the unstable pole on the system, because they explicitly show it.

Exercise 2. Which consequence does the application of a small disturbance d on the system have?

1.1.4 Relations to Performance

By looking at the feedback loop in Figure 1, one can introduce a new variable

$$\varepsilon(t) = r(t) - \eta(t), \tag{1.14}$$

which represents the error between the reference signal and the real plant output. One can show, that this error can be written as

$$\varepsilon(s) = S(s)R(s) - P(s)S(s)D(s) + T(s)N(s). \tag{1.15}$$

From this equation one can essentially read:

- For a good reference tracking and disturbance attenuation, one needs a small $S(s)$ (or high $L(s)$).
- For a good noise rejection, one needs a small $T(s)$ (or small $L(s)$).

Exercise 3. Derive Equation (1.15) with what you learned in this chapter.

1.1.5 Feed Forward

The feedforward technique complements the feedback one. If on one hand feedback is error based and tries to compensate unexpected or unmodeled phenomena, such as disturbances, noise and model uncertainty, the feedforward technique works well if we have some *knowledge* of the system (i.e. disturbances, plant, reference). Let's illustrate the main idea behind this concept through an easy example.

Example 2. The easiest example for this concept, is the one of perfect control. Imagine to have a system as the one depicted in Figure 2.
This is also known as perfect control/plant inversion, where we want to find an input $u(t)$, such that $y(t) = r(t)$. One can write

$$Y(s) = P(s)U(s) \tag{1.16}$$

and hence

$$R(s) = P(s)U(s) \Rightarrow U(s) = P(s)^{-1}R(s). \tag{1.17}$$

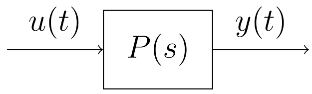

Figure 2: Standard perfect control system structure.

This is not possible when:

- The plant $P(s)$ has right-hand side poles (unstable inverse).
- There are time delays, (non causal inverse): *how much of the future output trajectory information we need in order to perform the desired output tracking?*
- More poles than zeros, (unrealizable inverse).
- Model uncertainty, (unknown inverse).

But what does it mean for a system to be realizable or causal? Let's illustrate this with an example.

Example 3. If one has a transfer function with a number of zeros bigger than the number of poles, this represents pure differentiators, which are not causal. Imagine to deal with the transfer function

$$P(s) = \frac{(s+2)(s+3)}{s+1}. \tag{1.18}$$

This transfer function has two zeros and one pole. This can be rewritten as

$$\begin{aligned} P(s) &= \frac{(s+2)(s+3)}{s+1} \\ &= \frac{s^2+5s+6}{s+1} \\ &= \frac{s(s+1)+4s+6}{s+1} \\ &= s + \frac{4s+6}{s+1}, \end{aligned} \tag{1.19}$$

where s is a pure differentiator. A pure differentiator's transfer function can be written as the ratio of an output and an input:

$$G(s) = s = \frac{Y(s)}{U(s)}, \tag{1.20}$$

which describes the time domain equation

$$\begin{aligned} y(t) &= \dot{u}(t) \\ &= \lim_{\delta_t \to 0} \frac{u(t+\delta_t) - u(t)}{\delta_t}, \end{aligned} \tag{1.21}$$

which confirms us that this transfer function must have knowledge of future values of the input $u(t)$ (from $u(t+\delta_t)$) in order to react with the current output $y(t)$. This is per definition not physical and hence not realizable, not causal.

1.2 General Control Objectives

In this section we are going to present the standard control objectives and relate them to what you learned in the course Control Systems I.
But what are the real objectives of a controller? We can subdivide them into four specific needs:

1. **Nominal Stability**: Is the closed-loop interconnection of a nominal plant and a controller stable?

2. **Nominal Performance**: Does the closed-loop interconnection of a nominal plant and a controller achieve specific performance objectives?

3. **Robust Stability**: Is the closed-loop interconnection of *any* disturbed nominal plant and a controller stable?

4. **Robust Performance**: Does the closed-loop interconnection of *any* plant and a controller achieve specific perfomance objectives?

One can essentially subdivide the job of a control engineer into two big tasks:

(I) **Analysis**: Given a controller, how can we check that the objectives above are satisfied?

(II) **Synthesis**: Given a plant, how can we design a controller that achieves the objectives above?

Let's analyse the objectives of a controller with respect to their relation to these two tasks.

1.2.1 Nominal Stability

During the course Control Systems I, you learned about different stability concepts. Moreover, you have learned the differences between internal and external stability: let's recall them here. Consider a generic nonlinear system defined by the dynamics

$$\dot{x}(t) = f(x(t)), \quad t \in \mathbb{R}, \quad x(t) \in \mathbb{R}^n, \quad f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n. \tag{1.22}$$

Definition 1. A state $\hat{x} \in \mathbb{R}^n$ is called an *equilibrium* of system (1.22) if and only if $f(\hat{x}) = 0 \quad \forall t \in \mathbb{R}$.

Internal/Lyapunov Stability

Internal stability, also called Lyapunov stability, characterises the stability of the trajectories of a dynamic system subject to a perturbation near the to equilibrium. Let now $\hat{x} \in \mathbb{R}^n$ be an equilibrium of system (1.22).

Definition 2. An equilibrium $\hat{x} \in \mathbb{R}^n$ is said to be *Lyapunov* stable if

$$\forall \varepsilon > 0, \quad \exists \delta > 0 \quad \text{s.t. } \|x(0) - \hat{x}\| < \delta \Rightarrow \|x(t) - \hat{x}\| < \varepsilon. \tag{1.23}$$

In words, an equilibrium is said to be *Lyapunov* stable if for any bounded initial condition and zero input, the state remains bounded.

Definition 3. An equilibrium $\hat{x} \in \mathbb{R}^n$ is said to be *asymptotically stable* in $\Omega \subseteq \mathbb{R}^n$ if it is Lyapunov stable and attractive, i.e. if

$$\lim_{t\to\infty} (x(t) - \hat{x}) = 0, \quad \forall x(0) \in \Omega. \tag{1.24}$$

In words, an equilibrium is said to be asymptotically stable if, for any bounded initial condition and zero input, the state converges to the equilibrium.

Definition 4. An equilibrium $\hat{x} \in \mathbb{R}^n$ is said to be *unstable* if it is not stable.

Remark. Note that stability is a property of the equilibrium and not of the system in general.

External/BIBO Stability

External stability, also called BIBO stability (Bounded Input-Bounded Output), characterises the stability of a dynamic system which for bounded inputs gives back bounded outputs.

Definition 5. A signal $s(t)$ is said to be bounded, if there exists a finite value $B > 0$ such that the signal magnitude never exceeds B, that is

$$|s(t)| \leq B \quad \forall t \in \mathbb{R}. \tag{1.25}$$

Definition 6. A system is said to be *BIBO-stable* if

$$\|u(t)\| \leq \varepsilon \quad \forall t \geq 0, \text{ and } x(0) = 0 \Rightarrow \|y(t)\| < \delta \quad \forall t \geq 0, \quad \varepsilon, \delta \in \mathbb{R}. \tag{1.26}$$

In words, for any bounded input, the output remains bounded.

Stability for LTI Systems

Above, we focused on general nonlinear system. However, in Control Systems I you learned that the output $y(t)$ for a LTI system of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.27}$$

can be written as

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t). \tag{1.28}$$

The transfer function relating input to output is a rational function

$$\begin{aligned} P(s) &= C(s\mathbb{I} - A)^{-1}B + D \\ &= \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} + d. \end{aligned} \tag{1.29}$$

Furthermore, it holds:

- The zeros of the numerator of Equation (1.29) are the *zeros* of the system, i.e. the values s_i which fulfill
$$P(s_i) = 0. \tag{1.30}$$

- The zeros of the denominator of Equation (1.29) are the *poles* of the system, i.e. the values s_i which fulfill $\det(s_i\mathbb{I} - A) = 0$, or, in other words, the eigenvalues of A.

One can show, that the following Theorem holds:

Theorem 1. The equilibrium $\hat{x} = 0$ of a linear time invariant system is stable if and only if the following two conditions are met:

1. For all $\lambda \in \sigma(A), \quad \text{Re}(\lambda) \leq 0$.
2. The algebraic and geometric multiplicity of all $\lambda \in \sigma(A)$ such that $\text{Re}(\lambda) = 0$ are equal.

Remark. We won't go into the proof of this theorem, because beyond the scope of the course. As an intuition, however, one can look at Equation (1.28). As you learned in Linear Algebra, the matrix exponential computation can be simplified with help of the diagonalization of a matrix. Moreover, if matrix A is diagonalizable, you can derive a form where you are left with exponential terms of the eigenvalues of A on the diagonal. If these eigenvalues are bigger than 0, the exponentials, which depends on time, diverge. If these eigenvalues are smaller than zero, the exponentials converge to 0 (asymptotically stable behaviour). In the case of zero eigenvalues, the exponentials converge, but not to 0 (stable behaviour). If the matrix A is not diagonalizable, i.e. the algebraic and the geometric multiplicity of an eigenvalue do not coincide, one should recall the Jordan form. In this case, some polynomial terms may be multiplied with the exponential ones in the diagonal: this could lead to unstable behaviour (stable vs. unstable because of 0 eigenvalue). For the rigorous proof of the Theorem, go to `https://en.wikibooks.org/wiki/Control_Systems/State-Space_Stability`.

1.2.2 Analysis

Which tools do we already know in order to analyze nominal stability? In the course Control Systems I you learned about

- **Root locus.** In order to recall the root locus method, have a look at Example 4.15, page 123 in [2].
- **Bode diagram:** the Bode diagram is a frequency explicit representation of the magnitude $|L(j\omega)|$ and the phase $\angle(L(j\omega))$ of a complex number $L(j\omega)$. Because of graphic reasons, one uses decibel (dB) as unit for the amplitude and degrees as unit for the phase. As a reminder, the conversion reads

$$\mathbb{X}_{\text{dB}} = 20 \cdot \log_{10}(\mathbb{X}), \quad \mathbb{X} = 10^{\frac{\mathbb{X}_{\text{dB}}}{20}}. \tag{1.31}$$

  Moreover, stable and unstable poles and zeros have specific consequences on the Bode diagram:

  - Poles cause a gradient of $-20\frac{\text{dB}}{\text{decade}}$ in the amplitude:

Pole	stable	unstable
Magnitude	$-20\frac{\text{dB}}{\text{decade}}$	$-20\frac{\text{dB}}{\text{decade}}$
Phase	$-90°$	$90°$

– Zeros cause a gradient of $20\frac{\text{dB}}{\text{decade}}$ in the amplitude:

Zero	stable	unstable
Magnitude	$20\frac{\text{dB}}{\text{decade}}$	$20\frac{\text{dB}}{\text{decade}}$
Phase	$90°$	$-90°$

An example of a Bode diagram is depicted in Figure 3.

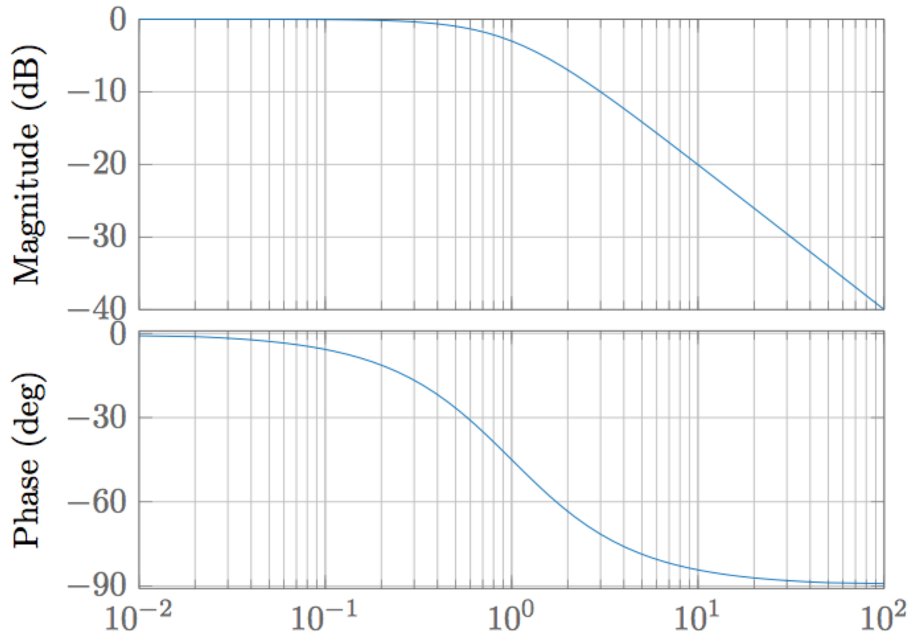

Figure 3: Example of a Bode diagram.

- **Nyquist diagram:** the Nyquist diagram is a frequency implicit representation of the complex number $L(j\omega)$ in the complex plane. An example of a Nyquist diagram is shown in Figure 4.

  Remark. In order to draw the Nyquist diagram, some useful limits can be computed:

$$\lim_{\omega\to 0} L(j\omega), \quad \lim_{\omega\to\infty} L(j\omega), \quad \lim_{\omega\to\infty} \angle L(j\omega). \tag{1.32}$$

- **Nyquist theorem:** a closed-loop system $T(s)$ is asymptotically stable if

$$n_c = n_+ + \frac{n_0}{2} \tag{1.33}$$

  holds, where

  – n_c: Number of mathematical positive encirclements of $L(s)$ about critical point -1 (counterclockwise).

  – n_+: Number of unstable poles of $L(s)$ ($Re(\pi) > 0$).

  – n_0: Number marginal stable poles of $L(s)$ ($Re(\pi) = 0$).

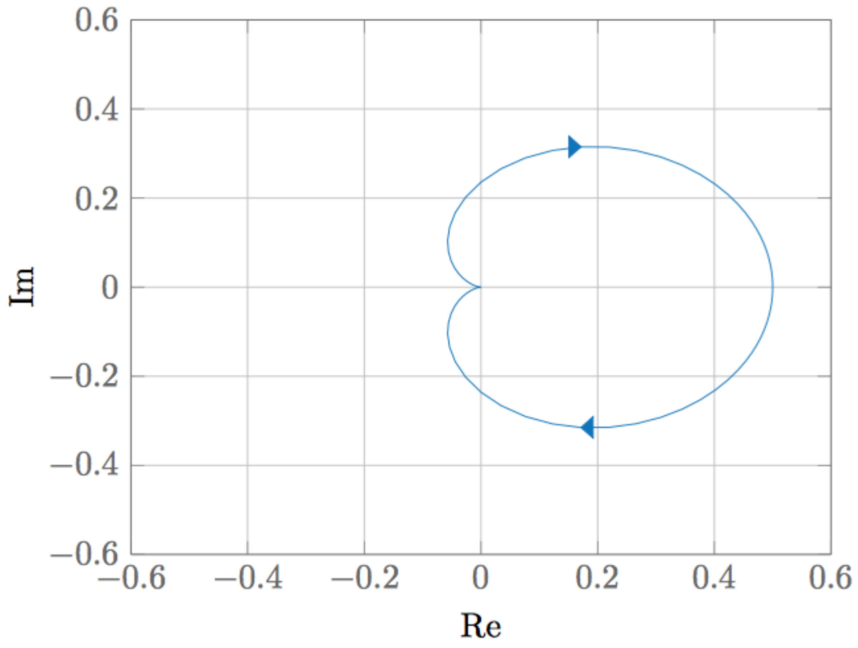

Figure 4: Example of a Nyquist diagram.

1.2.3 Synthesis: Loop Shaping

Plant inversion

As seen in the feedforward example, this method isn't indicated for non-minimum phase plants and for unstable plants: in those cases this would lead to non-minimum phase or unstable controllers. This method is indicated for *simple* systems for which it holds

- Plant is asymptotically stable.
- Plant is minimum phase.

The method is then based on a simple step:

$$L(s) = C(s) \cdot P(s) \Rightarrow C(s) = L(s) \cdot P(s)^{-1}. \tag{1.34}$$

The choice of the loop gain is free: it can be chosen such that it meets the desired specifications.

Loop shaping for Non-minimum Phase systems

A non-minimum phase system shows a wrong response: a change in the input results in a change in sign, that is, the system initially lies. Our controller should therefore be *patient* and for this reason one should use a *slow* control system. This is obtained by a crossover frequency that is smaller than the non-minimum phase zero. One begins to design the controller with a **PI-Controller**, which has the form

$$C(s) = k_p \cdot \frac{T_i \cdot s + 1}{T_i \cdot s}. \tag{1.35}$$

The parameters k_p and T_i can be chosen such that the loop gain $L(s)$ meets the known specifications. One can reach better robustness with **Lead/Lag** elements of the form

$$C(s) = \frac{T \cdot s + 1}{\alpha \cdot T \cdot s + 1}. \tag{1.36}$$

where $\alpha, T \in \mathbb{R}^+$. One can understand the Lead and the Lag elements as

- $\alpha < 1$: **Lead-Element**:
  - Phase margin increases.
  - Loop gain increases.
- $\alpha > 1$: **Lag-Element**:
  - Phase margin decreases.
  - Loop gain decreases.

As one can see in Figure 5 and Figure 6, the maximal benefits are reached at frequencies $(\hat{\omega})$, where the drawbacks are not yet fully developed.

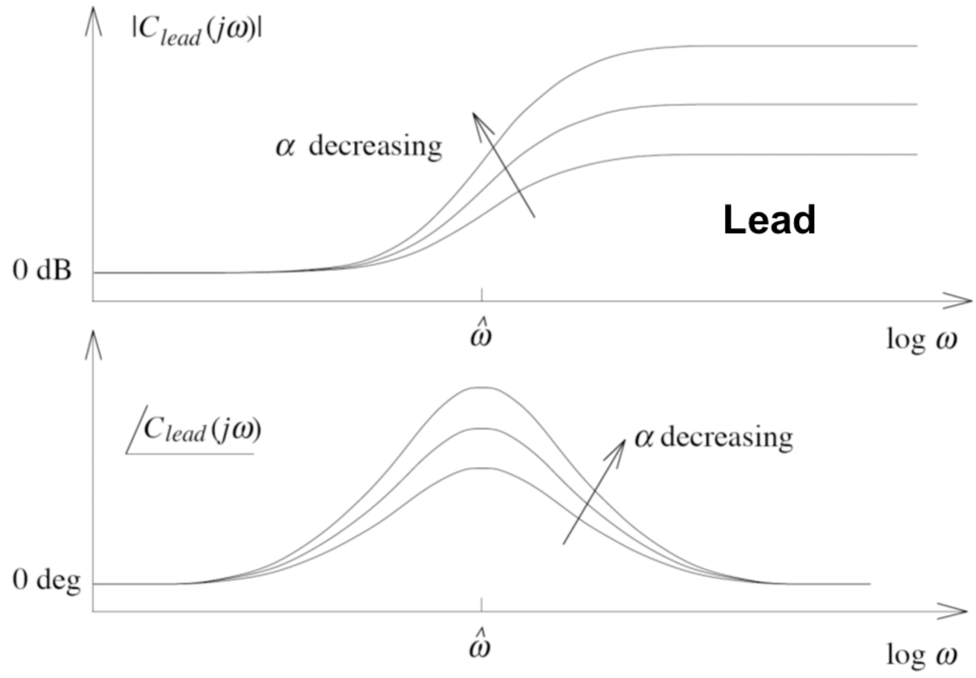

Figure 5: Bodeplot of the Lead Element

The element's parameters can be calculated as

$$\alpha = \left(\sqrt{\tan^2(\hat{\varphi}) + 1} - \tan(\varphi)\right)^2 = \frac{1 - \sin(\hat{\varphi})}{1 + \sin(\hat{\varphi})} \tag{1.37}$$

and

$$T = \frac{1}{\hat{\omega} \cdot \sqrt{\alpha}}. \tag{1.38}$$

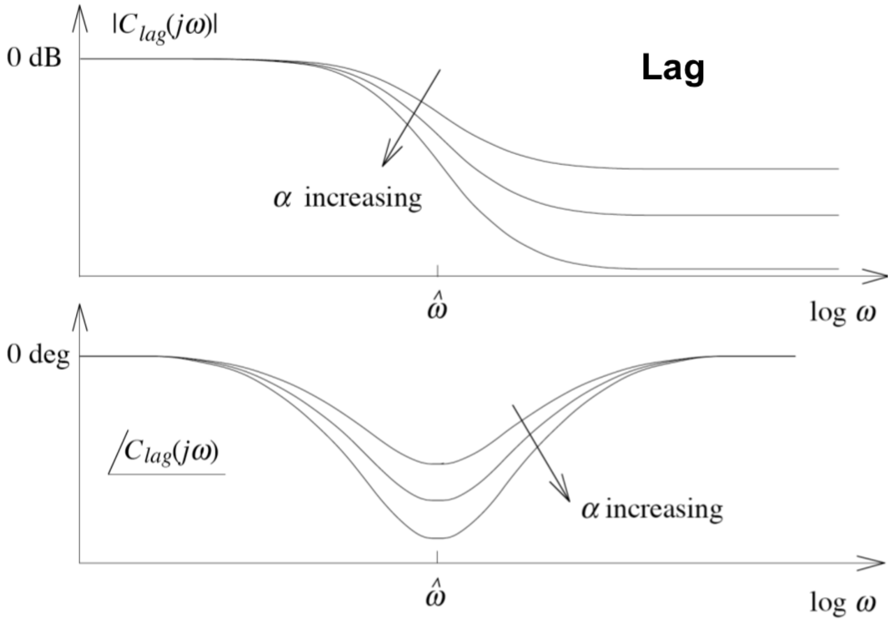

Figure 6: Bodeplot of the Lag Element

where $\hat{\omega}$ is the desired **center frequency** and $\hat{\varphi} = \varphi_{new} - \varphi$ is the desired **maximum phase shift** (in *rad*).

The classic loop-shaping method reads:

1. Design of a PI(D) controller.
2. Add Lead/Lag elements where needed[2]
3. Set the gain of the controller k_p such that we reach the desired crossover frequency.

Loop shaping for unstable systems

Since the Nyquist theorem should always hold, if it isn't the case, one has to design the controller such that $n_c = n_+ + \frac{n_0}{2}$ is valid. To remember is: stable poles *decrease* the phase by 90° and minimum phase zeros *increase* the phase by 90°.

Realizability

Once the controller is designed, one has to look if this is really feasible and possible to implement. That is, if the number of poles is equal or bigger than the number of zeros of the system. If that is not the case, one has to add poles at high frequencies, such that they don't affect the system near the crossover frequency. One could e.g. add to a PID controller a *Roll-Off Term* as

$$C(s) = \underbrace{k_p \cdot \left(1 + \frac{1}{T_i \cdot s} + T_d \cdot s\right)}_{\text{PID Controller}} \cdot \underbrace{\frac{1}{(\tau \cdot s + 1)^2}}_{\text{Roll-Off Term}} . \tag{1.39}$$

[2] $L(j\omega)$ often suits not the learned requirements

1.2.4 Performance

Under performance, one can understand two specific tasks:

- **Regulation/disturbance rejection:** Keep a setpoint despite disturbances, i.e. keep $y(t)$ at $r(t)$. As an example, you can imagine you try to keep driving your Duckiebot at a constant speed towards a cooling fan.
- **Reference Tracking**: Reference following, i.e. let $y(t)$ track $r(t)$. As an example, imagine a luxury Duckiebot which carries Duckiecustomers: a temperature controller tracks the different temperatures which the different Duckiecustomers may want to have in the Duckiebot.

1.2.5 Robustness

$$\textit{All models are wrong, but some of them are useful.} \tag{1.40}$$

A control system is said to be **robust** when it is insensive to model uncertainties. But why should a model have uncertainties? Essentially, for the following reasons:

- **Aging**: the model that was good a year ago, maybe is not good now. As an example, think of the wheel deterioration which could cause slip in a Duckiebot.
- **Poor system identification:** there are entire courses dedicated to the art of system modeling. It is not possible not to come to assumptions, which simplify your real system to something that does not *perfectly* describe that.

1.3 The Bode's Integral Formula

As we have learned in the previous section, a control systems must satisfy specific performance conditions on the sensitivity functions (also called Gang of Four). As we have seen, the sensitivity function S refers to the disturbance attenuation and relates the tracking error e to the reference signal. As stated in the previous section, one wants the sensitivity to be small over the range of frequencies where small tracking error and good disturbance rejection are desired. Let's introduce the next concepts with an example:

Example 4. (**11.10 Murray**) We consider a closed loop system with loop transfer function

$$L(s) = P(s)C(s) = \frac{k}{s+1}, \tag{1.41}$$

where k is the gain of the controller. Computing the sensitivity function for this loop transfer function results in

$$\begin{aligned} S(s) &= \frac{1}{1+L(s)} \\ &= \frac{1}{1+\frac{k}{s+1}} \\ &= \frac{s+1}{s+1+k}. \end{aligned} \tag{1.42}$$

By looking at the magnitude of the sensitivity function, one gets

$$|S(j\omega)| = \sqrt{\frac{1+\omega^2}{1+2k+k^2+\omega^2}}. \tag{1.43}$$

One notes, that this magnitude $|S(j\omega)| < 1$ for all finite frequencies and can be made as small as desired by choosing a sufficiently large k.

Theorem 2. Bode's integral formula. Assume that the loop transfer function $L(s)$ of a feedback system goes to zero faster than $\frac{1}{s}$ as $s \to \infty$, and let $S(s)$ be the sensitivity function. If the loop transfer function has poles p_k in the right-half-plane, then the sensitivity function satisfies the following integral:

$$\begin{aligned}\int_0^\infty \log|S(j\omega)|\mathrm{d}\omega &= \int_0^\infty \log\frac{1}{|1+L(j\omega)|}\mathrm{d}\omega \\ &= \pi\sum p_k.\end{aligned} \tag{1.44}$$

This is usually called the *principle of conservation of dirt.*

What does this mean?

- Low sensitivity is desirable across a broad range of frequencies. It implies disturbance rejection and good tracking.
- *So much dirt we remove at some frequency, that much we need to add at some other frequency.* This is also called the waterbed effect.

This can be resumed with Figure 7

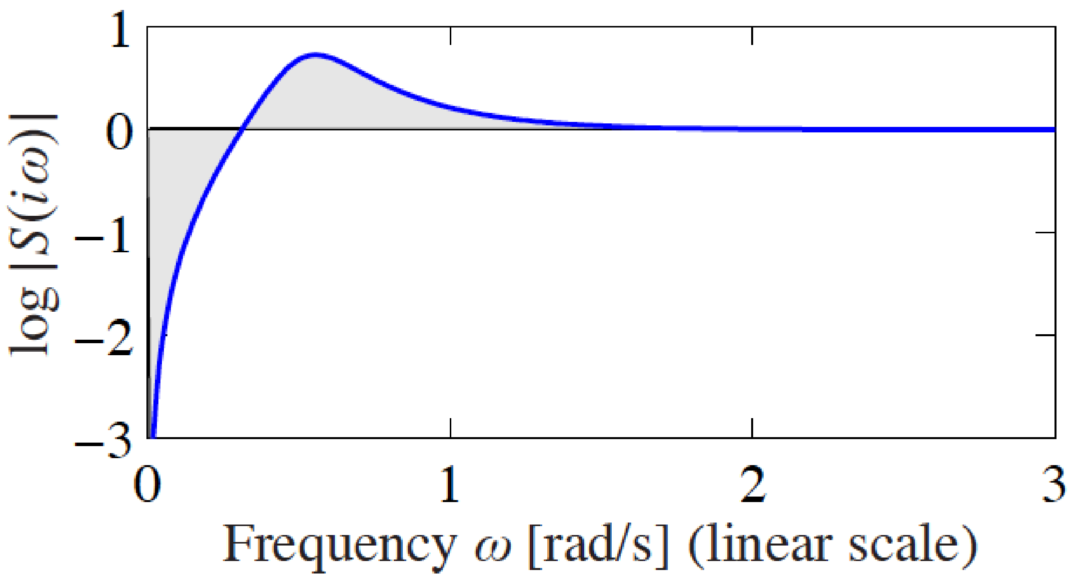

Figure 7: Waterbed Effect.

Theorem 3. (Second waterbed formula) Suppose that $L(s)$ has a single real RHP-zero z or a complex conjugate pair of zeros $z = x \pm jy$ and has N_p RHP-poles p_i. Let $\bar{p}_i$ denote the complex conjugate of p_i. Then for closed-loop stability, the sensitivity function must satisfy

$$\int_0^\infty \ln|S(j\omega)|\cdot w(z,\omega)\mathrm{d}\omega = \pi\prod_{I=1}^{Np}|\frac{p_i+z}{\bar{p}_i-z}|, \tag{1.45}$$

where

$$\begin{cases} w(z,\omega) = \frac{2z}{z^2+\omega^2}, & \text{if real zero} \\ w(z,\omega) = \frac{x}{x^2+(y-\omega)^2} + \frac{x}{x^2+(y+\omega)^2}, & \text{if complex zero.} \end{cases} \tag{1.46}$$

Summarizing, unstable poles close to RHP-zeros make a plant difficult to control. These weighting functions make the argument of the integral negligible at $\omega > z$. A RHP-zero reduces the frequency range where we can distribute dirt, which implies a higher peak for $S(s)$ and hence disturbance amplification.

1.4 Examples

Example 5. The dynamic equations of a system are given as

$$\begin{aligned} \dot{x}_1(t) &= x_1(t) - 5x_2(t) + u(t), \\ \dot{x}_2(t) &= -2x_1(t), \\ \dot{x}_3(t) &= -x_2(t) - 2x_3(t), \\ y(t) &= 3x_3(t). \end{aligned} \tag{1.47}$$

(a) Draw the **Signal Diagram** of the system.

(b) Find the **state space description** of the above system.

Solution.

(a) The **Signal Diagram** reads

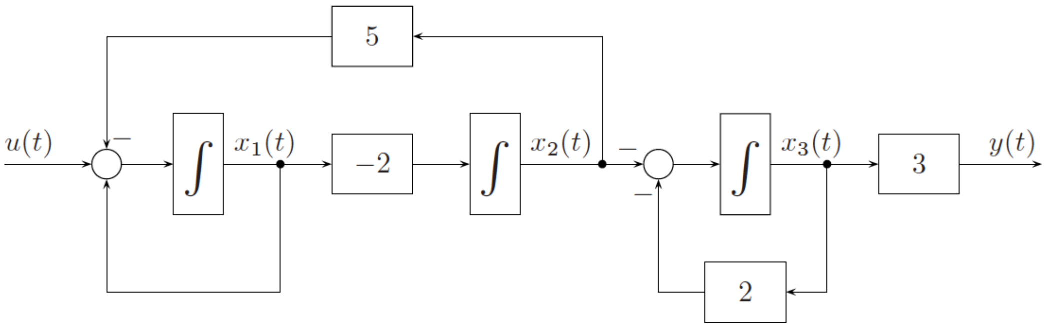

Figure 8: Signal Diagram of the system.

(b) The state space description has the following matrices:

$$A = \begin{pmatrix} 1 & -5 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & -2 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 & 0 & 3 \end{pmatrix}, \qquad d = 0. \tag{1.48}$$

Example 6. Some friends from a new startup of ETH called SpaghETH and want you to help them linearizing the system which describes their ultimate invention. The company is active in the market of the food trucks and sells pasta on the Polyterrasse everyday. They decided to automate the blending of the pasta in the water by carefully optimizing its operation. A sketch of the revolutionary system is shown in Figure 63.

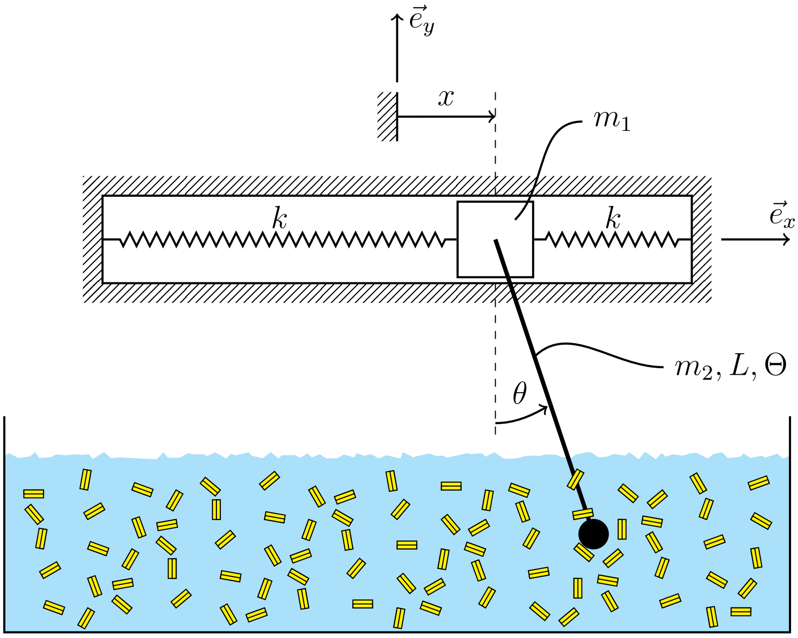

Figure 9: Sketch of the system.

The blender is modeled as a bar of mass m_2, length L, and moment of inertia (w.r.t. center of mass) $\Theta = \frac{1}{12}m_2L^2$. The blender is attached to a point mass m_1. In order to deal with possible vibrations that might occur in the system, the mass is attached to two springs with spring constant k.
The equations of motion of the system are given by

$$\begin{aligned}(m_1+m_2)\ddot{x} + \frac{1}{2}m_2L\left(-\dot{\theta}^2\sin(\theta)\right) + 2kx = 0\\ m_2\left(\frac{L^2}{12}\right)\ddot{\theta} + \frac{1}{2}m_2\dot{x}L\dot{\theta}\sin(\theta) + m_2g\frac{L}{2}\sin(\theta) = 0.\end{aligned} \tag{1.49}$$

a) How would you choose the state space vector in order to linearize this system? Write the system in a new form, with the chosen state space vector.

b) Linearize this system around the equilibrium point where all states are zero except $\dot{x}(0) = t \in \mathbb{R}$ is constat in order to find the matrix A.
Hint: Note that no input and no output are considered here, i.e. just the computations for the matrix A are required.

Solution.

a) The variables x and θ arise from the two equations of motion. Since the equation of motion have order two with respect to these variables, we augment the state and consider also $\dot{x}$ and $\dot{\theta}$. Hence, the state space vector $s(t)$ reads

$$s(t) = \begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ \theta(t) \\ \dot{x}(t) \\ \dot{\theta}(t) \end{pmatrix}. \tag{1.50}$$

By re-writing the equations of motion with the state space vector one gets

$$\dot{s}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{\theta}(t) \\ \ddot{x}(t) \\ \ddot{\theta}(t) \end{pmatrix} = \begin{pmatrix} s_3(t) \\ s_4(t) \\ \frac{1}{(m_1+m_2)} \cdot \left(\frac{1}{2} m_2 L s_4(t)^2 \sin(s_2(t)) - 2k s_1(t)\right) \\ -\frac{1}{L} \cdot (6 s_3(t) s_4(t) \sin(s_2(t)) + 6g \sin(s_2(t))) \end{pmatrix} := f \tag{1.51}$$

b) The linearization of the system reads

$$\begin{aligned} A = \frac{\partial f}{\partial s}\Big|_{s_{\text{eq.}}} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{2k}{(m_1+m_2)} & \frac{\frac{1}{2} m_2 L s_4(t)^2 \cos(s_2(t))}{(m_1+m_2)} & 0 & \frac{m_2 L s_4 \sin(s_2)}{(m_1+m_2)} \\ 0 & -\frac{6 s_3(t) s_4(t) \cos(s_2(t)) + 6g\cos(s_2(t))}{L} & -\frac{6 s_4(t) \sin(s_2(t))}{L} & -\frac{6 s_3(t) \sin(s_2(t))}{L} \end{pmatrix}\Bigg|_{s_{\text{eq}}} \\ &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{2k}{(m_1+m_2)} & 0 & 0 & 0 \\ 0 & -\frac{6g}{L} & 0 & 0 \end{pmatrix}. \end{aligned} \tag{1.52}$$

Example 7. You are given the following matrix:

$$A = \begin{pmatrix} -3 & 4 & -4 \\ 0 & 5 & -8 \\ 0 & 4 & -7 \end{pmatrix}. \tag{1.53}$$

a) Find the eigenvalues of A.

b) Find the eigendecomposition of matrix A, i.e. compute its eigenvectors and determine T and D s.t. $A = TDT^{-1}$.

c) You are given a system of the form

$$\begin{aligned} x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{1.54}$$

Can you conclude something about the stability of the system?

d) What if you have

$$A = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \tag{1.55}$$

instead?

Solution.

a) The eigenvalues of A should fulfill

$$\det(A - \lambda\mathbb{I}) = 0. \tag{1.56}$$

It follows

$$\begin{aligned}\det(A - \lambda\mathbb{I}) &= \det\begin{pmatrix} -3-\lambda & 4 & -4 \\ 0 & 5-\lambda & -8 \\ 0 & 4 & -7-\lambda \end{pmatrix} \\ &= (-3-\lambda)\det\begin{pmatrix} 5-\lambda & -8 \\ 4 & -7-\lambda \end{pmatrix} \\ &= -(3+\lambda)\cdot[(5-\lambda)\cdot(-7-\lambda)-(-24)] \\ &= -(3+\lambda)^2\cdot(\lambda-1).\end{aligned} \tag{1.57}$$

This means that matrix A has eigenvalues

$$\lambda_{1,2} = -3, \quad \lambda_3 = 1. \tag{1.58}$$

b) In order to compute the eigendecomposition of A, we need to compute its eigenvectors with respect to its eigenvalues. The eigenvectors v_i should fulfill

$$(A - \lambda\mathbb{I})v_i = 0. \tag{1.59}$$

It holds:

- $E_{\lambda_1} = E_{-3}$: From $(A - \lambda_1\mathbb{I})\cdot x = 0$ one gets the linear system of equations

$$\left(\begin{array}{ccc|c} 0 & 4 & -4 & 0 \\ 0 & 8 & -8 & 0 \\ 0 & 4 & -4 & 0 \end{array}\right).$$

  Using the first row as reference and subtracting it from the other two rows, one gets the form

$$\left(\begin{array}{ccc|c} 0 & 4 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

  Since one has two zero rows, one can introduce two free parameters. Let $x_1 = s$, $x_2 = t$, $s, t \in \mathbb{R}$. Using the first row, one can recover $x_3 = x_2 = t$. This defines the first eigenspace, which reads

$$E_{-3} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \tag{1.60}$$

  Note that since we have introduced two free parameters, the geometric multiplicity of $\lambda_{1,2} = -3$ is 2.

- $E_{\lambda_2} = E_1$: From $(A - \lambda_2\mathbb{I})\cdot x = 0$, one gets the linear system of equations

$$\left(\begin{array}{ccc|c} -4 & 4 & -4 & 0 \\ 0 & 4 & -8 & 0 \\ 0 & 4 & -8 & 0 \end{array}\right).$$

Subtracting the second row from the third row results in the form

$$\left(\begin{array}{ccc|c} -4 & 4 & -4 & 0 \\ 0 & 4 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Since one has a zero row, one can introduce a free parameter. Let $x_3 = u$, $u \in \mathbb{R}$. It follows $x_2 = 2t$ and $x_1 = t$. The second eigenspace hence reads

$$E_1 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\} \tag{1.61}$$

Note that since we have introduced one free parameter, the geometric multiplicity of $\lambda_3 = 1$ is 1.

Since the algebraic and geometric multiplicity concide for every eigenvalue of A, the matrix is diagonalizable. With the computed eigenspaces, one can build the matrix T as

$$T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \tag{1.62}$$

and D as a diagonal matrix with the eigenvalues on the diagonal:

$$D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.63}$$

These T and D are guaranteed to satisfy $A = TDT^{-1}$.

c) Because of $\lambda_3 = 1 > 0$ one can conclude that the system is unstable in the sense of Lyapunov.

d) Because the new matrix A contains only 0 elements above the diagonal, one can clearly see that its eigenvalues are

$$\lambda_1 = -1, \quad \lambda_{2,3} = 0. \tag{1.64}$$

The eigenvalue $\lambda_1 = -1$ leads to asymptotically stable behaviour. The eigenvalue $\lambda_{2,3} = 0$ has algebraic multiplicity of 2, which means that, in order to have a marginally stable system, its geometric multiplicity should be 2 as well. It holds

- $E_{\lambda_{2,3}} = E_0$: From $(A - \lambda_{2,3}\mathbb{I}) \cdot x = 0$, one gets the linear system of equations

$$\left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right),$$

which clearly has $x_1 = 0$, $x_2 = x_2 = 0$ and $x_3 = t \in \mathbb{R}$. Since we introduced only one free parameter, the geometric multiplicity of this eigenvalue is only 1, which means that the system is unstable.

Example 8. Assume that the standard feedback control system structure depicted in Figure 10 is given.

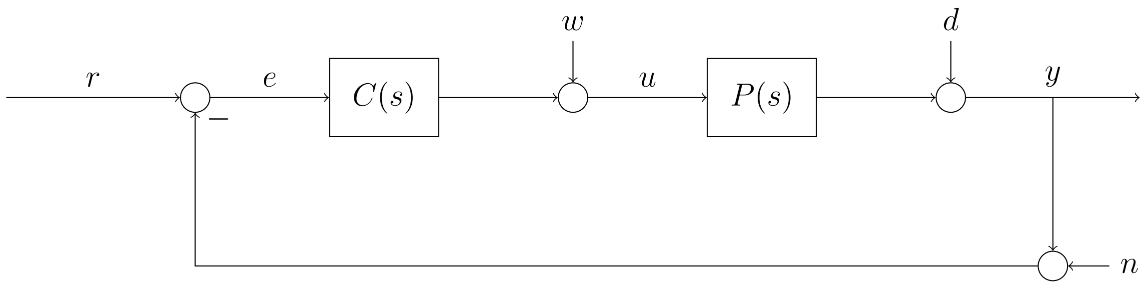

Figure 10: Standard feedback control system structure (simplified w.r.t. the lecture notes).

a) Assume that $R(s) = 0$. What $T(s)$ and $S(s)$ would you prefer? Is this choice possible?

Solution.

a) If the reference signal $R(s)$ is 0 for all times, the desired output signal $Y(s)$ should be zero for all times as well. Thus, referring to the closed loop dynamics

$$Y(s) = S(s) \cdot (D(s) + P(s) \cdot W(s)) + T(s) \cdot (R(s) - N(s)), \tag{1.65}$$

the preferred choice would be

$$T(s) = S(s) = 0. \tag{1.66}$$

In this case, all the disturbances and noise would be suppressed. However, this choice is **not** possible. This can be easily checked by looking at the constraint

$$S(s) + T(s) = \frac{1}{1 + L(s)} + \frac{L(s)}{1 + L(s)} = \frac{1 + L(s)}{1 + L(s)} = 1. \tag{1.67}$$

This result has a key importance for the following discussions. In fact, at a fixed frequency s, either $S(s)$ or $T(s)$ can be 0 but not both. In other words, it is not possible to suppress both disturbances *and* noise in the same frequency band.

2 Digital Control

2.1 Signals and Systems

A whole course is dedicated to this topic (see Signals and Systems of professor D'Andrea). A *signal* is a function of time that represents a physical quantity.
Continuous-time signals are described by a function $x(t)$ such that this takes continuous values.
Discrete-time Signals differ from continuous-time ones because of a sampling procedure. Computers don't understand the concept of continuous-time and therefore sample the signals, i.e. measure signal's informations at specific time instants. Discrete-time systems are described by a function

$$x[n] = x(n \cdot T_s), \tag{2.1}$$

where T_s is the **sampling time**. The **sampling frequency** is defined as $f_s = \frac{1}{T_s}$.
One can understand the difference between the two descriptions by looking at Figure 11.

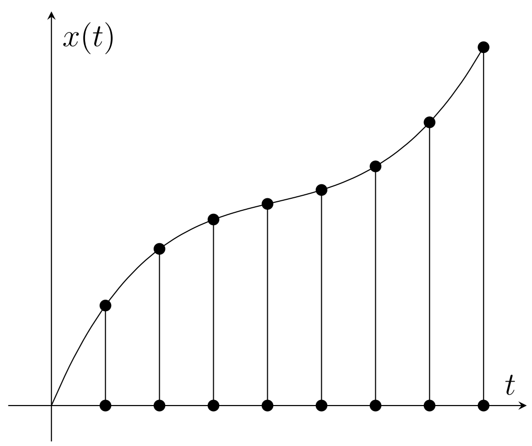

Figure 11: Continuous-Time versus Discrete-Time representation

Advantages of Discrete-Time analysis

- Calculations are easier. Moreover, integrals become sums and differentiations become finite differences.
- One can implement complex algorithms.

Disadvantages of Discrete-Time analysis

- The sampling introduces a delay in the signal ($\approx e^{-\frac{sT_s}{2}}$).
- The informations between two samplings, that is between $x[n]$ and $x[n+1]$, are lost.

Every controller which is implemented on a microprocessor is a discrete-time system.

2.2 Discrete-Time Control Systems

Nowadays, controls systems are implemented in microcontrollers or in microprocessors in discrete-time and really rarely (see the lecture *Elektrotechnik II* for an example) in continuous-time. As defined, although the processes are faster and easier, the informations are still sampled and there is a certain loss of data. But how are we flexible about information loss? What is acceptable and what is not? The concept of **aliasing** will help us understand that.

2.2.1 Aliasing

If the sampling frequency is chosen too low, i.e. one measures less times pro second, the signal can become poorly determined and the loss of information is too big to reconstruct it uniquely. This situation is called **aliasing** and one can find many examples of that in the real world. Let's have a look to an easy example:

Example 9. You are finished with your summer's exam session and you are flying to Ibiza, to finally enjoy the sun after a summer spent at ETH. You decide to film the turbine of the plane because, although you are on holiday, you have an engineer's spirit. You land in Ibiza and, as you get into your hotel room, you want you have a look at your film. The rotation of the turbine's blades you observe looks different to what it is supposed to be, and since you haven't drunk yet, there must be some scientific reason. In fact, the sampling frequency of your phone camera is much lower than the turning frequency of the turbine: this results in a loss of information and hence in a wrong perception of what is going on.

Let's have a more mathematical approach. Let's assume a signal

$$x_1(t) = \cos(\omega \cdot t). \tag{2.2}$$

After discretization, the sampled signal reads

$$x_1[n] = \cos(\omega \cdot T_s \cdot n) = \cos(\Omega \cdot n), \quad \Omega = \omega \cdot T_s. \tag{2.3}$$

Let's assume a second signal

$$x_2(t) = \cos\left(\left(\omega + \frac{2\pi}{T_s}\right) \cdot t\right), \tag{2.4}$$

where the frequency

$$\omega_2 = \omega + \frac{2\pi}{T_s}. \tag{2.5}$$

is given. Using the periodicity of the cos function, the discretization of this second signal reads

$$\begin{aligned} x_2[n] &= \cos\left(\left(\omega + \frac{2\pi}{T_s}\right) \cdot T_s \cdot n\right) \\ &= \cos\left(\omega \cdot T_s \cdot n + 2\pi \cdot n\right) \\ &= \cos(\omega \cdot T_s \cdot n) \\ &= x_1[n]. \end{aligned} \tag{2.6}$$

Although the two signals have different frequencies, they are equal when discretized. For this reason, one has to define an interval of *good* frequencies, where aliasing doesn't occur. In particular it holds

$$|\omega| < \frac{\pi}{T_s} \tag{2.7}$$

or

$$f < \frac{1}{2 \cdot T_s} \Leftrightarrow f_s > 2 \cdot f_{max}. \tag{2.8}$$

The maximal frequency accepted is $f = \frac{1}{2 \cdot T_s}$ and is called **Nyquist frequency**. In order to ensure good results, one uses in practice a factor of 10.

$$f < \frac{1}{10 \cdot T_s} \Leftrightarrow f_s > 10 \cdot f_{max}. \tag{2.9}$$

For control systems the crossover frequency should be

$$f_s \geq 10 \cdot \frac{\omega_c}{2\pi}. \tag{2.10}$$

2.2.2 Discrete-time Control Loop Structure

The discrete-time control loop structure is depicted in Figure 12. This is composed of

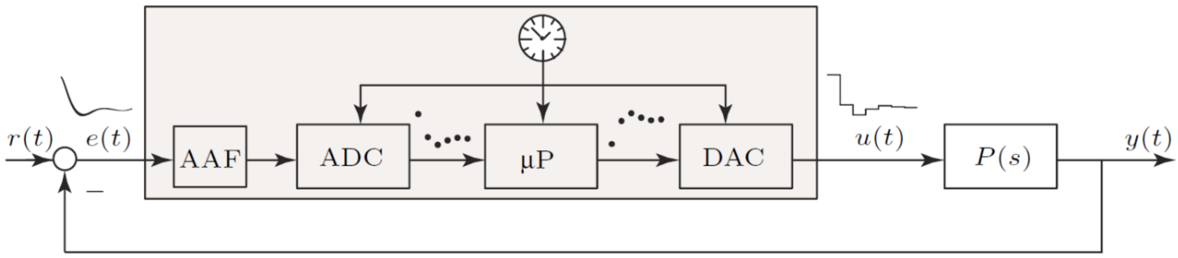

Figure 12: Control Loop with AAF.

different elements, which we list and describe in the following paragraphs.

Anti Aliasing Filter (AAF)

In order to solve this problem, an **Anti Aliasing Filter (AAF)** is used. The Anti Aliasing Filter is an **analog** filter and not a discrete one. In fact, we want to eliminate unwanted frequencies before sampling, because after that is *too late* (refer to Figure 12). But how can one define unwanted frequencies? Those frequencies are normally the higher frequencies of a signal [3]. Because of that, as AAF one uses normally a **low-pass filter**. This type of filter *lets* low frequencies *pass* and blocks higher ones[4]. The mathematic formulation of a first-order low-pass filter is given by

$$\mathrm{lp}(s) = \frac{k}{\tau \cdot s + 1}. \tag{2.11}$$

where k is the gain and τ is the time constant of the system. The drawback of such a filter is problematic: the filter introduces additional unwanted phase that can lead to unstable behaviours.

[3]Keep in mind: high signal frequency means problems by lower sampling frequency!

[4]This topic is exhaustively discussed in the course Signals and Systems, offered in the fifth semester by Prof. D'Andrea.

Analog to Digital Converter (ADC)

At each discrete time step $t = k \cdot T$ the **ADC** converts a voltage $e(t)$ to a digital number following a sampling frequency.

Microcontroller (μP)

This is a discrete-time controller that uses the sampled discrete-signal and gives back a dicrete output.

Digital to Analog Converter (DAC)

In order to convert back the signal, the **DAC** applies a **zero-order-hold (ZOH)**. This introduces an extra delay of $\frac{T}{2}$ (refer to Figure 13).

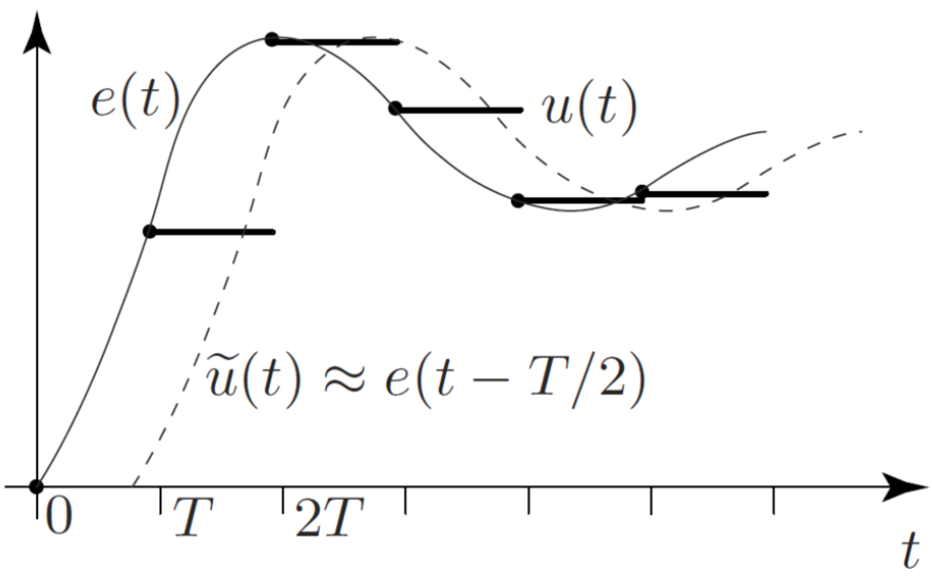

Figure 13: Zero-Order-Hold.

2.3 Controller Discretization/Emulation

In order to understand this concept, we have to introduce the concept of z-transform.

2.3.1 The z-Transform

From the Laplace Transform to the $z-$transform

The Laplace transform is an *integral transform* which takes a function of a real variable t to a function of a complex variable s. Intuitively, for control systems t represents time and s represents frequency.

Definition 7. The one-sided **Laplace transform** of a signal $x(t)$ is defined as

$$\begin{aligned}\mathcal{L}\left(x(t)\right) &= X(s)\\ &= \tilde{x}(s)\\ &= \int_0^{\infty} x(t)e^{-st}\mathrm{d}t.\end{aligned} \tag{2.12}$$

Because of its definition, the Laplace transform is used to consider continuous-time signals/systems. In order to deal with discrete-time system, one must derive its discrete analogon.

Example 10. Consider $x(t) = \cos(\omega t)$. The Laplace transform of such a signal reads

$$\begin{aligned}\mathcal{L}(\cos(\omega t)) &= \int_0^{\infty} e^{-st}\cos(\omega t)\mathrm{d}t\\ &= -\frac{1}{s}e^{-st}\cos(\omega t)\Big|_0^{\infty} - \frac{\omega}{s}\int_0^{\infty} e^{-st}\sin(\omega t)\mathrm{d}t\\ &= \frac{1}{s} - \frac{\omega}{s}\left(-\frac{1}{s}e^{-st}\sin(\omega t)\Big|_0^{\infty} + \frac{\omega}{s}\int_0^{\infty} e^{-st}\cos(\omega t)\mathrm{d}t\right)\\ &= \frac{1}{s} - \frac{\omega^2}{s^2}\mathcal{L}(\cos(\omega t)).\end{aligned} \tag{2.13}$$

From this equation, one has

$$\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2+\omega^2} \tag{2.14}$$

Some of the known Laplace transforms are listed in Table 1.
Laplace transforms receive as inputs functions, which are defined in continuous-time. In order to analyze discrete-time system, one must derive its discrete analogue. Discrete time signals $x(kT) = x[k]$ are obtained by sampling a continuous-time function $x(t)$. A sample of a function is its ordinate at a specific time, called the sampling instant, i.e.

$$x[k] = x(t_k),\ t_k = t_0 + kT, \tag{2.15}$$

where T is the sampling period. A sampled function can be expressed through the multiplication of a continuous funtion and a Dirac comb (see reference), i.e.

$$x[k] = x(t)\cdot D(t), \tag{2.16}$$

with $D(t)$ which is a Dirac comb.

Definition 8. A Dirac comb, also known as sampling function, is a periodic distribution constructed from Dirac delta functions and reads

$$D(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{2.17}$$

Remark. An intuitive explanation of this, is that this function is 1 for $t = kT$ and 0 for all other cases. Since k is a natural number, i.e. $k = -\infty, \ldots, \infty$, applying this function to a continuous-time signal consists in considering informations of that signal spaced with the sampling time T.

Imagine to have a continuous-time signal $x(t)$ and to sample it with a sampling period T. The sampled signal can be described with the help of a Dirac comb as

$$\begin{aligned} x_m(t) &= x(t) \cdot \sum_{k=-\infty}^{\infty} \delta(t - kT) \\ &= \sum_{k=-\infty}^{\infty} x(kT) \cdot \delta(t - kT) \\ &= \sum_{k=-\infty}^{\infty} x[k] \cdot \delta(t - kT), \end{aligned} \tag{2.18}$$

where we denote $x[k]$ as the k-th sample of $x(t)$. Let's compute the Laplace transform of the sampled signal:

$$\begin{aligned} X_m(s) &= \mathcal{L}\left(x_m(t)\right) \\ \text{(a)} \qquad &= \int_0^{\infty} x_m(t) e^{-st} \mathrm{d}t \\ &= \int_0^{\infty} \sum_{k=-\infty}^{\infty} x[k] \cdot \delta(t - kT) e^{-st} \mathrm{d}t \\ \text{(b)} \qquad &= \sum_{k=-\infty}^{\infty} x[k] \cdot \int_0^{\infty} \delta(t - kT) e^{-st} \mathrm{d}t \\ \text{(c)} \qquad &= \sum_{k=-\infty}^{\infty} x[k] e^{-ksT}, \end{aligned} \tag{2.19}$$

where we used

(a) This is an application of Definition 7.

(b) The sum and the integral can be switched because the function $f(t) = \delta(t - kT)e^{-st}$ is non-negative. This is a direct consequence of the Fubini/Tonelli's theorem. If you are interested in this, have a look at `https://en.wikipedia.org/wiki/Fubini%27s_theorem`.

(c) This result is obtained by applying the Dirac integral property, i.e.

$$\int_0^{\infty} \delta(t - kT) e^{-st} \mathrm{d}t = e^{-ksT}. \tag{2.20}$$

By introducing the variable $z = e^{sT}$, one can rewrite Equation 2.19 as

$$X_m(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k}, \tag{2.21}$$

which is defined as the $z-$transform of a discrete time system. We have now found the relation between the z transform and the Laplace transform and are able to apply the concept to any discrete-time signal.

Definition 9. The bilateral z-**transform** of a discrete-time signal $x[k]$ is defined as

$$X(z) = \mathcal{Z}\left((x[k]\right) = \sum_{k=-\infty}^{\infty} x[k]z^{-k}. \tag{2.22}$$

Some of the known z-transforms are listed in Table 1.

$x(t)$	$\mathcal{L}(x(t))(s)$	$x[k]$	$X(z)$
1	$\frac{1}{s}$	1	$\frac{1}{1-z^{-1}}$
e^{-at}	$\frac{1}{s+a}$	e^{-akT}	$\frac{1}{1-e^{-aT}z^{-1}}$
t	$\frac{1}{s^2}$	kT	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
t^2	$\frac{2}{s^3}$	$(kT)^2$	$\frac{T^2z^{-1}\left(1+z^{-1}\right)}{(1-z^{-1})^3}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\sin(\omega kT)$	$\frac{z^{-1}\sin(\omega T)}{1-2z^{-1}\cos(\omega T)+z^{-2}}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\cos(\omega kT)$	$\frac{1-z^{-1}\cos(\omega T)}{1-2z^{-1}\cos(\omega T)+z^{-2}}$

Table 1: Known Laplace and $z-$transforms.

Properties

In the following we list some of the most important properties of the $z-$transform. Let $X(z), Y(z)$ be the $z-$transforms of the signals $x[k], y[k]$.

1. **Linearity**

$$\mathcal{Z}\left(ax[k] + by[k]\right) = aX(z) + bY(z). \tag{2.23}$$

Proof. It holds

$$\begin{aligned}\mathcal{Z}\left(ax[k] + by[k]\right) &= \sum_{k=-\infty}^{\infty} \left(ax[k] + by[k]\right) z^{-k} \\ &= \sum_{k=-\infty}^{\infty} ax[k]z^{-k} + \sum_{k=-\infty}^{\infty} by[k]z^{-k} \\ &= aX(z) + bY(z).\end{aligned} \tag{2.24}$$

□

2. **Time shifting**
$$\mathcal{Z}\left(x[k-k_0]\right) = z^{-k_0}X(z). \tag{2.25}$$

Proof. It holds
$$\mathcal{Z}\left(x[k-k_0]\right) = \sum_{k=-\infty}^{\infty} x[k-k_0]z^{-k}. \tag{2.26}$$
Define $m = k - k_0$. It holds $k = m + k_0$ and
$$\begin{aligned}\sum_{k=-\infty}^{\infty} x[k-k_0]z^{-k} &= \sum_{k=-\infty}^{\infty} x[m]z^{-m}z^{-k_0} \\ &= z^{-k_0}X(z).\end{aligned} \tag{2.27}$$
□

3. **Convolution** $*$
$$\mathcal{Z}\left(x[k] * y[k]\right) = X(z)Y(z). \tag{2.28}$$

Proof. Follows directly from the definition of convolution. □

4. **Reverse time**
$$\mathcal{Z}\left(x[-k]\right) = X\left(\frac{1}{z}\right). \tag{2.29}$$

Proof. It holds
$$\begin{aligned}\mathcal{Z}\left(x[-k]\right) &= \sum_{k=-\infty}^{\infty} x[-k]z^{-k} \\ &= \sum_{r=-\infty}^{\infty} x[r]\left(\frac{1}{z}\right)^{-r} \\ &= X\left(\frac{1}{z}\right).\end{aligned} \tag{2.30}$$
□

5. **Scaling in z domain**
$$\mathcal{Z}\left(a^k x[k]\right) = X\left(\frac{z}{a}\right). \tag{2.31}$$

Proof. It holds
$$\begin{aligned}\mathcal{Z}\left(a^k x[k]\right) &= \sum_{k=-\infty}^{\infty} x[k]\left(\frac{z}{a}\right)^{-k} \\ &= X\left(\frac{z}{a}\right).\end{aligned} \tag{2.32}$$
□

6. **Conjugation**

$$\mathcal{Z}\left(x^*[k]\right) = X^*(z^*). \tag{2.33}$$

Proof. It holds

$$\begin{aligned} X^*(z) &= \left(\sum_{k=-\infty}^{\infty} x[k]z^{-k}\right)^* \\ &= \sum_{k=-\infty}^{\infty} x^*[k](z^*)^{-k}. \end{aligned} \tag{2.34}$$

Replacing z by z^* one gets the desired result. □

7. **Differentiation in z domain**

$$\mathcal{Z}\left(kx[k]\right) = -z\frac{\partial}{\partial z}X(z). \tag{2.35}$$

Proof. It holds

$$\begin{aligned} \frac{\partial}{\partial z}X(z) &= \frac{\partial}{\partial z}\sum_{k=-\infty}^{\infty} x[k]z^{-k} \\ \text{linearity of sum/derivative} \qquad &= \sum_{k=-\infty}^{\infty} x[k]\frac{\partial}{\partial z}z^{-k} \\ &= \sum_{k=-\infty}^{\infty} x[k](-k)z^{-k-1} \\ &= -\frac{1}{z}\sum_{k=-\infty}^{\infty} kx[k]z^{-k}, \end{aligned} \tag{2.36}$$

from which the statement follows. □

Approximations

In order to use this concept, often the exact solution is too complicated to compute and not needed for an acceptable result. In practice, approximations are used. Instead of considering the derivative as it is defined, one tries to approximate this via differences. Given $y(t) = \dot{x}(t)$, the three most used approximation methods are

- **Euler forward:**

$$y[k] \approx \frac{x[k+1]-x[k]}{T_s} \tag{2.37}$$

- **Euler backward:**

$$y[k] \approx \frac{x[k]-x[k-1]}{T_s} \tag{2.38}$$

- **Tustin method:**

$$\frac{y[k]-y[k-1]}{2} \approx \frac{x[k]-x[k-1]}{T_s} \tag{2.39}$$

Exact	$s = \frac{1}{T_s} \cdot \ln(z)$	$z = e^{s \cdot T_s}$
Euler forward	$s = \frac{z-1}{T_s}$	$z = s \cdot T_s + 1$
Euler backward	$s = \frac{z-1}{z \cdot T_s}$	$z = \frac{1}{1 - s \cdot T_s}$
Tustin	$s = \frac{2}{T_s} \cdot \frac{z-1}{z+1}$	$z = \frac{1 + s \cdot \frac{T_s}{2}}{1 - s \cdot \frac{T_s}{2}}$

Table 2: Discretization methods and substitution.

The meaning of the variable z can change with respect to the chosen discretization approach. Here, just discretization results are presented. You can try derive the following rules on your own. A list of the most used transformations is reported in Table 2. The different approaches are results of different Taylor's approximations[5]:

- **Euler Forward:**
$$z = e^{s \cdot T_s} \approx 1 + s \cdot T_s. \tag{2.40}$$
- **Euler Backward:**
$$z = e^{s \cdot T_s} = \frac{1}{e^{-s \cdot T_s}} \approx \frac{1}{1 - s \cdot T_s}. \tag{2.41}$$
- **Tustin:**
$$z = \frac{e^{s \cdot \frac{T_s}{2}}}{e^{-s \cdot \frac{T_s}{2}}} \approx \frac{1 + s \cdot \frac{T_s}{2}}{1 - s \cdot \frac{T_s}{2}}. \tag{2.42}$$

In practice, the most used approach is the Tustin transformation, but there are cases where the other transformations could be useful.

Example 11. You are given the differential relation

$$y(t) = \frac{\mathrm{d}}{\mathrm{d}t} x(t), \qquad x(0) = 0. \tag{2.43}$$

One can rewrite the relation in the frequency domain using the Laplace transform. Using the property for derivatives

$$\mathcal{L}\left(\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right) = s\mathcal{L}(f(t)) - f(0). \tag{2.44}$$

By Laplace transforming both sides of the relation and using the given initial condition, one gets

$$Y(s) = sX(s). \tag{2.45}$$

In order to discretize the relation, we sample with a generic sampling time T the signals. Forward Euler's method for the approximation of differentials reads

$$\dot{x}(kT) \approx \frac{x((k+1)T) - x(kT)}{T}. \tag{2.46}$$

[5]As reminder: $e^x \approx 1 + x$.

The discretized relation reads

$$y(kT) = \frac{x((k+1)T) - x(kT)}{T}. \tag{2.47}$$

In order to compute the z-transform of the relation, one needs to use its time shift property, i.e.

$$\mathcal{Z}\left(x((k-k_0)T)\right) = z^{-k_0}\mathcal{Z}(x(kT)). \tag{2.48}$$

In this case, the shift is of -1 and transforming both sides of the relation results in

$$Y(z) = \frac{zX(z) - X(z)}{T} = \frac{z-1}{T}X(z). \tag{2.49}$$

By using the relations of Equation (2.45) and Equation (2.49), one can write

$$s = \frac{z-1}{T}. \tag{2.50}$$

2.4 State Space Discretization

Starting from the continuous-time state space form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{2.51}$$

one wants to obtain the discrete-time state space representation

$$\begin{aligned} x[k+1] &= A_{\mathrm{d}}x[k] + B_{\mathrm{d}}u[k] \\ y[k] &= C_{\mathrm{d}}x[k] + D_{\mathrm{d}}u[k]. \end{aligned} \tag{2.52}$$

By recalling that $x[k+1] = x((k+1)T)$, one can start from the solution derived for continuous-time systems

$$x(t) = e^{At}x(0) + e^{At}\int_0^t e^{-A\tau}Bu(\tau)\mathrm{d}\tau. \tag{2.53}$$

By plugging into this equation $t = (k+1)T$, one gets

$$x((k+1)T) = e^{A(k+1)T}x(0) + e^{A(k+1)T}\int_0^{(k+1)T} e^{-A\tau}Bu(\tau)\mathrm{d}\tau \tag{2.54}$$

and hence

$$x(kT) = e^{AkT}x(0) + e^{AkT}\int_0^{kT} e^{-A\tau}Bu(\tau)\mathrm{d}\tau. \tag{2.55}$$

Since we want to write $x((k+1)T)$ in terms of $x(kT)$, we multiply all terms of Equation (2.55) by e^{AT} and rearrange the equation as

$$e^{A(k+1)T}x(0) = e^{AT}x(kT) - e^{A(k+1)T}\int_0^{kT} e^{-A\tau}Bu(\tau)\mathrm{d}\tau. \tag{2.56}$$

Substituting this result into Equation (2.54), one gets

$$
\begin{aligned}
x((k+1)T) &= e^{AT}x(kT) - e^{A(k+1)T}\int_0^{kT} e^{-A\tau}Bu(\tau)\mathrm{d}\tau + e^{A(k+1)T}\int_0^{(k+1)T} e^{-A\tau}Bu(\tau)\mathrm{d}\tau\\
&= e^{AT}x(kT) + e^{A(k+1)T}\int_{kT}^{(k+1)T} e^{-A\tau}Bu(\tau)\mathrm{d}\tau\\
&= e^{AT}x(kT) + \int_{kT}^{(k+1)T} e^{A[(k+1)T-\tau]}Bu(\tau)\mathrm{d}\tau\\
\text{(a)} \qquad &= e^{AT}x(kT) - \int_T^0 e^{A\alpha}B\mathrm{d}\alpha u(kT).\\
&= \underbrace{e^{AT}}_{A_\mathrm{d}}x[k] + \underbrace{\int_0^T e^{A\alpha}B\mathrm{d}\alpha}_{B_\mathrm{d}}\, u[k],
\end{aligned}
\tag{2.57}
$$

where we used

(a) $\alpha = (k+1)T - \tau, \quad \mathrm{d}\alpha = -\mathrm{d}\tau.$

It follows that

$$
\begin{aligned}
A_\mathrm{d} &= e^{AT},\\
B_\mathrm{d} &= \int_0^T e^{A\alpha}B\mathrm{d}\alpha,\\
C_\mathrm{d} &= C,\\
D_\mathrm{d} &= D.
\end{aligned}
\tag{2.58}
$$

Example 12. Given the general state space for in Equation (2.51), the forward Euler approach for differentials reads

$$
\dot{x} \approx \frac{x[k+1]-x[k]}{T_s}. \tag{2.59}
$$

Applying this to the generic state space formulation

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t)\\
y(t) &= Cx(t) + Du(t),
\end{aligned}
\tag{2.60}
$$

one gets

$$
\begin{aligned}
\frac{x[k]-x[k-1]}{T_s} &= Ax[k] + Bu[k]\\
y[k] &= Cx[k] + Du[k],
\end{aligned}
\tag{2.61}
$$

which results in

$$
\begin{aligned}
x[k+1] &= \underbrace{(\mathbb{I} + T_sA)}_{A_\mathrm{d,f}}\, x[k] + \underbrace{T_sB}_{B_\mathrm{d,f}}\, u[k]\\
y[k] &= \underbrace{C}_{C_\mathrm{d,f}}\, x[k] + \underbrace{D}_{D_\mathrm{d,f}}\, u[k].
\end{aligned}
\tag{2.62}
$$

Example 13. You are given the system

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix}1 & -1\\ 2 & 4\end{pmatrix} x(t) + \begin{pmatrix}1\\ 0\end{pmatrix} u(t)\\ y(t) &= \begin{pmatrix}1 & 1\end{pmatrix} x(t).\end{aligned} \tag{2.63}$$

(a) Find the discrete-time state space representation of the system using a sampling time $T_s = 1s$, i.e. find $A_{\mathrm{d}}, B_{\mathrm{d}}, C_{\mathrm{d}}, D_{\mathrm{d}}$

Solution. In order to compute the exact discretization, we use the formulas derived in class. For A_d, one has

$$A_\mathrm{d} = e^{AT_s} = e^{A}. \tag{2.64}$$

In order to compute the matrix exponential, one has to compute its eigenvalues, store them in a matrix D, find its eigenvectors, store them in matrix T, find the diagonal form and use the law

$$e^{A} = Te^{D}T^{-1}. \tag{2.65}$$

First, we compute the eigenvalues of A. It holds

$$\begin{aligned} P_A(\lambda) &= \det(A - \lambda\mathbb{I}) \\ &= \det\begin{pmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix} \\ &= \lambda^2 - 5\lambda + 6 \\ &= (\lambda - 2)\cdot(\lambda - 3). \end{aligned} \tag{2.66}$$

Therefore, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$ and they have algebraic multiplicity 1. We compute now the eigenvectors:

- $E_{\lambda_1} = E_2$: from $(A - \lambda_1\mathbb{I})x = 0$ one gets the system of equations

$$\left(\begin{array}{cc|c} -1 & -1 & 0 \\ 2 & 2 & 0 \end{array}\right)$$

One can note that the second row is linear dependent with the first. We therefore have a free parameter and the eigenspace for λ_1 reads

$$E_2 = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}. \tag{2.67}$$

E_2 has geometric multiplicity 1.

- $E_{\lambda_2} = E_3$: from $(A - \lambda_2\mathbb{I})x = 0$ one gets the system of equations

$$\left(\begin{array}{cc|c} -2 & -1 & 0 \\ 2 & 1 & 0 \end{array}\right)$$

One notes that the first and the second row are linearly dependent. We therefore have a free parameter and the eigenspace for λ_2 reads

$$E_3 = \left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\} \tag{2.68}$$

E_3 has geometric multiplicity 1. Since the algebraic and geometric multiplicity concide for every eigenvalue of A, the matrix is diagonalizable. With the computed eigenspaces, one can build the matrix T as

$$T = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}, \tag{2.69}$$

and D as a diagonal matrix with the eigenvalues on the diagonal:

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}. \tag{2.70}$$

It holds

$$\begin{aligned} T^{-1} &= \frac{1}{(-2+1)} \cdot \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}. \end{aligned} \tag{2.71}$$

Using Equation (2.65) one gets

$$\begin{aligned} A_\mathrm{d} &= e^A \\ &= Te^DT^{-1} \\ &= \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} e^2 & 0 \\ 0 & e^3 \end{pmatrix} \cdot \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2e^2 - e^3 & e^2 - e^3 \\ -2e^2 + 2e^3 & -e^2 + 2e^3 \end{pmatrix}. \end{aligned} \tag{2.72}$$

For B_d holds

$$\begin{aligned} B_\mathrm{d} &= \int_0^{T_s} e^{A\tau} B \mathrm{d}\tau \\ &= \int_0^1 e^{A\tau} B \mathrm{d}\tau \\ &= \int_0^1 \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} e^{2\tau} & 0 \\ 0 & e^{3\tau} \end{pmatrix} \cdot \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}\tau \\ &= \int_0^1 \begin{pmatrix} -e^{2\tau} & -e^{3\tau} \\ e^{2\tau} & 2e^{3\tau} \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} \mathrm{d}\tau \\ &= \int_0^1 \begin{pmatrix} 2e^{2\tau} - e^{3\tau} \\ -2e^{2\tau} + 2e^{3\tau} \end{pmatrix} \mathrm{d}\tau \\ &= \begin{pmatrix} e^2 - \frac{e^3}{3} - \frac{2}{3} \\ -e^2 + \frac{2}{3}e^3 + \frac{1}{3} \end{pmatrix}. \end{aligned} \tag{2.73}$$

Furthermore, one has $C_\mathrm{d} = C$ and $D_\mathrm{d} = D = 0$.

2.5 Discrete-time Systems Stability

We want to investigate the stability conditions for the discrete-time system given as

$$\begin{aligned} x[k+1] &= A_\mathrm{d}x[k] + B_\mathrm{d}u[k] \\ y[k] &= C_\mathrm{d}x[k] + D_\mathrm{d}u[k], \quad x[0] = x_0. \end{aligned} \tag{2.74}$$

As usual, we want to find conditions for wich the state $x[k]$ does not diverge. The free evolution (free means without input) can be written as

$$x[k+1] = A_\mathrm{d}x[k]. \tag{2.75}$$

Starting from the initial state, one can write

$$\begin{aligned} x[1] &= A_\mathrm{d}x_0 \\ x[2] &= A_\mathrm{d}^2x_0 \\ &\vdots \\ x[k] &= A_\mathrm{d}^kx_0. \end{aligned} \tag{2.76}$$

In order to analyze the convergence of this result, let's assume that $A_\mathrm{d} \in \mathbb{R}^{n\times n}$ is diagonalizable and let's rewrite A_d with the help of its diagonal form:

$$\begin{aligned} x[k] &= A_\mathrm{d}^kx_0 \\ &= \left(TDT^{-1}\right)^k x_0 \\ &= \left(TD\underbrace{T^{-1}T}_{\mathbb{I}}DT^{-1}\ldots TDT^{-1}\right) \\ &= TD^kT^{-1}x_0. \end{aligned} \tag{2.77}$$

where D is the matrix containing the eigenvalues of A_d and T is the matrix containing the relative eigenvectors. One can rewrite this using the modal decomposition as

$$TD^kT^{-1}x_0 = \sum_{i=1}^{n} \alpha_i\lambda_i^k v_i, \tag{2.78}$$

where v_i are the eigenvectors relative to the eigenvalues λ_i and $\alpha_i = T^{-1}x_0$ some coefficients depending on the initial condition x_0. Considering any possible eigenvalue, i.e. $\lambda_i = \rho_i e^{j\phi_i}$, one can write

$$TD^kT^{-1}x_0 = \sum_{i=1}^{n} \alpha_i\rho_i^k e^{j\phi_i k} v_i. \tag{2.79}$$

It holds

$$\begin{aligned} |\lambda_i| &= \rho_i^k|e^{j\phi_i k}| \\ &= \rho_i^k. \end{aligned} \tag{2.80}$$

This helps us defining the following cases:

- $|\lambda_i| < 1 \ \forall i = 1, \ldots, n$: the free evolution converges to 0 and the system is **asymptotically stable**.

- $|\lambda_i| \leq 1 \; \forall i = 1, \dots, n$ and eigenvalues with unit modulus have equal geometric and algebraic multiplicity: the free evolution converges (but not to 0) and the system is **marginally stable** or **stable**.
- $\exists i$ s.t. $|\lambda_i| > 1$: the free evolution diverges and the system is **unstable**.

Remark. The same analysis can be performed for non diagonalizable matrices A_d. The same conditions can be derived, with the help of the Jordan diagonal form of A_d.

Example 14. You are given the dynamics

$$x[k+1] = \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & \frac{1}{2} \end{pmatrix}}_{A_\mathrm{d}} x[k], \quad x[0] = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}. \tag{2.81}$$

Since A_d is a lower diagonal matrix, its eigenvalues lie in the diagonal, i.e. $\lambda_1 = 0$, $\lambda_2 = \frac{1}{2}$. Since both eigenvalues satisfy $|\lambda_i| < 1$, the system is asymptotically stable.

2.6 Discrete Time Controller Synthesis

As you learned in class, there are two ways to discretize systems. The scheme in Figure 14 resumes them.

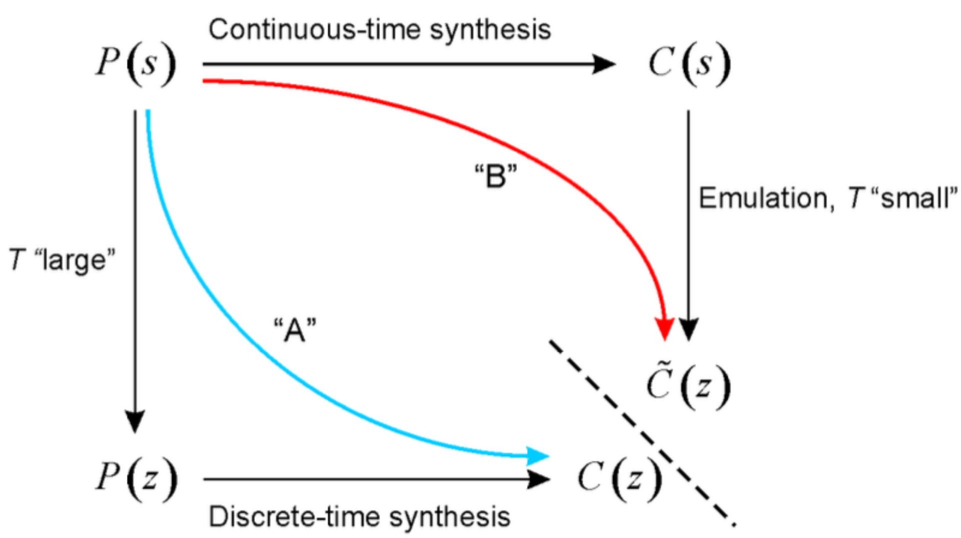

Figure 14: Emulation and Discrete time controller synthesis.

2.6.1 Emulation

In the previous chapter, we learned how to emulate a system. Let's define the recipe for such a procedure: Given a continous-time plant $P(s)$:

1. Design a continuous-time controller for the continuous-time plant.

2. Choose a sampling rate that is at least twice (ten times in practice) the crossover frequency.

3. If required, design an anti aliasing filter (AAF) to remove high frequency components of the continuous signal that is going to be sampled.

4. Modify your controller to take into accout the phase lag introduced by the discretization (up to a sampling period delay) and the AAF.

5. Discretize the controller (e.g. use the Tustin method for best accuracy).

6. Check open loop stability. If the system is unstable, change the emulation method, choose a faster sampling rate, or increase margin of phase.

7. Implement the controller.

2.6.2 Discrete-Time Synthesis

In this chapter we learn how to perform discrete time controller synthesis (also called direct synthesis). The general situation is the following: a control loop is given as in Figure 15. The continuous time transfer function $G(s)$ is given.

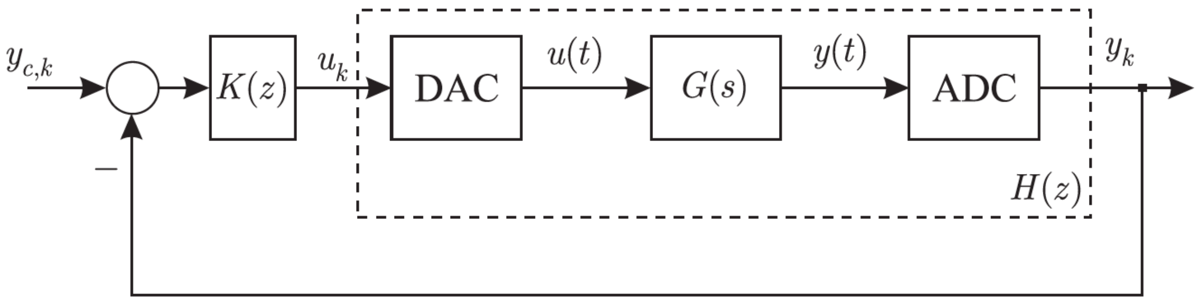

Figure 15: Discrete-time control loop.

We want to compute the equivalent discrete-time transfer function $H(z)$. The loop is composed of a Digital-to-Analog Converter (DAC), the continuous-time transfer function $G(s)$ and of an Analog to Digital Converter (ADC). We aim reaching a structure that looks like the one depicted in Figure 16.

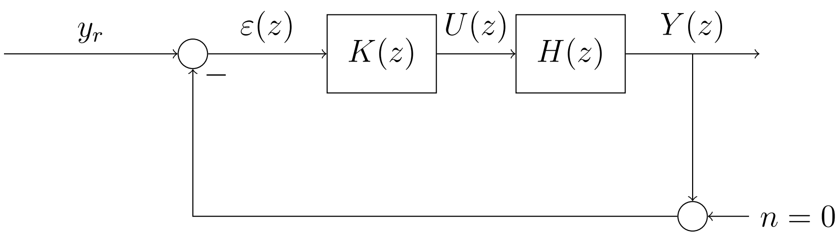

Figure 16: Discrete Synthesis.

The first thing to do, is to consider an input to analyze. The usual choice for this type of analysis is a **unit-step** of the form

$$u(kT) = \{\ldots, 0, 1, 1, \ldots\}. \tag{2.82}$$

Since the z-Transform is defined as

$$X(z) = \sum_{n=0}^{\infty} x(n) \cdot z^{-n} \tag{2.83}$$

one gets for u

$$U(z) = 1 + z^{-1} + z^{-2} + z^{-3} + \ldots + z^{-n}. \tag{2.84}$$

This sum can be written as (see geometric series)

$$U(z) = \frac{1}{1 - z^{-1}}. \tag{2.85}$$

For $U(z)$ to be defined, this sum must converge. This can be verified by exploring the properties of the geometric series.

Remark. **Recall: sum of geometric series.** Let S_n denote the sum over the first n elements of a geometric series:

$$\begin{aligned} S_n &= U_0 + U_0 \cdot a + U_0 \cdot a^2 + \ldots + U_0 \cdot a^{n-1} \\ &= U_0 \cdot (1 + a + a^2 + \ldots + a^{n-1}). \end{aligned} \tag{2.86}$$

Then

$$a \cdot S_n = U_0 \cdot (a + a^2 + a^3 + \ldots + a^n) \tag{2.87}$$

and

$$S_n - a \cdot S_n = U_0 \cdot (1 - a^n), \tag{2.88}$$

which leads to

$$S_n = U_0 \cdot \frac{1 - a^n}{1 - a}. \tag{2.89}$$

From here, it can be shown that the limit for n going to infinity convergenges if and only if the absolute value of a is smaller than one, i.e.

$$\lim_{n \to \infty} S_n = U_0 \cdot \frac{1}{1 - a}, \ \textbf{iff} \ |a| < 1. \tag{2.90}$$

Therefore the limiting case $|a| = 1 =: r$ is called **radius of convergence**. The according **convergence criterion** is $|a| < r$.

$H(z)$ contains the converters: at first, we have the *digital -to-analog* converter. The *Laplace-Transform* of the unit-step reads generally

$$\frac{1}{s}. \tag{2.91}$$

Hence, the transfer function before the *analog-to-digital* converter reads

$$\frac{G(s)}{s}. \tag{2.92}$$

In order to consider the *analog-to-digital* converter, we have to apply the inverse Laplace transfrom to get

$$y(t) = \mathcal{L}^{-1}\left(\frac{G(s)}{s}\right). \tag{2.93}$$

Through a $z-$ transform one can now get $Y(z)$. It holds

$$\begin{aligned} Y(z) &= \mathcal{Z}\left(y(kT)\right) \\ &= \mathcal{Z}\left(L^{-1}\left(\frac{G(s)}{s}\right)\right). \end{aligned} \tag{2.94}$$

The transfer function is then given as

$$H(z) = \frac{Y(z)}{U(z)}. \tag{2.95}$$

2.7 Examples

Example 15. A continuous-time system with the following transfer function is considered:

$$G(s) = \frac{9}{s+3}. \tag{2.96}$$

(a) Calculate the equivalent discrete-time transfer function $H(z)$. The latter is composed of a Digital-to-Analog Converter (DAC), the continuous-time transfer function $G(s)$ and an Analog-to-Digital Converter (ADC). Both converters, i.e. the DAC and ADC, have a sampling time $T_s = 1s$.

(b) Calculate the static error if a proportional controller $K(z) = k_p$ is used and the reference input $y_{c,k}$ is a step signal.
Hint: Heavyside with amplitude equal to 1.

Solution.

(a) Rather than taking into account all the individual elements which make up the continuous-time part of the system (DAC, plant, ADC), in a first step, these elements are lumped together and are represented by the discrete-time description $H(z)$. In this case, the discrete-time output of the system is given by

$$Y(z) = H(z) \cdot U(z), \tag{2.97}$$

where $U(z)$ is the $z-$transform of the discrete input u_k given to the system. Therefore, the discrete-time representation of the plant is given by the ratio of the output to the input

$$H(z) = \frac{Y(z)}{U(z)}. \tag{2.98}$$

For the sake of convenience, u_k is chosen to be the discrete-time *Heaviside* function

$$\begin{cases} u_k[k] = 1, & k \geq 0 \\ 0, & \text{else.} \end{cases} \tag{2.99}$$

This input function needs to be $z-$transformed. Recall the definition of the z-transform

$$X(z) = \sum_{n=0}^{\infty} x(n) \cdot z^{-n}. \tag{2.100}$$

and applying it to the above equation, with the input u_k gives ($u_k[k] = 1$ for $k \geq 0$)

$$\begin{aligned} U(z) &= X(u_k) \\ &= \sum_{k=0}^{\infty} z^{-k} \\ &= \sum_{k=0}^{\infty} (z^{-1})^k. \end{aligned} \tag{2.101}$$

For $U(z)$ to be defined, this sum must converge. Recalling the properties of geometric series one can see

$$U(z) = \frac{1}{a - z^{-1}}, \tag{2.102}$$

as long as the convergence criterion is satisfied, i.e. as long as $|z^{-1}| < 1$ or better $|z| > 1$ ($a = 1$). This signal is then transformed to continuous time using a zero-order hold DAC. The output of this transformation is again a Heaviside function $u_h(t)$. Since the signal is now in continuous time, the Laplace transform is used to for the analysis. The Laplace transform of the step function is well known to be

$$\mathcal{L}(u_h(t))(s) = \frac{1}{s} = U(s). \tag{2.103}$$

The plant output in continuous time is given by

$$\begin{aligned} Y(s) &= G(s) \cdot U(s) \\ &= \frac{G(s)}{s}. \end{aligned} \tag{2.104}$$

After the plant $G(s)$, the signal is sampled and transformed into discrete time once more. Therefore, the $z-$transform of the output has to be calculated. However, the signal $Y(s)$ cannot be transformed directly, since it is expressed in the frequency domain. Thus, first, it has to be transformed back into the time domain (i.e. into $y(t)$) using the inverse Laplace transform, where it is then sampled every $t = k \cdot T$. The resulting series of samples $\{y[k]\}$ is then transformed back into the $z-$domain, i.e.

$$Y(z) = X\left(\{\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)(kT)\}\right). \tag{2.105}$$

To find the inverse Laplace transform of the output, its frequency domain representation is decomposed into a sum of simpler functions

$$\begin{aligned} \frac{G(s)}{s} &= \frac{9}{s \cdot (s+3)} \\ &= \frac{\alpha}{s} + \frac{\beta}{s+3} \\ &= \frac{s \cdot (\alpha + \beta) + 3 \cdot \alpha}{s \cdot (s+3)}. \end{aligned} \tag{2.106}$$

The comparison of the numerators yields

$$\alpha = 3, \quad \beta = -3. \tag{2.107}$$

and thus

$$\begin{aligned} \frac{G(s)}{s} &= \frac{3}{s} - \frac{3}{s+3} \\ &= 3 \cdot \left(\frac{1}{s} - \frac{1}{s+3}\right). \end{aligned} \tag{2.108}$$

Now the terms can be individually transformed with the result

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{G(s)}{s}\right) &= 3 \cdot \left(1 - e^{-3t}\right) \cdot u_h(t) \\ &= y(t). \end{aligned} \tag{2.109}$$

The $z-$transform of the output sampled at discrete time istants $y(kT)$ is given by

$$\begin{aligned} X(\{y(kT)\}) &= 3 \cdot \left[\sum_{k=0}^{\infty} z^{-k} - \sum_{k=0}^{\infty} e^{-3kT} \cdot z^{-k}\right] \\ &= 3 \cdot \left[\sum_{k=0}^{\infty} z^{-1k} - \sum_{k=0}^{\infty} (e^{-3T} \cdot z^{-1})^k\right] \\ &= Y(z). \end{aligned} \tag{2.110}$$

From above, the two necessary convergence criteria are known:

$$\begin{aligned} |z^{-1}| < 1 &\Rightarrow |z| > 1 \\ |e^{-3T} \cdot z^{-1}| &\Rightarrow |z| > |e^{-3T}|. \end{aligned} \tag{2.111}$$

Using the above equations the output transfrom converges to (given that the two convergence criteria are satisfied)

$$Y(z) = 3 \cdot \left(\frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-3T} \cdot z^{-1}} \right). \tag{2.112}$$

Finally, the target transfer function $H(z)$ is given by

$$\begin{aligned} H(z) &= \frac{Y(z)}{U(z)} \\ &= (1 - z^{-1}) \cdot Y(z) \\ &= 3 \cdot \left(1 - \frac{1 - z^{-1}}{1 - e^{-3T} \cdot z^{-1}} \right) \\ &= 3 \cdot \frac{(1 - e^{-3T}) \cdot z^{-1}}{1 - e^{-3T} \cdot z^{-1}}. \end{aligned} \tag{2.113}$$

(b) From the signal flow diagram, it can be seen that the error $\varepsilon(z)$ is composed of

$$\begin{aligned} \varepsilon(z) &= Y_c(z) - Y(z) \\ &= Y_c(z) - H(z) \cdot K(z) \cdot \varepsilon(z) \\ &= \frac{Y_c(z)}{1 + k_p \cdot H(z)}. \end{aligned} \tag{2.114}$$

The input $y_c(t)$ is a discrete step signal, for which the z-transform was calculated in (a):

$$Y_c(z) = \frac{1}{1 + z^{-1}}. \tag{2.115}$$

Therefore, the error signal reads

$$\varepsilon(z) = \frac{\frac{1}{1+z^{-1}}}{1 + 3 \cdot k_p \cdot \frac{(1-e^{-3T}) \cdot z^{-1}}{1-e^{-3T} \cdot z^{-1}}}. \tag{2.116}$$

To calculate the steady-state error, i.e. the error after infinite time, the discrete-time final value theorem[6] is used:

$$\lim_{t \to \infty} \varepsilon(t) = \lim_{z \to 1} (1 - z^{-1}) \cdot \varepsilon(z), \tag{2.117}$$

but as z goes to 1, so does z^{-1} and thereforem 1 is substituted for each z^{-1} in $\varepsilon(z)$ and the stati error becomes

$$\varepsilon_\infty = \frac{1}{1 + 3 \cdot k_p}. \tag{2.118}$$

Note that the error does not completely vanish but can be made smaller by increasing k_p. This is the same behaviour which would have been expected from a purely proportional controller in continuous time. To drive the static error to zero, a discrete-time integrator of the form $\frac{1}{T_i \cdot (1 - z^{-1})}$ would be necessary.

[6] $\lim_{t \to \infty} \varepsilon(t) = \lim_{s \to 0} s \cdot \varepsilon(s)$

Example 16. Together with your friends, you founded the new startup TACSII, which is active in self-driving taxi services. You are given the dynamics of the demand for the two different types of vehicles you offer: small vehicles ($x_1(t)$) and large vehicles ($x_2(t)$). TACSI, a startup founded a year ago, helps you when you don't have enough vehicles, by giving you a multiple of $e(t)$ extra vehicles when needed. Due to software limitations, for the moment you can only mesaure the sum of small and large vehicles that are requested to you. The dynamics read

$$\begin{aligned} \dot{x}_1(t) &= 5x_1(t) - 6x_2(t) + e(t) \\ \dot{x}_2(t) &= 3x_1(t) - 4x_2(t) + 2e(t) \end{aligned} \tag{2.119}$$

a) Write down the state space description for the system, considering the extra vehicles as an input to the system, i.e. find the system matrices A, B, C and D.

b) Discretize the system using the forward Euler approach and a sampling time $T_s = 2s$, i.e. find $A_{\text{d,f}}$, $B_{\text{d,f}}$, $C_{\text{d,f}}$ and $D_{\text{d,f}}$.

c) Discretize the system using exact discretization and a sampling time of $T_s = 1s$, i.e. find A_{d}, B_{d}, C_{d} and D_{d}.

Solution.

a) The state space description reads

$$\begin{aligned}\dot{x} = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} &= \underbrace{\begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}}_{A} x(t) + \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{B} E(t) \\ y(t) &= \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} x(t).\end{aligned} \tag{2.120}$$

One notes that for this example $D = 0$.

b) The forward Euler approach for differentials reads

$$\dot{x} \approx \frac{x[k+1] - x[k]}{T_s}. \tag{2.121}$$

Applying this to the generic state space formulation

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t),\end{aligned} \tag{2.122}$$

one gets

$$\begin{aligned}\frac{x[k+1] - x[k]}{T_s} &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k],\end{aligned} \tag{2.123}$$

which results in

$$\begin{aligned}x[k+1] &= \underbrace{(\mathbb{I} + T_s A)}_{A_{\mathrm{d,f}}} x[k] + \underbrace{T_s B}_{B_{\mathrm{d,f}}} u[k] \\ y[k] &= \underbrace{C}_{C_{\mathrm{d,f}}} x[k] + \underbrace{D}_{D_{\mathrm{d,f}}} u[k].\end{aligned} \tag{2.124}$$

For our special case, it holds

$$\begin{aligned}A_{\mathrm{d,f}} &= (\mathbb{I} + T_s A) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 10 & -12 \\ 6 & -8 \end{pmatrix} \\ &= \begin{pmatrix} 11 & -12 \\ 6 & -7 \end{pmatrix}, \\ B_{\mathrm{d,f}} &= T_s B \\ &= \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \\ C_{\mathrm{d,f}} &= C \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \\ D_{\mathrm{d,f}} &= D = 0,\end{aligned} \tag{2.125}$$

where we used $T_s = 2s$.

c) In order to compute the exact discretization, we use the formulas derived in class. For A_d, one has

$$A_\text{d} = e^{AT_s} = e^{A}. \tag{2.126}$$

In order to compute the matrix exponential, one has to compute its eigenvalues, store them in a matrix D, find its eigenvectors, store them in matrix T, find the diagonal form and use the law

$$e^{A} = Te^{D}T^{-1}. \tag{2.127}$$

First, we compute the eigenvalues of A. It holds

$$\begin{aligned} P_A(\lambda) &= \det(A - \lambda\mathbb{I}) \\ &= \det\begin{pmatrix} 5-\lambda & -6 \\ 3 & -4-\lambda \end{pmatrix} \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda - 2)\cdot(\lambda + 1). \end{aligned} \tag{2.128}$$

Therefore, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$ and they have algebraic multiplicity 1. We compute now the eigenvectors:

- $E_{\lambda_1} = E_2$: from $(A - \lambda_1\mathbb{I})x = 0$ one gets the system of equations

$$\left(\begin{array}{cc|c} 3 & -6 & 0 \\ 3 & -6 & 0 \end{array}\right)$$

One can note that the second row is identical to the first. We therefore have a free parameter and the eigenspace for λ_1 reads

$$E_2 = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}. \tag{2.129}$$

E_2 has geometric multiplicity 1.

- $E_{\lambda_2} = E_{-1}$: from $(A - \lambda_2\mathbb{I})x = 0$ one gets the system of equations

$$\left(\begin{array}{cc|c} 6 & -6 & 0 \\ 3 & -3 & 0 \end{array}\right)$$

One notes that the first and the second row are linearly dependent. We therefore have a free parameter and the eigenspace for λ_2 reads

$$E_{-1} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \tag{2.130}$$

E_{-1} has geometric multiplicity 1. Since the algebraic and geometric multiplicity concide for every eigenvalue of A, the matrix is diagonalizable. With the computed eigenspaces, one can build the matrix T as

$$T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \tag{2.131}$$

and D as a diagonal matrix with the eigenvalues on the diagonal:

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.132}$$

It holds

$$\begin{aligned} T^{-1} &= \frac{1}{(2-1)} \cdot \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \end{aligned} \tag{2.133}$$

Using Equation (2.127) one gets

$$\begin{aligned} A_\mathrm{d} &= e^A \\ &= Te^DT^{-1} \\ &= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^2 & 0 \\ 0 & e^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2e^2 - e^{-1} & -2e^2 + 2e^{-1} \\ e^2 - e^{-1} & 2e^{-1} - e^2 \end{pmatrix}. \end{aligned} \tag{2.134}$$

For B_d holds

$$\begin{aligned} B_\mathrm{d} &= \int_0^{T_s} e^{A\tau} B \mathrm{d}\tau \\ &= \int_0^1 e^{A\tau} B \mathrm{d}\tau \\ &= \int_0^1 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{2\tau} & 0 \\ 0 & e^{-\tau} \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mathrm{d}\tau \\ &= \int_0^1 \begin{pmatrix} 2e^{2\tau} & e^{-\tau} \\ e^{2\tau} & e^{-\tau} \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \end{pmatrix} \mathrm{d}\tau \\ &= \int_0^1 \begin{pmatrix} -2e^{2\tau} + 3e^{-\tau} \\ -e^{2\tau} + 3e^{-\tau} \end{pmatrix} \mathrm{d}\tau \\ &= \begin{pmatrix} 4 - e^2 - 3e^{-1} \\ \frac{7}{2} - \frac{e^2}{2} - 3e^{-1} \end{pmatrix}. \end{aligned} \tag{2.135}$$

Furthermore, one has $C_\mathrm{d} = C$ and $D_\mathrm{d} = D = 0$.

Example 17.

(a) Choose all the signals that can be sampled without aliasing. The sampling time is $T_s = 1s$.

- ☐ $x(t) = \cos(4\pi \cdot t)$.
- ☐ $x(t) = \cos(4\pi \cdot t + \pi)$.
- ☐ $x(t) = 2 \cdot \cos(4\pi \cdot t + \pi)$.
- ☐ $x(t) = \cos(0.2\pi \cdot t)$.
- ☐ $x(t) = \cos(0.2\pi \cdot t + \pi)$.
- ☐ $x(t) = 3 \cdot \cos(0.2\pi \cdot t + \pi)$.
- ☐ $x(t) = \cos(\pi \cdot t)$.
- ☐ $x(t) = \cos(\pi \cdot t + \pi)$.
- ☐ $x(t) = 2 \cdot \cos(\pi \cdot t + \pi)$.
- ☐ $x(t) = \cos(0.2\pi \cdot t) + \cos(4\pi \cdot t)$.
- ☐ $x(t) = \sin(0.2\pi \cdot t) + \sin(0.4\pi \cdot t)$.
- ☐ $x(t) = \sum_{i=1}^{100} \cos\left(\frac{2\pi}{i+1} \cdot t\right)$.
- ☐ $x(t) = \sum_{i=1}^{100} \cos\left(\frac{2\pi}{i+2} \cdot t\right)$.

(b) The signal

$$x(t) = 2 \cdot \cos(20 \cdot \pi \cdot t + \pi) + \cos(40 \cdot \pi \cdot t) + \cos(30 \cdot \pi \cdot t).$$

is sampled with sampling frequency f_s. What is the minimal f_s such that no aliasing occurs?

Solution.

(a) ☐ $x(t) = \cos(4\pi \cdot t)$.
☐ $x(t) = \cos(4\pi \cdot t + \pi)$.
☐ $x(t) = 2 \cdot \cos(4\pi \cdot t + \pi)$.
☑ $x(t) = \cos(0.2\pi \cdot t)$.
☑ $x(t) = \cos(0.2\pi \cdot t + \pi)$.
☑ $x(t) = 3 \cdot \cos(0.2\pi \cdot t + \pi)$.
☐ $x(t) = \cos(\pi \cdot t)$.
☐ $x(t) = \cos(\pi \cdot t + \pi)$.
☐ $x(t) = 2 \cdot \cos(\pi \cdot t + \pi)$.
☐ $x(t) = \cos(0.2\pi \cdot t) + \cos(4\pi \cdot t)$.
☑ $x(t) = \sin(0.2\pi \cdot t) + \sin(0.4\pi \cdot t)$.
☐ $x(t) = \sum_{i=1}^{100} \cos\left(\frac{2\pi}{i+1} \cdot t\right)$.
☑ $x(t) = \sum_{i=1}^{100} \cos\left(\frac{2\pi}{i+2} \cdot t\right)$.

Explanation:
If one goes back to the definition of the ranges to ensure no aliasing occurs, one gets the formula
$$f < \frac{1}{2 \cdot T_s}.$$
In this case the condition reads
$$f < \frac{1}{2 \cdot 1s} = 0.5Hz.$$
One can read the frequency of a signal from its formula: the value that multiplies t is ω and
$$\frac{\omega}{2\pi} = f.$$
The first three signals have
$$f = \frac{4\pi}{2\pi} = 2Hz.$$
which is greater than $0.5Hz$. Moreover, additional phase and gain don't play an important role in this sense. The next three signals have a frequency of
$$f = \frac{0.2\pi}{2\pi} = 0.1Hz.$$
that is lower than $0.5Hz$ and hence perfect, in order to not encounter aliasing. The next three signals have the critical frequency and theoretically speaking, one sets this as already aliased. The reason for that is that the Nyquist theorem sets a strict $<$ in the condition.
For the next two signals a special procedure applies: if one has a combination of signals, one has to look at the bigger frequency of the signal. In the first case this reads $2Hz$, that exceeds the limit frequency. In the second case this reads $0.2Hz$, that is acceptable. The last two cases are a general form of combination

of signals. The leading frequency of the first sum, decreases with $\frac{1}{i+1}$ and has its biggest value with $i = 1$, namely $0.5Hz$. This is already at the limit frequency, hence not acceptable. The leading frequency of the second sum, decreases with $\frac{1}{i+2}$ and has its biggest value with $i = 1$, namely $0.33Hz$. This is lower that the limit frequency, hence acceptable.

(b) The general formula reads

$$f_s > 2 \cdot f_{max}.$$

Here it holds

$$f_{max} = \frac{40\pi}{2\pi} = 20Hz.$$

It follows

$$f_s > 2 \cdot 20Hz = 40Hz.$$

Example 18.

(a) For which A and B is the system asymptotically stable?

- ☐ $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- ☐ $A = \begin{pmatrix} -1 & -2 \\ -1 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- ☐ $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}$.
- ☐ $A = \begin{pmatrix} -1 & -2 \\ 1 & -0.5 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- ☐ $A = \begin{pmatrix} -1 & -2 \\ 0 & -0.5 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- ☐ $A = \begin{pmatrix} 1 & -2 \\ 0 & 0.5 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- ☐ $A = \begin{pmatrix} -0.1 & -2 \\ 0 & -0.5 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- ☐ $A = \begin{pmatrix} -0.1 & -2 \\ 0 & -0.5 \end{pmatrix}$, $B = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}$.
- ☐ $A = \begin{pmatrix} 0.1 & -2 \\ 0 & 0.5 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- ☐ $A = \begin{pmatrix} 0.1 & -2 \\ 0 & 0.5 \end{pmatrix}$, $B = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}$.

(b) The previous exercise can be solved independently of B.

- ☐ True.
- ☐ False.

Solution.

(a)
- ☐ $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- ☐ $A = \begin{pmatrix} -1 & -2 \\ -1 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- ☑ $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}$.
- ☐ $A = \begin{pmatrix} -1 & -2 \\ 1 & -0.5 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- ☐ $A = \begin{pmatrix} -1 & -2 \\ 0 & -0.5 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- ☑ $A = \begin{pmatrix} -0.1 & -2 \\ 0 & -0.5 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- ☑ $A = \begin{pmatrix} -0.1 & -2 \\ 0 & -0.5 \end{pmatrix}$, $B = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}$.
- ☑ $A = \begin{pmatrix} 0.1 & -2 \\ 0 & 0.5 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- ☑ $A = \begin{pmatrix} 0.1 & -2 \\ 0 & 0.5 \end{pmatrix}$, $B = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}$.

(b)
- ☑ True.
- ☐ False.

Explanation:
The eigenvalues of A should fulfill

$$\lambda_i| < 1. \tag{2.136}$$

Furthermore. this has nothing to do with B.

Example 19. One of your colleagues has developed the following continuous-time controller for a continuous-time plant

$$C(s) = \frac{2s+1}{s+\alpha} \tag{2.137}$$

where $\alpha \in \mathbb{R}$ is a tuning factor.

(a) You now want to implement the controller on a microprocessor using the **Euler backward** emulation approach. What is the resulting function $C(z)$ for a generic sampling time $T \in \mathbb{R}^+$?

(b) What is the range of the tuning factor α that produces an asymptotically stable discrete.time controller when applying the Euler backward emulation approach?

(c) What is the condition on α to obtain an asymptotically stable continuous-time controller $C(s)$ and an asymptotically stable discrete-time controller $C(z)$ using the Euler backward emulation approach?

Solution.

(a) The Euler backward emulation approach reads

$$s = \frac{z-1}{T \cdot z}. \tag{2.138}$$

If one substitutes this into the transfer function of the continuous-time controller, one gets

$$\begin{aligned} C(z) &= \frac{2 \cdot \frac{z-1}{T \cdot z} + 1}{\frac{z-1}{T \cdot z} + \alpha} \\ &= \frac{z \cdot (2+T) - 2}{z \cdot (1 + \alpha \cdot T) - 1}. \end{aligned} \tag{2.139}$$

(b) The controller $C(z)$ is asymptotically stable if its pole π_d fulfills the condition

$$|\pi_d| < 1.$$

The pole reads

$$\begin{aligned} z \cdot (1 + \alpha \cdot T) - 1 = 0 \\ \pi_d = \frac{1}{1 + \alpha \cdot T}. \end{aligned} \tag{2.140}$$

This, together with the condition for stability gives

$$-1 < \frac{1}{1 + \alpha \cdot T} < 1 \Rightarrow \alpha > 0 \text{ or } \alpha < -\frac{2}{T}. \tag{2.141}$$

(c) For $C(s)$ to be asymptotically stable, its pole $\pi_c = -\alpha$ must lie in the left half of the complex plane:

$$\mathrm{Re}\{\pi_c\} < 0 \Rightarrow \alpha > 0. \tag{2.142}$$

Together with the results from (b), the condition on α is $\alpha > 0$.

3 Introduction to MIMO Systems

MIMO systems are systems with multiple inputs and multiple outputs. In this chapter we will introduce some analytical tools.

3.1 System Description

3.1.1 State Space Description

The state-space description of a MIMO system is very similar to the one of a SISO system. For a linear, time invariant MIMO system with m input signals and p output signals, it holds

$$\begin{aligned} \dot{x}(t) &= A \cdot x(t) + B \cdot u(t), \qquad & x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \\ y(t) &= C \cdot x(t) + D \cdot u(t), \qquad & y(t) \in \mathbb{R}^p \end{aligned} \tag{3.1}$$

where

$$x(t) \in \mathbb{R}^{n \times 1},\ u(t) \in \mathbb{R}^{m \times 1},\ y(t) \in \mathbb{R}^{p \times 1},\ A \in \mathbb{R}^{n \times n},\ B \in \mathbb{R}^{n \times m},\ C \in \mathbb{R}^{p \times n},\ D \in \mathbb{R}^{p \times m}. \tag{3.2}$$

Remark. The dimensions of the matrices A, B, C, D are very important and they are a key concept to understand problems.

The big difference from SISO systems is that $u(t)$ and $y(t)$ are here vectors and not scalars anymore. For this reason B, C, D are now matrices.

3.1.2 Transfer Function

One can compute the transfer function of a MIMO system with the well known formula

$$P(s) = C \cdot (s \cdot \mathbb{I} - A)^{-1} \cdot B + D. \tag{3.3}$$

This is no more a scalar, but a $p \times m$-matrix. The elements of that matrix are rational functions. Mathematically:

$$P(s) = \begin{pmatrix} P_{11}(s) & \cdots & P_{1m}(s) \\ \vdots & \ddots & \vdots \\ P_{p1}(s) & \cdots & P_{pm}(s) \end{pmatrix}, \qquad P_{ij}(s) = \frac{b_{ij}(s)}{a_{ij}(s)}. \tag{3.4}$$

Here $P_{ij}(s)$ is the transfer function from the j-th input to the i-th output.

Remark. In the SISO case, the only matrix we had to care about was A. Since for the MIMO case B,C,D are matrices, one has to pay attention to a fundamental mathematical property: the matrix multiplication is not **commutative** (i.e. $A \cdot B \neq B \cdot A$). Since now $P(s)$ and $C(s)$ are matrices, considering Figure 17, it holds

$$L_{\mathrm{O}}(s) = P(s) \cdot C(s) \neq C(s) \cdot P(s) = L_{\mathrm{I}}(s), \tag{3.5}$$

where $L_{\mathrm{O}}(s)$ is the outer loop transfer function and $L_{\mathrm{I}}(s)$ is the inner loop transfer function. Moreover, one can no more define the complementary sensitivity and the sensitivity as

$$T(s) = \frac{L(s)}{1 + L(s)},\ S(s) = \frac{1}{1 + L(s)}. \tag{3.6}$$

because no matrix division is defined. There are however similar expressions to describe those transfer functions:

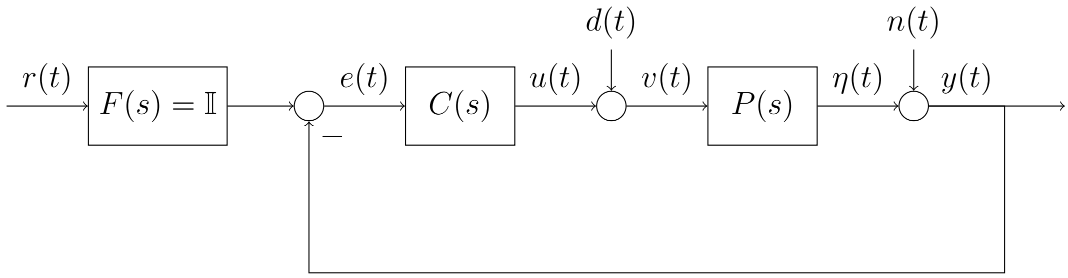

Figure 17: Standard feedback control system structure.

Output Sensitivity Functions

Referring to Figure 17, one can write

$$\begin{aligned} Y(s) &= N(s) + \eta(s) \\ &= N(s) + P(s)V(s) \\ &= N(s) + P(s)\left(D(s) + U(s)\right) \\ &= N(s) + P(s)\left(D(s) + C(s)E(s)\right) \\ &= N(s) + P(s)\left(D(s) + C(s)(R(s) - Y(s))\right), \end{aligned} \tag{3.7}$$

from which follows

$$\begin{aligned} (\mathbb{I} + P(s)C(s))Y(s) &= N(s) + P(s)D(s) + P(s)C(s)R(s) \\ Y(s) &= (\mathbb{I} + P(s)C(s))^{-1}\left(N(s) + P(s)D(s) + P(s)C(s)R(s)\right). \end{aligned} \tag{3.8}$$

It follows

- **Output sensitivity function** $(n \to y)$

$$S_{\mathrm{O}}(s) = (\mathbb{I} + L_{\mathrm{O}}(s))^{-1}. \tag{3.9}$$

- **Output complementary sensitivity function** $(r \to y)$

$$T_{\mathrm{O}}(s) = (\mathbb{I} + L_{\mathrm{O}}(s))^{-1}L_{\mathrm{O}}(s). \tag{3.10}$$

Input Sensitivity Functions

Referring to Figure 18, one can write

$$\begin{aligned} U(s) &= C(s)E(s) \\ &= C(s)\left(R(s) - Y(s)\right) \\ &= C(s)R(s) - C(s)\left(N(s) + \eta(s)\right) \\ &= C(s)R(s) - C(s)N(s) - C(s)P(s)V(s) \\ &= C(s)R(s) - C(s)N(s) - C(s)P(s)\left(D(s) + U(s)\right), \end{aligned} \tag{3.11}$$

from which it follows

$$\begin{aligned} (\mathbb{I} + C(s)P(s))U(s) &= C(s)R(s) - C(s)N(s) - C(s)P(s)D(s) \\ -U(s) &= (\mathbb{I} + C(s)P(s))^{-1}\left(-C(s)R(s) + C(s)N(s) + C(s)P(s)D(s)\right). \end{aligned} \tag{3.12}$$

It follows

- **Input sensitivity function** $(d \to v)$

$$S_{\mathrm{I}}(s) = (\mathbb{I} + L_{\mathrm{I}}(s))^{-1}. \tag{3.13}$$

- **Input complementary sensitivity function** $(d \to -u)$

$$T_{\mathrm{I}}(s) = (\mathbb{I} + L_{\mathrm{I}}(s))^{-1} L_{\mathrm{I}}(s). \tag{3.14}$$

Example 20. In order to understand how to work with these matrices, let's analyze the following problem. We start from the standard control system's structure (see Figure 18). To keep things general, let's say the plant $P(s) \in \mathbb{C}^{p\times m}$ and the controller $C(s) \in \mathbb{C}^{m\times p}$. The reference $r \in \mathbb{R}^p$, the input $u \in \mathbb{R}^m$ and the disturbance $d \in \mathbb{R}^p$. The output $Y(s)$ can as always be written as

$$Y(s) = T(s) \cdot R(s) + S(s) \cdot D(s). \tag{3.15}$$

where $T(s)$ is the transfer function of the complementary sensitivity and $S(s)$ is the transfer function of the sensitivity.

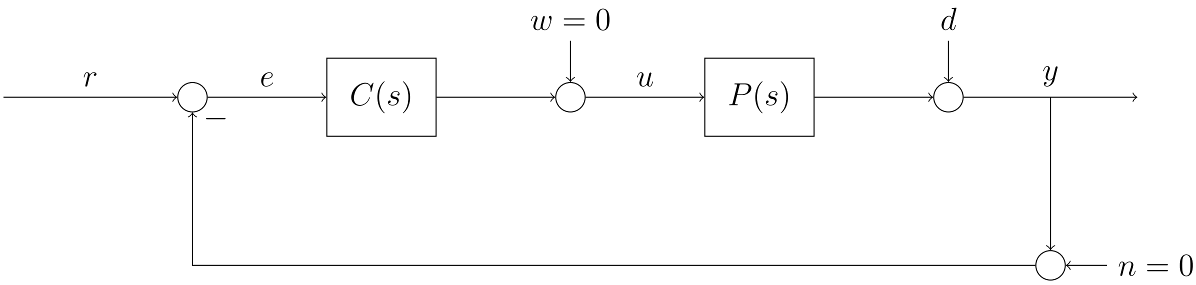

Figure 18: Standard feedback control system structure.

Starting from the error $E(s)$

If one wants to determine the matrices of those transfer functions, one can start writing (by paying attention to the direction of multiplications) with respect to $E(s)$

$$\begin{aligned} E(s) &= R(s) - P(s) \cdot C(s) \cdot E(s) - D(s), \\ Y(s) &= P(s) \cdot C(s) \cdot E(s) + D(s). \end{aligned} \tag{3.16}$$

This gives in the first place

$$E(s) = (\mathbb{I} + P(s) \cdot C(s))^{-1} \cdot (R(s) - D(s)). \tag{3.17}$$

Inserting and writing the functions as $F(s) = F$ for simplicity, one gets

$$\begin{aligned} Y &= P \cdot C \cdot E + D \\ &= P \cdot C \cdot (\mathbb{I} + P \cdot C)^{-1} \cdot (R - D) + D \\ &= P \cdot C \cdot (\mathbb{I} + P \cdot C)^{-1} \cdot R - P \cdot C\,(\mathbb{I} + P \cdot C)^{-1} \cdot D + \underbrace{(\mathbb{I} + P \cdot C)\,(\mathbb{I} + P \cdot C)^{-1}}_{\mathbb{I}} \cdot D \\ &= P \cdot C \cdot (\mathbb{I} + P \cdot C)^{-1} \cdot R + (\mathbb{I} + P \cdot C - P \cdot C) \cdot (\mathbb{I} + P \cdot C)^{-1} \cdot D \\ &= P \cdot C \cdot (\mathbb{I} + P \cdot C)^{-1} \cdot R + (\mathbb{I} + P \cdot C)^{-1} \cdot D. \end{aligned} \tag{3.18}$$

Recalling the general equation (4.39) one gets the two transfer functions:

$$\begin{aligned} T_1(s) &= P(s) \cdot C(s) \cdot (\mathbb{I} + P(s) \cdot C(s))^{-1}, \\ S_1(s) &= (\mathbb{I} + P(s) \cdot C(s))^{-1}. \end{aligned} \tag{3.19}$$

Starting from the input $U(s)$

If one starts with respect to $U(s)$, one gets

$$\begin{aligned} U(s) &= C(s) \cdot (R(s) - D(s)) - C(s) \cdot P(s) \cdot U(s), \\ Y(s) &= P(s) \cdot U(s) + D(s). \end{aligned} \tag{3.20}$$

This gives in the first place

$$U(s) = (\mathbb{I} + C(s) \cdot P(s))^{-1} \cdot C(s) \cdot (R(s) - D(s)). \tag{3.21}$$

Inserting and writing the functions as $F(s) = F$ for simplicity, one gets

$$\begin{aligned} Y &= P \cdot U + D \\ &= P \cdot (\mathbb{I} + C \cdot P)^{-1} \cdot C \cdot (R - D) + D \\ &= P \cdot (\mathbb{I} + C \cdot P)^{-1} \cdot C \cdot R + \left(\mathbb{I} - P \cdot (\mathbb{I} + C \cdot P)^{-1} \cdot C\right) \cdot D. \end{aligned} \tag{3.22}$$

Recalling the general equation (4.39) one gets the two transfer functions:

$$\begin{aligned} T_2(s) &= P(s) \cdot (\mathbb{I} + C(s) \cdot P(s))^{-1} \cdot C(s), \\ S_2(s) &= \mathbb{I} - P(s) \cdot (\mathbb{I} + C(s) \cdot P(s))^{-1} \cdot C(s). \end{aligned} \tag{3.23}$$

It can be shown that this two different results actually are the equivalent. It holds

$$\begin{aligned} S_1 &= S_2 \\ (\mathbb{I} + P \cdot C)^{-1} &= \mathbb{I} - P \cdot (\mathbb{I} + C \cdot P)^{-1} \cdot C \\ \mathbb{I} &= \mathbb{I} + P \cdot C - P \cdot (\mathbb{I} + C \cdot P)^{-1} \cdot C \cdot (\mathbb{I} + P \cdot C) \\ \mathbb{I} &= \mathbb{I} + P \cdot C - P \cdot (\mathbb{I} + C \cdot P)^{-1} \cdot (C + C \cdot P \cdot C) \\ \mathbb{I} &= \mathbb{I} + P \cdot C - P \cdot (\mathbb{I} + C \cdot P)^{-1} \cdot (\mathbb{I} + C \cdot P) \cdot C \\ \mathbb{I} &= \mathbb{I} + P \cdot C - P \cdot C \\ \mathbb{I} &= \mathbb{I} \\ \\ T_1 &= T_2 \\ P \cdot C \cdot (\mathbb{I} + P \cdot C)^{-1} &= P \cdot (\mathbb{I} + C \cdot P)^{-1} \cdot C \\ P \cdot C &= P \cdot (\mathbb{I} + C \cdot P)^{-1} \cdot C \cdot (\mathbb{I} + P \cdot C) \\ P \cdot C &= P \cdot (\mathbb{I} + C \cdot P)^{-1} \cdot (C + C \cdot P \cdot C) \\ P \cdot C &= P \cdot (\mathbb{I} + C \cdot P)^{-1} \cdot (\mathbb{I} + C \cdot P) \cdot C \\ P \cdot C &= C \cdot P \\ \mathbb{I} &= \mathbb{I}. \end{aligned} \tag{3.24}$$

Finally, one can show that

$$\begin{aligned} S(s) + T(s) &= (\mathbb{I} + P \cdot C)^{-1} + P \cdot C \cdot (\mathbb{I} + P \cdot C)^{-1} \\ &= (\mathbb{I} + P \cdot C) \cdot (\mathbb{I} + P \cdot C)^{-1} \\ &= \mathbb{I}. \end{aligned} \tag{3.25}$$

3.2 Poles and Zeros

Since we have to deal with matrices, one has to use the theory of *minors* (see *Lineare Algebra I/II*) in order to compute the zeros and the poles of a transfer function.
The first step of this computation is to calculate all the minors of the transfer function $P(s)$. The minors of a matrix $F \in \mathbb{R}^{n\times m}$ are the determinants of all square submatrices. By **maximal minor** it is meant the minor with the biggest dimension.

Example 21. The minors of a given matrix

$$P(s) = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \tag{3.26}$$

are:
First order:

$$a, b, c, d, e, f \tag{3.27}$$

Second order (maximal minors):

$$det \begin{pmatrix} a & b \\ d & e \end{pmatrix}, \; det \begin{pmatrix} a & c \\ d & f \end{pmatrix}, \; det \begin{pmatrix} b & c \\ e & f \end{pmatrix}. \tag{3.28}$$

From the minors one can calculate the poles and the zeros as follows:

3.2.1 Zeros

The zeros are the zeros of the numerator's *greatest common divisor* of the maximal minors, after their normalization with respect to the same denominator (*polepolynom*).

3.2.2 Poles

The poles are the zeros of the *least common denominator* of all the minors of $P(s)$.

3.2.3 Directions

In MIMO systems, the poles and the zeros are related to a **direction**. Moreover, a zero-pole cancellation occurs only if zero and pole have the same magnitude and input-output direction. The directions $\delta_{\pi,i}^{in,out}$ associated with a pole π_i are defined by

$$P(s)\Big|_{s=\pi_i} \cdot \delta_{\pi,i}^{\text{in}} = \infty \cdot \delta_{\pi,i}^{\text{out}}. \tag{3.29}$$

The directions $\delta_{\xi,i}^{\text{in,out}}$ associated with a zero ξ_i are defined by

$$P(s)\Big|_{s=\xi_i} \cdot \delta_{\xi,i}^{\text{in}} = 0 \cdot \delta_{\xi,i}^{\text{out}}. \tag{3.30}$$

The directions can be computed with the **singular value decomposition** (see next chapters) of the matrix $P(s)$.

3.3 Examples

Example 22. One wants to find the poles and the zeros of the given transfer function

$$P(s) = \begin{pmatrix} \frac{s+2}{s+3} & 0 \\ 0 & \frac{(s+1)\cdot(s+3)}{s+2} \end{pmatrix}. \tag{3.31}$$

Solution. First of all, we list **all** the minors of the transfer function:

Minors:

- First order: $\frac{s+2}{s+3}$, $\frac{(s+1)\cdot(s+3)}{s+2}$, 0, 0 ;
- Second order: $s+1$.

Poles:

The least common denominator of all the minors is

$$(s+3)\cdot(s+2) \tag{3.32}$$

This means that the poles are

$$\begin{aligned}\pi_1 &= -2\\ \pi_2 &= -3.\end{aligned} \tag{3.33}$$

Zeros:

The maximal minor is $s+1$ and we have to normalize it with respect to the polepolynom $(s+3)\cdot(s+2)$. It holds

$$(s+1) \;\Rightarrow\; \frac{(s+1)\cdot(s+2)\cdot(s+3)}{(s+2)\cdot(s+3)} \tag{3.34}$$

The numerator reads

$$(s+1)\cdot(s+2)\cdot(s+3) \tag{3.35}$$

and so the zeros are

$$\begin{aligned}\zeta_1 &= -1\\ \zeta_2 &= -2\\ \zeta_3 &= -3.\end{aligned} \tag{3.36}$$

Example 23. One wants to find the poles and the zeros of the given transfer function

$$P(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{1}{s+2} & \frac{2\cdot(s+1)}{(s+2)\cdot(s+3)} \\ 0 & \frac{s+3}{(s+1)^2} & \frac{s+4}{s+1} \end{pmatrix}. \tag{3.37}$$

Solution. First of all, we list **all** the minors of the transfer function:

Minors:

- First order: $\frac{1}{s+1}, \frac{1}{s+2}, \frac{2\cdot(s+1)}{(s+2)\cdot(s+3)}, 0, \frac{s+3}{(s+1)^2}, \frac{s+4}{s+1}$;
- Second order: $\frac{s+3}{(s+1)^3}, \frac{s+4}{(s+1)\cdot(s+2)} - \frac{2}{(s+2)\cdot(s+1)} = \frac{1}{s+1}, -\frac{s+4}{(s+1)^2}$.

Poles:

The least common denominator of all the minors is

$$(s+1)^3 \cdot (s+2) \cdot (s+3). \tag{3.38}$$

This means that the poles are

$$\begin{aligned} \pi_1 &= -1 \\ \pi_2 &= -1 \\ \pi_3 &= -1 \\ \pi_4 &= -2 \\ \pi_5 &= -3. \end{aligned} \tag{3.39}$$

Zeros:

The numerators of the maximal minors are $(s+3)$, 1 and $-(s+4)$. We have to normalize them with respect to the polepolynom $(s+1)^3 \cdot (s+2) \cdot (s+3)$. It holds

$$(s+3) \quad \Rightarrow \quad \frac{(s+3)^2 \cdot (s+2)}{(s+1)^3 \cdot (s+2) \cdot (s+3)}, \tag{3.40}$$

$$1 \quad \Rightarrow \quad \frac{(s+1)^2 \cdot (s+2) \cdot (s+3)}{(s+1)^3 \cdot (s+2) \cdot (s+3)}, \tag{3.41}$$

$$-(s+4) \quad \Rightarrow \quad -\frac{(s+4) \cdot (s+1) \cdot (s+2) \cdot (s+3)}{(s+1)^3 \cdot (s+2) \cdot (s+3)}. \tag{3.42}$$

The greatest common divisor of these is

$$(s+3) \cdot (s+2). \tag{3.43}$$

Hence, the zeros are

$$\begin{aligned} \zeta_1 &= -2, \\ \zeta_2 &= -3. \end{aligned} \tag{3.44}$$

Example 24. The following transfer function matrix is given:

$$P(s) = \begin{pmatrix} \frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\ \\ \frac{-1}{s-1} & \frac{1}{s+2} & \frac{1}{s+2} \end{pmatrix}. \tag{3.45}$$

Compute the poles and the zeros of the system.

Solution. First order minors are:

$$\frac{1}{s+1}, \quad \frac{s-1}{(s+1)(s+2)}, \quad \frac{-1}{s-1}, \quad \frac{1}{s+2}, \quad \frac{1}{s+2} \tag{3.46}$$

Minors of second order are:

$$\frac{-(s-1)}{(s+1)(s+2)^2}, \quad \frac{2}{(s+1)(s+2)}, \quad \frac{1}{(s+1)(s+2)} \tag{3.47}$$

The least common denominator – the pole-polynom – is $(s+1)(s+2)^2(s-1)$, from which the system's poles and their multiplicities can be read: $s = -1$ (multiplicity = 1), $s = 1$ (multiplicity = 1) und $s = -2$ (multiplicity = 2).
Normalizing the minors of second order with the denominator $(s+1)(s+2)^2(s-1)$ yields

$$\frac{-(s-1)^2}{(s+1)(s+2)^2(s-1)}, \quad \frac{2(s-1)(s+2)}{(s+1)(s+2)^2(s-1)}, \quad \frac{(s-1)(s+2)}{(s+1)(s+2)^2(s-1)} \tag{3.48}$$

The greatest common divisor of these minors is $(s-1)$ and therefore the MIMO–system has its only zero at $s = 1$.

4 Analysis of MIMO Systems

4.1 Norms

The concept of norm will be extremely useful for evaluating signals and systems quantitatively during this course. In the following, we will present vector norms and matrix norms.

4.1.1 Vector Norms

Definition 10. A **norm** on a linear space (V, F) is a function $\|\cdot\| : V \to \mathbb{R}_+$ such that

a) $\forall v_1, v_2 \in V, \|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$ (triangle inequality).

b) $\forall v \in V, \forall \alpha \in F, \|\alpha v\| = |\alpha| \cdot \|v\|$.

c) $\|v\| = 0 \Leftrightarrow v = 0$.

Remark. Note that norms are always non-negative. This can be noticed by seeing that a norm always maps to $\mathbb{R}_+$.

Considering $x \in \mathbb{C}^n$, i.e. $V = \mathbb{C}^n$, one can define the $p-$**norm** as

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}, \quad p = 1, 2, \ldots \tag{4.1}$$

The easiest example of such a norm is the case where $p = 2$, i.e. the **euclidean norm** (shortest distance between two points):

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}. \tag{4.2}$$

Another important example of such a norm is the **infinity norm** (largest element in the vector):

$$\|x\|_\infty = \max_i |x_i|. \tag{4.3}$$

Example 25. You are given the vector

$$x = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}. \tag{4.4}$$

Compute the $\|x\|_1$,$\|x\|_2$ and $\|x\|_\infty$ norms of x.

Solution. It holds

$$\begin{aligned} \|x\|_1 &= |-1| + |2| + |3| \\ &= 6. \\ \|x\|_2 &= \sqrt{1+4+9} \\ &= \sqrt{14}. \\ \|x\|_\infty &= \max\{1, 2, 3\} \\ &= 3. \end{aligned} \tag{4.5}$$

4.1.2 Matrix Norms

In addition to the defined axioms for norms, matrix norms fulfill

$$\|A \cdot B\| \leq \|A\| \cdot \|B\|. \tag{4.6}$$

Considering the linear space (V, F), with $V = \mathbb{C}^{m\times n}$, and assuming a matrix $A \in \mathbb{C}^{m\times n}$ is given, one can define the **Frobenius norm** (euclidean matrix norm) as:

$$\|A\|_F = \|A\|_2 = \left(\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}^2\right)^{\frac{1}{2}}, \tag{4.7}$$

where a_{ij} are the elements of A. This can also be written as

$$\|A\|_F = \sqrt{\mathrm{tr}\,(A^*A)}, \tag{4.8}$$

where tr is the trace (i.e. sum of eigenvalues or diagonal elements) of the matrix and A^* is the Hermitian transpose of A (complex transpose), i.e.

$$A^* = (conj(A))^T. \tag{4.9}$$

Example 26. Let

$$A = \begin{pmatrix} 1 & -2-i \\ 1+i & i \end{pmatrix}. \tag{4.10}$$

Then it holds

$$\begin{aligned} A^* &= (conj(A))^T \\ &= \begin{pmatrix} 1 & -2+i \\ 1-i & -i \end{pmatrix}^T \\ &= \begin{pmatrix} 1 & 1-i \\ -2+i & -i \end{pmatrix}. \end{aligned} \tag{4.11}$$

The **maximum matrix norm** is the largest element of the matrix and is defined as

$$\|A\|_\infty = \max_{i=1,\dots,m}\max_{j=1,\dots,n} |a_{ij}|. \tag{4.12}$$

Matrix Norms as Induced Norms

Matrix norms can always be defined as induced norms.

Definition 11. Let $A \in \mathbb{C}^{m\times n}$. Then, the **induced norm** of matrix A can be written as

$$\begin{aligned} \|A\|_p &= \sup_{x\in\mathbb{C}^n,\ x\neq 0}\left(\frac{\|Ax\|_p}{\|x\|_p}\right) \\ &= \sup_{x\in\mathbb{C}^n,\ x=1}(\|Ax\|_p) \end{aligned} \tag{4.13}$$

Remark. At this point, one would ask what is the difference between sup and max. A maximum is the largest number **within** a set. A sup is a number that bounds a set. A sup may or may not be part of the set itself (0 is not part of the set of negative numbers, but it is a sup because it is the least upper bound). If the sup **is** part of the set, it is also the max.

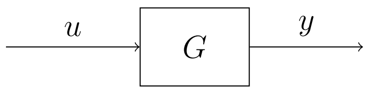

Figure 19: Interpretation of induced norm.

The definition of induced norm is interesting because one can interpret this as in Figure 19. In fact, using the induced norm

$$\|G\|_p = \sup_{y \neq 0} \left(\frac{\|Gu\|}{\|u\|} \right), \tag{4.14}$$

one can quantify the **maimum gain** (amplification) of an output vector for any possible input direction at a given frequency. This turns out to be extremely useful for evaluating system interconnections. Referring to Figure 20 and using the multiplication property for norms, it holds

$$\begin{aligned} \|y\|_p = \|G_2 G_1 u\|_p &\leq \|G_2\|_p \cdot \|G_1\|_p \cdot \|u\|_p. \\ \Rightarrow \frac{\|y\|_p}{\|u\|_p} &\leq \|G_2\|_p \cdot \|G_1\|_p. \end{aligned} \tag{4.15}$$

In words, the input-output gain of a system series is upper bounded by the product of the induced matrix norms.

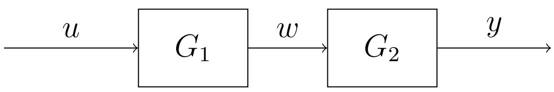

Figure 20: Interpretation of induced norm, system series.

Properties of the Euclidean Norm

We can list a few useful properties for the Euclidean norm, intended as induced norm:

(i) If A is squared (i.e. $m = n$), the norm is defined as

$$\begin{aligned} \|A\|_2 &= \sqrt{\mu_{max}} \\ &= \sqrt{\text{maximal eigenvalue of } A^* \cdot A}. \end{aligned} \tag{4.16}$$

(ii) If A is **orthogonal**:

$$\|A\|_2 = 1. \tag{4.17}$$

Note that this is the case because orthogonal matrices always have eigenvalues with magnitude 1.

(iii) If A is **symmetric** (i.e. $A^\intercal = A$):

$$\|A\|_2 = \max_i(|\lambda_i|), \tag{4.18}$$

where λ_i are the eigenvalues of A.

(iv) If A is **invertible**:

$$\begin{aligned}\|A^{-1}\|_2 &= \frac{1}{\sqrt{\mu_{min}}} \\ &= \frac{1}{\sqrt{\text{minimal eigenvalue of } A^* \cdot A}}.\end{aligned} \tag{4.19}$$

(v) If A is **invertible and symmetric**:

$$\|A^{-1}\|_2 = \frac{1}{\min_i(|\lambda_i|)}. \tag{4.20}$$

Remark. Remember: the matrix A^*A is always a square matrix.

4.1.3 Signal Norms

The norms we have seen so far are **space** measures. **Temporal** norms (signal norms), take into account the variability of signals in time/frequency. Let

$$e(t) = \begin{pmatrix} e_1(t) & \dots e_n(t) \end{pmatrix}^\intercal, \quad e_i(t) \in \mathbb{C}, \quad i = 1, \dots, n. \tag{4.21}$$

The p**-norm** is defined as

$$\|e(t)\|_p = \left(\int_{-\infty}^{\infty} \sum_{i=1}^{n} |e_i(\tau)|^p \mathrm{d}\tau \right)^{\frac{1}{p}}. \tag{4.22}$$

A special case of this is the two-norm, also called euclidean norm, integral square error, energy of a signal:

$$\|e(t)\|_2 = \sqrt{\left(\int_{-\infty}^{\infty} \sum_{i=1}^{n} |e_i(\tau)|^2 \mathrm{d}\tau \right)}. \tag{4.23}$$

The **infinity norm** is defined as

$$\|e(t)\|_\infty = \sup_\tau \left(\max_i |e_i(\tau)| \right). \tag{4.24}$$

4.1.4 System Norms

Considering linear, time-invariant, causal systems of the form depicted in Figure 19, one can write the relation

$$\begin{aligned} y(t) &= G * u(t) \\ &= \int_{-\infty}^{\infty} G(t-\tau) u(\tau) \mathrm{d}\tau. \end{aligned} \tag{4.25}$$

The **two-norm** for the transfer function $\hat{G}$ reads

$$\begin{aligned} \|\hat{G}(s)\|_2 &= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 \mathrm{d}\omega \right)^{\frac{1}{2}} \\ &= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{tr}\left(\hat{G}^*(j\omega) \hat{G}(j\omega) \right) \mathrm{d}\omega \right)^{\frac{1}{2}} \\ &= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{i,j=1} |g_{ij}|^2 \mathrm{d}\omega \right)^{\frac{1}{2}}. \end{aligned} \tag{4.26}$$

Remark. Note that this norm is a measure of the combination of system gains in all directions, over all frequency. This is not an induced norm, as it does not respect the multiplicative property.

The **infinity norm** is

$$\|\hat{G}(s)\|_\infty = \sup_\omega \|\hat{G}(j\omega)\|_2 \tag{4.27}$$

Remark. This norm is a measure of the peak of the maximum singular value, i.e. the biggest amplification the system may bring at any frequency, for any input direction (worst case scenario). This is an induced norm and respects the multiplicative property.

4.1.5 Examples

Example 27.

a) Calculate the euclidean **and** the maximum norm of the matrix:

$$A = \begin{pmatrix} 1 & 4 & 4 \\ -7 & 2 & -4 \\ 2 & 1 & 6 \end{pmatrix} \tag{4.28}$$

b) Find the euclidean **and** the maximum signal norm of the following signals, for $t \in (0, \infty)$:

   i) $u(t) = e^{-2t}$.

   ii) $v(t) = \cos(5t)$.

Solution.

a) The euclidean norm of a matrix is defined as:

$$\|G\|_2 = \sqrt{\sum_{i,j} |g_{i,j}|^2}. \tag{4.29}$$

The 2-norm of matrix A is therefore:

$$\|A\|_2 = \sqrt{1+16+16+49+4+16+4+1+36} \approx 12. \tag{4.30}$$

The maximum norm is defined by:

$$\|G\|_{max} = \max_{i,j} |g_{i,j}|. \tag{4.31}$$

For this reason we have:

$$\|A\|_{max} = \max\{1,4,4,7,2,4,2,1,6\} = 7. \tag{4.32}$$

b) The requested norms are defined as:

$$\begin{aligned} \|e(t)\|_2 &= \sqrt{\int_{-\infty}^{\infty} \sum_{i=1}^{n} |e_i(\tau)|^2 d\tau}. \\ \|e(t)\|_\infty &= \sup_\tau(\max_i |e_i(\tau)|). \end{aligned} \tag{4.33}$$

The given input signals are only one dimensional, so $n = 1$ for both cases. As written in the exercise description, only times from 0 to ∞ are to be considered.

i)

$$\begin{aligned} \|u(t)\|_2 &= \sqrt{\int_0^{\infty} |u(\tau)|^2 d\tau} \\ &= \sqrt{\int_0^{\infty} |e^{-2\tau}|^2 d\tau} \\ &= \sqrt{\int_0^{\infty} e^{-4\tau} d\tau} \\ &= \sqrt{\frac{-1}{4}[e^{-4\tau}]_0^{\infty}} \\ &= \sqrt{-\frac{1}{4}[0-1]} \\ &= \frac{1}{2}. \end{aligned} \tag{4.34}$$

$$\begin{aligned} \|u(t)\|_\infty &= \sup_\tau(|u(\tau)|) \\ &= \sup_\tau(|e^{-2\tau}|) \text{ with } \tau \in (0,\infty) \\ &= 1. \end{aligned} \tag{4.35}$$

ii)

$$\begin{aligned}\|v(t)\|_2 &= \sqrt{\int_0^\infty |v(\tau)|^2 d\tau} \\ &= \sqrt{\int_0^\infty |\cos(5\tau)|^2 d\tau},\end{aligned} \tag{4.36}$$

which is ∞.

$$\begin{aligned}\|v(t)\|_\infty &= \sup_\tau(|v(\tau)|) \\ &= \sup_\tau(|\cos(5\tau)|) \\ &= 1.\end{aligned} \tag{4.37}$$

4.2 Singular Value Decomposition (SVD)

The Singular Value Decomposition plays a central role in MIMO frequency response analysis. Let's recall some concepts from the course *Lineare Algebra I/II*:

4.2.1 Preliminary Definitions

The **induced** norm $\|A\|$ of a matrix that describes a linear function like

$$y = A \cdot u \tag{4.38}$$

is defined as

$$\begin{aligned} \|A\| &= \max_{u \neq 0} \frac{\|y\|}{\|u\|} \\ &= \max_{\|u\|=1} \|y\|. \end{aligned} \tag{4.39}$$

Let's recall Equation 4.16, and let's notice that if $A \in \mathbb{R}^{n \times m}$ it holds

$$A^* = A^\mathsf{T}. \tag{4.40}$$

In order to define the SVD we have to go a step further. Let's consider a Matrix A and the linear function given in Equation 4.38. It holds

$$\begin{aligned} \|A\|_2^2 &= \max_{\|=1} y^* \cdot y \\ &= \max_{\|=1} (A \cdot u)^* \cdot (A \cdot u) \\ &= \max_{\|=1} u^* \cdot A^* \cdot A \cdot u \\ &= \max_i \mu(A^* \cdot A) \\ &= \max_i \sigma_i^2. \end{aligned} \tag{4.41}$$

where σ_i are the **singular values** of matrix A. They are defined as

$$\sigma_i = \sqrt{\mu_i} \tag{4.42}$$

where μ_i are the eigenvalues of $A^* \cdot A$.
Combining Equations 4.39 and 4.42 one gets

$$\sigma_{min}(A) \leq \frac{\|y\|}{\|u\|} \leq \sigma_{max}(A). \tag{4.43}$$

4.2.2 Singular Value Decomposition

Our goal is to write a general matrix $A \in \mathbb{C}^{p \times m}$ as product of three matrices: U, Σ and V^*. It holds

$$A = U \cdot \Sigma \cdot V^* \text{ with } U \in \mathbb{C}^{p \times p},\ \Sigma \in \mathbb{R}^{p \times m},\ V \in \mathbb{C}^{m \times m}. \tag{4.44}$$

Remark. U and V are orthogonal, Σ is a diagonal matrix.

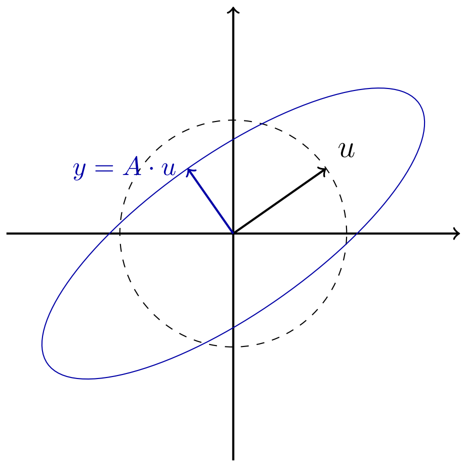

Figure 21: Illustration of the singular values.

Kochrezept:

Let $A \in \mathbb{C}^{p\times m}$ be given:

(I) Compute all the eigenvalues and eigenvectors of the matrix

$$A^* \cdot A \in \mathbb{C}^{m\times m}. \tag{4.45}$$

and sort them as

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_r > \mu_{r+1} = \ldots = \mu_m = 0 \tag{4.46}$$

(II) Compute an orthogonal basis from the eigenvectors v_i and write it in a matrix as

$$V = (v_1 \ \ldots \ v_m) \in \mathbb{C}^{m\times m}. \tag{4.47}$$

(III) We have already found the singular values: they are defined as

$$\sigma_i = \sqrt{\mu_i} \text{ for } i = 1, \ldots, \min\{p, m\}. \tag{4.48}$$

By ordering them from the biggest to the smallest, we can then write Σ as

$$\Sigma = \begin{pmatrix} \sigma_1 & & & 0 & \ldots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & \sigma_m & 0 & \ldots & 0 \end{pmatrix} \in \mathbb{R}^{p\times m}, \ p < m \tag{4.49}$$

$$\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{pmatrix} \in \mathbb{R}^{p\times m}, \ p > m. \tag{4.50}$$

(IV) One finds $u_1, \ldots, u_r$ from

$$u_i = \frac{1}{\sigma_i} \cdot A \cdot v_i \text{ for all } i = 1, \ldots, r \text{ (for } \sigma_i \neq 0) \tag{4.51}$$

(V) If $r < p$ one has to complete the basis $u_1, \dots, u_r$ (with ONB from Gram-Schmid) to obtain an orthogonal basis, with U orthogonal.

(VI) If you followed the previous steps, you can write

$$A = U \cdot \Sigma \cdot V^*. \tag{4.52}$$

Motivation for the computation of Σ, U und V.

$$\begin{aligned} A^* \cdot A &= (U \cdot \Sigma \cdot V^*)^* \cdot (U \cdot \Sigma \cdot V^*) \\ &= V \cdot \Sigma^* \cdot U^* \cdot U \cdot \Sigma \cdot V^* \\ &= V \cdot \Sigma^* \cdot \Sigma \cdot V^* \\ &= V \cdot \Sigma^2 \cdot V^*. \end{aligned} \tag{4.53}$$

This is nothing else than the **diagonalization** of the matrix $A^* \cdot A$. The columns of V are the eigenvectors of $A^* \cdot A$ and the σ_i^2 the eigenvalues.
For U:

$$\begin{aligned} A \cdot A^* &= (U \cdot \Sigma \cdot V^*) \cdot (U \cdot \Sigma \cdot V^*)^* \\ &= U \cdot \Sigma \cdot V^* \cdot V \cdot \Sigma \cdot U^* \\ &= U \cdot \Sigma^* \cdot \Sigma \cdot U^* \\ &= U \cdot \Sigma^2 \cdot U^*. \end{aligned} \tag{4.54}$$

This is nothing else than the **diagonalization** of the matrix $A \cdot A^*$. The columns of U are the eigenvectors of $A \cdot A^*$ and the σ_i^2 the eigenvalues.

Remark. In order to derive the previous two equations I used that:

- The matrix $A^* \cdot A$ is symmetric, i.e.

$$\begin{aligned} (A^* \cdot A)^* &= A^* \cdot (A^*)^* \\ &= A^* \cdot A. \end{aligned} \tag{4.55}$$

- $U^{-1} = U^*$ (because U is unitary).
- $V^{-1} = V^*$ (because V is unitary).

Remark. Since the matrix $A^* \cdot A$ is always symmetric and positive semidefinite, the singular values are always real numbers.

Remark. The MATLAB command for the singular value decomposition is

```
[U,S,V]=svd
```

One can write $A^\intercal$ as `A.'=transpose(A)` and A^* as `A'=conj(transpose(A))`. Those two are equivalent for real numbers.

4.2.3 Intepretation

Considering the system depicted in Figure 19, one rewrite the system G as $G = U\Sigma V^*$. The matrix V is **orthogonal** and contains the **input directions** of the system. The matrix U is **orthogonal** as well and contains the **output directions** of the system (unfortunate notation). It holds

$$\begin{aligned} G &= U\Sigma V^* \\ GV &= U\Sigma \\ Gv_i &= \sigma_i u_i, \quad \forall i, \end{aligned} \tag{4.56}$$

which is similar to an eigenvalue equation. This can be rewritten as

$$\sigma_i = \frac{\|Gv_i\|}{\|u_i\|}. \tag{4.57}$$

For a unitary input, i.e. $\|u\|_2 = 1$, one has

$$\begin{aligned} y_1 &= \sigma_1 u_1, \\ y_m &= \sigma_m u_m, \end{aligned} \tag{4.58}$$

where σ_m is the last (and hence the smallest) singular value. This can be interpreted using Figure 21 and interpreting the circle as a unit circle.

Example 28. Let u be

$$u = \begin{pmatrix} \cos(x) \\ \sin(x) \end{pmatrix} \tag{4.59}$$

with $\|u\| = 1$. The matrix M is given as

$$M = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{4.60}$$

We know that the product of M and u defines a linear function

$$\begin{aligned} y &= M \cdot u \\ &= \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \cos(x) \\ \sin(x) \end{pmatrix} \\ &= \begin{pmatrix} 2 \cdot \cos(x) \\ \frac{1}{2} \cdot \sin(x) \end{pmatrix}. \end{aligned} \tag{4.61}$$

We need the maximum of $\|y\|$. In order to avoid square roots, one can use that the x that maximizes $\|y\|$ should also maximize $\|y\|^2$.

$$\|y\|^2 = 4 \cdot \cos^2(x) + \frac{1}{4} \cdot \sin^2(x) \tag{4.62}$$

has maximum

$$\begin{aligned} \frac{\mathrm{d}\|y\|^2}{\mathrm{d}x} &= -8 \cdot \cos(x) \cdot \sin(x) + \frac{1}{2} \cdot \sin(x) \cdot \cos(x) \overset{!}{=} 0 \\ \Rightarrow \quad x_{\max} &= \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}. \end{aligned} \tag{4.63}$$

Inserting back for the maximal $\|y\|$ one gets:

$$\|y\|_{\max} = 2, \qquad \|y\|_{\max} = \frac{1}{2}. \tag{4.64}$$

The singular values can be calculated with $M^* \cdot M$:

$$\begin{aligned} M^* \cdot M &= M^\intercal \cdot M \\ &= \begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \quad \Rightarrow \quad \lambda_i = \left\{4, \frac{1}{4}\right\} \quad \Rightarrow \quad \sigma_i = \left\{2, \frac{1}{2}\right\}. \end{aligned} \tag{4.65}$$

As stated before, one can see that $\|y\| \in [\sigma_{\min}, \sigma_{\max}]$. The matrix U has eigenvectors of $M \cdot M^\intercal$ as coulmns and the matrix V has eigenvectors of $M^\intercal \cdot M$ as columns.
In this case

$$M \cdot M^\intercal = M^\intercal \cdot M,$$

hence the two matrices are equal. Since their product is a diagonal matrix one should recall from the theory that the eigenvectors are easy to determine: they are nothing else than the standard basis vectors. This means

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.66}$$

Interpretation:

Referring to Figure 22, let's interprete these calculations. One can see that the maximal *amplification* occurs at $v = V(:,1)$ and has direction $u = U(:,1)$, i.e. the vector u is doubled (σ_{max}). The minimal *amplification* occurs at $v = V(:,2)$ and has direction $u = U(:,2)$, i.e. the vector u is halved (σ_{min}).

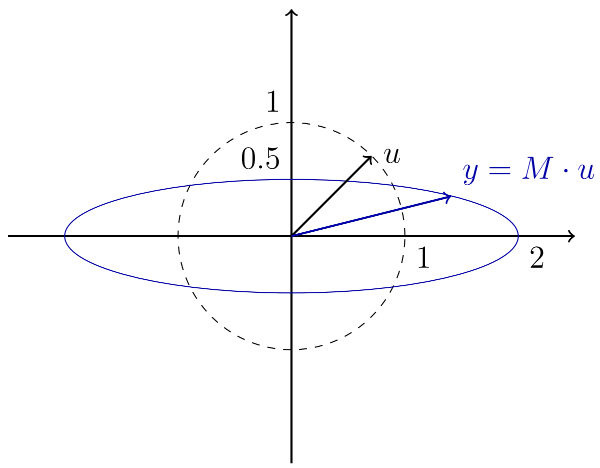

Figure 22: Illustration of the singular value decomposition.

Example 29. Let

$$A = \begin{pmatrix} -3 & 0 \\ 0 & 3 \\ \sqrt{3} & 2 \end{pmatrix} \tag{4.67}$$

be given.
Question: Find the singular values of A and write down the matrix Σ.

Solution. Let's compute $A^\intercal A$:

$$A^\intercal A = \begin{pmatrix} -3 & 0 & \sqrt{3} \\ 0 & 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} -3 & 0 \\ 0 & 3 \\ \sqrt{3} & 2 \end{pmatrix} = \begin{pmatrix} 12 & 2\sqrt{3} \\ 2\sqrt{3} & 13 \end{pmatrix} \tag{4.68}$$

One can see easily that the eigenvalues are

$$\lambda_1 = 16, \ \lambda_2 = 9. \tag{4.69}$$

The singular values are

$$\sigma_1 = 4, \ \sigma_2 = 3. \tag{4.70}$$

One writes in this case

$$\Sigma = \begin{pmatrix} 4 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix}. \tag{4.71}$$

Example 30. A transfer function $G(s)$ is given as

$$\begin{pmatrix} \frac{1}{s+3} & \frac{s+1}{s+3} \\ \frac{s+1}{s+3} & \frac{1}{s+3} \end{pmatrix} \tag{4.72}$$

Find the singular values of $G(s)$ at $\omega = 1\frac{\text{rad}}{s}$.

Solution. The transfer function $G(s)$ evaluated at $\omega = 1\frac{\text{rad}}{s}$ has the form

$$G(j) = \begin{pmatrix} \frac{1}{j+3} & \frac{j+1}{j+3} \\ \frac{j+1}{j+3} & \frac{1}{j+3} \end{pmatrix} \tag{4.73}$$

In order to calculate the singular values, we have to compute the eigenvalues of $H = G^* \cdot G$:

$$\begin{aligned} H &= G^* \cdot G \\ &= \begin{pmatrix} \frac{1}{-j+3} & \frac{-j+1}{-j+3} \\ \frac{-j+1}{-j+3} & \frac{1}{-j+3} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{j+3} & \frac{j+1}{j+3} \\ \frac{j+1}{j+3} & \frac{1}{j+3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{10} & \frac{2}{10} \\ \frac{2}{10} & \frac{3}{10} \end{pmatrix} \\ &= \frac{1}{10} \cdot \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}. \end{aligned} \tag{4.74}$$

For the eigenvalues it holds

$$\begin{aligned} \det(H - \lambda \cdot \mathbb{I}) &= \det \begin{pmatrix} \frac{3}{10} - \lambda & \frac{2}{10} \\ \frac{2}{10} & \frac{3}{10} - \lambda \end{pmatrix} \\ &= \left(\frac{3}{10} - \lambda\right)^2 - \left(-\frac{2}{10}\right)^2 \\ &= \lambda^2 - \frac{6}{10}\lambda + \frac{5}{100} \\ &= \left(\lambda - \frac{1}{10}\right) \cdot \left(\lambda - \frac{5}{10}\right). \end{aligned} \tag{4.75}$$

It follows

$$\begin{aligned} \lambda_1 &= \frac{1}{10} \\ \lambda_2 &= \frac{1}{2} \end{aligned} \tag{4.76}$$

and so

$$\begin{aligned} \sigma_1 &= \sqrt{\frac{1}{10}} \\ &\approx 0.3162. \\ \sigma_2 &= \sqrt{\frac{1}{2}} \\ &\approx 0.7071. \end{aligned} \tag{4.77}$$

Example 31. Let be

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ B &= \begin{pmatrix} j & 1 \end{pmatrix}. \end{aligned} \tag{4.78}$$

Find the singular values of the two matrices.

Solution.

- Let's begin with matrix A. It holds

$$\begin{aligned} H &= A^* \cdot A \\ &= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}. \end{aligned} \tag{4.79}$$

In order to find the eigenvalues of H we compute

$$\begin{aligned} \det(H - \lambda \cdot \mathbb{I}) &= \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 5-\lambda \end{pmatrix} \\ &= (1-\lambda) \cdot (5-\lambda) - 4 \\ &= \lambda^2 - 6\lambda + 1. \end{aligned} \tag{4.80}$$

This means that the eigenvalues are

$$\begin{aligned} \lambda_1 &= 3 + 2\sqrt{2} \\ \lambda_2 &= 3 - 2\sqrt{2}. \end{aligned} \tag{4.81}$$

The singular values are then

$$\begin{aligned} \sigma_1 &\approx 2.4142 \\ \sigma_2 &\approx 0.4142. \end{aligned} \tag{4.82}$$

- Let's look at matrix B. It holds

$$\begin{aligned} F &= B^* \cdot B \\ &= \begin{pmatrix} -j \\ 1 \end{pmatrix} \cdot \begin{pmatrix} j & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix}. \end{aligned} \tag{4.83}$$

In order to find the eigenvalues of F we compute

$$\begin{aligned} \det(F - \lambda \cdot \mathbb{I}) &= \det \begin{pmatrix} 1-\lambda & -j \\ j & 1-\lambda \end{pmatrix} \\ &= (1-\lambda)^2 - 1 \\ &= \lambda^2 - 2\lambda \\ &= \lambda \cdot (\lambda - 2). \end{aligned} \tag{4.84}$$

This means that the eigenvalues are

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 2. \end{aligned} \tag{4.85}$$

The singular values are then

$$\begin{aligned} \sigma_1 &= 0 \\ \sigma_2 &= \sqrt{2}. \end{aligned} \tag{4.86}$$

4.2.4 Directions of poles and zeros

Directions of zeros

Assume a system $G(s)$ has a zero at $s = z$. Then, it must holds

$$G(z)u_z = 0, \qquad y_z^* G(z) = 0, \tag{4.87}$$

where u_z is the **input zero direction** and y_z is the **output zero direction**. Furthermore, it holds

$$\|u_z\|_2 = 1, \qquad \|y_z\|_2 = 1. \tag{4.88}$$

Directions of poles

Assume a system $G(s)$ has a pole at $s = p$. Then, it must holds

$$G(p)u_p \to \infty, \qquad y_p^* G(p) \to \infty, \tag{4.89}$$

where u_p is the **input pole direction** and y_p is the **output pole direction**. Furthermore, it holds

$$\|u_p\|_2 = 1, \qquad \|y_p\|_2 = 1. \tag{4.90}$$

Remark. In both cases, if the zero/pole causes an unfeasible calculation, one consider feasible variations, i.e. $z + \varepsilon$, $p + \varepsilon$.

4.2.5 Frequency Responses

As we learned for SISO systems, if one excites a system with an *harmonic* signal

$$u(t) = h(t) \cdot \cos(\omega \cdot t), \tag{4.91}$$

the answer after a big amount of time is still an harmonic function with equal frequency ω:

$$y_\infty(t) = |P(j \cdot \omega)| \cos(\omega \cdot t + \angle(P(j \cdot \omega))). \tag{4.92}$$

One can generalize this and apply it to MIMO systems. With the assumption of $p = m$, i.e. equal number of inputs and outputs, one excite a system with

$$u(t) = \begin{pmatrix} \mu_1 \cdot \cos(\omega \cdot t + \phi_1) \\ \vdots \\ \mu_m \cdot \cos(\omega \cdot t + \phi_m) \end{pmatrix} \cdot h(t) \tag{4.93}$$

and get

$$y_\infty(t) = \begin{pmatrix} \nu_1 \cdot \cos(\omega \cdot t + \psi_1) \\ \vdots \\ \nu_m \cdot \cos(\omega \cdot t + \psi_m) \end{pmatrix}. \tag{4.94}$$

Let's define two diagonal matrices

$$\begin{aligned} \Phi &= \mathrm{diag}(\phi_1, \ldots, \phi_m) \in \mathbb{R}^{m \times m}, \\ \Psi &= \mathrm{diag}(\psi_1, \ldots, \psi_m) \in \mathbb{R}^{m \times m} \end{aligned} \tag{4.95}$$

and two vectors

$$\begin{aligned} \mu &= \begin{pmatrix} \mu_1 & \dots & \mu_m \end{pmatrix}^T, \\ \nu &= \begin{pmatrix} \nu_1 & \dots & \nu_m \end{pmatrix}^T. \end{aligned} \tag{4.96}$$

With these one can compute the Laplace Transform of the two signals as:

$$U(s) = e^{\frac{\Phi \cdot s}{\omega}} \cdot \mu \cdot \frac{s}{s^2+\omega^2}. \tag{4.97}$$

and

$$Y(s) = e^{\frac{\Psi \cdot s}{\omega}} \cdot \nu \cdot \frac{s}{s^2+\omega^2}. \tag{4.98}$$

With the general equation for a systems one gets

$$\begin{aligned} Y(s) &= P(s) \cdot U(s) \\ e^{\frac{\Psi \cdot s}{\omega}} \cdot \nu \cdot \frac{s}{s^2+\omega^2} &= P(s) \cdot e^{\frac{\Phi \cdot s}{\omega}} \cdot \mu \cdot \frac{s}{s^2+\omega^2} \\ e^{\frac{\Psi \cdot j \cdot \omega}{\omega}} \cdot \nu &= P(s) \cdot e^{\frac{\Phi \cdot j \cdot \omega}{\omega}} \cdot \mu \\ e^{\Psi \cdot j} \cdot \nu &= P(s) \cdot e^{\Phi \cdot j} \cdot \mu. \end{aligned} \tag{4.99}$$

We then recall that the **induced norm** for the matrix of a linear transformation $y = A \cdot u$ from 4.39. Here it holds

$$\begin{aligned} \|P(j \cdot \omega)\| &= \max_{e^{\Phi \cdot j} \cdot \mu \neq 0} \frac{\|e^{\Psi \cdot j} \cdot \nu\|}{\|e^{\Phi \cdot j} \cdot \mu\|} \\ &= \max_{\|e^{\Phi \cdot j} \cdot \mu\| = 1} \|e^{\Psi \cdot j} \cdot \nu\|. \end{aligned} \tag{4.100}$$

Since

$$\|e^{\Phi \cdot j} \cdot \mu\| = \|\mu\| \tag{4.101}$$

and

$$\|e^{\Psi \cdot j} \cdot \nu\| = \|\nu\|. \tag{4.102}$$

One gets

$$\begin{aligned} \|P(j \cdot \omega)\| &= \max_{\mu \neq 0} \frac{\|\nu\|}{\|\mu\|} \\ &= \max_{\|\mu\|=1} \|\nu\|. \end{aligned} \tag{4.103}$$

Here one should get the feeling of why we introduced the singular value decomposition. From the theory we've learned, it is clear that

$$\sigma_{min}(P(j \cdot \omega)) \leq \|\nu\| \leq \sigma_{max}(P(j \cdot \omega)). \tag{4.104}$$

and if $\|\mu\| \neq 1$

$$\sigma_{min}(P(j \cdot \omega)) \leq \frac{\|\nu\|}{\|\mu\|} \leq \sigma_{max}(P(j \cdot \omega)). \tag{4.105}$$

with σ_i singular values of $P(j \cdot \omega)$. These two are worst case ranges and is important to notice that there is no exact formula for $\nu = f(\mu)$.

Maximal and minimal Gain

You are given a singular value decomposition

$$P(j \cdot \omega) = U \cdot \Sigma \cdot V^*. \tag{4.106}$$

One can read out from this decomposition several informations: the maximal/minial gain will be reached with an excitation in the direction of the column vectors of V. The response of the system will then be in the direction of the coulmn vectors of U.
Let's look at an example and try to understand how to use these informations:

Example 32. We consider a system with $m = 2$ inputs and $p = 3$ outputs. We are given its singular value decomposition at $\omega = 5\frac{\text{rad}}{\text{s}}$:

$$\begin{aligned}
\Sigma &= \begin{pmatrix} 0.4167 & 0 \\ 0 & 0.2631 \\ 0 & 0 \end{pmatrix}, \\
V &= \begin{pmatrix} 0.2908 & 0.9568 \\ 0.9443 - 0.1542 \cdot j & -0.2870 + 0.0469 \cdot j \end{pmatrix}, \\
U &= \begin{pmatrix} -0.0496 - 0.1680 \cdot j & 0.1767 - 0.6831 \cdot j & -0.6621 - 0.1820 \cdot j \\ 0.0146 - 0.9159 \cdot j & -0.1059 + 0.3510 \cdot j & -0.1624 + 0.0122 \cdot j \\ 0.0349 - 0.3593 \cdot j & 0.1360 - 0.5910 \cdot j & 0.6782 + 0.2048 \cdot j \end{pmatrix}.
\end{aligned} \tag{4.107}$$

For the singular value $\sigma_{\max} = 0.4167$ the eigenvectors are $V(:,1)$ and $U(:,1)$:

$$V_1 = \begin{pmatrix} 0.2908 \\ 0.9443 - 0.1542 \cdot j \end{pmatrix}, \quad |V_1| = \begin{pmatrix} 0.2908 \\ 0.9568 \end{pmatrix}, \quad \angle(V_1) = \begin{pmatrix} 0 \\ -0.1618 \end{pmatrix}, \tag{4.108}$$

$$U_1 = \begin{pmatrix} -0.0496 - 0.1680 \cdot j \\ 0.0146 - 0.9159 \cdot j \\ 0.0349 - 0.3593 \cdot j \end{pmatrix}, \quad |U_1| = \begin{pmatrix} 0.1752 \\ 0.9160 \\ 0.3609 \end{pmatrix}, \quad \angle(U_1) = \begin{pmatrix} -1.8581 \\ -1.5548 \\ -1.4741 \end{pmatrix}. \tag{4.109}$$

The maximal gain is then reached with

$$u(t) = \begin{pmatrix} 0.2908 \cdot \cos(5 \cdot t) \\ 0.9568 \cdot \cos(5 \cdot t - 0.1618) \end{pmatrix}. \tag{4.110}$$

The response of the system is then

$$y(t) = \sigma_{\max} \cdot \begin{pmatrix} 0.1752 \cdot \cos(5 \cdot t - 1.8581) \\ 0.9160 \cdot \cos(5 \cdot t - 1.5548) \\ 0.3609 \cdot \cos(5 \cdot t - 1.4741) \end{pmatrix} = 0.4167 \cdot \begin{pmatrix} 0.1752 \cdot \cos(5 \cdot t - 1.8581) \\ 0.9160 \cdot \cos(5 \cdot t - 1.5548) \\ 0.3609 \cdot \cos(5 \cdot t - 1.4741) \end{pmatrix}. \tag{4.111}$$

Since the three signals $y_1(t)$, $y_2(t)$ and $y_3(t)$ are not in phase, the maximal gain will never be reachen. One can show that

$$\max_t \|y(t)\| \approx 0.4160 < 0.4167 = \sigma_{\max} \tag{4.112}$$

The reason for this difference stays in the phase deviation between $y_1(t)$, $y_2(t)$ and $y_3(t)$. The same analysis can be computed for $\sigma_{\min}$.

Example 33. Given the MIMO system

$$P(s) = \begin{pmatrix} \frac{1}{s+3} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{3}{s+1} \end{pmatrix}. \tag{4.113}$$

Starting at $t = 0$, the system is excited with the following input signal:

$$u(t) = \begin{pmatrix} \cos(t) \\ \mu_2 \cos(t + \varphi_2) \end{pmatrix}. \tag{4.114}$$

Find the parameters φ_2 and μ_2 such that for steady-state conditions the output signal

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \tag{4.115}$$

has $y_1(t)$ equal to zero.

Solution. For a system excited using a harmonic input signal

$$u(t) = \begin{pmatrix} \mu_1 \cos(\omega t + \varphi_1) \\ \mu_2 \cos(\omega t + \varphi_2) \end{pmatrix} \tag{4.116}$$

the output signal $y(t)$, after a transient phase, will also be a harmonic signal and hence have the form

$$y(t) = \begin{pmatrix} \nu_1 \cos(\omega t + \psi_1) \\ \nu_2 \cos(\omega t + \psi_2) \end{pmatrix}. \tag{4.117}$$

As we have learned, it holds

$$e^{\Psi \cdot j} \cdot \nu = P(j\omega) \cdot e^{\Phi \cdot j} \cdot \mu. \tag{4.118}$$

One gets

$$\begin{pmatrix} e^{\psi_1 \cdot j} & 0 \\ 0 & e^{\psi_2 \cdot j} \end{pmatrix} \cdot \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} P_{11}(j\omega) & P_{12}(j\omega) \\ P_{11}(j\omega) & P_{21}(j\omega) \end{pmatrix} \cdot \begin{pmatrix} e^{\varphi_1 \cdot j} & 0 \\ 0 & e^{\varphi_2 \cdot j} \end{pmatrix} \cdot \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}. \tag{4.119}$$

For the first component one gets

$$e^{\psi_1 \cdot j} \cdot \nu_1 = P_{11}(j\omega) \cdot e^{\varphi_1 \cdot j} \cdot \mu_1 + P_{12}(j\omega) \cdot e^{\varphi_2 \cdot j} \cdot \mu_2. \tag{4.120}$$

For $y_1(t) = 0$ to hold we must have $\nu_1 = 0$. In the given case, some parameters can be easily copied from the signals:

$$\begin{aligned} \mu_1 &= 1 \\ \varphi_1 &= 0 \\ \omega &= 1. \end{aligned} \tag{4.121}$$

With the given transfer functions, one gets

$$\begin{aligned} 0 &= \frac{1}{j+3} + \mu_2 \cdot \frac{1}{j+1} \cdot e^{\varphi_2 \cdot j} \\ 0 &= \frac{3-j}{10} + \mu_2 \cdot \frac{1-j}{2} \cdot e^{\varphi_2 \cdot j} \\ 0 &= \frac{3-j}{10} + \mu_2 \cdot \frac{1-j}{2} \cdot (\cos(\varphi_2) + j \sin(\varphi_2)) \\ 0 &= \frac{3}{10} + \mu_2 \cdot \frac{1}{2} \cdot (\cos(\varphi_2) + \sin(\varphi_2)) + j \cdot \left(\mu_2 \cdot \frac{1}{2} \cdot (\sin(\varphi_2) - \cos(\varphi_2)) - \frac{1}{10} \right). \end{aligned} \tag{4.122}$$

Splitting the real to the imaginary part, one can get two equations that are easily solvable:

$$\begin{aligned} \mu_2 \cdot \frac{1}{2} \cdot (\cos(\varphi_2) + \sin(\varphi_2)) + \frac{3}{10} &= 0 \\ \mu_2 \cdot \frac{1}{2} \cdot (\sin(\varphi_2) - \cos(\varphi_2)) - \frac{1}{10} &= 0. \end{aligned} \tag{4.123}$$

Adding and subtracting the two equations one can reach two better equations:

$$\begin{aligned} \mu_2 \cdot \sin(\varphi_2) + \frac{1}{5} &= 0 \\ \mu_2 \cdot \cos(\varphi_2) + \frac{2}{5} &= 0. \end{aligned} \tag{4.124}$$

One of the solutions (periodicity) reads

$$\begin{aligned} \mu_2 &= \frac{1}{\sqrt{5}} \\ \varphi_2 &= \arctan\left(\frac{1}{2}\right) + \pi. \end{aligned} \tag{4.125}$$

Example 34. A 2×2 linear time invariant MIMO system with transfer function

$$P(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{2}{s+1} \\ \frac{s^2+1}{s+10} & \frac{1}{s^2+2} \end{pmatrix} \tag{4.126}$$

is excited with the signal

$$u(t) = \begin{pmatrix} \mu_1 \cdot \cos(\omega \cdot t + \varphi_1) \\ \mu_2 \cdot \cos(\omega \cdot t + \varphi_2) \end{pmatrix}. \tag{4.127}$$

Because we bought a cheap signal generator, we cannot know exactly the constants $\mu_{1,2}$ and $\varphi_{1,2}$. A friend of you just found out with some measurements, that the excitation frequency is $\omega = 1\frac{\text{rad}}{\text{s}}$. The cheap generator, cannot produce signals with magnitude of μ bigger than 10, i.e. $\sqrt{\mu_1^2 + \mu_2^2} \leq 10$. This works always at maximal power, i.e. at 10. Choose all possible responses of the system after infinite time.

☐ $y_\infty(t) = \begin{pmatrix} 5 \cdot \sin(t + 0.114) \\ \cos(t) \end{pmatrix}$.

☐ $y_\infty(t) = \begin{pmatrix} 5 \cdot \sin(t + 0.114) \\ \cos(2 \cdot t) \end{pmatrix}$.

☐ $y_\infty(t) = \begin{pmatrix} \sin(t + 0.542) \\ \sin(t + 0.459) \end{pmatrix}$.

☐ $y_\infty(t) = \begin{pmatrix} 19 \cdot \cos(t + 0.114) \\ \cos(t + 1.124) \end{pmatrix}$.

☐ $y_\infty(t) = \begin{pmatrix} 5 \cdot \cos(t + 0.114) \\ 5 \cdot \cos(t) \end{pmatrix}$.

☐ $y_\infty(t) = \begin{pmatrix} 10 \cdot \sin(t + 2.114) \\ 11 \cdot \sin(t + 1.234) \end{pmatrix}$.

Solution.

☑ $y_\infty(t) = \begin{pmatrix} 5 \cdot \sin(t + 0.114) \\ \cos(t) \end{pmatrix}$.

☐ $y_\infty(t) = \begin{pmatrix} 5 \cdot \sin(t + 0.114) \\ \cos(2 \cdot t) \end{pmatrix}$.

☐ $y_\infty(t) = \begin{pmatrix} \sin(t + 0.542) \\ \sin(t + 0.459) \end{pmatrix}$.

☐ $y_\infty(t) = \begin{pmatrix} 19 \cdot \cos(t + 0.114) \\ \cos(t + 1.124) \end{pmatrix}$.

☑ $y_\infty(t) = \begin{pmatrix} 5 \cdot \cos(t + 0.114) \\ 5 \cdot \cos(t) \end{pmatrix}$.

☑ $y_\infty(t) = \begin{pmatrix} 10 \cdot \sin(t + 2.114) \\ 11 \cdot \sin(t + 1.234) \end{pmatrix}$.

Explanation

We have to compute the singular values of the matrix $P(j \cdot 1)$. These are

$$\begin{aligned} \sigma_{max} &= 1.8305 \\ \sigma_{min} &= 0.3863. \end{aligned} \tag{4.128}$$

With what we have learned it follows

$$10 \cdot \sigma_{min} = 3.863 \leq \|\nu\|\| \leq 18.305 = 10 \cdot \sigma_{max}. \tag{4.129}$$

The first response has $\|\nu\| = \sqrt{26}$ that is in this range. The second response also has $\|\nu\| = \sqrt{26}$ but the frequency in its second element changes and that isn't possible for linear systems. The third response has $\|\nu\| = \sqrt{2}$ that is too small to be in the range. The fourth response has $\|\nu\| = \sqrt{362}$ that is too big to be in the range. The fifth response has $\|\nu\| = \sqrt{50}$ that is in the range. The sixth response has $\|\nu\| = \sqrt{221}$ that is in the range.

Example 35. A 3×2 linear time invariant MIMO system is excited with the input

$$u(t) = \begin{pmatrix} 3 \cdot \sin(30 \cdot t) \\ 4 \cdot \cos(30 \cdot t) \end{pmatrix}. \tag{4.130}$$

You have forgot your PC and you don't know the transfer function of the system. Before coming to school, however, you have saved the MATLAB plot of the singular values of the system on your phone (see Figure 23. Choose all the possible responses of the system.

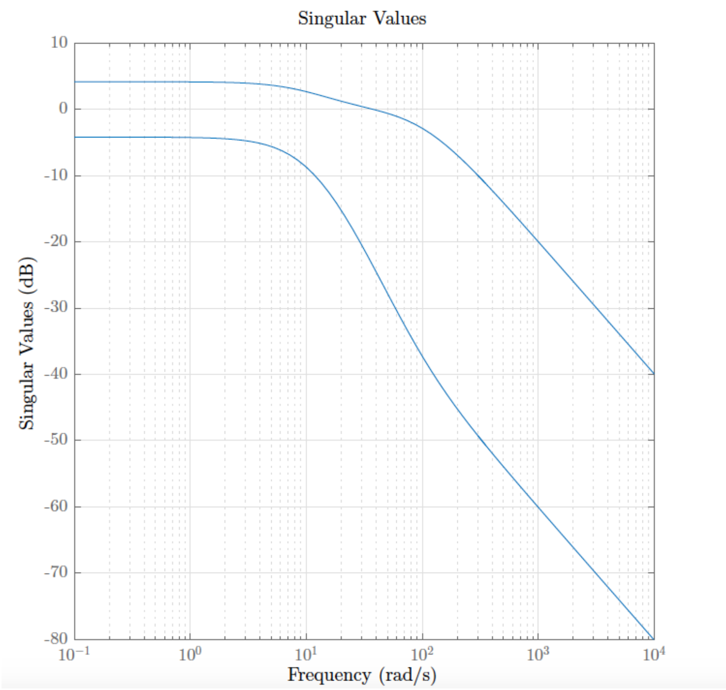

Figure 23: Singular values behaviour.

☐ $y_\infty(t) = \begin{pmatrix} 0.5 \cdot \sin(30 \cdot t + 0.314) \\ 0.5 \cdot \cos(30 \cdot t) \\ 0.5 \cdot \cos(30 \cdot t + 1) \end{pmatrix}.$

☐ $y_\infty(t) = \begin{pmatrix} 4 \cdot \sin(30 \cdot t + 0.314) \\ 3 \cdot \cos(30 \cdot t) \\ 2 \cdot \cos(30 \cdot t + 1) \end{pmatrix}.$

☐ $y_\infty(t) = \begin{pmatrix} 0.1 \cdot \sin(30 \cdot t + 0.314) \\ 0.1 \cdot \cos(30 \cdot t) \\ 0.1 \cdot \cos(30 \cdot t + 1) \end{pmatrix}.$

☐ $y_\infty(t) = \begin{pmatrix} 0 \\ 4 \cdot \cos(30 \cdot t) \\ 2 \cdot \cos(30 \cdot t + 1) \end{pmatrix}.$

☐ $y_\infty(t) = \begin{pmatrix} 2 \cdot \cos(30 \cdot t + 0.243) \\ 2 \cdot \cos(30 \cdot t + 0.142) \\ 2 \cdot \cos(30 \cdot t + 0.252) \end{pmatrix}.$

Solution.

- ☑ $y_\infty(t) = \begin{pmatrix} 0.5\cdot\sin(30\cdot t+0.314)\\ 0.5\cdot\cos(30\cdot t)\\ 0.5\cdot\cos(30\cdot t+1)\end{pmatrix}$.
- ☐ $y_\infty(t) = \begin{pmatrix} 4\cdot\sin(30\cdot t+0.314)\\ 3\cdot\cos(30\cdot t)\\ 2\cdot\cos(30\cdot t+1)\end{pmatrix}$.
- ☐ $y_\infty(t) = \begin{pmatrix} 0.1\cdot\sin(30\cdot t+0.314)\\ 0.1\cdot\cos(30\cdot t)\\ 0.1\cdot\cos(30\cdot t+1)\end{pmatrix}$.
- ☑ $y_\infty(t) = \begin{pmatrix} 0\\ 4\cdot\cos(30\cdot t)\\ 2\cdot\cos(30\cdot t+1)\end{pmatrix}$.
- ☑ $y_\infty(t) = \begin{pmatrix} 2\cdot\cos(30\cdot t+0.243)\\ 2\cdot\cos(30\cdot t+0.142)\\ 2\cdot\cos(30\cdot t+0.252)\end{pmatrix}$.

Explanation

From the given input one can read

$$\|\mu\| = \sqrt{3^2+4^2} = 5. \tag{4.131}$$

From the plot one can read at $\omega = 30\frac{\text{rad}}{\text{s}}$ $\sigma_{min} = 0.1$ and $\sigma_{max} = 1$. It follows

$$5\cdot\sigma_{min} = 0.5 \leq \|\nu\| \leq 5 = 5\cdot\sigma_{max}. \tag{4.132}$$

The first response has $\|\nu\| = \sqrt{0.75}$ that is in the range. The second response has $\|\nu\| = \sqrt{29}$ that is too big to be in the range. The third response has $\|\nu\| = \sqrt{0.03}$ that is to small to be in the range. The fourth response has $\|\nu\| = \sqrt{20}$ that is in the range. The fifth response has $\|\nu\| = \sqrt{12}$ that is in the range.

Example 36. You are given the transfer function

$$P(s) = \begin{pmatrix} 1 & \frac{1}{s-3} \\ 1 & 1 \end{pmatrix}. \tag{4.133}$$

a) Find the poles of the system.

b) Find the zeros of the system.

c) Find the directions of the zeros of the system.

d) How could you compute the directions of the poles of the system? Note that you don't need to compute them by hand.

Solution.

a) In order to compute the poles of the transfer function, one needs to compute its minors. First order minors are

$$1, \quad \frac{1}{s-3}, \quad 1, \quad 1. \tag{4.134}$$

Second order minor is

$$1 - \frac{1}{s-3} = \frac{s-4}{s-3}. \tag{4.135}$$

The least common denominator (i.e. the pole-polynom) is $(s-3)$, from which it is clear that the only pole of the system is $\pi_1 = 3$.

b) Normalizing the second order minor with the pole-polynom results in

$$\frac{s-4}{s-3}. \tag{4.136}$$

Since we have only one element, the greatest common divisor is $(s-4)$, from which it is clear that the only zero of the system is $z_1 = 4$.

c) With the method learned in class, in order to find the direction of the zero $z_1 = 4$, one needs to compute the singular value decomposition of $P(4)$. It holds

$$P(4) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad P^* = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{4.137}$$

In the following, we refer to the SVD recipe provided in the lecture.

1) As a first step, we compute the eigenvalues of $M = P^*(4)P(4)$. It holds

$$\begin{aligned} M = P^*P &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}. \end{aligned} \tag{4.138}$$

In order to find the eigenvalues, we compute

$$\begin{aligned} \det(M - \lambda \mathbb{I}) &= \det \begin{pmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{pmatrix} \\ &= (2-\lambda)^2 - 4 \\ &= \lambda(\lambda - 4). \end{aligned} \tag{4.139}$$

From this characteristic polynom one can read the eigenvalues

$$\lambda_1 = 4, \quad \lambda_2 = 0. \tag{4.140}$$

2) The singular values are the positive square roots of the found eigenvalues. It holds

$$\sigma_1 = 2, \quad \sigma_2 = 0. \tag{4.141}$$

3) The singular value matrix is

$$\Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.142}$$

4) The right singular vectors v_i are the eigenvectors of M with respect to the found eigenvalues λ_i. It holds

- $E_{\lambda_1} = E_4$: From $(M - \lambda_1 \mathbb{I}) \cdot x = 0$ one gets the linear system of equations

$$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right).$$

Using the first row as reference and summing it to the second row, one gets the form

$$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Since one has a zero row, one can introduce a free parameter. Let $x_1 = s \in \mathbb{R}$. Using the first row, one can recover $x_2 = x_1$. This defines the first eigenspace, which reads

$$E_4 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.143}$$

The eigenvectors have to be normalized. The magnitude of this eigenvector is

$$\|v_1\|_2 = \sqrt{1+1} = \sqrt{2}. \tag{4.144}$$

It holds

$$v_{1,n} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.145}$$

- $E_{\lambda_2} = E_0$: From $(M - \lambda_2 \mathbb{I}) \cdot x = 0$ one gets the linear system of equations

$$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right).$$

Using the first row as reference and subtracting it from the second row, one gets the form

$$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Since one has a zero row, one can introduce a free parameter. Let $x_1 = s \in \mathbb{R}$. Using the first row, one can recover $x_2 = -x_1$. This defines the first eigenspace, which reads

$$E_{\frac{1}{2}} = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{4.146}$$

The eigenvectors have to be normalized. The magnitude of this eigenvector is

$$\|v_2\|_2 = \sqrt{1+1} = \sqrt{2}. \tag{4.147}$$

It holds

$$v_{2,n} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{4.148}$$

The matrix V is then

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{4.149}$$

5) The left singular vectors can be computed with

$$u_{i,n} = \frac{1}{\sigma_i} P v_{i,n}. \tag{4.150}$$

- For the first vector it holds

$$\begin{aligned} u_{1,n} &= \frac{1}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned} \tag{4.151}$$

- For the second vector, because of the 0 singular value the formula does not hold anymore. However, from the lecture we know that one can find u_i as the eigenvector with respect of λ_i of matrix PP^*. It holds

$$\begin{aligned} P(4)P^*(4) &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}. \end{aligned} \tag{4.152}$$

The eigenspace for eigenvalue $\lambda_2 = 0$ is the same as the one computed in Equation 4.148 (same matrices). It holds then

$$u_{2,n} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{4.153}$$

The matrix U is then

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{4.154}$$

One can now write the singular value decomposition

$$\begin{aligned} P &= U \cdot \Sigma \cdot V^* \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^*. \end{aligned} \tag{4.155}$$

In order to find the input zero direction, we want to find a direction u_z s.t.

$$P(z) \cdot u_z = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.156}$$

This corresponds to the vector

$$u_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{4.157}$$

which is the second column of matrix V (relative to the 0 singular value). In order to find the output zero direction, we want to find a direction y_z s.t.

$$y_z^* P(z) = \begin{pmatrix} 0 & 0 \end{pmatrix}. \tag{4.158}$$

This corresponds to the vector

$$y_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{4.159}$$

which is the second column of matrix U (relative to the 0 singular value).

d) In order to use the same method used for the zeros, we consider a deviation $\varepsilon = 0.001$ from the pole. One can write

$$\begin{aligned} P(p_1 + \varepsilon) &= P(3 + 0.001) \\ &= \begin{pmatrix} 1 & 1000 \\ 1 & 1 \end{pmatrix}. \end{aligned} \tag{4.160}$$

The singular value decomposition of $P(3.001)$ is

$$P(3.001) = \begin{pmatrix} -1 & -0.001 \\ -0.001 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1000 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -0.001 & 1 \\ -1 & -0.001 \end{pmatrix}^*, \tag{4.161}$$

from which one can identify

$$u_p = \frac{1}{\sqrt{1 + 0.001^2}} \begin{pmatrix} -0.001 \\ -1 \end{pmatrix}, \quad y_p = \frac{1}{\sqrt{1 + 0.001^2}} \begin{pmatrix} -1 \\ -0.001 \end{pmatrix}, \tag{4.162}$$

where we took the directions relative to the biggest singular value (pole causes increase of the value of the transfer function), i.e. first columns of U and V. Note that in order to find this decomposition, the MATLAB® command `[U,V,D]=svd(P)` has been used.
For more informations and examples, please have a look at `http://karimpor.profcms.um.ac.ir/imagesm/354/stories/mul_con/multivariable_lec4.pdf`.

4.3 MIMO Stability

4.3.1 External Stability

The **input-output** stability (also known as external stability) describes the stability properties of a system with respect to its input-output behaviour. Let's consider the system interaction depicted in Figure 24.

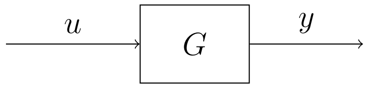

Figure 24: Interpretation of induced norm.

Definition 12. A MIMO system $y = Gu$ is said to be **BIBO stable** (i.e. bounded input bounded output) if there exists a finite constant $k \in \mathbb{R}$ such that

$$\|y\|_\infty \leq k\|u\|_\infty. \tag{4.163}$$

Remark. A **necessary and sufficient** condition for BIBO stability is: the closed loop transfer function

$$P(s) = C(s\mathbb{I} - A)^{-1} + D \tag{4.164}$$

has all poles in the open left-half of the complex plane (all poles have real part strictly smaller than 0).

4.3.2 Internal Stability

Consider the linear time invariant system

$$\begin{aligned} \dot{x}(t) &= A \cdot x(t) + B \cdot u(t), \qquad & x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \\ y(t) &= C \cdot x(t) + D \cdot u(t), \qquad & y(t) \in \mathbb{R}^p \end{aligned} \tag{4.165}$$

where

$$x(t) \in \mathbb{R}^{n\times 1},\ u(t) \in \mathbb{R}^{m\times 1},\ y(t) \in \mathbb{R}^{p\times 1},\ A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}. \tag{4.166}$$

Such a system is internally stable if for all initial conditions, and all bounded signals injected at any place in the system, all states remain bounded for all future time.

Definition 13. The MIMO linear time invariant system described in Equation 4.165 is BIBO stable if and only if $C(s\mathbb{I} - A)^{-1} + D$ has all poles on the open left-half of the complex plane (all poles have real part strictly smaller than 0).

Remark.

- Internal stability **implies** BIBO stability. The converse is not true.
- BIBO stability with controllability and observability **imply** internal stability.

This is a crucial concept: it is not sufficient for the input-output transfer function of the system to be stable. In fact, internal transfer functions, related to the sensitivity functions, must be stable as well to prevent pole/zero cancellations, which could hide instabilities.

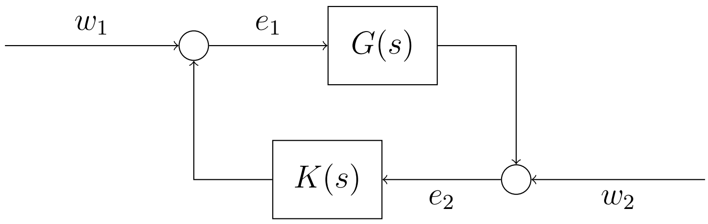

Figure 25: MIMO Loop.

Internal Stability Check

Assume a MIMO loop as the one depiced in Figure 25. It holds

$$\begin{aligned} E_1(s) &= W_1(s) + K(s)E_2(s) \\ &= W_1(s) + K(s)\left[G(s)E_1(s) + W_2(s)\right] \\ &= W_1(s) + K(s)G(s)E_1(s) + K(s)W_2(s), \end{aligned} \tag{4.167}$$

from which it follows

$$\begin{aligned} \left(\mathbb{I} - K(s)G(s)\right) E_1(s) &= W_1(s) + K(s)W_2(s) \\ E_1(s) &= \left(\mathbb{I} - K(s)G(s)\right)^{-1} W_1(s) + \left(\mathbb{I} - K(s)G(s)\right)^{-1} K(s)W_2(s). \end{aligned} \tag{4.168}$$

Similarly, one can write

$$\begin{aligned} E_2(s) &= W_2(s) + G(s)E_1(s) \\ &= W_2(s) + G(s)\left[K(s)E_2(s) + W_1(s)\right] \\ &= W_2(s) + G(s)K(s)E_2(s) + G(s)W_1(s), \end{aligned} \tag{4.169}$$

from which it follows

$$\begin{aligned} \left(\mathbb{I} - G(s)K(s)\right) E_2(s) &= W_2(s) + G(s)W_1(s) \\ E_2(s) &= \left(\mathbb{I} - G(s)K(s)\right)^{-1} W_2(s) + \left(\mathbb{I} - G(s)K(s)\right)^{-1} G(s)W_1(s). \end{aligned} \tag{4.170}$$

Resuming the calculations into matrix form, one gets

$$\begin{pmatrix} E_1(s) \\ E_2(s) \end{pmatrix} = \begin{pmatrix} \left(\mathbb{I} - K(s)G(s)\right)^{-1} & \left(\mathbb{I} - K(s)G(s)\right)^{-1} K(s) \\ \left(\mathbb{I} - G(s)K(s)\right)^{-1} G(s) & \left(\mathbb{I} - G(s)K(s)\right)^{-1} \end{pmatrix} \cdot \begin{pmatrix} W_1(s) \\ W_2(s) \end{pmatrix}. \tag{4.171}$$

The necessary and sufficient condition for **internal stability** is: each of the four transfer functions in relation 4.171 must be stable, i.e. smaller than 0 . Note: even if three of four are stable, the system is not internally stable.

4.3.3 Lyapunov Stability

The Lyapunov stability theorem analyses the behaviour of a system near to its equilibrium points when $u(t) = 0$. Because of this, we don't care if the system is MIMO or SISO. The three cases are

- Asymptotically stable: $\lim_{t\to\infty} \|x(t)\| = 0$;

- Stable: $\|x(t)\| < \infty \, \forall \, t \geq 0$;
- Unstable: $\lim_{t\to\infty} \|x(t)\| = \infty$.

As it was done for the SISO case, one can show by using $x(t) = e^{A\cdot t} \cdot x_0$ that the stability can be related to the eigenvalues of A through:

- Asymptotcally stable: $\text{Re}(\lambda_i) < 0 \, \forall \, i$;
- (Marginally) Stable: $\text{Re}(\lambda_i) \leq 0 \, \forall \, i$;
- Unstable: $\text{Re}(\lambda_i) > 0$ for at least one i.

4.3.4 Examples

Example 37. You are given the feedback control loop depicted in Figure 26.

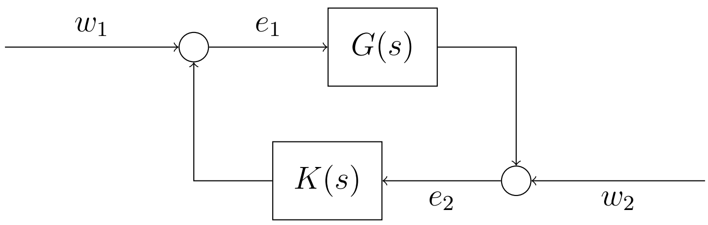

Figure 26: MIMO Loop.

a) Derive the internal stability criterion, i.e. write the error vector $\begin{pmatrix} e_1 & e_2 \end{pmatrix}^\mathsf{T}$ as a function of $\begin{pmatrix} w_1 & w_2 \end{pmatrix}^\mathsf{T}$.

b) You are given

$$G(s) = \frac{s-1}{s+1}, \quad K(s) = -\frac{1}{s-1}. \tag{4.172}$$

Is the resulting system internally stable?

c) You are given

$$G(s) = \begin{pmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s+1} \end{pmatrix}, \quad K(s) = \begin{pmatrix} \frac{1-s}{s+1} & -1 \\ 0 & -1 \end{pmatrix}. \tag{4.173}$$

Is the resulting system internally stable?

Solution.

a) Considering Figure 26, one can write

$$\begin{aligned} E_1(s) &= W_1(s) + K(s)E_2(s) \\ &= W_1(s) + K(s)\left[G(s)E_1(s) + W_2(s)\right] \\ &= W_1(s) + K(s)G(s)E_1(s) + K(s)W_2(s), \end{aligned} \tag{4.174}$$

from which it follows

$$\begin{aligned} \left(\mathbb{I} - K(s)G(s)\right)E_1(s) &= W_1(s) + K(s)W_2(s) \\ E_1(s) &= \left(\mathbb{I} - K(s)G(s)\right)^{-1} W_1(s) + \left(\mathbb{I} - K(s)G(s)\right)^{-1} K(s)W_2(s). \end{aligned} \tag{4.175}$$

Similarly, one can write

$$\begin{aligned} E_2(s) &= W_2(s) + G(s)E_1(s) \\ &= W_2(s) + G(s)\left[K(s)E_2(s) + W_1(s)\right] \\ &= W_2(s) + G(s)K(s)E_2(s) + G(s)W_1(s), \end{aligned} \tag{4.176}$$

from which it follows

$$\begin{aligned} \left(\mathbb{I} - G(s)K(s)\right)E_2(s) &= W_2(s) + G(s)W_1(s) \\ E_2(s) &= \left(\mathbb{I} - G(s)K(s)\right)^{-1} W_2(s) + \left(\mathbb{I} - G(s)K(s)\right)^{-1} G(s)W_1(s). \end{aligned} \tag{4.177}$$

Resuming the calculations into matrix form, one gets

$$\begin{pmatrix} E_1(s) \\ E_2(s) \end{pmatrix} = \begin{pmatrix} \left(\mathbb{I} - K(s)G(s)\right)^{-1} & \left(\mathbb{I} - K(s)G(s)\right)^{-1} K(s) \\ \left(\mathbb{I} - G(s)K(s)\right)^{-1} G(s) & \left(\mathbb{I} - G(s)K(s)\right)^{-1} \end{pmatrix} \cdot \begin{pmatrix} W_1(s) \\ W_2(s) \end{pmatrix}. \tag{4.178}$$

The necessary and sufficient condition for **internal stability** is: each of the four transfer functions in relation 4.178 must be stable. (Note: even if three of four are stable, the system is not internally stable).

b) It holds

$$\begin{aligned} \left(\mathbb{I} - K(s)G(s)\right)^{-1} &= \left(1 - \left(-\frac{1}{s+1}\right)\right)^{-1} \\ &= \left(\frac{s+2}{s+1}\right)^{-1} \\ &= \frac{s+1}{s+2}. \\ \left(\mathbb{I} - K(s)G(s)\right)^{-1} K(s) &= -\frac{s+1}{(s+2)(s-1)}. \\ \left(\mathbb{I} - G(s)K(s)\right)^{-1} G(s) &= \frac{s-1}{s+2}. \\ \left(\mathbb{I} - G(s)K(s)\right)^{-1} &= \frac{s+1}{s+2}. \end{aligned} \tag{4.179}$$

Rewriting Equation 4.178 for this specific problem, one gets

$$\begin{aligned}\begin{pmatrix}E_1(s)\\E_2(s)\end{pmatrix}&=\begin{pmatrix}(\mathbb{I}-K(s)G(s))^{-1} & (\mathbb{I}-K(s)G(s))^{-1}K(s)\\(\mathbb{I}-G(s)K(s))^{-1}G(s) & (\mathbb{I}-G(s)K(s))^{-1}\end{pmatrix}\cdot\begin{pmatrix}W_1(s)\\W_2(s)\end{pmatrix}\\&=\begin{pmatrix}\frac{s+1}{s+2} & -\frac{s+1}{(s+2)(s-1)}\\\frac{s-1}{s+2} & \frac{s+1}{s+2}\end{pmatrix}\cdot\begin{pmatrix}W_1(s)\\W_2(s)\end{pmatrix}.\end{aligned}\tag{4.180}$$

One can notice hat the element in the first row, second column has a pole at $s = 1 > 0$, which causes the system to be not internally stable.

c) We start by computing the term in the first column, first row of Equation 4.178:

$$\begin{aligned}(\mathbb{I}-K(s)G(s))^{-1}&=\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}-\frac{s-1}{s+1}&-1\\0&-1\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{s-1}&0\\0&\frac{1}{s+1}\end{pmatrix}\right)^{-1}\\&=\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}-\frac{1}{s+1}&-\frac{1}{s+1}\\0&-\frac{1}{s+1}\end{pmatrix}\right)^{-1}\\&=\begin{pmatrix}\frac{s+2}{s+1}&\frac{1}{s+1}\\0&\frac{s+2}{s+1}\end{pmatrix}^{-1}\\&=\frac{(s+1)^2}{(s+2)^2}\begin{pmatrix}\frac{s+2}{s+1}&-\frac{1}{s+1}\\0&\frac{s+2}{s+1}\end{pmatrix}\\&=\begin{pmatrix}\frac{s+1}{s+2}&-\frac{s+1}{(s+2)^2}\\0&\frac{s+1}{s+2}\end{pmatrix}.\end{aligned}\tag{4.181}$$

The term in the first row,second column of Equation 4.178 is

$$\begin{aligned}(\mathbb{I}-K(s)G(s))^{-1}K(s)&=\begin{pmatrix}\frac{s+1}{s+2}&-\frac{s+1}{(s+2)^2}\\0&\frac{s+1}{s+2}\end{pmatrix}\cdot\begin{pmatrix}-\frac{s-1}{s+1}&-1\\0&-1\end{pmatrix}\\&=\begin{pmatrix}-\frac{s-1}{s+2}&-\frac{s+1}{s+2}+\frac{s+1}{(s+2)^2}\\0&-\frac{s+1}{s+2}\end{pmatrix}\\&=\begin{pmatrix}-\frac{s-1}{s+2}&-\frac{(s+1)^2}{(s+2)^2}\\0&-\frac{s+1}{s+2}\end{pmatrix}.\end{aligned}\tag{4.182}$$

The term in the second row, second column of Equation 4.178 is

$$\begin{aligned}(\mathbb{I}-G(s)K(s))^{-1}&=\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}\frac{1}{s-1}&0\\0&\frac{1}{s+1}\end{pmatrix}\cdot\begin{pmatrix}-\frac{s-1}{s+1}&-1\\0&-1\end{pmatrix}\right)^{-1}\\&=\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}-\frac{1}{s+1}&-\frac{1}{s-1}\\0&-\frac{1}{s+1}\end{pmatrix}\right)^{-1}\\&=\begin{pmatrix}\frac{s+2}{s+1}&\frac{1}{s-1}\\0&\frac{s+2}{s+1}\end{pmatrix}^{-1}\\&=\frac{(s+1)^2}{(s+2)^2}\begin{pmatrix}\frac{s+2}{s+1}&-\frac{1}{s-1}\\0&\frac{s+2}{s+1}\end{pmatrix}\\&=\begin{pmatrix}\frac{s+1}{s+2}&-\frac{(s+1)^2}{(s-1)(s+2)^2}\\0&\frac{s+1}{s+2}\end{pmatrix}.\end{aligned}\tag{4.183}$$

The term in the second row,first column of Equation 4.178 is

$$
\begin{aligned}
\left(\mathbb{I}-G(s)K(s)\right)^{-1}G(s) &= \begin{pmatrix} \frac{s+1}{s+2} & -\frac{(s+1)^2}{(s-1)(s+2)^2} \\ 0 & \frac{s+1}{s+2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s+1} \end{pmatrix} \\
&= \begin{pmatrix} \frac{s+1}{(s-1)(s+2)} & -\frac{s+1}{(s-1)(s+2)^2} \\ 0 & \frac{1}{s+2} \end{pmatrix}.
\end{aligned}
\tag{4.184}
$$

It is easy to see, that the last two terms we computed contain a pole at $s = 1 > 0$, which causes the system to be not internally stable.

4.4 MIMO Controllability and Observability

4.4.1 Controllability

Controllable: is it possible to control all the states of a system with an input $u(t)$?
Mathematically, a linear time invariant system is controllable if, for every state $x^*(t)$ and every finite time $T > 0$, there exists an input function $u(t)$, $0 < t \leq T$ such that the system can be driven from the initial state $x(0) = x_0$ to $x(T) = x^*(t)$.
A system of the form of the one represented in Equation 4.165 is said to be **completely controllable**, if the **controllability Matrix**

$$\mathcal{R} = \begin{pmatrix} B & A \cdot B & A^2 \cdot B & \dots & A^{n-1} \cdot B \end{pmatrix} \in \mathbb{R}^{n \times (n \cdot m)}. \tag{4.185}$$

has full rank n (easy by checking row rank).

4.4.2 Observability

Observable: is it possible to reconstruct the initial conditions of all the states of a system from the output $y(t)$?
A system is said to be **completely observable**, if the **observability Matrix**

$$O = \begin{pmatrix} C \\ C \cdot A \\ C \cdot A^2 \\ \vdots \\ C \cdot A^{n-1} \end{pmatrix} \in \mathbb{R}^{(n \cdot p) \times n}. \tag{4.186}$$

has full rank n (easy by checking column rank).

Example 38. The dynamics of a system are given as

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot x(t) + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot u(t) \\ y(t) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot x(t).\end{aligned} \tag{4.187}$$

Moreover the transfer function of the system is given as

$$P(s) = \begin{pmatrix} \frac{(s-2)}{s^2-6s+9} & 0 \\ \frac{-1}{s^2-6s+9} & \frac{1}{s-2} \end{pmatrix}. \tag{4.188}$$

(a) Is the system Lyapunov stable, asymptotically stable or unstable?

(b) Is the system completely controllable?

(c) Is the system completely observable?

(d) The poles of the system are $\pi_1 = 2$ and $\pi_{2,3} = 3$. The zero of the system is $\zeta_1 = 2$. Are there any zero-pole cancellations?

Solution.

(a) First of all, one identifies the matrices as:

$$\begin{aligned}\dot{x}(t) &= \underbrace{\begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{A} \cdot x(t) + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}_{B} \cdot u(t) \\ y(t) &= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}}_{C} \cdot x(t) + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{D} \cdot u(t).\end{aligned} \tag{4.189}$$

We have to compute the eingevalues of A. It holds

$$\begin{aligned}det(A - \lambda \cdot \mathbb{I}) &= \left| \begin{pmatrix} 4-\lambda & 1 & 0 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} \right| \\ &= (2-\lambda) \cdot \left| \begin{pmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{pmatrix} \right| \\ &= (2-\lambda) \cdot ((4-\lambda) \cdot (2-\lambda) + 1) \\ &= (2-\lambda) \cdot (\lambda^2 - 6\lambda + 9) \\ &= (2-\lambda) \cdot (\lambda - 3)^2.\end{aligned} \tag{4.190}$$

Since all the three eigenvalues are bigger than zero, the system is Lyapunov **unstable**.

(b) The controllability matrix can be found with the well-known multiplications:

$$\begin{aligned}A \cdot B &= \begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 0 \\ -1 & 0 \\ 0 & 2 \end{pmatrix}, \\ A^2 \cdot B &= \begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ -1 & 0 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 15 & 0 \\ -6 & 0 \\ 0 & 4 \end{pmatrix}.\end{aligned} \tag{4.191}$$

Hence, the controllability matrix reads

$$R = \begin{pmatrix} 1 & 0 & 4 & 0 & 15 & 0 \\ 0 & 0 & -1 & 0 & -6 & 0 \\ 0 & 1 & 0 & 2 & 0 & 4 \end{pmatrix}. \tag{4.192}$$

This has full rank 3: the system ist completely controllable.

(c) The observability matrix can be found with the well-known multiplications:

$$\begin{aligned} C \cdot A &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 2 \end{pmatrix}, \\ C \cdot A^2 &= \begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 15 & 6 & 0 \\ -6 & 3 & 4 \end{pmatrix}. \end{aligned} \tag{4.193}$$

Hence, the observability matrix reads

$$O = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 4 & 1 & 0 \\ -1 & 2 & 2 \\ 15 & 6 & 0 \\ -6 & 3 & 4 \end{pmatrix}. \tag{4.194}$$

This has full rank 3: the system is completely observable.

(d) Although $\zeta_1 = 2$ and $\pi_1 = 2$ have the same magnitude, they don't cancel out. Why? Since the system ist completely controllable and completely observable, we have already the minimal realization of the system. This means that no more cancellation is possible. The reason for that is that the directions of the two don't coincide. We will learn more about this in the next chapter.

Example 39. Given is the system

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 5 \\ 0 & -1 & 0 \end{pmatrix} \cdot x(t) + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \cdot u(t) \\ y(t) &= \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot x(t),\end{aligned} \tag{4.195}$$

with two inputs $u_1(t)$ and $u_2(t)$ and two outputs $y_1(t)$ and $y_2(t)$. The transfer function of the system reads

$$P(s) = \begin{pmatrix} \frac{2s-1}{(s^2+2s+5)\cdot(s+2)} & \frac{s+2}{s^2+2s+5} \\ \frac{5}{s^2+2s+5} & \frac{5}{s^2+2s+5} \end{pmatrix}. \tag{4.196}$$

(a) How many state variables are needed, in order to describe the input/output behaviour of the system?

(b) How many outputs are needed, in order to reconstruct the initial state $x(0)$?

(c) For every $x(0) \neq 0$ we have to ensure $\lim_{t\to\infty} x(t) = 0$. How many inputs are needed, in order to ensure this condition?

Solution.

(a) We want to find the minimal order of the system, that is, the number of poles.

Minors 1st. Order

$$\frac{2s-1}{(s^2+2s+5)\cdot(s+2)},\ \frac{s+2}{s^2+2s+5},\ \frac{5}{s^2+2s+5}. \tag{4.197}$$

Minors 2nd. Order

$$\begin{aligned}
&\frac{2s-1}{(s^2+2s+5)\cdot(s+2)}\cdot\frac{5}{s^2+2s+5}-\frac{s+2}{s^2+2s+5}\cdot\frac{5}{s^2+2s+5}\\
&=\frac{5}{s^2+2s+5}\cdot\left(\frac{2s-1}{(s^2+2s+5)\cdot(s+2)}-\frac{s+2}{s^2+2s+5}\right)\\
&=\frac{5}{s^2+2s+5}\cdot\frac{-s^2-2s-5}{(s^2+2s+5)\cdot(s+2)}\\
&=\frac{-5}{(s^2+2s+5)\cdot(s+2)}.
\end{aligned} \tag{4.198}$$

The pole-polynom reads

$$(s^2+2s+5)\cdot(s+2). \tag{4.199}$$

and the poles

$$\begin{aligned}\pi_1&=-2\\ \pi_{2,3}&=-1\pm 2j.\end{aligned} \tag{4.200}$$

The minimal order of the system is $n=3$, since we have 3 poles. The given description is already in the minimal realization.

Alternative

As alternative one can compute the controllability matrix and the observability matrix as follows: for the controllability matrix it holds:

$$\begin{aligned}
A\cdot B&=\begin{pmatrix}-2&0&0\\0&-2&5\\0&-1&0\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&0\\1&1\end{pmatrix}\\
&=\begin{pmatrix}-2&0\\5&5\\0&0\end{pmatrix}.\\
A^2\cdot B&=\begin{pmatrix}-2&0&0\\0&-2&5\\0&-1&0\end{pmatrix}\cdot\begin{pmatrix}-2&0\\5&5\\0&0\end{pmatrix}\\
&=\begin{pmatrix}4&0\\-10&-10\\-5&-5\end{pmatrix}.
\end{aligned} \tag{4.201}$$

The matrix reads

$$R = \begin{pmatrix} 1 & 0 & -2 & 0 & 4 & 0 \\ 0 & 0 & 5 & 5 & -10 & -10 \\ 1 & 1 & 0 & 0 & -5 & -5 \end{pmatrix}. \tag{4.202}$$

This matrix has full rank $r = 3 = n$: the system is completely controllable.
For the observability matrix it holds:

$$\begin{aligned} C \cdot A &= \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 5 \\ 0 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 & 0 \\ 0 & -2 & 5 \end{pmatrix}. \\ C \cdot A^2 &= \begin{pmatrix} 2 & -1 & 0 \\ 0 & -2 & 5 \end{pmatrix} \cdot \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 5 \\ 0 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 2 & -5 \\ 0 & -1 & -10 \end{pmatrix}. \end{aligned} \tag{4.203}$$

The matrix reads

$$O = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & -2 & 5 \\ -4 & 2 & -5 \\ 0 & -1 & -10 \end{pmatrix}. \tag{4.204}$$

This matrix has full rank $r = 3 = n$: the system is completely observable.
Since the system is completely observable and controllable, the given state-space description is already in its minimal realization, which means that the minimal order is $n = 3$.

(b) If we take into account just the first output $y_1(t)$, we get

$$C_1 = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} \tag{4.205}$$

with its observability matrix

$$\begin{pmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \\ -4 & 2 & -5 \end{pmatrix}. \tag{4.206}$$

This matrix has full rank and the partial system is completely observable: this means that $x(0)$ cab be reconstructed from the first output.

Remark. This holds e.g. not for the second output $y_2(t)$. One would get

$$C_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \tag{4.207}$$

with its observability matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 5 \\ 0 & -1 & -10 \end{pmatrix}. \tag{4.208}$$

This observability matrix has rank $r = 2$ and so the system is not completely observable: $x(0)$ cannot be reconstructed from the second output only.

(c) None. The eigenvalues of the system are

$$\begin{aligned} \lambda_1 &= -2 \\ \lambda_{2,3} &= -1 \pm 2j. \end{aligned} \tag{4.209}$$

These eigenvalues are all asymptotically stable: this means that no matter which input is given, the state will come back to its initial state.

Remark. If this wouldn't be the case, we would proceed like in task (b): if one takes into account just input $u_1(t)$, one gets

$$B_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{4.210}$$

and its controllability matrix

$$\mathcal{C} = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 5 & -10 \\ 1 & 0 & -5 \end{pmatrix}. \tag{4.211}$$

This controllability matrix has full rank and would satisfy the condition. You can check yourselves that this wouldn't be the case for $u_2(t)$.

4.5 MIMO Performance Analysis

A good performance means

- good disturbance rejection,
- good noise attenuation,
- good reference tracking,

at input and output. Recalling the general MIMO loop depicted in Figure 27, we defined

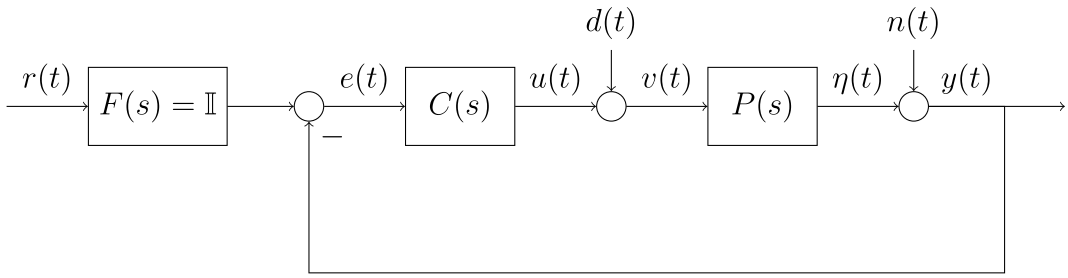

Figure 27: Standard feedback control system structure.

inner and outer loop transfer functions

$$L_{\mathrm{O}}(s) = P(s) \cdot C(s) \neq C(s) \cdot P(s) = L_{\mathrm{I}}(s), \tag{4.212}$$

and the input/output sensitivity functions, i.e.

- **Output sensitivity function** $(n \to y)$

$$S_{\mathrm{O}}(s) = (\mathbb{I} + L_{\mathrm{O}}(s))^{-1}. \tag{4.213}$$

- **Output complementary sensitivity function** $(r \to y)$

$$T_{\mathrm{O}}(s) = (\mathbb{I} + L_{\mathrm{O}}(s))^{-1} L_{\mathrm{O}}(s). \tag{4.214}$$

- **Input sensitivity function** $(d \to v)$

$$S_{\mathrm{I}}(s) = (\mathbb{I} + L_{\mathrm{I}}(s))^{-1}. \tag{4.215}$$

- **Input complementary sensitivity function** $(d \to -u)$

$$T_{\mathrm{I}}(s) = (\mathbb{I} + L_{\mathrm{I}}(s))^{-1} L_{\mathrm{I}}(s). \tag{4.216}$$

4.5.1 Output Conditions

Referring to Figure 27, one can write

$$\begin{aligned} Y(s) &= N(s) + \eta(s) \\ &= N(s) + P(s)V(s) \\ &= N(s) + P(s)\left(D(s) + U(s)\right) \\ &= N(s) + P(s)\left(D(s) + C(s)E(s)\right) \\ &= N(s) + P(s)\left(D(s) + C(s)(R(s) - Y(s))\right), \end{aligned} \tag{4.217}$$

from which follows

$$\begin{aligned}(\mathbb{I}+P(s)C(s))Y(s) &= N(s)+P(s)D(s)+P(s)C(s)R(s)\\ Y(s) &= (\mathbb{I}+P(s)C(s))^{-1}\left(N(s)+P(s)D(s)+P(s)C(s)R(s)\right).\end{aligned} \tag{4.218}$$

Using the defined sensitivity functions, one can write

$$Y(s) = S_{\mathrm{O}}(s)N(s)+S_{\mathrm{O}}(s)P(s)D(s)+S_{\mathrm{O}}(s)L_{\mathrm{O}}(s)R(s). \tag{4.219}$$

Disturbance Rejection

Equation 4.274 shows that the effects of the disturbance $D(s)$ on the output can be rejected by making the output sensitivity function $S_{\mathrm{O}}(s)$ small. Since typically disturbances occur at *low frequencies*, one needs to do that only for this frequency range. How can we relate this to what we have learned about singular values? It must hold

$$\begin{aligned}\bar{\sigma}\left(S_{\mathrm{O}}(j\omega)P(j\omega)\right) &= \bar{\sigma}\left((\mathbb{I}+P(j\omega)C(j\omega))^{-1}P(j\omega)\right)\\ \text{push-through rule} \qquad &= \bar{\sigma}\left(P(j\omega)(\mathbb{I}+C(j\omega)P(j\omega))^{-1}\right)\\ &= \bar{\sigma}(P(j\omega)S_{\mathrm{I}}(j\omega))\\ &\ll 1,\end{aligned} \tag{4.220}$$

where we used the push-through rule

$$G_1(\mathbb{I}-G_2G_1)^{-1} = (\mathbb{I}-G_1G_2)^{-1}G_1, \tag{4.221}$$

and $\bar{\sigma}(H(j\omega)$ refers to the maximum singular value of $H(j\omega)$.

Noise Attenuation

Similarly, Equation 4.274 shows that the effects of the noise $N(s)$ on the output can be attenuated by making the output sensitivity function $S_{\mathrm{O}}(s)$ small. Since typically noise occurs at *high frequencies*, one needs to do that only for this frequency range. It holds

$$\begin{aligned}\bar{\sigma}\left(S_{\mathrm{O}}(j\omega)\right) &= \bar{\sigma}\left((\mathbb{I}+P(j\omega)C(j\omega))^{-1}\right)\\ &\ll 1.\end{aligned} \tag{4.222}$$

4.5.2 Input Conditions

Referring to Figure 27, one can write

$$\begin{aligned}V(s) &= D(s)+U(s)\\ &= D(s)+C(s)E(s)\\ &= D(s)+C(s)\left(R(s)-Y(s)\right)\\ &= D(s)+C(s)\left(R(s)-N(s)-\eta(s)\right)\\ &= D(s)+C(s)\left(R(s)-N(s)-P(s)V(s)\right)\\ &= D(s)+C(s)R(s)-C(s)N(s)-C(s)P(s)V(s),\end{aligned} \tag{4.223}$$

from which follows

$$\begin{aligned}\left(\mathbb{I}+C(s)P(s)\right)V(s) &= D(s)+C(s)R(s)-C(s)N(s)\\ V(s) &= (\mathbb{I}+C(s)P(s))^{-1}\left(D(s)+C(s)R(s)-C(s)N(s)\right)\end{aligned} \tag{4.224}$$

Using the defined sensitivity functions, one can write

$$V(s) = S_{\mathrm{I}}(s)D(s)+S_{\mathrm{I}}(s)C(s)R(s)-S_{\mathrm{I}}(s)C(s)N(s). \tag{4.225}$$

Disturbance Rejection

Equation 4.225 shows that the effects of the disturbance $D(s)$ on the input can be rejected by making the input sensitivity function $S_{\mathrm{I}}(s)$ small. Since typically disturbances occurr at *low frequencies*, one needs to do that only for this frequency range. It must hold

$$\begin{aligned}\bar{\sigma}\left(S_{\mathrm{I}}(j\omega)\right) &= \bar{\sigma}\left((\mathbb{I}+C(j\omega)P(j\omega))^{-1}\right)\\ &\ll 1.\end{aligned} \tag{4.226}$$

Noise Attenuation

Similarly, Equation 4.225 shows that the effects of the noise $N(s)$ on the input can be attenuated by making the input sensitivity function $S_{\mathrm{I}}(s)$ small. Since typically noise occurs at *high frequencies*, one needs to do that only for this frequency range. It holds

$$\begin{aligned}\bar{\sigma}\left(S_{\mathrm{I}}(j\omega)C(j\omega)\right) &= \bar{\sigma}\left((\mathbb{I}+C(j\omega)P(j\omega))^{-1}C(j\omega)\right)\\ &\ll 1.\end{aligned} \tag{4.227}$$

4.5.3 Reference Tracking

Referring to Figure 27, one can write

$$\begin{aligned}E(s) &= R(s) - Y(s)\\ &= R(s) - N(s) - \eta(s)\\ &= R(s) - N(s) - P(s)V(s)\\ &= R(s) - N(s) - P(s)\left(D(s)+U(s)\right)\\ &= R(s) - N(s) - P(s)D(s) - P(s)C(s)E(s),\end{aligned} \tag{4.228}$$

from which follows

$$\begin{aligned}\left(\mathbb{I}+P(s)C(s)\right)E(s) &= R(s) - N(s) - P(s)D(s)\\ E(s) &= \left(\mathbb{I}+P(s)C(s)\right)^{-1}\left(R(s)-N(s)-P(s)D(s)\right).\end{aligned} \tag{4.229}$$

Using the defined sensitivity functions, one can write

$$E(s) = S_{\mathrm{O}}(s)(R(s)-N(s)) - S_{\mathrm{O}}(s)P(s)D(s) \tag{4.230}$$

Disturbance Rejection

Equation 4.230 shows that the effects of the disturbance $D(s)$ on the error can be rejected by making the output sensitivity function $S_{\mathrm{O}}(s)$ small. Since typically disturbances occurr at *low frequencies*, one needs to do that only for this frequency range. It must hold

$$\begin{aligned}\bar{\sigma}\left(S_{\mathrm{O}}(j\omega)P(j\omega)\right) &= \bar{\sigma}\left((\mathbb{I}+P(j\omega)C(j\omega))^{-1}P(j\omega)\right)\\ \text{push-through rule}\qquad &= \bar{\sigma}\left(P(j\omega)(\mathbb{I}+C(j\omega)P(j\omega))^{-1}\right)\\ &= \bar{\sigma}\left(P(j\omega)S_{\mathrm{I}}(j\omega)\right)\\ &\ll 1.\end{aligned} \tag{4.231}$$

Noise Attenuation

Similarly, Equation 4.230 shows that the effects of the noise $N(s)$ on the error can be attenuated by making the output sensitivity function $S_\mathrm{O}(s)$ small. Since typically noise occurs at *high frequencies*, one needs to do that for this frequency range **and** for reference relevant frequencies (we have $R(s)$ in the term). It holds

$$\begin{aligned}\bar{\sigma}\left(S_\mathrm{O}(j\omega)\right) &= \bar{\sigma}\left((\mathbb{I}+P(j\omega)C(j\omega))^{-1}\right)\\ &\ll 1.\end{aligned} \tag{4.232}$$

Remark. One can note that the reference tracking case resumes the other two cases.

4.5.4 Useful Properties

Given an invertible matrix A and a matrix B, it holds

(I) **Inverse:**

$$\bar{\sigma}\left(A^{-1}\right) = \frac{1}{\underline{\sigma}\left(A\right)}, \tag{4.233}$$

where $\underline{\sigma}(A)$ represents the smallest singular value of A.

(II) **Sum:**

$$\begin{aligned}\sigma_i(A)-\bar{\sigma}(B) &\leq \sigma_i(A+B)\\ &\leq \sigma_i(A)+\bar{\sigma}(B).\end{aligned} \tag{4.234}$$

In particular, it holds

$$\begin{aligned}\underline{\sigma}(A)-1 &\leq \underline{\sigma}(\mathbb{I}+A)\\ &\leq \underline{\sigma}(A)+1.\end{aligned} \tag{4.235}$$

(III) **Product:**

$$\begin{aligned}\bar{\sigma}(AB) &\leq \bar{\sigma}(A)\bar{\sigma}(B)\\ \underline{\sigma}(AB) &\leq \underline{\sigma}(A)\underline{\sigma}(B).\end{aligned} \tag{4.236}$$

4.5.5 Towards Clearer Bounds

Assuming $P(s)$ and $C(s)$ are invertible, one can use the defined properties to write

$$\begin{aligned}\underline{\sigma}(P(j\omega)C(j\omega))-1 \leq \underline{\sigma}(\mathbb{I}+P(j\omega)C(j\omega)) \leq \underline{\sigma}(P(j\omega)C(j\omega))+1\\ \underline{\sigma}(C(j\omega)P(j\omega))-1 \leq \underline{\sigma}(\mathbb{I}+C(j\omega)P(j\omega)) \leq \underline{\sigma}(C(j\omega)P(j\omega))+1\end{aligned} \tag{4.237}$$

For disturbance rejection, using Equations 4.220, 4.226, 4.231 one can write

$$\begin{aligned}\bar{\sigma}(P(j\omega)S_\mathrm{I}(j\omega)) &\ll 1\\ \bar{\sigma}(S_\mathrm{I}(j\omega)) &\ll 1.\end{aligned} \tag{4.238}$$

For noise attenuation, using Equations 4.222, 4.227, 4.232 one can write

$$\begin{aligned}\bar{\sigma}(S_\mathrm{O}(j\omega)) &\ll 1\\ \bar{\sigma}(S_\mathrm{I}(j\omega)C(j\omega)) &\ll 1.\end{aligned} \tag{4.239}$$

With the inverse property of singular values, we know that

$$\begin{aligned}\bar{\sigma}(S_{\mathrm{I}}(j\omega)) &= \bar{\sigma}((\mathbb{I}+C(j\omega)P(j\omega))^{-1}) \\ &= \frac{1}{\underline{\sigma}(\mathbb{I}+C(j\omega)P(j\omega))}, \\ \bar{\sigma}(S_{\mathrm{O}}(j\omega)) &= \bar{\sigma}((\mathbb{I}+P(j\omega)C(j\omega))^{-1}) \\ &= \frac{1}{\underline{\sigma}(\mathbb{I}+P(j\omega)C(j\omega))}.\end{aligned} \tag{4.240}$$

With Equation 4.237 and $\underline{\sigma}(C(j\omega)P(j\omega)) > 1$, $\underline{\sigma}(P(j\omega)C(j\omega)) > 1$, one can write

$$\begin{aligned}\frac{1}{\underline{\sigma}(C(j\omega)P(j\omega))+1} \leq \bar{\sigma}(S_{\mathrm{I}}(j\omega)) \leq \frac{1}{\underline{\sigma}(C(j\omega)P(j\omega))-1} \\ \frac{1}{\underline{\sigma}(P(j\omega)C(j\omega))+1} \leq \bar{\sigma}(S_{\mathrm{O}}(j\omega)) \leq \frac{1}{\underline{\sigma}(P(j\omega)C(j\omega))-1}.\end{aligned} \tag{4.241}$$

This implies

$$\begin{aligned}\bar{\sigma}(S_{\mathrm{I}}(j\omega)) \ll 1 \Leftrightarrow \underline{\sigma}(C(j\omega)P(j\omega)) \gg 1 \\ \bar{\sigma}(S_{\mathrm{O}}(j\omega)) \ll 1 \Leftrightarrow \underline{\sigma}(P(j\omega)C(j\omega)) \gg 1\end{aligned} \tag{4.242}$$

Disturbance Rejection

Suppose that $P(s)$ and $C(s)$ are invertible.

- **Output:** It holds

$$\begin{aligned}\underline{\sigma}(P(j\omega)C(j\omega)) \gg 1 \Leftrightarrow \bar{\sigma}(S_{\mathrm{O}}(j\omega)P(j\omega)) &= \bar{\sigma}((\mathbb{I}+P(j\omega)C(j\omega))^{-1}P(j\omega)) \\ &\approx \bar{\sigma}((P(j\omega)C(j\omega))^{-1}P(j\omega)) \\ &= \bar{\sigma}(C(j\omega)^{-1}) \\ &= \frac{1}{\underline{\sigma}(C(j\omega))}.\end{aligned} \tag{4.243}$$

  This implies:

$$\bar{\sigma}(S_{\mathrm{O}}(j\omega)P(j\omega)) \ll 1 \Leftrightarrow \underline{\sigma}(C(j\omega)) \gg 1, \qquad \forall \omega \in (0, \omega_{\mathrm{low}}). \tag{4.244}$$

- **Input:** Considering Equation 4.241, one can write

$$\bar{\sigma}(S_{\mathrm{I}}(j\omega)) \geq \frac{1}{\underline{\sigma}(C(j\omega)P(j\omega))+1}. \tag{4.245}$$

  This implies

$$\bar{\sigma}(S_{\mathrm{I}}(j\omega)) \ll 1 \Leftrightarrow \underline{\sigma}(C(j\omega)P(j\omega)) \gg 1, \qquad \forall \omega \in (0, \omega_{\mathrm{low}}). \tag{4.246}$$

Noise Attenuation

Suppose that $P(s)$ and $C(s)$ are invertible.

- **Output:** Using Equation 4.241, one can write

$$\frac{1}{\underline{\sigma}(P(j\omega)C(j\omega))+1} \leq \bar{\sigma}(S_{\mathrm{O}}(j\omega)) \tag{4.247}$$

  This implies

$$\bar{\sigma}(S_{\mathrm{O}}(j\omega)) \ll 1 \Leftrightarrow \underline{\sigma}(P(j\omega)C(j\omega)) \gg 1, \qquad \forall \omega \in (\omega_{\text{high}}, \infty). \tag{4.248}$$

- **Input:** It holds

$$\begin{aligned}\underline{\sigma}(C(j\omega)P(j\omega)) \gg 1 \Leftrightarrow \bar{\sigma}(C(j\omega)S_{\mathrm{O}}(j\omega)) &= \bar{\sigma}(C(j\omega)(\mathbb{I}+P(j\omega)C(j\omega))^{-1}) \\ &\approx \bar{\sigma}(C(j\omega)(P(j\omega)C(j\omega))^{-1}) \\ &= \bar{\sigma}(P(j\omega)^{-1}) \\ &= \frac{1}{\underline{\sigma}(P(j\omega))}.\end{aligned} \tag{4.249}$$

  This implies

$$\bar{\sigma}(C(j\omega)S_{\mathrm{O}}(j\omega)) \ll 1 \Leftrightarrow \underline{\sigma}(P(j\omega)) \gg 1, \qquad \forall \omega \in (\omega_{\text{high}}, \infty). \tag{4.250}$$

4.5.6 Is this the whole Story? Tradeoffs

Robust Stability

One defines **robust stability** to be the stability in the presence of *model uncertainty*. Let Δ be a stable uncertainty matrix, such that

$$P_{\text{real}}(s) = (\mathbb{I}+\Delta)\, P_{\text{nominal}}(s) \tag{4.251}$$

The perturbed closed loop transfer function is then characterized by

$$\det(\mathbb{I}+P(s)C(s)) \rightarrow \det(\mathbb{I}+(\mathbb{I}+\Delta)P(s)C(s)) = \det(\mathbb{I}+P(s)C(s))\det(\mathbb{I}+\Delta T_{\mathrm{O}}), \tag{4.252}$$

where we used

$$\det(X+AB) = \det(X)\det(\mathbb{I}+BX^{-1}A), \forall X : \exists X^{-1} \tag{4.253}$$

Since

$$\det(\mathbb{I}+\Delta T_{\mathrm{O}}) \approx 1, \tag{4.254}$$

it holds

$$\|\Delta T_{\mathrm{O}}\| \ll 1. \tag{4.255}$$

This implies

$$\bar{\sigma}(T_{\mathrm{O}}(j\omega)) \ll 1 \Rightarrow \bar{\sigma}(L_{\mathrm{O}}(j\omega)) \ll 1, \qquad \forall \omega \in (\omega_{\text{high}}, \infty). \tag{4.256}$$

Remark. Note that typically Δ becomes important at high frequencies.

Actuator Saturation

Using Figure 27, one can derive

$$U(s) = C(s)S_{\mathrm{O}}(s)R(s) - T_{\mathrm{I}}(s)D(s) - C(s)S_{\mathrm{O}}(s)N(s). \tag{4.257}$$

With the defined conditions, it holds

$$U(s) \approx C(s)\left(R(s) - N(s)\right). \tag{4.258}$$

In order to avoid the actuator saturation, the controller gain cannot be chosen too big, i.e.

$$\bar{\sigma}(C(j\omega)) \leq M, \qquad \forall \omega \in (\omega_{\text{high}}, \infty). \tag{4.259}$$

4.5.7 Summary

The specifications we derived are resumed in Figure 28. Mathematically, we have found:

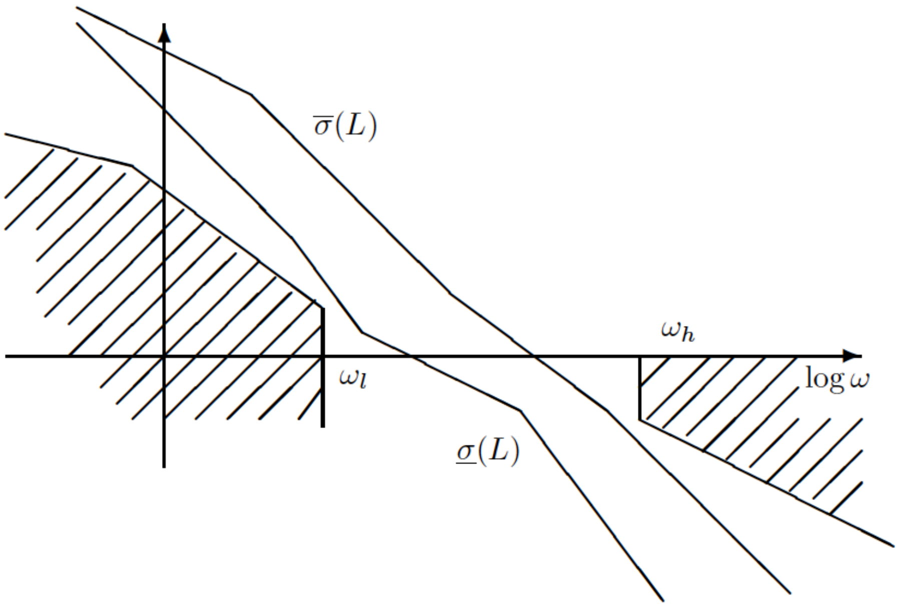

Figure 28: Desired Loop Gain

Disturbance Rejection

At frequency $\omega \in (0, \omega_{\text{low}})$ holds

$$\begin{aligned} \underline{\sigma}(C(j\omega)) &\gg 1, \\ \underline{\sigma}(C(j\omega)P(j\omega) &\gg 1, \\ \underline{\sigma}(P(j\omega)C(j\omega)) &\gg 1. \end{aligned} \tag{4.260}$$

Noise Attenuation

At frequency $\omega \in (\omega_{\text{high}}, \infty)$ holds

$$\begin{aligned} \bar{\sigma}(C(j\omega)) &\ll M, \\ \bar{\sigma}(C(j\omega)P(j\omega) &\ll 1, \\ \bar{\sigma}(P(j\omega)C(j\omega)) &\ll 1. \end{aligned} \tag{4.261}$$

4.6 MIMO Robust Stability

4.6.1 MIMO Robustness

$$\textit{All models are wrong, but some are useful.} \tag{4.262}$$

A model maps inputs into outputs and we consider good a model which predicts the outputs accurately. The difference between a model prediction and reality (which is never 0) is referred to as **model uncertainty**.

Modeling Uncertainty

Let $P(s)$, $C(s)$ be the nominal MIMO plant and an internally stabilizing controller, respectively. Let's define $\Delta(s)$, $W_1(s)$, $W_2(s)$ to be stable, rational and proper transfer matrices. We call $W_1(s)$ and $W_2(s)$ *weighting functions*. $\Delta(s)$ is the *modeling error*. We represent uncertainty as

$$W_1(s)\Delta(s)W_2(s). \tag{4.263}$$

Let's define $\Pi(s)$ to be the set of perturbed plants such that $P(s) \in \Pi(s)$.

Unstructured Uncertainty

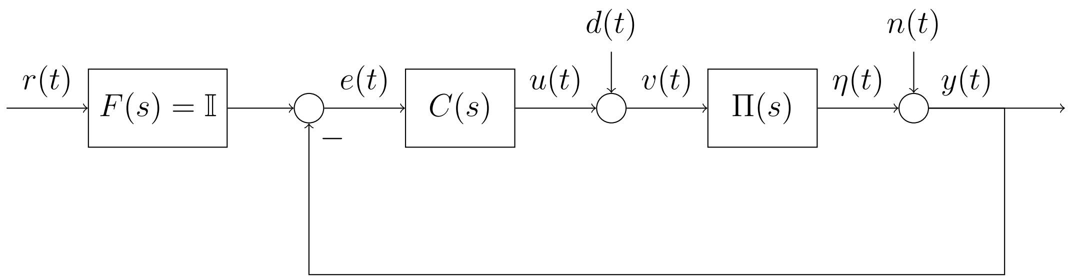

Figure 29: Standard feedback control system structure.

Using Figure 29 and the Equations for internal stability, one can write

$$\begin{pmatrix} E_1(s) \\ E_2(s) \end{pmatrix} = \begin{pmatrix} (\mathbb{I} - C(s)\Pi(s))^{-1} & (\mathbb{I} - C(s)\Pi(s))^{-1} C(s) \\ (\mathbb{I} - \Pi(s)C(s))^{-1}\Pi(s) & (\mathbb{I} - \Pi(s)C(s))^{-1} \end{pmatrix} \cdot \begin{pmatrix} W_1(s) \\ W_2(s) \end{pmatrix}. \tag{4.264}$$

For unstructured uncertainty, nothing more can be said without writing the relation between the uncertainty and the plant.

Additive Uncertainty

Theorem 4. (robust stability under additive uncertainty). Let

$$\Pi(s) = \{P + W_1(s)\Delta(s)W_2(s) : \Delta \text{ rational, proper and stable}\} \tag{4.265}$$

and let $C(s)$ be a stabilizing controller for the nominal plant $P(s)$. Then, the closed loop system is well-posed (i.e., realizable) and internally stable for all $\|\Delta\|_\infty < 1$ if and only if $\|W_2(s)C(s)S_O(s)W_1(s)\|_\infty \leq 1$.

Multiplicative Uncertainty

Theorem 5. (robust stability under multiplicative uncertainty). Let

$$\Pi(s) = (\mathbb{I} + W_1(s)\Delta(s)W_2(s))\,P(s) : \Delta \text{ rational, proper and stable}\} \tag{4.266}$$

and let $C(s)$ be a stabilizing controller for the nominal plant $P(s)$. Then, the closed loop system is well-posed (i.e., realizable) and internally stable for all $\|\Delta\|_\infty < 1$ if and only if $\|W_2(s)T_O W_1(s)\|_\infty \leq 1$.

Definition 14. Robust stability: Given a controller C, one determines whether the system remains stable for all possible plants P in the uncertaint set.

4.6.2 SISO Case

In order to understand what we will address in this section, let's have a look at the SISO case. Let's assume *multiplicative uncertainty*, i.e.

$$P(s) = P_0(s)\left(1 + W(s)\Delta(s)\right), \qquad \text{with } |\Delta(j\omega)| \leq 1 \forall\omega, \tag{4.267}$$

where $P(s)$ represents the perturbed plant and $P_0(s)$ the nominal plant. Assuming a controller which *stabilizes* the nominal plant, one has

$$|1 + L_0(j\omega)| > 0. \tag{4.268}$$

If one looks at the perturbed plant, instead, one has

$$\begin{aligned} 1 + L(j\omega) &= 1 + P(j\omega)C(j\omega) \\ &= 1 + \underbrace{P_0(j\omega)C(j\omega)}_{L_0(j\omega)} + W(j\omega)\Delta(j\omega)P_0(j\omega)C(j\omega). \end{aligned} \tag{4.269}$$

In order to ensure stability even in the worst case scenario, it should hold

$$\begin{aligned} & |1 + L(j\omega)| > |1 + L_0(j\omega)| - |L_0(j\omega)W(j\omega)\Delta(j\omega)| \\ |\Delta| \leq 1 \quad & \qquad\qquad\quad > |1 + L_0(j\omega)| - |L_0(j\omega)W(j\omega)| \\ & \qquad\qquad\quad > 0. \end{aligned} \tag{4.270}$$

From this it follows

$$\left|\frac{W(j\omega)L_0(j\omega)}{1 + L_0(j\omega)}\right| < 1. \tag{4.271}$$

What we want to do, is to be able to write such relations for MIMO systems.

4.6.3 Linear Fractional Transform (LFT)

In order to analyze robust stability, it is worth first to separate the nominal plant from the uncertainty which affects it. Assuming a nominal plant $P_0(s)$ and a feedback controller $C(s)$ that stabilizes $P_0(s)$, one can write the problem as in Figure 30. Note that the generalization for an uncertainty block $W_1(s)\Delta(s)W_2(s)$ instead of $\Delta(s)$ is trivial and can be used as well. Note that $w(t) = (r(t), d(t), n(t))$ represents the exogenous inputs and $z(t) = (y(t), u(t), e(t))$ represents the regulated variables. One can write

$$\begin{pmatrix} U_\Delta(s) \\ Z(s) \end{pmatrix} = \begin{pmatrix} M(s) & N(s) \\ J(s) & L(s) \end{pmatrix} \cdot \begin{pmatrix} Y_\Delta(s) \\ W(s) \end{pmatrix}. \tag{4.272}$$

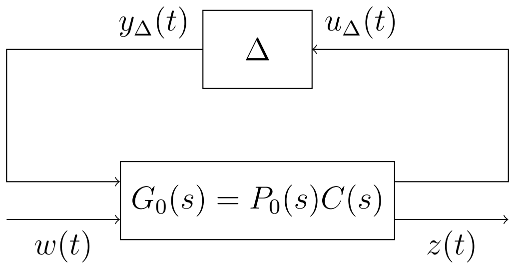

Figure 30: Standard feedback control system structure.

Note that $\begin{pmatrix} U_\Delta(s) \\ Z(s) \end{pmatrix}$ represents the plant outputs and $\begin{pmatrix} Y_\Delta(s) \\ W(s) \end{pmatrix}$ the plant inputs, where $Y_\Delta(s) = \Delta(s) \cdot U_\Delta(s)$. In order for the system to be **internally stable**, each element of the matrix

$$\begin{pmatrix} M(s) & N(s) \\ J(s) & L(s) \end{pmatrix} \tag{4.273}$$

must be stable itself. By looking at the transfer function which relates $z(t)$ to $w(t)$ one has

$$\begin{aligned} Z(s) &= J(s)Y_\Delta(s) + L(s)W(s) \\ &= J(s)\Delta(s)U_\Delta(s) + L(s)W(s) \end{aligned} \tag{4.274}$$

Furthermore, the first equation of the system reads

$$\begin{aligned} U_\Delta(s) &= M(s)Y_\Delta(s) + N(s)W(s) \\ W(s) &= N^{-1}\left[\left(\mathbb{I} - M(s)\Delta(s)\right)U_\Delta(s)\right]. \end{aligned} \tag{4.275}$$

By pluggin Equation 4.275 in Equation 4.274 one gets

$$\begin{aligned} Z(s) &= J(s)\Delta(s)U_\Delta(s) + L(s)W(s) \\ &= \left[J(s)\Delta(s)U_\Delta(s)W^{-1}(s) + L(s)\right]W(s) \\ &= \left[J(s)\Delta(s)U_\Delta(s)U_\Delta(s)^{-1}\left(\mathbb{I} - M(s)\Delta(s)\right)^{-1}N(s) + L(s)\right]W(s) \\ &= \left[J(s)\Delta(s)\left(\mathbb{I} - M(s)\Delta(s)\right)^{-1}N(s) + L(s)\right]W(s). \end{aligned} \tag{4.276}$$

This means that the transfer function from $w(t)$ to $z(t)$ is

$$G_{zw}(s) = J(s)\Delta(s)\left(\mathbb{I} - M(s)\Delta(s)\right)^{-1}N(s) + L(s). \tag{4.277}$$

The **internal stability** of the perturbed closed-loop system requires this transfer function to be stable for all possible perturbations $\Delta(s)$. Since from above $M(s), N(s), J(s), L(s)$ are stable, $G_{zw}(s)$ is stable for all stable $\left(\mathbb{I} - M(s)\Delta(s)\right)^{-1}$.

4.6.4 Unstructured Small Gain Theorem

Theorem 6. Let the set of allowable model uncertainties be

$$\tilde{\Delta} = \{\Delta : \|\Delta\|_\infty \leq 1\} \tag{4.278}$$

and let M be stable. Then, $\left(\mathbb{I} - M(s)\Delta(s)\right)^{-1}$ and $\Delta\left(\mathbb{I} - M(s)\Delta(s)\right)^{-1}$ are stable, for all $\Delta \in \tilde{\Delta}$, **if and only if** $\|M\|_\infty < 1$.

Proof. We first prove **sufficiency** and then **necessity**.

(I) **Sufficiency:** we show that $(\mathbb{I} - M(s)\Delta(s))$ has no zeros ζ in the right-half plane. In particular, we show

$$\|M\|_\infty < 1 \Rightarrow (\mathbb{I} - M(s)\Delta(s))^{-1} \text{ stable.} \tag{4.279}$$

It holds

$$\begin{aligned}
& \|(\mathbb{I} - M(\zeta)\Delta(\zeta))x\|_2 > 0, \qquad x \neq 0, \forall \Delta \in \tilde{\Delta} \\
\textit{triangle inequality} \quad & \|(\mathbb{I} - M(\zeta)\Delta(\zeta))x\|_2 \geq \|x\|_2 - \|M(\zeta)\Delta(\zeta)x\|_2 \\
\textit{induced matrix norm} \quad & \geq \|x\|_2 - \bar{\sigma}\left(M(\zeta)\Delta(\zeta)\right)\|x\|_2 \\
& \geq \|x\|_2 - \underbrace{\|M(\zeta)\|_\infty \|\Delta(\zeta)\|_\infty}_{\leq 1} \|x\|_2 \\
& > 0,
\end{aligned} \tag{4.280}$$

where in the last step we used the fact that $\bar{\sigma}(H(s)) \leq \|H(s)\|_\infty$ for stable and causal $H(s)$.

(II) **Necessity**: we show by construction, that if $\bar{\sigma}(M(j\omega_0)) > 1$, there exists a $\Delta \in \tilde{\Delta}$ such that $(\mathbb{I} - M\Delta)^{-1}$ is unstable, i.e.

$$\det(\mathbb{I} - M(j\omega_0)\Delta(j\omega_0) = 0. \tag{4.281}$$

In particular, we show

$$\neg\|M\|_\infty < 1 \Rightarrow \neg\left(\mathbb{I} - M(s)\Delta(s)\right)^{-1} \text{ stable.} \tag{4.282}$$

Let's write the singular value decomposition of M as

$$M(j\omega_0) = U \begin{pmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_p \end{pmatrix} V^*, \; \sigma_1 > 1. \tag{4.283}$$

We choose a Δ such that

$$\Delta(j\omega_0) = V \begin{pmatrix} \sigma_1^{-1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} U^*, \; \|\Delta\|_\infty < 1. \tag{4.284}$$

It holds then

$$
\begin{aligned}
(\mathbb{I}-M(j\omega_0)\Delta(j\omega_0)) &= \mathbb{I}-U\begin{pmatrix}\sigma_1 & 0 & \dots & 0\\ 0 & \sigma_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \sigma_p\end{pmatrix}V^*V\begin{pmatrix}\sigma_1^{-1} & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0\end{pmatrix}U^*\\
&= U\left(\mathbb{I}-\begin{pmatrix}1 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0\end{pmatrix}\right)U^*\\
&= U\underbrace{\begin{pmatrix}0 & 1 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 1\end{pmatrix}}_{I}U^*,
\end{aligned}
\tag{4.285}
$$

where I is clearly **not invertible**.

□

4.6.5 From the Block-Diagram to the LFT

One follows usually this procedure

1. Define the input and the output of each perturbation block Δ_i as $(u_{\Delta,i}, y_{\Delta,i})$ and let

$$u_\Delta = \begin{pmatrix}u_{\Delta,1} & \dots & u_{\Delta,q}\end{pmatrix}^\mathsf{T}, \qquad y_\Delta = \begin{pmatrix}y_{\Delta,1} & \dots & y_{\Delta,q}\end{pmatrix}^\mathsf{T}, \tag{4.286}$$

   where q is the number of uncertainties in the loop.

2. Compute each component of the transfer matrix M as the map between the (i,j)-th inputs and outputs to each uncertainty block, assuming $\Delta_i = \mathbb{I}\ \forall i = 1,\dots,q$, i.e.

$$M(s) = \begin{pmatrix}M_{1,1}(s) & M_{1,2}(s) & \dots & M_{1,q}(s)\\ M_{2,1}(s) & \ddots & \ddots & \vdots\\ \vdots & \vdots & \ddots & \vdots\\ M_{q,1}(s) & \dots & \dots & M_{q,q}(s)\end{pmatrix}, \qquad M_{i,j} = \frac{U_{\Delta,i}}{Y_{\Delta,j}}. \tag{4.287}$$

3. The uncertainty block will be block diagonal in the MIMO case and diagonal in the SISO one:

$$\Delta = \mathrm{diag}\left(\Delta_1,\dots,\Delta_q\right), \quad \|\Delta_i\|_\infty < 1. \tag{4.288}$$

Example 40. (**Additive Uncertainty**) You are given the system depicted in Figure 34 and the input output behaviour depicted in Figure 32, where

$$\eta(s) = P_0(s)U(s) + W_1(s)\Delta(s)W_2(s)U(s). \tag{4.289}$$

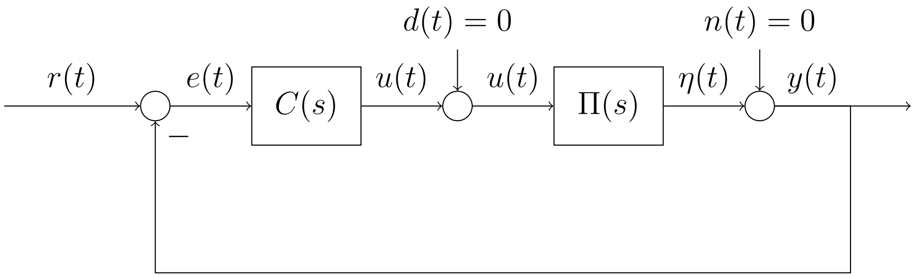

Figure 31: Additive Uncertainty Control System Loop

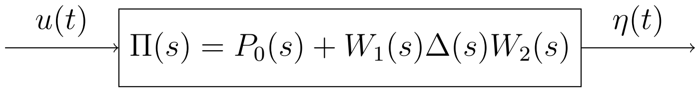

Figure 32: Input/Output Behaviour

In order to find the transfer function M, one rewrites the problem as depicted in Figure 33. It holds

$$U_\Delta(s) = W_2(s)U(s) \tag{4.290}$$

and

$$\begin{aligned}
U(s) &= C(s)(R(s) - Y(s)) \\
&= -C(s)\left(P_0(s)U(s) + W_1(s)Y_\Delta(s)\right) \\
&= -C(s)W_1(s)Y_\Delta(s) - C(s)P_0(s)U(s) \\
(\mathbb{I} + C(s)P_0(s))U(s) &= -C(s)W_1(s)Y_\Delta(s) \\
U(s) &= -(\mathbb{I} + C(s)P_0(s))^{-1}C(s)W_1(s)Y_\Delta(s),
\end{aligned} \tag{4.291}$$

from which it follows

$$\begin{aligned}
U_\Delta(s) &= W_2(s)U(s) \\
&= -\underbrace{W_2(s)(\mathbb{I} + C(s)P_0(s))^{-1}C(s)W_1(s)}_{M(s)}Y_\Delta(s).
\end{aligned} \tag{4.292}$$

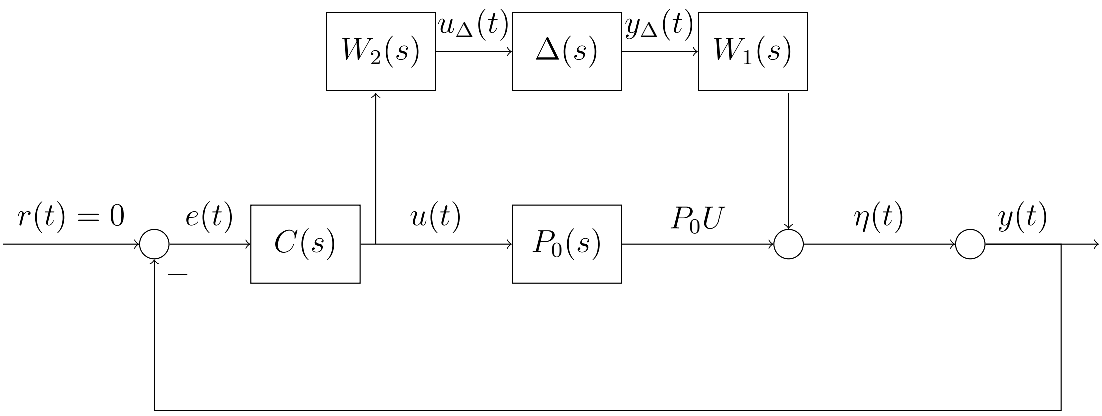

Figure 33: Additive Uncertainty Control System Loop

Example 41. (**Multiplicative Uncertainty**) You are given the system depicted in Figure 34 and the input output behaviour depicted in Figure 35, where

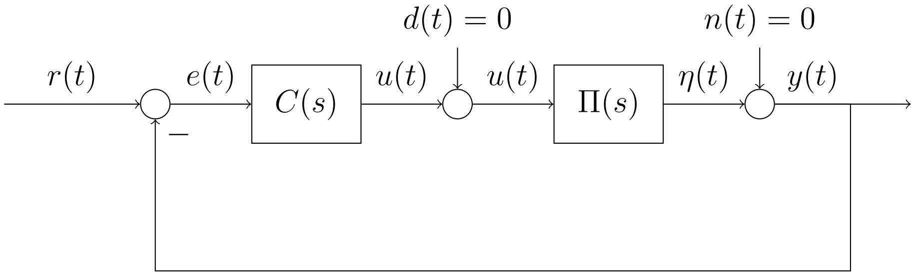

Figure 34: Multiplicative Uncertainty Control System Loop

$$\eta(s) = P_0(s)U(s) + P_0(s)W_1(s)\Delta(s)W_2(s)U(s). \tag{4.293}$$

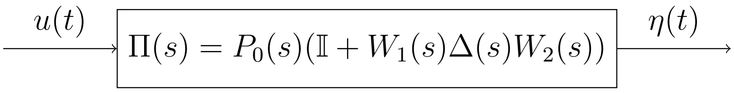

Figure 35: Input/Output Behaviour

In order to find the transfer function M, one rewrites the problem as depicted in Figure 36. It holds

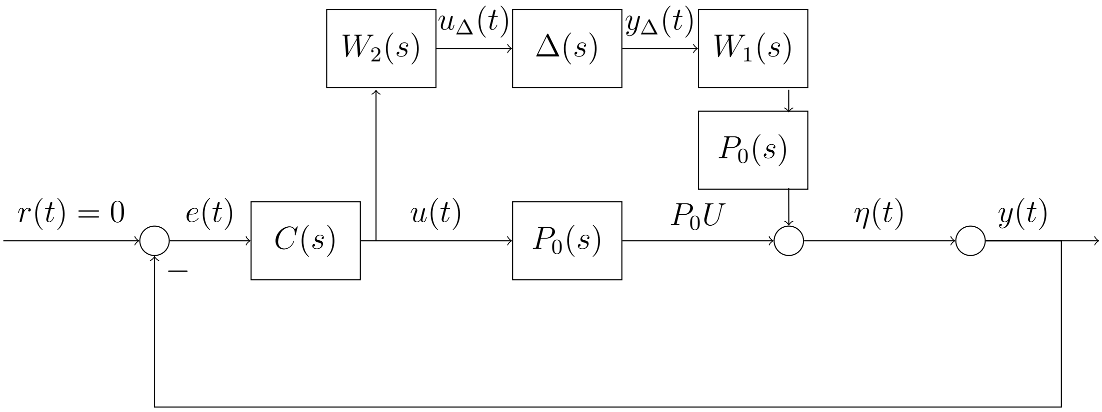

Figure 36: Multiplicative Uncertainty Control System Loop

$$U_\Delta(s) = W_2(s)U(s) \tag{4.294}$$

and

$$\begin{aligned}
U(s) &= C(s)(R(s) - Y(s)) \\
&= -C(s)\left(P_0(s)U(s) + P_0(s)W_1(s)Y_\Delta(s)\right) \\
&= -C(s)P_0(s)W_1(s)Y_\Delta(s) - C(s)P_0(s)U(s) \\
(\mathbb{I} + C(s)P_0(s))U(s) &= -C(s)P_0(s)W_1(s)Y_\Delta(s) \\
U(s) &= -(\mathbb{I} + C(s)P_0(s))^{-1}C(s)P_0(s)W_1(s)Y_\Delta(s),
\end{aligned} \tag{4.295}$$

from which it follows

$$\begin{aligned} U_\Delta(s) &= W_2(s)U(s) \\ &= -\underbrace{W_2(s)(\mathbb{I}+C(s)P_0(s))^{-1}C(s)P_0(s)W_1(s)}_{M(s)}Y_\Delta(s). \end{aligned} \tag{4.296}$$

4.6.6 Recasting Performance in a Robust Stability Problem

One can summarize robust stability conditions in bounding the infinity norm of selected functions. We are here assuming that we want to attenuate noise on the output. Let $\|n\|_2$¡1 be the norm of the noise signal and $W_n(s)$ a weighting function to rescale and shape the frequency content of the signal. We define the norm of the noise to be like this in order to use the form of the problem we derived previously: the weighting function W_n exists exactly for re-modulating the importance of n to its correct value. Using the nominal performance approach, one uses the diagram depicted in Figure 37 and writes

$$Y(s) = (\mathbb{I}+P(s)C(s))^{-1}W_n(s)N(s)+\dots \qquad \Rightarrow \|S_0(s)W_n(s)\|_\infty \ll 1 \tag{4.297}$$

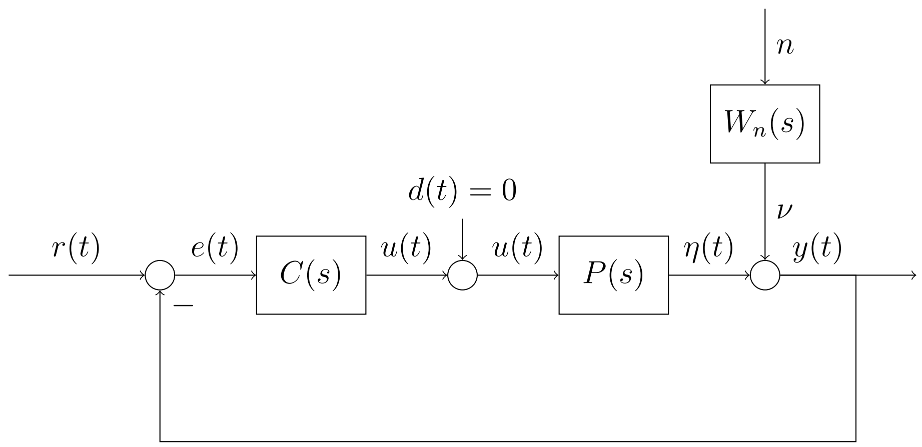

Figure 37: Robust Performance Problem

The same result is obtained considering the following loop and treating it as a robust stability problem, as in Figure 38. One can then identify the transfer function

$$M(s) = \frac{Y}{N} = S_O(s)W_n(s) \Rightarrow \|M\|_\infty. \tag{4.298}$$

4.7 MIMO Robust Performance

Definition 15. Robust Performance: The effect of exogenous signals in presence of plant uncertainty can degrade performance to unacceptable levels before the system goes unstable. We need a *robust performance* test to evaluate the worst case effect of performance, given uncertainty.

Before having a closer look to the problem, let's recall what we have seen so far:

- **Nominal Stability (NS)**: The controller internally stabilizes the (nominal) plant.

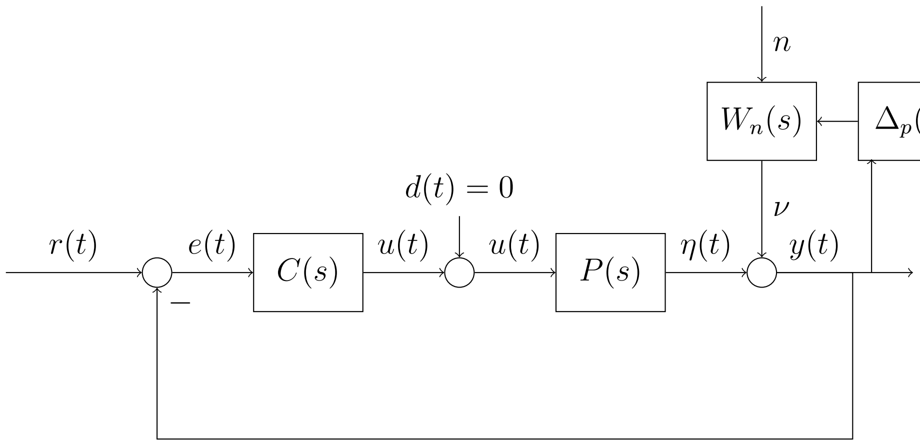

Figure 38: Robust Stability Problem

- **Robust Stability (RS)** The controller internally stabilizes all plants parametrized through model uncertainty.
- **Nominal Performance (NP)**: is guaranteed by imposing constraints on the infinity norm of some sensitivity function, given nominal stability.
- **Robust Performance (RP)**: like NP, but for all plants within a given model set.

4.7.1 Problem Definition

Given a nominal plant $P_0(s)$ and a model uncertainty parametrization $\Delta(s)$, find conditions on the nominal closed loop system, such that

1. The controller $C(s)$ stabilizes the closed loop system for all $P \in \Pi$ with

$$\Pi = \{(\mathbb{I} + W_1(s)\Delta(s)W_2(s))P_0(s) : \; W_1(s), W_2(s), \Delta, \quad \text{rational, proper, stable}\} \tag{4.299}$$

2. A performance metric on some relevant transfer function is satisfied for all $P \in \Pi$.

4.7.2 M-Delta Approach: from RP to RS

A robust performance as the one depicted in Figure 39 (with $\|\Delta_r\|_\infty < 1$ and $\|\Delta_p\|_\infty < 1$), can be transformed in a robust stability problem as the one depicted in Figure 40.
In particular, one can show

$$\begin{pmatrix} U_{\Delta,1} \\ U_{\Delta,2} \end{pmatrix} = \underbrace{\begin{pmatrix} -W_2(s)T_O(s)W_1(s) & -W_2(s)T_O(s)W_n(s) \\ S_O(s)W_1(s) & S_O(s)W_n(s) \end{pmatrix}}_{M} \begin{pmatrix} Y_{\Delta,1} \\ Y_{\Delta,2}. \end{pmatrix} \tag{4.300}$$

Using the small gain theorem, a sufficient condition for robust performance is

$$\|M\|_\infty < 1. \tag{4.301}$$

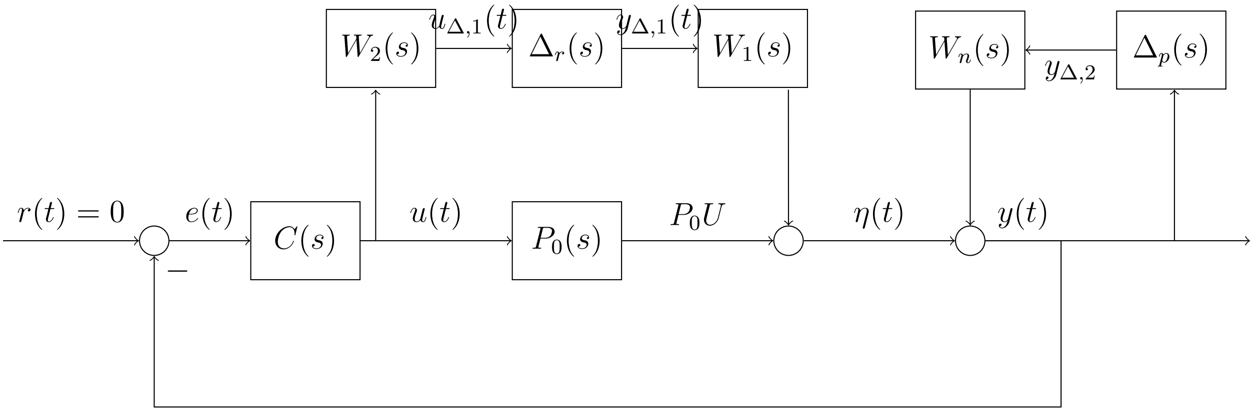

Figure 39: M-Delta Approach

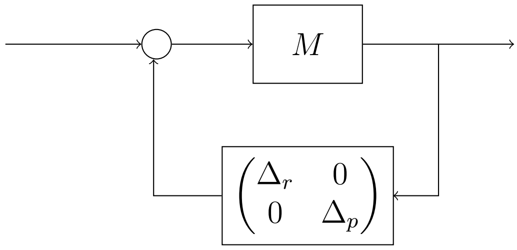

Figure 40: M-Delta Approach

4.7.3 Structured Singular Value

Definition

The approach we have seen in the previous section applies to a diagonal uncertainty. How can we handle any uncertainty $\Delta \in \tilde{\Delta}$?

Intuition: The Structured Singular Value is a generalization of the maximum singular value and the spectral radius. Through SSV, a *generalized small gain theorem* is obtained. This accounts for the structure of uncertainty.

Definition 16. Mu: Given Δ, find the smallest (in terms of $\bar{\sigma}(\Delta)$) Δ which makes

$$\det(\mathbb{I} - M(s)\Delta(s)) = 0. \tag{4.302}$$

Then:

$$\mu(M) = \frac{1}{\bar{\sigma}(\Delta)}. \tag{4.303}$$

If $\det(\mathbb{I} - M(s)\Delta(s)) \neq 0 \ \forall \Delta \in \tilde{\Delta}$, then $\mu(M) = 0$.

Theorem 7. (**SSV Robust Stability**) The $M - \Delta$ system is stable for all $\Delta \in \tilde{\Delta}$ with $\|\Delta\|_\infty < 1$ **if and only if**

$$\sup_\omega \mu(M(j\omega)) < 1. \tag{4.304}$$

Remark. Mu is a measure of the smallest perturbation that sends the system unstable.

Properties

(I)
$$\mu(M) \geq 0. \tag{4.305}$$

(II) It holds
$$\tilde{\Delta} = \{\Delta | \Delta \in \mathbb{C}^{p\times q},\ \text{full matrix}\} \Rightarrow \mu(M) = \bar{\sigma}(M). \tag{4.306}$$

(III) It holds
$$\tilde{\Delta} = \{\lambda \mathbb{I} | \lambda \in \mathbb{C}\} \Rightarrow \mu(M) = \rho(M) = |\lambda_{max}(M)|. \tag{4.307}$$
because $\inf \lambda^{-1}(M) = \rho(M)$.

(IV) It holds
$$\tilde{\Delta} = \{\text{diag}(\Delta_1, \ldots, \Delta_q) | \Delta_i \text{ is complex}\} \Rightarrow \rho(M) \leq \mu(M) \leq \bar{\sigma}(M) \tag{4.308}$$

(V) It holds
$$\tilde{\Delta} = \{\text{diag}(\Delta_1, \ldots, \Delta_q) | \Delta_i \text{ is complex}\} \Rightarrow \mu(M) = \mu(D^{-1}MD),\ \forall D \in \mathcal{D}, \tag{4.309}$$
where $\mathcal{D} = \{D = \text{diag}(d_1, \ldots, d_n) | d_i > 0\}$, $D\Delta = \Delta D$.

Remark.

- SSV provides a necessary and sufficient condition for RS (and thus RP), provided mu. This leads to a less conservative bound than the infinity norm condition.
- Computing mu is very tricky. There exist numerical approaches to refine upper and lower bounds for mu.
- The bounds are defined as
$$\mu(M) = \mu(D^{-1}MD) \leq \inf_{D\in\mathcal{D}} \bar{\sigma}(D^{-1}MD) \tag{4.310}$$

Robust Performance Noise Rejection: SISO Case

One can recover the structure defined in the previous chapters, but the SISO case offers some simplifications:
$$W_1(s) = 1, \quad S_0, T_0 \rightarrow S, T. \tag{4.311}$$
It follows
$$M = \begin{pmatrix} -W_2(s)T(s) & -W_2(s)T(s)W_n(s) \\ S(s) & S(s)W_n(s) \end{pmatrix}. \tag{4.312}$$
Let $D = \text{diag}(d_1, d_2)$ and $\alpha = \frac{d_2}{d_1}$ with $|d_1|, |d_2| < 1$. It must hold
$$\begin{aligned} \mu(M(j\omega)) &= \mu(\underbrace{D^{-1}M(j\omega)D}_{A(\alpha)}) \\ &\leq \inf_{|\alpha|>0} \lambda_{\max}^{\frac{1}{2}}(A^*(\alpha)A(\alpha)) \\ &< 1. \end{aligned} \tag{4.313}$$

We perform the analysis following specific steps:

1. We fix ω and find $A(\alpha)$ and $A^*(\alpha)A(\alpha)$.
   In the SISO case, the matrix Δ is diagonal and we define

$$\Delta = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \qquad d_1, d_2 < 1, \qquad \alpha = \frac{d_2}{d_1}. \tag{4.314}$$

   Since $\mu(M) = \mu(D^{-1}MD)$, let's set $D = \Delta$ and write

$$\begin{aligned} A(\alpha) = D^{-1}MD &= \begin{pmatrix} \frac{1}{d_1} & 0 \\ 0 & \frac{1}{d_2} \end{pmatrix} \begin{pmatrix} -W_2(s)T(s) & -W_2(s)T(s)W_n(s) \\ S(s) & S(s)W_n(s) \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{W_2(s)T(s)}{d_1} & -\frac{W_2(s)T(s)W_n(s)}{d_1} \\ \frac{S(s)}{d_2} & \frac{S(s)W_n(s)}{d_2} \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \\ &= \begin{pmatrix} -W_2(s)T(s) & -\alpha W_2(s)T(s)W_n(s) \\ \frac{1}{\alpha}S(s) & S(s)W_n(s) \end{pmatrix}. \end{aligned} \tag{4.315}$$

   Furthermore, it holds (by dropping the s in the notation for simplicity)

$$\begin{aligned} A^*(\alpha)A(\alpha) &= \begin{pmatrix} -\bar{W}_2\bar{T}(s) & \frac{1}{\alpha}\bar{S} \\ -\alpha\bar{W}_2\bar{T}\bar{W}_n & \bar{S}\bar{W}_n \end{pmatrix} \begin{pmatrix} -W_2T & -\alpha W_2TW_n \\ \frac{1}{\alpha}S & SW_n \end{pmatrix} \\ &= \begin{pmatrix} \|W_2\|_2^2\|T\|_2^2 + \frac{\|S\|_2^2}{\alpha^2} & \alpha\|W_2\|_2^2\|T\|_2^2W_n + \frac{1}{\alpha}\|S\|_2^2W_n \\ \alpha\|W_2\|_2^2\|T\|_2^2\bar{W}_n + \frac{1}{\alpha}\|S\|_2^2\bar{W}_n & \alpha^2\|W_2\|_2^2\|T\|_2^2\|W_n\|_2^2 + \|S\|_2^2\|W_n\|_2^2 \end{pmatrix}. \end{aligned} \tag{4.316}$$

2. We find $\lambda_{\max}(\alpha)$, i.e. the biggest λ from $\det(\underbrace{A^*(\alpha)A(\alpha) - \lambda\mathbb{I}}_{I}) = 0$. It holds

$$\begin{aligned} \det(I) &= \det\begin{pmatrix} \|W_2\|_2^2\|T\|_2^2 + \frac{\|S\|_2^2}{\alpha^2} - \lambda & \alpha\|W_2\|_2^2\|T\|_2^2W_n + \frac{1}{\alpha}\|S\|_2^2W_n \\ \alpha\|W_2\|_2^2\|T\|_2^2\bar{W}_n + \frac{1}{\alpha}\|S\|_2^2\bar{W}_n & \alpha^2\|W_2\|_2^2\|T\|_2^2\|W_n\|_2^2 + \|S\|_2^2\|W_n\|_2^2 - \lambda \end{pmatrix} \\ &= \alpha^2\|W_2\|_2^4\|T\|_2^4\|W_n\|_2^2 + \|W_2\|_2^2\|W_2\|_2^2\|T\|_2^2\|S\|_2^2\|W_n\|_2^2 - \lambda\|W_2\|_2^2\|T\|_2^2 \\ &+ \|W_2\|_2^2\|T\|_2^2\|S\|_2^2\|W_n\|_2^2 + \frac{\|S\|_2^4\|W_n\|_2^2}{\alpha^2} - \frac{\lambda}{\alpha^2}\|S\|_2^2 - \lambda\alpha^2\|W_2\|_2^2\|T\|_2^2\|W_n\|_2^2 \\ &- \lambda\|S\|_2^2\|W_n\|_2^2 + \lambda^2 - \alpha^2\|W_2\|_2^4\|T\|_2^4\|W_n\|_2^2 - 2\|W_2\|_2^2\|W_2\|_2^2\|T\|_2^2\|S\|_2^2\|W_n\|_2^2 \\ &- \frac{1}{\alpha^2}\|S\|_2^4\|W_n\|_2^2 \\ &= \lambda^2 - \lambda\left(\|W_2\|_2^2\|T\|_2^2 + \alpha^2\|W_2\|_2^2\|T\|_2^2\|W_n\|_2^2 + \|S\|_2^2\|W_n\|_2^2 + \frac{1}{\alpha^2}\|S\|_2^2\right), \end{aligned} \tag{4.317}$$

   from which it follows

$$\lambda_{\max}(A^*(\alpha)A(\alpha)) = \|W_2\|_2^2\|T\|_2^2 + \alpha^2\|W_2\|_2^2\|T\|_2^2\|W_n\|_2^2 + \|S\|_2^2\|W_n\|_2^2 + \frac{1}{\alpha^2}\|S\|_2^2. \tag{4.318}$$

3. We now want to minimize this with respect to α. It holds

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\lambda_{\max}\right) &= 0\\
2\alpha\|W_2\|_2^2\|T\|_2^2\|W_n\|_2^2 - 2\frac{1}{\alpha^3}\|S\|_2^2 &= 0\\
2\alpha^4\|W_2\|_2^2\|T\|_2^2\|W_n\|_2^2 - 2\|S\|_2^2 &= 0\\
\alpha^2 &= \frac{\|S\|_2}{\|W_2\|_2\|T\|_2\|W_n\|_2}.
\end{aligned} \tag{4.319}$$

4. By plugging this into the original equation one gets

$$\begin{aligned}
\mu(M) = \lambda_{\max} &= \|W_2\|_2^2\|T\|_2^2 + \frac{\|S\|_2}{\|W_2\|_2\|T\|_2\|W_n\|_2}\|W_2\|_2^2\|T\|_2^2\|W_n\|_2^2 + \|S\|_2^2\|W_n\|_2^2\\
&+ \frac{\|W_2\|_2\|T\|_2\|W_n\|_2}{\|S\|_2}\|S\|_2^2\\
&= \|W_2\|_2^2\|T\|_2^2 + 2\|S\|_2\|W_2\|_2\|T\|_2\|W_n\|_2 + \|S\|_2^2\|W_n\|_2^2\\
&= \left(\|S\|_2\|W_n\|_2 + \|W_2\|_2\|T\|_2\right)^2
\end{aligned} \tag{4.320}$$

5. The condition on μ implies

$$\|S\|_2\|W_n\|_2 + \|W_2\|_2\|T\|_2 < 1. \tag{4.321}$$

5 MIMO Control Fundamentals

5.1 Decentralized Control

5.1.1 Idea and Definitions

As we have introduced in previous lectures, the generalization from SISO to MIMO systems adds crosscouplings and complexities to the control problem. In general, one can divide the control strategies into two philosophies:

1. Avoid the MIMO complexity by trying to use SISO controllers. How?
   - *Decentralized control:* every input signal is determined only by a feedback from **one** output.
   - *Pairing problem:* choose use of input-output paris for feedback.
   - *Decoupled control:* change of variables to facilitate input-output pairing.
2. Centralized multivariable control, optimizing some cost function, e.g.
   - Linear Quadratic Regulator (LQR, next episode).
   - H-infinity control

The first philosophy results in suboptimal solutions and requires less modeling effort. The second philosophy results in optimal results, but the modeling effort increases. Let's address the problem more spefifically:

Definition 17. Decentralized control: when the control systems consists of independent feedback controllers which interconnect a subset of the output measurements with a subset of manipulated inputs. These subsets should not be used by any other controller.

This represents a good strategy if the the MIMO system shows a **low degree of interaction** between inputs and outputs. How can we evaluate this property? Let's have a look at a generic 2×2 MIMO system with full rank and same number of inputs $u_i(t)$ and outputs $u_i(t)$. For the **coupled** system one can write

$$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} P_1 1(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{pmatrix} \begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix} = \begin{pmatrix} \sum_i P_{1i}(s)U_i(s) \\ \sum_i P_{2i}(s)U_i(s) \end{pmatrix}, \tag{5.1}$$

i.e. each input affects each output. For a **decoupled** system, one can e.g. write

$$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} P_{11}(s) & 0 \\ 0 & P_{22}(s) \end{pmatrix} \begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix} = \begin{pmatrix} P_{11}(s)U_1(s) \\ P_{22}(s)U_2(s) \end{pmatrix}, \tag{5.2}$$

i.e. the system behaves like a union of *non interacting* SISO systems. Furthermore, if one assumes a non-square system, for a general system $P(s) \in \mathbb{R}^{l \times n}$, one can meet the following two cases:

1. **Tall system**($l > m$): we have more outputs than inputs, i.e. not all outputs are affected by an input. Which outputs are best controlled with which inputs?
2. **Fat system**($l < m$): we have more inputs than outputs. How to distribute control action over the inputs?

In the next section, we will introduce a systematic way to address this kind of problems.

5.1.2 Relative-Gain Array (RGA)

As introduced in the previous section, if a system has a specific decoupled form, one can avoid complex control strategies and use independent SISO controllers. In some cases, this reasoning is actually the good one, but how can one distinguish when to use this approach?
The RGA-matrix tells us how the different subplants of a MIMO plant interact: this matrix is a good indicator of *how SISO* a system is.
This matrix can be generally calculated as

$$\text{RGA}(s) = P(s). \times P(s)^{-T} \tag{5.3}$$

where

$$P(s)^{-T} = (P(s)^T)^{-1}. \tag{5.4}$$

and $A. \times A$ represents the **element-wise, Shur** multiplication (`A.*A` in MATLAB). If $P(s)$ is not invertible (recall tall, fat and non inverbile square systems), one needs to generalize the inverse with the *Moore-Penrose* Inverse. Recalling $P(s) \in \mathbb{R}^{l\times m}$ one can define two cases:

- **Tall system**($l > m$): if $\text{rank}(P(s)) = m$,

$$A^\dagger = (A^*A)^{-1}A^*, \quad A^\dagger A = \mathbb{I}_m. \tag{5.5}$$

- **Fat system**($l < m$): if $\text{rank}(P(s)) = l$,

$$A^\dagger = A^*(AA^*)^{-1}, \quad AA^\dagger = \mathbb{I}_l. \tag{5.6}$$

In general, each element of the matrix gives us a special information:

$$[\text{RGA}]_{ab} = \frac{\text{gain from } u_a \text{ to } y_b \text{ with all other loops open}}{\text{gain from } u_a \text{ to } y_b \text{ with all other loops closed (perfect control)}}. \tag{5.7}$$

Remark. It's intuitive to notice, that if

$$[\text{RGA}]_{ab} \approx 1 \tag{5.8}$$

the numerator and the denominator are equal, i.e. SISO control is enough to bring u_a at y_b.

Remark. The theory behind the relative-gain array goes far beyond the aim of this course and one should be happy with the given examples. If however you are interested in this topic, you can have a look here.

Let's take the example of a 2×2 plant: in order to compute the first element $(1, 1)$ of the $\text{RGA}(s)$ we consider the system depicted in Figure 41. We close with a SISO controller $C_{22}(s)$ the loop from $y_2(t)$ to $u_2(t)$ and try to compute the transfer function from $u_1(t)$ to $y_1(t)$.
Everyone has his special way to decouple a MIMO system. I've always used this procedure: starting from the general equation in frequency domain

$$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{pmatrix} \cdot \begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix}, \tag{5.9}$$

one can read

$$\begin{aligned} Y_1(s) &= P_{11}(s) \cdot U_1(s) + P_{12}(s) \cdot U_2(s) \\ Y_2(s) &= P_{21}(s) \cdot U_1(s) + P_{22}(s) \cdot U_2(s). \end{aligned} \tag{5.10}$$

Since we want to relate $u_1(t)$ and $y_1(t)$ let's express $u_2(t)$ as something we know. Using the controller $C_{22}(s)$ we see

$$\begin{aligned} U_2(s) &= -C_{22}(s) \cdot Y_2(s) \\ &= -C_{22}(s) \cdot P_{21}(s) \cdot U_1(s) - C_{22}(s) \cdot P_{22}(s) \cdot U_2(s) \\ \Rightarrow U_2(s) &= \frac{-C_{22}(s) \cdot P_{21}(s) \cdot U_1(s)}{1 + P_{22}(s) \cdot C_{22}(s)}. \end{aligned} \tag{5.11}$$

With the general equation one can then write

$$\begin{aligned} Y_1(s) &= P_{11}(s) \cdot U_1(s) + P_{12}(s) \cdot U_2(s) \\ &= P_{11}(s) \cdot U_1(s) + P_{12}(s) \cdot \frac{-C_{22}(s) \cdot P_{21}(s) \cdot U_1(s)}{1 + P_{22}(s) \cdot C_{22}(s)} \\ &= \frac{P_{11}(s) \cdot (1 + P_{22}(s) \cdot C_{22}(s)) - P_{12}(s) \cdot C_{22}(s) \cdot P_{21}(s)}{1 + P_{22}(s) \cdot C_{22}(s)} \cdot U_1(s). \end{aligned} \tag{5.12}$$

We have found the general transfer function that relates $u_1(t)$ to $y_1(t)$. We now consider two extreme cases:

- We assume open loop conditions, i.e. *all other loops open*: $C_{22} \approx 0$. One gets

$$Y_1(s) = P_{11}(s) \cdot U_1(s). \tag{5.13}$$

- We assume high controller gains, i.e. *all other loops closed*: $P_{22}(s) \cdot C_{22}(s) \gg 1$. One gets

$$\begin{aligned} &\lim_{C_{22}(s) \to \infty} \frac{P_{11}(s) \cdot (1 + P_{22}(s) \cdot C_{22}(s)) - P_{12}(s) \cdot C_{22}(s) \cdot P_{21}(s)}{1 + P_{22}(s) \cdot C_{22}(s)} \\ &= \frac{P_{11}(s) \cdot P_{22}(s) - P_{12}(s) \cdot P_{21}(s)}{P_{22}(s)}. \end{aligned} \tag{5.14}$$

As stated before, the first element of the RGA is the division of these two. It holds

$$\begin{aligned} [\mathrm{RGA}]_{11} &= \frac{P_{11}(s)}{\frac{P_{11}(s) \cdot P_{22}(s) - P_{12}(s) \cdot P_{21}(s)}{P_{22}(s)}} \\ &= \frac{P_{11}(s) \cdot P_{22}(s)}{P_{11}(s) \cdot P_{22}(s) - P_{12}(s) \cdot P_{21}(s)}. \end{aligned} \tag{5.15}$$

Remark. As you can see, the definition of the element of the RGA matrix does not depend on the chosen controller $C_{22}(s)$. This makes this method extremely powerful.

By repeating the procedure one can try to find $[\mathrm{RGA}]_{22}$. In order to do that one has to close the loop from y_1(t) to $u_1(t)$: the result will be exactly the same:

$$[\mathrm{RGA}]_{11} = [\mathrm{RGA}]_{22}. \tag{5.16}$$

Let's go a step further. In order to compute the element $[\mathrm{RGA}]_{21}$, one has to close the loop from $y_1(t)$ to $u_2(t)$ and find the transfer function from $u_1(t)$ to $y_2(t)$.

Remark. This could be a nice exercise to test your understanding!

With a similar procedure one gets

$$[\text{RGA}]_{21} = \frac{-P_{12}(s) \cdot P_{21}(s)}{P_{22}(s) \cdot P_{11}(s) - P_{21}(s) \cdot P_{12}(s)}. \tag{5.17}$$

and as before

$$[\text{RGA}]_{21} = [\text{RGA}]_{12}. \tag{5.18}$$

How can we now use this matrix, to know if SISO control would be enough? As already stated before, $[\text{RGA}]_{ab} \approx 1$ means SISO control is enough. Moreover, if the diagonal terms differ substantially from 1, the MIMO interactions (also called *cross couplings*) are too important and a SISO control is no more recommended.

If

$$\text{RGA} \approx \mathbb{I} \tag{5.19}$$

evaluated at the relevant frequencies of the system, i.e. at $\omega_c \pm$ **one decade**, one can ignore the cross couplings and can control the system with SISO tools *one loop at time.* If this is not the case, one has to design a MIMO controller. A bunch of observations could be useful by calculations:

1. Rows and columns of the RGA matrix add up to 1. This means one can write the matrix as

$$\begin{pmatrix} [\text{RGA}]_{11} & [\text{RGA}]_{12} \\ [\text{RGA}]_{21} & [\text{RGA}]_{22} \end{pmatrix} = \begin{pmatrix} [\text{RGA}]_{11} & 1 - [\text{RGA}]_{11} \\ 1 - [\text{RGA}]_{11} & [\text{RGA}]_{11} \end{pmatrix}. \tag{5.20}$$

   This allows to calculate just one element of the matrix.

2. If one looks at $\text{RGA}(s = 0)$ and the diagonal entries of the matrix are positive, SISO control is possible.

3. The RGA of a triangular matrix $P(s)$ is the identity matrix.

4. The RGA is invariant to scaling, i.e. for every diagonal matrix D_i it holds

$$[\text{RGA}](P(s)) = [\text{RGA}](D_1 \cdot P(s) \cdot D_2). \tag{5.21}$$

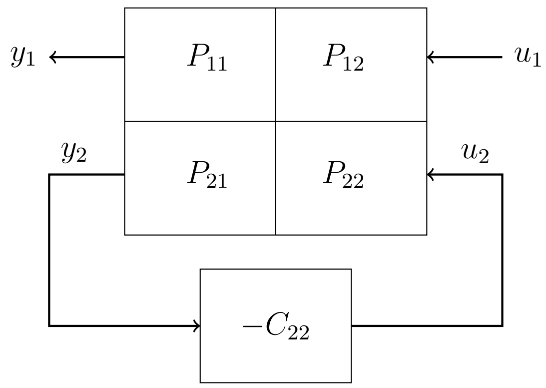

Figure 41: Derivation of the RGA-Matrix for the 2×2 case.

Example 42. For a MIMO system with two inputs and two outputs just the first element of the RGA matrix is given. This is a function of a system parameter p and is given as

$$[\text{RGA}(s)]_{11} = \frac{1}{ps^2 + 2ps + 1}. \tag{5.22}$$

(a) Find the other elements of the RGA matrix.

(b) For which values of p is the system for all frequencies $\omega \in [0, \infty)$ controllable with two independent SISO control loops (*one loop at the time*)?

Now, you are given the following transfer function of another MIMO system:

$$P(s) = \begin{pmatrix} \frac{1}{s} & \frac{s+2}{s+1} \\ 1 & -\frac{1}{s+1} \end{pmatrix}. \tag{5.23}$$

(c) Find the RGA matrix of this MIMO system.

(c) Use the computed matrix to see if for frequencies in the range $\omega \in [3, 10]$ rad/s the system is controllable with two separate SISO controllers.

Solution.

(a) Using the theory we learned, it holds

$$[\text{RGA}(s)]_{11} = [\text{RGA}(s)]_{22} = \frac{1}{ps^2 + 2ps + 1} \tag{5.24}$$

and

$$\begin{aligned}
[\text{RGA}(s)]_{12} &= [\text{RGA}(s)]_{21} \\
&= 1 - [\text{RGA}(s)]_{11} \\
&= 1 - \frac{1}{ps^2 + 2ps + 1} \\
&= \frac{ps \cdot (s+2)}{ps^2 + 2ps + 1}.
\end{aligned} \tag{5.25}$$

(b) In order to use two independend SISO control loops, the diagonal elements of the RGA matrix should be ≈ 1 and the anti diagonal elements should be ≈ 0. It's easy to see that this is the case for $p = 0$. In fact, if one sets $p = 0$ one gets

$$\text{RGA}(s) = \mathbb{I}. \tag{5.26}$$

Hence, independently of the frequency one has, i.e. $\omega \in [0, \infty)$, the control problem can be solved with two independent SISO controllers.

(c) Using the learned theory, it holds

$$\begin{aligned}
[\text{RGA}(s)]_{11} &= [\text{RGA}(s)]_{22} \\
&= \frac{P_{11}(s) \cdot P_{22}(s)}{P_{11}(s) \cdot P_{22}(s) - P_{12}(s) \cdot P_{21}(s)} \\
&= \frac{-\frac{1}{s\cdot(s+1)}}{-\frac{1}{s\cdot(s+1)} - \frac{s+2}{s+1}} \\
&= \frac{1}{1 + s \cdot (s+2)} \\
&= \frac{1}{s^2 + 2s + 1} \\
&= \frac{1}{(s+1)^2}.
\end{aligned} \tag{5.27}$$

and

$$\begin{aligned}
[\text{RGA}(s)]_{12} &= [\text{RGA}(s)]_{21} \\
&= 1 - [\text{RGA}(s)]_{11} \\
&= 1 - \frac{1}{(s+1)^2} \\
&= \frac{s \cdot (s+2)}{(s+1)^2}.
\end{aligned} \tag{5.28}$$

(d) In order to evaluate the RGA matrix in this range, we have to express it with it's frequency dependence, i.e. $s = j\omega$. For the magnitudes it holds

$$\begin{aligned} |[\mathrm{RGA}(j\omega)]_{11}| &= |[\mathrm{RGA}(j\omega)]_{22}| \\ &= \frac{1}{|j\omega+1|^2} \\ &= \frac{1}{1+\omega^2}. \end{aligned} \tag{5.29}$$

and

$$\begin{aligned} |[\mathrm{RGA}(j\omega)]_{12}| &= |[\mathrm{RGA}(j\omega)]_{21}| \\ &= \frac{1}{|j\omega+1|^2}\cdot|j\omega|\cdot|j\omega+2| \\ &= \frac{\omega\cdot\sqrt{4+\omega^2}}{1+\omega^2}. \end{aligned} \tag{5.30}$$

We can know insert the two limit values of the given range and get

$$\begin{aligned} |[\mathrm{RGA}(j\cdot 3)]_{11}| &= |[\mathrm{RGA}(j\cdot 3)]_{22}| \\ &= \frac{1}{10} \\ &= 0.10. \\ |[\mathrm{RGA}(j\cdot 3)]_{12}| &= |[\mathrm{RGA}(j\cdot 3)]_{21}| \\ &= \frac{3\cdot\sqrt{13}}{10} \\ &\approx 1.08. \end{aligned} \tag{5.31}$$

and

$$\begin{aligned} |[\mathrm{RGA}(j\cdot 10)]_{11}| &= |[\mathrm{RGA}(j\cdot 10)]_{22}| \\ &= \frac{1}{101} \\ &= 0.01. \\ |[\mathrm{RGA}(j\cdot 10)]_{12}| &= |[\mathrm{RGA}(j\cdot 10)]_{21}| \\ &= \frac{10\cdot\sqrt{104}}{101} \\ &\approx 1.01. \end{aligned} \tag{5.32}$$

In both cases the diagonal elements are close to 0 and the antidiagonal elements are close to 1. This means that the system is **diagonal dominant** and SISO control *one loop at time* is permitted. We just need to pay attention to what should be controlled: since the antidiagonal elements are close to 1, we need to use u_1 for y_2 and u_2 for y_1.

Example 43. Figure 42 shows a 2×2 MIMO system. Sadly, we don't know anything about the transfer functions $P_{ij}(s)$ but

$$P_{12}(s) = 0. \tag{5.33}$$

Your boss wants you to use a *one loop at the time* approach as you see in the picture.

(a) Why is your boss' suggestion correct?

(b) Just a reference r_i is affecting both outputs y_i, which one?

(c) Compute the transfer function $r_i \rightarrow y_j$ for $i \neq j$?

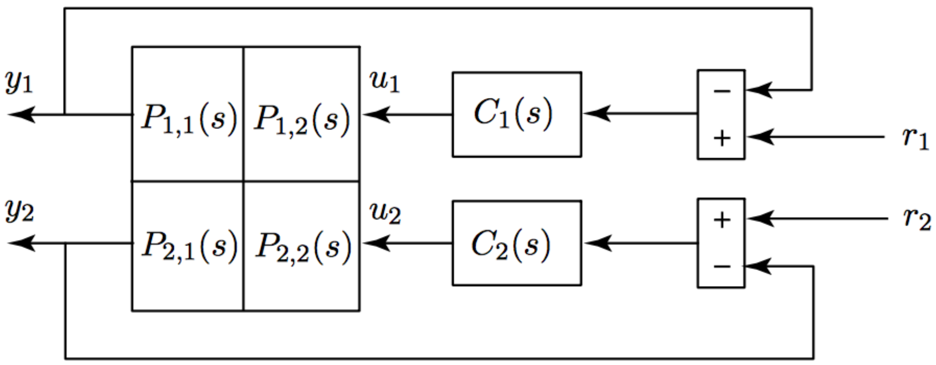

Figure 42: Structure of MIMO system.

Solution.

(a) To check if the suggestion is correct let's have a look at the RGA matrix: it holds

$$\begin{aligned}[\mathrm{RGA}]_{11} &= [\mathrm{RGA}]_{22} \\ &= \frac{P_{11}(s)\cdot P_{22}(s)}{P_{11}(s)\cdot P_{22}(s) - P_{12}(s)\cdot P_{21}(s)} \\ &= 1. \\ [\mathrm{RGA}]_{12} &= [\mathrm{RGA}]_{21} \\ &= 1-[\mathrm{RGA}]_{11} \\ &= 0.\end{aligned} \tag{5.34}$$

since $P_{12}(s) = 0$. This means that the RGA matrix is identical to the identity matrix, resulting in a perfect diagonal dominant system, which can be controlled with the *one loop at the time* approach.

(b) Let's analyze the signals from Figure 42. Since $P_{12}(s) = 0$, the output y_1 is not affected from u_2. Moreover, this means that the reference signal r_2, which influences u_2, cannot affect the output y_1. The only reference that acts on both y_1 and y_2 is r_1: directly through $C_1(s)$ on y_1 and with crosscouplings through $P_{21}(s)$ on y_2.

(c) As usual we set to 0 the reference values we don't analyze: here $r_2 = 0$. Starting from the general equation in frequency domain

$$\begin{aligned}\begin{pmatrix}Y_1(s)\\ Y_2(s)\end{pmatrix} &= \begin{pmatrix}P_{11}(s) & P_{12}(s)\\ P_{21}(s) & P_{22}(s)\end{pmatrix}\cdot\begin{pmatrix}U_1(s)\\ U_2(s)\end{pmatrix} \\ &= \begin{pmatrix}P_{11}(s) & 0\\ P_{21}(s) & P_{22}(s)\end{pmatrix}\cdot\begin{pmatrix}U_1(s)\\ U_2(s)\end{pmatrix}.\end{aligned} \tag{5.35}$$

one can read

$$\begin{aligned}Y_1(s) &= P_{11}(s)\cdot U_1(s) \\ Y_2(s) &= P_{21}(s)\cdot U_1(s) + P_{22}(s)\cdot U_2(s).\end{aligned} \tag{5.36}$$

Since we want to relate r_1 and y_2 let's express u_1 as something we know.
Using Figure 42 one gets

$$\begin{aligned}R_1(s)\cdot C_1(s) &= U_1(s) + P_{11}(s)\cdot C_1(s)\cdot U_1(s) \\ U_1 &= \frac{R_1(s)\cdot C_1(s)}{1+P_{11}(s)\cdot C_1(s)}.\end{aligned} \tag{5.37}$$

Inserting this into the second equation one gets

$$Y_2(s) = P_{21}(s)\cdot\frac{R_1(s)\cdot C_1(s)}{1+P_{11}(s)\cdot C_1(s)} + P_{22}(s)\cdot U_2(s). \tag{5.38}$$

One have to find an expression for $U_2(s)$. To do that, we look at the second loop in Figure 42 an see

$$\begin{aligned}\underbrace{R_2(s)}_{=0}\cdot C_2(s) - Y_2(s)\cdot C_2(s) &= U_2(s) \\ U_2(s) &= -Y_2(s)\cdot C_2(s).\end{aligned} \tag{5.39}$$

Inserting this into the second equation one gets

$$
\begin{aligned}
Y_2(s) &= P_{21}(s) \cdot \frac{R_1(s) \cdot C_1(s)}{1 + P_{11}(s) \cdot C_1(s)} + P_{22}(s) \cdot U_2(s) \\
&= P_{21}(s) \cdot \frac{R_1(s) \cdot C_1(s)}{1 + P_{11}(s) \cdot C_1(s)} + P_{22}(s) \cdot (-Y_2(s) \cdot C_2(s)) \\
Y_2(s) \cdot (1 + P_{22}(s) \cdot C_2(s)) &= P_{21}(s) \cdot \frac{R_1(s) \cdot C_1(s)}{1 + P_{11}(s) \cdot C_1(s)} \\
Y_2(s) &= \underbrace{\frac{P_{21}(s) \cdot C_1(s)}{(1 + P_{11}(s) \cdot C_1(s)) \cdot (1 + P_{22}(s) \cdot C_2(s))}}_{F(s)} \cdot R_1(s).
\end{aligned}
\tag{5.40}
$$

where $F(s)$ is the transfer function we wanted.

Example 44. Figure 43 shows the structure of a MIMO system, composed of three subsystems $P_1(s), P_2(s)$ and $P_3(s)$. It has inputs u_1 and u_2 and outputs y_1 and y_2. The

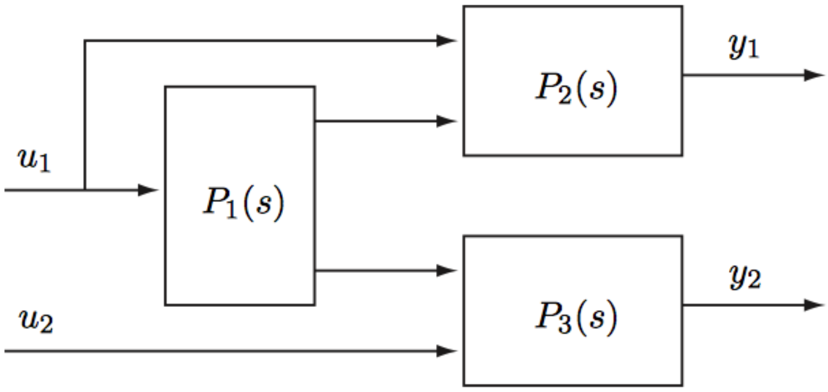

Figure 43: Structure of MIMO system.

three subsystems are given as

$$P_1(s) = \begin{pmatrix} \frac{s-5}{s+3} \\ \frac{1}{s+4} \end{pmatrix}, \quad P_2(s) = \begin{pmatrix} \frac{1}{s+3} & \frac{s+4}{s-5} \end{pmatrix}, \quad P_3(s) = \begin{pmatrix} \frac{s+2}{s+5} & \frac{1}{s+1} \end{pmatrix}. \tag{5.41}$$

Compute the transfer function of the whole system.

Solution. One should think with matrix dimensions here. Let's redefine the subsystem's matrices more generally:

$$P_1(s) = \begin{pmatrix} P_1^{11} \\ P_1^{21} \end{pmatrix}, \quad P_2(s) = \begin{pmatrix} P_2^{11} & P_2^{12} \end{pmatrix}, \quad P_3(s) = \begin{pmatrix} P_3^{11} & P_3^{12} \end{pmatrix} \tag{5.42}$$

Together with the structure of the system one gets

$$\begin{aligned} Y_1 &= P_2^{11} \cdot U_1 + P_2^{12} \cdot P_1^{11} \cdot U_1, \\ Y_2 &= P_3^{11} \cdot P_1^{21} \cdot U_1 + P_3^{12} \cdot U_2. \end{aligned} \tag{5.43}$$

This can be written in the general matrix for the transfer function:

$$\begin{aligned} P(s) &= \begin{pmatrix} P_2^{11} + P_2^{12} \cdot P_1^{11} & 0 \\ P_3^{11} \cdot P_1^{21} & P_3^{12} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{s+3} + \frac{s+4}{s-5} \cdot \frac{s-5}{s+3} & 0 \\ \frac{s+2}{s+5} \cdot \frac{1}{s+4} & \frac{1}{s+1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{s+5}{s+3} & 0 \\ \frac{s+2}{(s+5)\cdot(s+4)} & \frac{1}{s+1} \end{pmatrix}. \end{aligned} \tag{5.44}$$

Example 45. Figure 44 shows the structure of a MIMO system, composed of two subsystems $P_1(s), P_2(s)$. It has inputs u_1 and u_2 and outputs y_1 and y_2. The subsystem $P_1(s)$

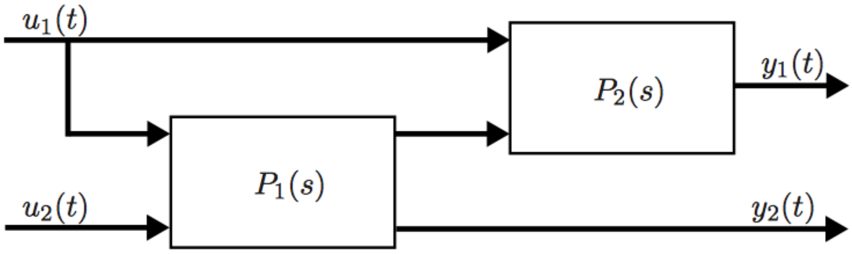

Figure 44: Structure of MIMO system.

is given with its state space description:

$$A_1 = \begin{pmatrix} -3 & 0 \\ 2 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} \quad D_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{5.45}$$

and the subsystem $P_2(s)$ is given as

$$P_2(s) = \begin{pmatrix} \frac{1}{s-2} & \frac{s-1}{(s+4)\cdot(s-2)} \end{pmatrix}. \tag{5.46}$$

Compute the transfer function of the whole system.

Solution. First of all, we compute the transfer function in frequency domain of the first subsystem $P_1(s)$. It holds

$$\begin{aligned}
P_1(s) &= C_1 \cdot (s \cdot \mathbb{I} - A_1)^{-1} \cdot B_1 + D_1 \\
&= \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} s+3 & 0 \\ -2 & s-1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} \cdot \frac{1}{(s+3)\cdot(s-1)} \cdot \begin{pmatrix} s-1 & 0 \\ 2 & s+3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \frac{1}{(s+3)\cdot(s-1)} \cdot \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} s-1 & 0 \\ 2s+8 & s+3 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \frac{1}{(s+3)\cdot(s-1)} \cdot \begin{pmatrix} 4s+16 & (s+1)(s+3) \\ 5s+5 & s+3 \end{pmatrix} \\
&= \begin{pmatrix} \frac{4s+16}{(s+3)\cdot(s-1)} & \frac{s+1}{s-1} \\ \frac{5s+5}{(s+3)\cdot(s-1)} & \frac{1}{s-1} \end{pmatrix}.
\end{aligned} \tag{5.47}$$

One should think with matrix dimensions here. Let's redefine the subsystem's matrices more generally:

$$P_1(s) = \begin{pmatrix} P_1^{11} & P_1^{12} \\ P_1^{21} & P_1^{22} \end{pmatrix}, \quad P_2(s) = \begin{pmatrix} P_2^{11} & P_2^{12} \end{pmatrix}. \tag{5.48}$$

Together with the structure of the system one gets

$$\begin{aligned}
Y_1 &= P_2^{11} \cdot U_1 + P_1^{11} \cdot P_2^{12} \cdot U_1 + P_1^{12} \cdot P_2^{12} \cdot U_2, \\
Y_2 &= P_1^{21} \cdot U_1 + P_1^{22} \cdot U_2.
\end{aligned} \tag{5.49}$$

This can be written in the general matrix for the transfer function:

$$\begin{aligned}
P(s) &= \begin{pmatrix} P_2^{11} + P_1^{11} \cdot P_2^{12} & P_1^{12} \cdot P_2^{12} \\ P_1^{21} & P_1^{22} \end{pmatrix} \\
&= \dots \\
&= \begin{pmatrix} \frac{s+7}{(s+3)\cdot(s-2)} & \frac{s+1}{(s+4)(s-2)} \\ \frac{5s+5}{(s+3)\cdot(s-1)} & \frac{1}{s-1} \end{pmatrix}.
\end{aligned} \tag{5.50}$$

Example 46. The system in Figure 45 can be well controlled with two separate SISO controllers.

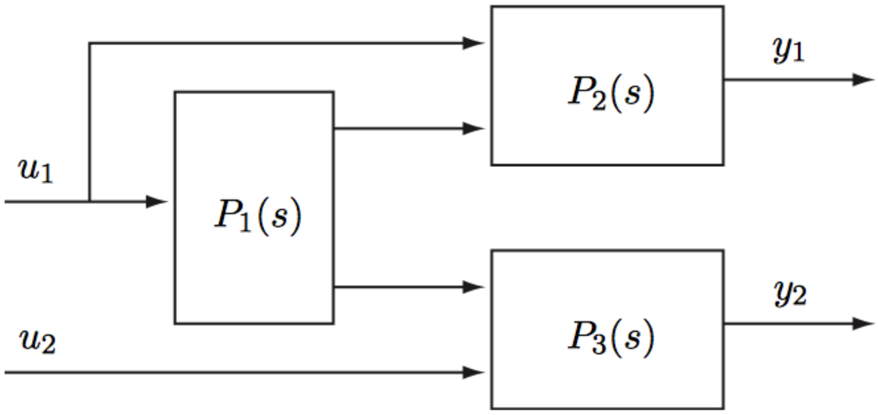

Figure 45: Structure of MIMO system.

☐ True.

☐ False.

Solution.

- ☑ True.
- ☐ False.

Explanation:

One can observe that the input u_2 affects only the output y_2. This means that the transfer function matrix has a **triangular** form and hence, that the RGA matrix is identical to the identity matrix: this means that we can reach good control with two separate SISO controllers.

5.1.3 Q Parametrization

Recalling the standard control Loop repicted in Figure 46, one can write

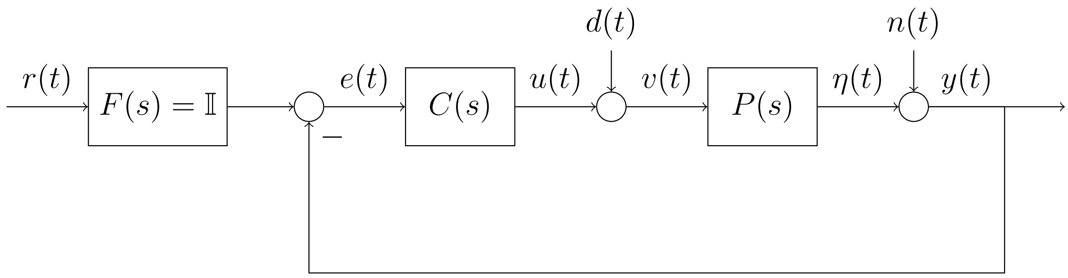

Figure 46: Standard feedback control system structure.

$$\begin{aligned}\begin{pmatrix} Y(s) \\ U(s) \end{pmatrix} &= \begin{pmatrix} (\mathbb{I}+P(s)C(s))^{-1}P(s)C(s) & (\mathbb{I}+P(s)C(s))^{-1}P(s) \\ (\mathbb{I}-C(s)P(s))^{-1}C(s) & -(\mathbb{I}+C(s)P(s))^{-1}C(s)P(s) \end{pmatrix} \begin{pmatrix} R(s) \\ D(s) \end{pmatrix} \\ &= \begin{pmatrix} T_O(s) & S_O(s)P(s) \\ S_I(s)C(s) & -T_I(s) \end{pmatrix} \begin{pmatrix} R(s) \\ D(s) \end{pmatrix}.\end{aligned} \tag{5.51}$$

In order for the system to be internally stable, $T_O(s), S_O(s)P(s), S_I(s)C(s)$ and $T_I(s)$ must be stable. Ideally, we would like to translate this properties on direct consequences for $C(s)$. However, relations are not linear and it is not obvious how to find direct translations. One defined

$$Q(s) = C(s)(\mathbb{I}+P(s)C(s))^{-1}. \tag{5.52}$$

Then, one can write

$$\begin{aligned}(\mathbb{I}+P(s)C(s))^{-1} &= (\mathbb{I}+P(s)C(s)-P(s)C(s))(\mathbb{I}+P(s)C(s))^{-1} \\ &= (\mathbb{I}+P(s)C(s))(\mathbb{I}+P(s)C(s))^{-1} - P(s)C(s)(\mathbb{I}+P(s)C(s))^{-1} \\ &= \mathbb{I} - P(s)Q(s),\end{aligned} \tag{5.53}$$

and

$$\begin{aligned}(\mathbb{I}+C(s)P(s))^{-1} &= (\mathbb{I}+C(s)P(s)-C(s)P(s))(\mathbb{I}+C(s)P(s))^{-1}\\ &= (\mathbb{I}+C(s)P(s))(\mathbb{I}+C(s)P(s))^{-1}-C(s)P(s)(\mathbb{I}+C(s)P(s))^{-1}\\ &= \mathbb{I}-Q(s)P(s).\end{aligned} \tag{5.54}$$

It follows

$$\begin{aligned}T_O(s) &= (\mathbb{I}+P(s)C(s))^{-1}P(s)C(s)\\ &= P(s)C(s)(\mathbb{I}+P(s)C(s))\\ &= P(s)Q(s).\\ S_O(s)P(s) &= (\mathbb{I}-P(s)Q(s))P(s)\\ S_I(s)C(s) &= Q(s)\\ T_I(s) &= Q(s)P(s).\end{aligned} \tag{5.55}$$

Theorem 8. Q internal stability: Let $P(s)$ be a stable plant of a negative feedback system, then the closed loop system is internally stable if and only if $Q(s)$ is stable.

This makes the tuning of the controller extremely easier: the sensitivity functions depend **linearly** on Q. Moreover, it holds:

- Supposing that the plant is stable: $Q(s)$ can be any transfer matrix that satisfies the definition.
- If only proper controllers are taken into account, then $Q(s)$ must be proper.
- Finding a $Q(s)$ is equivalent to finding the controller $C(s)$.
- As long as $Q(s)$ is stable, it can vary freely and internal stability will be guaranteed. Even if $Q(s)$ maps to an unstable controller $C(s)$.
- Starting from the formula for $Q(s)$, one can write

$$C(s) = (\mathbb{I}-Q(s)P(s))^{-1}Q(s) = Q(s)(\mathbb{I}-P(s)Q(s))^{-1}. \tag{5.56}$$

5.2 Internal Model Control (IMC)

5.2.1 Principle

Principle: *Accurate control can be achieved only if the control system encapsulates some representation of the controlled process.*
Approach: We feedback only the mismatch between the model prediction and the actual measured output, i.e. the uncertainty in the control loop.

Connection with Q Parametrization

The control system structure for IMC is depicted in Figure 47. $P(s)$ denotes the plant of the system and $P_0(s)$ the plant model. The measurement $y(t)$ is corrupted by a measurement noise $n(t)$. The signal $y_0(t)$ represents the predicted output. The signal i represents the signal mismatch between the measured and the predicted outputs. The controller $Q(s)$ (please refer to the previous section for its form) produces the input $u(t)$. Relating this structure with the classic one, one can write:

$$C(s) = Q(s)\left(I(s)-P_0(s)Q(s)\right)^{-1}. \tag{5.57}$$

Remark. Note that the controller $C(s)$ can be defined with the orange region in Figure 47.

Analysis

By trying to relate the output signal $y(t)$ to the other signals available in the loop, one can write

$$\begin{aligned}
Y(s) &= N(s) + \eta(s) \\
&= N(s) + P(s)V(s) \\
&= N(s) + P(s)\left(D(s) + U(s)\right) \\
&= N(s) + P(s)D(s) + P(s)Q(s)E(s) \\
&= N(s) + P(s)D(s) + P(s)Q(s)\left(R(s) - I(s)\right) \\
&= N(s) + P(s)D(s) + P(s)Q(s)R(s) - P(s)Q(s)\left(Y(s) - Y_0(s)\right) \\
&= N(s) + P(s)D(s) + P(s)Q(s)R(s) - P(s)Q(s)Y(s) + P(s)Q(s)P_0(s)U(s) \\
&= N(s) + P(s)D(s) + P(s)Q(s)R(s) - P(s)Q(s)\left(N(s) + P(s)V(s)\right) + P(s)Q(s)P_0(s)U(s) \\
&= \underbrace{(\mathbb{I} - PQ)}_{S_O(s)} N + \underbrace{P\left(\mathbb{I} - QP\right)}_{PS_I(s)} D + \underbrace{PQ}_{T_O(s)} R + PQ\left(P - P_0\right)U,
\end{aligned} \tag{5.58}$$

where in the last line we dropped the s dependency for simplicity reasons. In the case where $P(s) = P_0(s)$ and $Q(s) = P^{-1}(s)$, one gets

$$Y(s) = R(s). \tag{5.59}$$

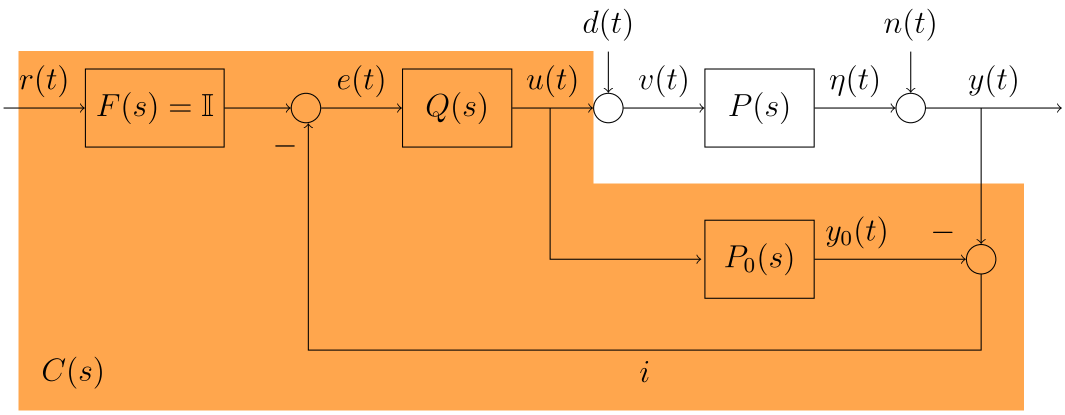

Figure 47: Internal Model Control System Structure.

5.2.2 Example: Predictive Control

Why predictive control

If a SISO system has *substantial* delays, it is very difficult to control it with a normal PID controller. The I part of the controller causes *impatience*, that is, integrates over time. As a practical example think of taking a shower in the morning: one let the water flow and of course this hasn't the desired temperature. For this reason one chooses warmer water by turning the temperature controller; the water becomes too hot and so one turns it on the other side to have colder water and so on, resulting in a non optimal strategy. Moreover, the D part of the controller is practically useless[7]. What does the expression *substantial* delays mean? As indicative range one can say that it is worth using predictive control if

$$\frac{T}{T+\tau} > 0.3, \tag{5.60}$$

where T is the delay and τ is the time constant of the system. Other prerequisites are

- The plant must be asymptotically stable.
- A good model of the plant should be available.

The Smith Predictor

One can see the two *equivalent* structures of the Smith Predictor in Figure 48 and Figure 49.

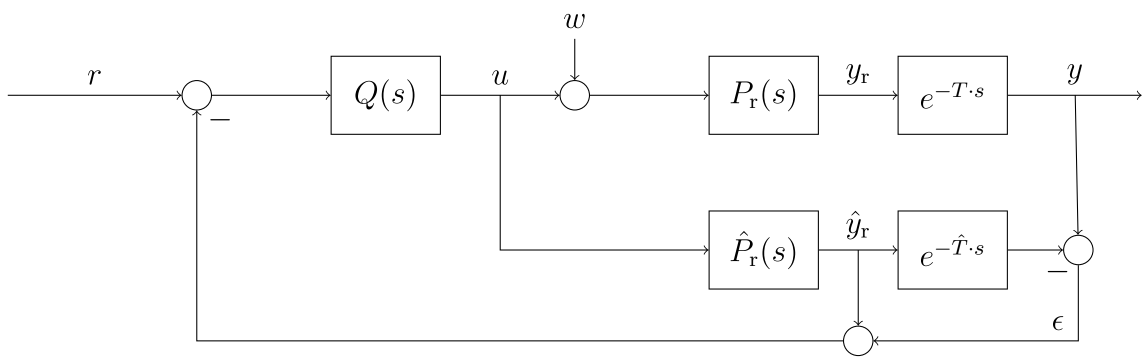

Figure 48: Structure of the Smith predictor.

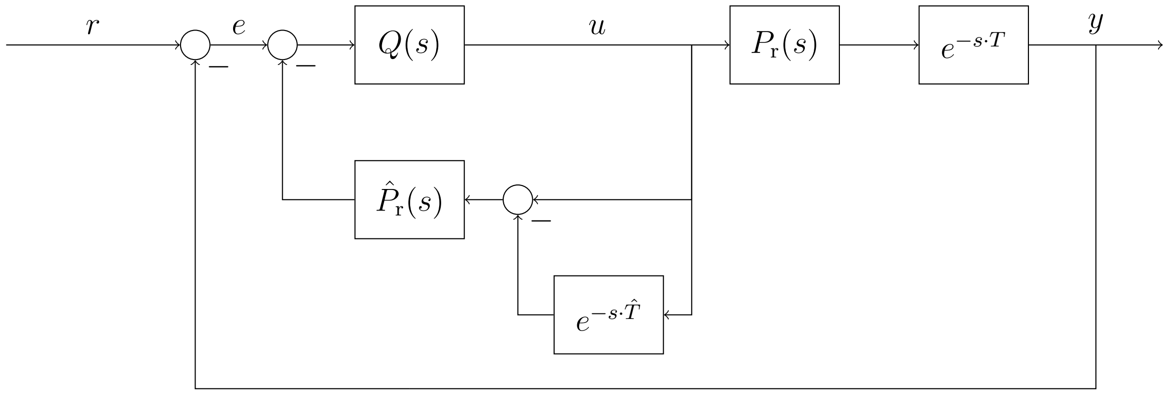

Figure 49: Structure of the Smith predictor.

If the system has big delays, one can assume that it is possible to write the delay element and the nondelayed plant as a product in the frequency domain: that's what is done in

[7]Taking the derivative of a delay element doesn't help to control it

the upper right side of Figure 48. This means that the transfer function $u(t) \to y(t)$ can be written as

$$P(s) = P_r(s) \cdot e^{-sT}. \tag{5.61}$$

Main Idea:

As long as we have no disturbance $d(t)$ (i.e. $d(t) = 0$) and our model is good enough (this means $P_r(s) = \hat{P}_r(s),\ T = \hat{T}$)[8], we can model a non delayed plant and get the non delayed output $\hat{y}_r(t)$ (one can see this on the lower right side of Figure 48). The feedback signal results from the sum of $\hat{y}_r(t)$ and the correction signal ϵ.

Analysis

The controller of the system is the transfer function $e(t) \to u(t)$, which can be computed as

$$\begin{aligned} U(s) &= Q(s)\left(E(s) - \hat{P}_{\mathrm{r}}(s)\left(U(s) - U(s)e^{-s\hat{T}}\right)\right) \\ &= Q(s)E(s) - Q(s)\hat{P}_{\mathrm{r}}(s)\left(1 - e^{-s\hat{T}}\right)U(s), \end{aligned} \tag{5.62}$$

from which it follows

$$\begin{aligned} C(s) &= \frac{U(s)}{E(s)} \\ &= \frac{Q(s)}{1 + Q(s)\hat{P}_{\mathrm{r}}(s)\left(1 - e^{-s\hat{T}}\right)}. \end{aligned} \tag{5.63}$$

This means that the loop gain transfer function is

$$\begin{aligned} L(s) &= P(s) \cdot C(s) \\ &= \frac{Q(s)P_{\mathrm{r}}(s)e^{-sT}}{1 + Q(s)\hat{P}_{\mathrm{r}}(s)\left(1 - e^{-s\hat{T}}\right)}. \end{aligned} \tag{5.64}$$

If one assumes as stated, that the model is good enough s.t. $P_r(s) = \hat{P}_r(s),\ T = \hat{T}$, one gets

$$\begin{aligned} T(s) &= \frac{L(s)}{1 + L(s)} \\ &= \frac{\frac{Q(s)\cdot P_{\mathrm{r}}(s)\cdot e^{-s\cdot T}}{1+Q(s)\cdot P_{\mathrm{r}}(s)\cdot(1-e^{-s\cdot T})}}{1 + \frac{Q(s)\cdot P_{\mathrm{r}}(s)\cdot e^{-s\cdot T}}{1+Q(s)\cdot P_{\mathrm{r}}(s)\cdot(1-e^{-s\cdot T})}} \\ &= \frac{Q(s)\cdot P_{\mathrm{r}}(s)\cdot e^{-s\cdot T}}{1 + Q(s)\cdot P_{\mathrm{r}}(s)\cdot(1 - e^{-s\cdot T}) + Q(s)\cdot P_{\mathrm{r}}(s)\cdot e^{-s\cdot T}} \\ &= \frac{Q(s)\cdot P_{\mathrm{r}}(s)}{1 + Q(s)\cdot P_{\mathrm{r}}(s)}\cdot e^{-s\cdot T} \\ &= T_{\mathrm{ref}}(s)\cdot e^{-s\cdot T}. \end{aligned} \tag{5.65}$$

[8] We use $.\hat{.}.$ to identify the parameters of the model

Remark.

- This result is very important: we have shown that the delay cannot be completely eliminated and that every tranfer function (here $T(s)$ but also $S(s)$) will have the same delay as the plant $P(s)$.
- Advantages of the prediction are:
  - Very fast.
  - Same Robustness.
- Disadvantages of the prediction are:
  - Very difficult to implement.
  - Very difficult to analyze.
  - Problems if there are model errors.

5.3 Examples

Example 47. Consider the control problem depicted in Figure 50, designed for the stable, linear, SISO system $P_0(s)$. Let's consider $w(t)$ to be the vector which describes the inputs to the system (disturbances and reference), $z(t)$ to be the vector which describes all the interesting signals for control, $u(t)$ to be the control signal from $C(s)$ and $y(t)$ to be signal used by the controller.

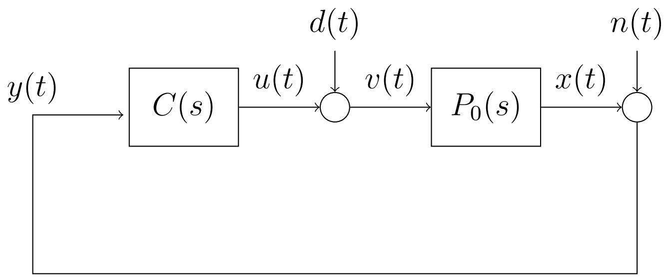

Figure 50: Control System Loop

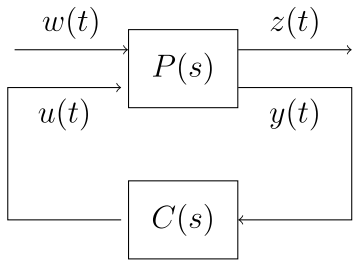

Figure 51: Control System Loop Generalization

a) Choose

$$w(t) = \begin{pmatrix} d(t) \\ n(t) \end{pmatrix}, \qquad z(t) = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}. \tag{5.66}$$

Derive the transfer function $P(s)$ such that the system can be rewritten in the form depicted in Figure 51.
Hint: $P(s)$ can be rewritten in terms of the contributions of its signals, i.e.

$$P(s) = \begin{pmatrix} P_{zw}(s) & P_{zu}(s) \\ P_{yw}(s) & P_{yu}(s) \end{pmatrix}, \tag{5.67}$$

where $P_{ij}(s)$ is the transfer function between signal i and signal j.

b) We denote the closed-loop system (i.e. from w to z) $H(s)$. Show that

$$H(s) = P_{zw}(s) + P_{zu}(s)C(s)\left(1 - P_{yu}(s)C(s)\right)^{-1} P_{yw}(s). \tag{5.68}$$

c) Determine $H(s)$ for the system depicted in Figure 50 and rewrite it in terms of the output sensitivity function $S_O(s) = (1 - P_{yu}(s)C(s))^{-1}$ and output complementary sensitivity function $T_O(s) = -(1 - P_{yu}(s)C(s))^{-1}P_{yu}(s)C(s)$.
Hint: Use the formula you derived in b).

d) Rewrite $H(s)$ using the Q parametrization $Q = C(s)\left(1 - P_{yu}(s)C(s)\right)^{-1}$.

e) Given $P_0(s) = \frac{1}{s+1}$ and $Q(s) = \frac{1}{s+10}$, the closed-loop system is internally stable.

- ☐ True.
- ☐ False.

Solution.

a) As a first step, one needs to identify the dimensions of the transfer functions in $P(s)$ (from the hint):

- $P_{zw}(s)$: it holds
$$Z(s) = P_{zw}(s)W(s). \tag{5.69}$$
Since $Z(s) \in \mathbb{C}^{2\times 1}$ and $W(s) \in \mathbb{C}^{2\times 1}$, matrix $P_{zw}(s) \in \mathbb{C}^{2\times 2}$. Moreover, one can write the signal dependencies as
$$P_{zw}(s) = \begin{pmatrix} P_{xd}(s) & P_{xn}(s) \\ P_{vd}(s) & P_{vn}(s) \end{pmatrix}. \tag{5.70}$$

- $P_{zu}(s)$: it holds
$$Z(s) = P_{zu}(s)U(s). \tag{5.71}$$
Since $Z(s) \in \mathbb{C}^{2\times 1}$ and $U(s) \in \mathbb{C}^{1\times 1}$, matrix $P_{zu}(s) \in \mathbb{C}^{2\times 1}$. Moreover, one can write the signal dependencies as
$$P_{zu}(s) = \begin{pmatrix} P_{xu}(s) \\ P_{vu}(s) \end{pmatrix}. \tag{5.72}$$

- $P_{yw}(s)$: it holds
$$Y(s) = P_{yu}(s)W(s). \tag{5.73}$$
Since $Y(s) \in \mathbb{C}^{1\times 1}$ and $W(s) \in \mathbb{C}^{2\times 1}$, matrix $P_{yw}(s) \in \mathbb{C}^{1\times 2}$. Moreover, one can write the signal dependencies as
$$P_{yw}(s) = \begin{pmatrix} P_{yd}(s) & P_{yn}(s) \end{pmatrix}. \tag{5.74}$$

- $P_{yu}(s)$: it holds
$$Y(s) = P_{yu}(s)U(s). \tag{5.75}$$
Since $Y(s) \in \mathbb{C}^{1\times 1}$ and $U(s) \in \mathbb{C}^{1\times 1}$, matrix $P_{yu}(s) \in \mathbb{C}^{1\times 1}$. Moreover, one can write the signal dependencies as
$$P_{yu}(s) = P_{yu}(s). \tag{5.76}$$

Referring to the given flow diagram, one can write the relations for $V(s)$:
$$V(s) = D(s) + U(s). \tag{5.77}$$

From Equation 5.77 one recovers
$$P_{vd}(s) = 1, \quad P_{vu}(s) = 1, \quad P_{vn}(s) = 0. \tag{5.78}$$

The relations for $X(s)$ are:
$$\begin{aligned} X(s) &= P_0(s)V(s) \\ &= P_0(s)D(s) + P_0(s)U(s). \end{aligned} \tag{5.79}$$

From Equation 9.92 one deduces
$$P_{xd}(s) = P_0(s), \quad P_{xu}(s) = P_0(s), \quad P_{xn}(s) = 0. \tag{5.80}$$

The relations for $Y(s)$ are:

$$\begin{aligned} Y(s) &= N(s) + X(s) \\ &= N(s) + P_0(s)V(s) \\ &= N(s) + P_0(s)D(s) + P_0(s)U(s). \end{aligned} \tag{5.81}$$

From Equation 5.81 one recovers

$$P_{yd}(s) = P_0(s), \quad P_{yu}(s) = P_0(s), \quad P_{yn}(s) = 1. \tag{5.82}$$

Putting everything together, one gets

$$P_{zw}(s) = \begin{pmatrix} P_0(s) & 0 \\ 1 & 0 \end{pmatrix}, \quad P_{zu}(s) = \begin{pmatrix} P_0(s) \\ 1 \end{pmatrix}, \quad P_{yw}(s) = \begin{pmatrix} P_0(s) & 1 \end{pmatrix}, \quad P_{yu}(s) = P_0(s), \tag{5.83}$$

and hence

$$P(s) = \begin{pmatrix} P_0(s) & 0 & P_0(s) \\ 1 & 0 & 1 \\ P_0(s) & 1 & P_0(s) \end{pmatrix}. \tag{5.84}$$

b) It holds

$$U(s) = C(s)Y(s), \tag{5.85}$$

and

$$\begin{aligned} Y(s) &= P_{yu}(s)U(s) + P_{yw}(s)W(s) \\ &= P_{yu}(s)C(s)Y(s) + P_{yw}(s)W(s) \\ \Rightarrow Y(s) &= (1 - P_{yu}(s)C(s))^{-1}P_{yw}(s)W(s). \end{aligned} \tag{5.86}$$

With these two informations, we compute the relation between $Z(s)$ and $W(s)$:

$$\begin{aligned} Z(s) &= P_{zw}(s)W(s) + P_{zu}(s)U(s) \\ &= P_{zw}(s)W(s) + P_{zu}(s)C(s)Y(s) \\ &= \underbrace{\left(P_{zw}(s) + P_{zu}(s)C(s)\left(1 - P_{yu}(s)C(s)\right)^{-1} P_{yw}(s)\right)}_{H(s)} W(s). \end{aligned} \tag{5.87}$$

c) Using the formula derived in b), we get

$$\begin{aligned} H(s) &= P_{zw}(s) + P_{zu}(s)C(s)\left(1 - P_{yu}(s)C(s)\right)^{-1} P_{yw}(s) \\ &= \begin{pmatrix} P_0(s) & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} P_0(s) \\ 1 \end{pmatrix} C(s)\left(1 - P_0(s)C(s)\right)^{-1} \begin{pmatrix} P_0(s) & 1 \end{pmatrix} \\ &= \begin{pmatrix} P_0(s) & 0 \\ 1 & 0 \end{pmatrix} + \frac{C}{1 - P_0(s)C(s)} \begin{pmatrix} P_0(s)^2 & P_0(s) \\ P_0(s) & 1 \end{pmatrix} \\ &= \frac{1}{1 - P_0(s)C(s)} \begin{pmatrix} P_0(s) & P_0(s)C(s) \\ 1 & C(s) \end{pmatrix} \\ &= \begin{pmatrix} P_0(s)S_O(s) & -T_O(s) \\ S_O(s) & C(s)S_O(s) \end{pmatrix}. \end{aligned} \tag{5.88}$$

d) Using the formula derived in b) and plugging in Q one gets

$$\begin{aligned} H(s) &= P_{zw}(s) + P_{zu}(s)C(s)\left(1 - P_{yu}(s)C(s)\right)^{-1} P_{yw}(s) \\ &= P_{zw}(s) + P_{zu}(s)Q(s)P_{yw}(s) \\ &= \begin{pmatrix} P_0(s) & 0 \\ 1 & 0 \end{pmatrix} + Q(s)\begin{pmatrix} P_0(s)^2 & P_0(s) \\ P_0(s) & 1 \end{pmatrix} \\ &= \begin{pmatrix} P_0(s) + P_0(s)^2 Q(s) & P_0(s)Q(s) \\ 1 + P_0(s)Q(s) & Q(s) \end{pmatrix}. \end{aligned} \tag{5.89}$$

Note that each element of $H(s)$ is linear in $Q(s)$. This simplifies the tuning of the controller.

e)

☑ True.

☐ False.

Solution: Since $Q(s)$ and $P_0(s)$ are both asymptotically stable, the closed-loop system is internally stable.

Remark. **Theorem** (Q internal stability): let $P(s)$ be a stable plant of a negative feedback system, then the closed loop system is internally stable if and only if $Q(s)$ is stable.

6 State Feedback

Motivation: Each control strategy we analyzed so far, was based on output feedback. In fact, the main analysis has been based on the fact that system outputs are available through measurements. Imagine now to have the *states* of the system available. Would the control problem benefit from this new information? Intuitively, outputs are nothing else than a linear combination of the states (one can always write this through the dynamics of the system), hence contain less *information.*

6.1 Concept

The big difference to what we have seen so far, is that we are looking at a continuous time control system, which operates in time domain and no more in frequency domain. The basic state feedback control structure is depicted in Figure 52. The basic idea is: we

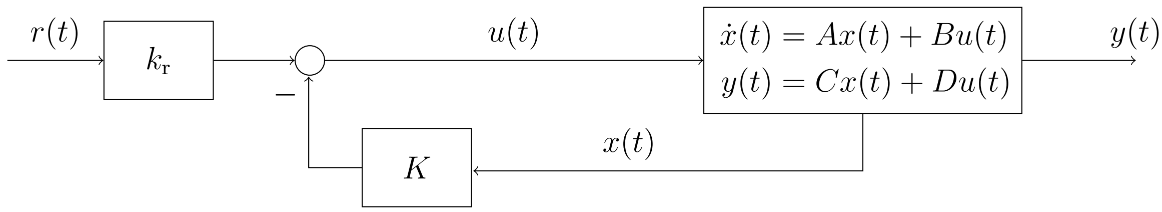

Figure 52: Basic State Feedback Control Structure.

have the dynamics in the loop, with the input $u(t)$ and the output $y(t)$. We negatively feedback the state $x(t)$ with a controller K and add to a reference (or a multiple of it, k_r). In words, we try to keep the state where we want it to be. Assuming for simplicity $D = 0$ (the same analysis can be performed), one gets

$$u(t) = k_\mathrm{r} r(t) - Kx(t), \tag{6.1}$$

from which it follows

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ &= Ax(t) + Bk_\mathrm{r} r(t) - BKx(t) \\ &= \underbrace{(A - BK)}_{A_\mathrm{cl}} x(t) + Bk_\mathrm{r} r(t). \end{aligned} \tag{6.2}$$

We get a new closed loop matrix A_cl, i.e., state feedback **affects** the poles of the closed loop transfer function.

6.2 Reachability

A key property of a control system is reachability. Which set of points in the state space of the system can be reached through the choice of a specific control input? Reachability plays a central role in deciding if state feedback is a good strategy for the control of a specific dynamic system. Let's assume that a dynamic system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = Ax(t) + Bu(t). \tag{6.3}$$

is given, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$. We deine the reachable set as the set of all points x_f such that there exists an input $u(t)$ with $0 \leq t \leq T$ that steers the system from $x(0) = x_0$ to $x(T) = x_\mathrm{f}$.

Definition 18. A linear system is **reachable** if for any $x_0, x_\mathrm{f} \in \mathbb{R}^n$, there exists a $T > 0$ and $u(t) : [0, T] \to \mathbb{R}$ such that the corresponding solution satisfies $x(0) = x_0$ and $x(T) = x_\mathrm{f} \in \mathbb{R}^n$.

Reachability test: A system (A, B) is reachable if and only if $\mathrm{rank}(\mathcal{R}) = n$, where $x \in \mathbb{R}^n$ and

$$\mathcal{R} = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}. \tag{6.4}$$

6.2.1 Reachable Canonical Form

Given a transfer function matrix

$$\underbrace{P(s)}_{\in\mathbb{R}^{l\times m}} = \frac{B_0 + B_1 s + \ldots + B_{n-1}s^{n-1}}{a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + s^n}, \tag{6.5}$$

one wants to find the system matrices

$$(A, B, C, 0), \tag{6.6}$$

such that

$$P(s) = C(s\mathbb{I} - A)^{-1}B. \tag{6.7}$$

A possible solution is the *reachable canonical form*:

$$A = \begin{pmatrix} 0_m & \mathbb{I}_m & 0_m & \dots & 0_m \\ 0_m & 0_m & \mathbb{I}_m & \dots & 0_m \\ \vdots & \vdots & 0_m & \ddots & \vdots \\ 0_m & 0_m & 0_m & \dots & \mathbb{I}_m \\ -a_0\mathbb{I}_m & -a_1\mathbb{I}_m & -a_2\mathbb{I}_m & \dots & -a_{n-1}\mathbb{I}_m \end{pmatrix}, \quad B = \begin{pmatrix} 0_m \\ 0_m \\ \vdots \\ 0_m \\ \mathbb{I}_m \end{pmatrix} \tag{6.8}$$
$$C = \begin{pmatrix} B_0 & B_1 & \dots & B_{n-2} & B_{n-1} \end{pmatrix},$$

where 0_m is the $m \times m$ zero matrix and $\mathbb{I}_m$ is the $m \times m$ identity matrix.

Remark. Note that this is the result of a possible change of coordinates. This solution is not unique.

Theorem 9. A system in the reachable canonical form is always reachable.

Theorem 10. Let A and B be the dynamics of a reachable system. Then there exists a transformation $z(t) = Tx(t)$ such that in the transformed coordinates the dynamics matrices are in reachable canonical form and the characteristic polynomial for A is given by

$$\det(s\mathbb{I} - A) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n. \tag{6.9}$$

Example 48. You are given the system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \underbrace{\begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix}}_{A} x(t) + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u(t). \tag{6.10}$$

The aim of the exercise, is to find the reachable canonical form of the system.

a) Which structure should the reachable canonical form have?

b) Compute the unknowns in the form you found in a).

c) Compute the reachability matrix for the original system.

d) Compute the reachability matrix for the general form you found in a).

e) Find a transformation $z(t) = Tx(t)$ which brings the original system in the reachable form.

Solution.

a) We wish to find a transformation which converts the system into the reachable canonical form:

$$\tilde{A} = \begin{pmatrix} 0 & 1 \\ -a_1 & -a_2 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{6.11}$$

b) A and $\tilde{A}$ must have the same eigenvalues (they describe the same system). It holds

$$\begin{aligned} \det(A - \lambda \mathbb{I}) &= \det \begin{pmatrix} \alpha - \lambda & \omega \\ -\omega & \alpha - \lambda \end{pmatrix} \\ &= \alpha^2 - 2\lambda\alpha + \lambda^2 + \omega^2 \\ \det(\tilde{A} - \lambda \mathbb{I}) &= \det \begin{pmatrix} -\lambda & 1 \\ -a_1 & -a_2 - \lambda \end{pmatrix} \\ &= \lambda^2 + \lambda a_2 + a_1. \end{aligned} \tag{6.12}$$

The comparison of the two reveals

$$\begin{aligned} a_1 &= \alpha^2 + \omega^2 \\ a_2 &= -2\alpha \end{aligned} \tag{6.13}$$

c) The reachability matrix for the original system can be computed as follows:

$$\begin{aligned} B &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ AB &= \begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \omega \\ \alpha \end{pmatrix}. \end{aligned} \tag{6.14}$$

This implies

$$\mathcal{R} = \begin{pmatrix} 0 & \omega \\ 1 & \alpha \end{pmatrix}. \tag{6.15}$$

d) The reachability matrix for the general form can be computed as follows:

$$\begin{aligned} \tilde{B} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ \tilde{A}\tilde{B} &= \begin{pmatrix} 0 & 1 \\ -a_1 & -a_2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -a_2 \end{pmatrix}. \end{aligned} \tag{6.16}$$

This implies

$$\tilde{\mathcal{R}} = \begin{pmatrix} 0 & 1 \\ 1 & -a_2 \end{pmatrix}. \tag{6.17}$$

e) Such a transformation results in $\dot{x} = T^{-1}\dot{z}$, which implies

$$\begin{aligned} \dot{x}(t) &= T^{-1}\dot{z}(t) = AT^{-1}z(t) + Bu(t) \\ \Rightarrow \dot{z}(t) &= \underbrace{TAT^{-1}}_{\tilde{A}} z(t) + \underbrace{TB}_{\tilde{B}} u(t). \end{aligned} \tag{6.18}$$

Since $\mathcal{R}$ contains

$$\begin{aligned}\tilde{B} &= TB \\ \tilde{A}\tilde{B} &= TAT^{-1}TB = TAB,\end{aligned} \tag{6.19}$$

one can rewrite the reachability matrix as

$$\begin{aligned}\tilde{\mathcal{R}} &= \begin{pmatrix} TB & TAB \end{pmatrix} \\ &= T\begin{pmatrix} B & AB \end{pmatrix} \\ &= T\mathcal{R},\end{aligned} \tag{6.20}$$

which leads to

$$\begin{aligned}T &= \tilde{\mathcal{R}}\mathcal{R}^{-1} \\ &= -\frac{1}{\omega}\begin{pmatrix} 0 & 1 \\ 1 & -a_2 \end{pmatrix}\begin{pmatrix} \alpha & -\omega \\ -1 & 0 \end{pmatrix} \\ &= \frac{1}{\omega}\begin{pmatrix} 1 & 0 \\ -\alpha - a_2 & \omega \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\omega} & 0 \\ \frac{\alpha}{\omega} & 1 \end{pmatrix}.\end{aligned} \tag{6.21}$$

6.3 Pole Placement

The system dynamics are determined by the poles of the closed loop transfer function. Recalling what we have seen so far, we write the plant transfer function as

$$P(s) = C(s\mathbb{I} - A)^{-1}B, \tag{6.22}$$

the characteristic polynomial as

$$p(s) = \det(s\mathbb{I} - A), \quad p(\pi_i) = 0, \tag{6.23}$$

where π_i are the poles of the system.
Problem definition: We want to find through state feedback a controller K such that the closed loop system has a desired characteristic polynomial

$$p^*_{\text{cl}}(s) = \det(s\mathbb{I} - A_{\text{cl}}) \tag{6.24}$$

But is it always possible to find a solution for the pole placement problem?

Theorem 11. The problem of pole placement has a solution if and only if the system is reachable.

6.3.1 Direct Method

The direct method consists in introducing a matrix K with the correct dimensions and forcing the eigenvalues of the closed loop system matrix $A - BK$ to be the desired ones.

6.3.2 Ackermann Formula

Placing poles by hand is tedious and tricky if the state space dimension grows. The Ackermann's formula provides a one step procedure for calculating the controller K. It holds

$$K = \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix} \mathcal{R}^{-1} p^*_{\text{cl}}(A) = \gamma p^*_{\text{cl}}(A), \tag{6.25}$$

where

- $\mathcal{R}$ is the reachability matrix. Note that this must be invertible (hence, the system is reachable).
- γ is the last row of the inverse of $\mathcal{R}$.
- $p^*_{\text{cl}}(A)$ is the desired closed loop characteristic polynomial evaluated at $s = A$.

Check in the Problem Set for the derivation of the Ackermann's formula.

Example 49. Your task is to keep a space shuttle in its trajectory . The deviations from the desired trajectory are well described by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + bu(t) \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot u(t),\end{aligned} \tag{6.26}$$

where $x_1(t)$ is the position of the space shuttle and $x_2(t)$ is its velocity. Moreover, $u(t)$ represents the propulsion. The position and the velocity are known for every moment and for this reason we can use a state-feedback regulator. You want to find a state feedback controller using pole placement. The specifications for the system are

- The system should not overshoot.
- The error of the regulator should decrease with $e^{-3 \cdot t}$.

a) Find the poles such that the specifications are met.

b) Find the new state feedback matrix K_2.

c) Use the Ackermann formula to get the same result.

Solution

a) Overshoot or in general oscillations, are due to the complex part of some poles. The real part of these poles is given by the decrease function of the error. Since the system must have two poles (A is 2×2), it holds

$$\pi_1 = \pi_2 = -3. \tag{6.27}$$

b) The closed loop has feedback matrix

$$A - B \cdot K. \tag{6.28}$$

We have to choose K such that the eigenvalues of the state feedback matrix are both -3. The dimensions of K must make the matrix multiplication with B and the subtraction with A feasible. It holds $K \in \mathbb{C}^{1\times 2}$. It holds

$$\begin{aligned} A - B \cdot K &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} k_1 & k_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ k_1 & k_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -k_1 & -k_2 \end{pmatrix}. \end{aligned} \tag{6.29}$$

The eigenvalues of this matrix are

$$\pi_{1,2} = \frac{-k_2 \pm \sqrt{k_2^2 - 4k_1}}{2}. \tag{6.30}$$

Since the two eigenvalues should have the same value, we know the the part under the square rooth has to vanish. This means that $-\frac{k_2}{2} = -3 \Rightarrow k_2 = 6$. Moreover:

$$\begin{aligned} k_1 &= \frac{k_2^2}{4} \\ &= 9. \end{aligned} \tag{6.31}$$

The matrix finally reads

$$K_2 = \begin{pmatrix} 9 & 6 \end{pmatrix}. \tag{6.32}$$

c) The Ackermann formula for this problem reads

$$K = \begin{pmatrix} 0 & 1 \end{pmatrix} \mathcal{R}^{-1} p_{\text{cl}}^*(A), \tag{6.33}$$

where $\mathcal{R}$ is the system reachability matrix and $p_{\text{cl}}^*(A)$ is the desired closed loop characteristic polynomial evaluated at $s = A$. For our system it holds

$$\begin{aligned} B &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ AB &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \Rightarrow \mathcal{R} = \mathcal{R}^{-1} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{6.34}$$

With the given poles, it holds

$$\begin{aligned} p_{\text{cl}}^*(s) &= (s+3)^2 \\ p_{\text{cl}}^*(A) &= (A+3\mathbb{I})^2 \\ &= \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 9 & 6 \\ 0 & 9 \end{pmatrix}. \end{aligned} \tag{6.35}$$

Putting everything together in Equation 6.33, one gets

$$\begin{aligned} K &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 9 & 6 \\ 0 & 9 \end{pmatrix} \\ &= \begin{pmatrix} 9 & 6 \end{pmatrix}, \end{aligned} \tag{6.36}$$

which confirms our previous result.

6.4 LQR

6.4.1 Motivation

In previous sections, we introduced the concept of state feedback, with the idea of using the states of the system and its dynamics to synthesize a controller using desired pole-placement. This week we introduce Linear Quadratic control (LQ) and the special case of the Linear Quadratic Regulator (LQR). The concept behind this control strategy has a key role for control theory and is worth a detailed explanation.
Moreover, pole placement has some drawbacks:

- Does not work well with model uncertainty.
- Does not allow specific tuning of desired trade-offs (e.g. cost vs. performance).

6.4.2 Problem Definition

Given the dynamics of a system

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = A \cdot x(t) + B \cdot u(t), \qquad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m \tag{6.37}$$

find a controller

$$u(t) = f(x(t), t), \quad t \in [0, \infty] \tag{6.38}$$

that brings $x(t)$ asymptotically to zero (with $x(0) \neq 0$). In other words it should hold:

$$\lim_{t\to\infty} x(t) = 0, \tag{6.39}$$

i.e. the cost

$$J_{\mathrm{LQR}}(x(t), u(t)) = \int_0^\infty \|z(t)\|_2^2 + \rho\|u(t)\|_2^2 \mathrm{d}t, \quad \rho \in \mathbb{R}^+ \tag{6.40}$$

is minimized, where ρ allows trade-off between energy of the input and energy of the controlled signal and $z(t) = Ex(t) + Fu(t)$ can be chosen to contain the state variables of interest. The LQR standard control loop is reported in Figure 53.

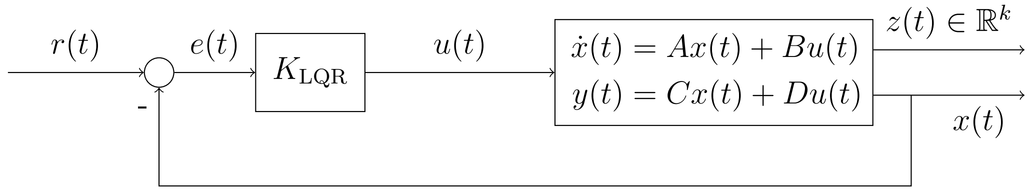

Figure 53: LQR Problem: Closed loop system.

6.4.3 General Form

Preliminary Definitions

Since the cost function introduced in Equation 6.40 contains the euclidean norm, one recalls its definition:

$$\begin{aligned}\|u(t)\|_2^2 &= u(t)^\intercal u(t)\\ &= \sum_i u_i(t)^2.\end{aligned} \tag{6.41}$$

The *weighted* euclidean norm reads

$$\|u(t)\|_{R,2}^2 = u(t)^\intercal R u(t). \tag{6.42}$$

Definition 19. A pair (A, C) is observable if and only if $\text{rank}(\mathcal{O}) = n = \dim(A)$, where

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}. \tag{6.43}$$

Definition 20. A pair (A, C) is detectable if all the unobservable modes are stable.

Definition 21. A real matrix M is said to be **positive definite** (denoted as $M > 0$) when the associated *quadratic form* $V(z)$ is non-negative, i.e.

$$V(z) = \sum_{i,j} m_{i,j} z_i z_j = z^\intercal M z \geq 0, \quad \forall z \neq 0. \tag{6.44}$$

Definition 22. A real matrix M is said to be **positive semi-definite** (denoted as $M \geq 0$) when the associated *quadratic form* $V(z)$ is positive, i.e.

$$V(z) = \sum_{i,j} m_{i,j} z_i z_j = z^\intercal M z > 0, \quad \forall z \neq 0. \tag{6.45}$$

How can we check if a matrix is positive definite?

Eigenvalue Test

A real, symmetric matrix is (semi)-positive definite if and only if it has all (non-negative) positive eigenvalues.

Sylvester's Criterion

An symmetric matrix $M \in \mathbb{R}^{m \times m}$ is positive definite if and only if all the *upper-left* $i \times i$ *submatrices (principal minors)*, $i \in 1, \ldots, m$ have positive determinant. In the case of

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \tag{6.46}$$

the conditions are

1. $a > 0$.
2. $ad - b^2 > 0$.

6.4.4 Weighted LQR

With these definitions, one can write (by dropping the time dependency for simplicity) the weighted LQR problem as

$$\begin{aligned}
J_{\text{LQR}}(x(t),u(t)) &= \int_0^\infty \|z(t)\|^2_{\bar{Q},2} + \rho\|u(t)\|^2_{\bar{R},2}\mathrm{d}t \\
&= \int_0^\infty z^\intercal \bar{Q} z + \rho u^\intercal \bar{R} u \mathrm{d}t \\
&= \int_0^\infty (Ex+Fu)^\intercal \bar{Q}\,(Ex+Fu) + \rho u^\intercal \bar{R} u \mathrm{d}t \\
&= \int_0^\infty (x^\intercal E^\intercal + u^\intercal F^\intercal)\left(\bar{Q}Ex + \bar{Q}Fu\right) + \rho u^\intercal \bar{R} u \mathrm{d}t \\
&= \int_0^\infty x^\intercal E^\intercal \bar{Q} E x + x^\intercal E^\intercal \bar{Q} F u + u^\intercal F^\intercal \bar{Q} E x + u^\intercal F^\intercal \bar{Q} F u + \rho u^\intercal \bar{R} u \mathrm{d}t \\
&= \int_0^\infty x^\intercal\left(E^\intercal \bar{Q} E\right)x + u^\intercal\left(F^\intercal \bar{Q} F + \rho \bar{R}\right)u + 2x^\intercal\left(E^\intercal \bar{Q} F\right)u \mathrm{d}t \\
&= \int_0^\infty x^\intercal Q x + u^\intercal R u + 2x^\intercal N u \mathrm{d}t,
\end{aligned} \tag{6.47}$$

where $\bar{Q},\bar{R}$ are symmetric and positive definite, $\rho \in \mathbb{R}^+$, $u(t) \in \mathbb{R}^{m\times 1}$, $z(t) \in \mathbb{R}^{k\times 1}$, $x \in \mathbb{R}^{n\times 1}$ and

$$\begin{aligned}
&R = F^\intercal \bar{Q} F + \rho \bar{R}, \quad R \in \mathbb{R}^{m\times m} \\
&Q = E^\intercal \bar{Q} E, \quad Q \in \mathbb{R}^{n\times n} \\
&N = E^\intercal \bar{Q} F.
\end{aligned} \tag{6.48}$$

Example 50. You are given the criterion

$$J(x,u) = \int_0^\infty \left(x_1^2 + 6\cdot x_1\cdot x_2 + 100\cdot x_2^2 + 6\cdot u_1^2 + 10\cdot u_2^2\right)\mathrm{d}t. \tag{6.49}$$

Matrix N is the zero matrix and matrices Q and R are

$$Q = \begin{pmatrix} 1 & 3 \\ 3 & 100 \end{pmatrix} \tag{6.50}$$

and

$$R = \begin{pmatrix} 6 & 0 \\ 0 & 10 \end{pmatrix}. \tag{6.51}$$

6.4.5 Solution

If

- The system (A,B) is stabilizable (all unstable modes are reachable). **Intuition:** this is necessary for state feedback to work (stabilizable means that the unstable modes must be controllable). Controllability is the same as reachability for continuous time systems).

- the pair $(\tilde{A}, \tilde{Q}) = (A - BR^{-1}N^\intercal, Q - NR^{-1}N^\intercal)$ is detectable. **Intuition:** this is necessary to ensure internal stability, i.e. that the closed loop is asymptotically stable. If the system input were known a priori to stabilize the closed loop, then this condition would not be necessary. For now, just remember that internal stability corresponds to input-output stability when no unstable zero/pole cancellations occur.

then

$$u_{\text{LQR}}(t) = -\underbrace{R^{-1}(N + PB)^\intercal}_{K_{\text{LQR}}} x(t), \tag{6.52}$$

where P is the **real, symmetric, positive definite** solution of the algebraic Riccati equation

$$(N + P \cdot B) \cdot R^{-1} \cdot (N^\intercal + B^\intercal \cdot P) - P \cdot A - A^\intercal \cdot P - Q = 0. \tag{6.53}$$

Solving the ARE

Hamiltonian Method

Starting from Equation 6.53, one can rearrange as

$$\begin{aligned}
(N + P \cdot B) \cdot R^{-1} \cdot (N^\intercal + B^\intercal \cdot P) - P \cdot A - A^\intercal \cdot P - Q &= 0 \\
(NR^{-1} + PBR^{-1})(N^\intercal + B^\intercal P) - P \cdot A - A^\intercal \cdot P - Q &= 0 \\
NR^{-1}N^\intercal + NR^{-1}B^\intercal P + PBR^{-1}N^\intercal + PBR^{-1}B^\intercal P - P \cdot A - A^\intercal \cdot P - Q &= 0 \\
-\underbrace{(A - BR^{-\intercal}N^\intercal)^\intercal}_{\tilde{A}^\intercal} P - P\underbrace{(A - BR^{-1}N^\intercal)}_{\tilde{A}} + P\underbrace{(BR^{-1}B^\intercal)}_{\tilde{R}} P - \underbrace{(Q - NR^{-1}N^\intercal)}_{\tilde{Q}} &= 0.
\end{aligned} \tag{6.54}$$

Hence, one gets the Riccati equation

$$\tilde{A}^* P + P\tilde{A}^\intercal - P\tilde{R}P + \tilde{Q} = 0, \tag{6.55}$$

with the unknown quadratic matrix P. Note that this equation can be rewritten as

$$\begin{pmatrix} P & -\mathbb{I} \end{pmatrix} \underbrace{\begin{pmatrix} \tilde{A} & \tilde{R} \\ -\tilde{Q} & -\tilde{A}^* \end{pmatrix}}_{H \in \mathbb{R}^{2n \times 2n}} \begin{pmatrix} \mathbb{I} \\ P \end{pmatrix} = 0, \tag{6.56}$$

where H is the hamiltonian matrix. In order to find the solution of the ARE, we assume two things:

1. H has no eigenvalues on the imaginary axis, i.e. it wil have n eigenvalues in the LHP and n in the RHP. Let the subspace spanned by the eigenvectors associated to the stable eigenvalues (i.e. in the LHP) be

$$X_H = \text{Im} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \tag{6.57}$$

   where $X_1, X_2 \in \mathbb{C}^{n \times n}$.

2. X_1 is invertible.

If the two assumptions are met, one says that *the Hamiltonian belongs to the domain of the Riccati operator*, i.e. $H \in \text{dom}(\text{Ric})$. With these two assumptions, the solution of the ARE can be computed as

$$P = X_2 X_1^{-1}. \tag{6.58}$$

Remark. The ARE has in general more than one solution, but only one is stabilizing, i.e. it makes the closed loop asymptotically stable (see assumption before).

Theorem 12. $H \in \text{dom}(\text{Ric})$ if there exists symmetri matrices $P, H \in \mathbb{R}^{n\times n}$ with stable H such that

$$H \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} H, \tag{6.59}$$

where

a) $P = X_2 X_1^{-1}$ is real and symmetric.

b) P satisfies the ARE.

c) The matrix $\tilde{A} + \tilde{R}P$ is stable (all eigenvalues are in the open LHP).

6.4.6 Direct Method

The direct method consists in introducing a matrix P with the correct dimensions and unknowns, apply the Riccati equation and solve the system of equations.

6.4.7 Examples

Example 51. You have to design a LQ regulator for a plant with 2 inputs, 3 outputs and 6 state variables.

(a) What are the dimensions of A, B, C and D?

(b) What is the dimension of the transfer function $u \to y$?

(c) What is the dimension of the matrix Q of J_{LQR}?

(d) What is the dimension of the matrix R of J_{LQR}?

(e) What is the dimension of the matrix K?

Solution.

(a) One can find the solution by analyzing the meaning of the matrices:

- Since we are given 6 states variables, the matrix A should have 6 rows and 6 columns, i.e. $A \in \mathbb{R}^{6\times 6}$.
- Since we are given 2 inputs, the matrix B should have 2 columns and 6 rows, i.e. $B \in \mathbb{R}^{6\times 2}$.
- Since we are given 3 outputs, the matrix C should have 6 columns and 3 rows, i.e. $C \in \mathbb{R}^{3\times 6}$.
- Since we are given 2 inputs and 3 outputs, the matrix D should have 2 columns and 3 rows, i.e. $D \in \mathbb{R}^{3\times 2}$.

(b) Since we are dealing with a system with 2 inputs and 3 outputs, $P(s) \in \mathbb{R}^{3\times 2}$. Moreover, $P(s)$ should have the same dimensions of D because of its formula.

(c) From the formulation of Q one can easily see that its dimensions are the same of the dimensions of A, i.e. $Q \in \mathbb{R}^{6\times 6}$.

(d) From

$$u(t) = -K \cdot x(t).$$

we can see that K should have 6 columns and 2 rows, i.e. $K \in \mathbb{R}^{2\times 6}$.

Example 52. A system is given as

$$\begin{aligned}\dot{x}_1(t) &= 3 \cdot x_2(t)\\ \dot{x}_2(t) &= 3 \cdot x_1(t) - 2 \cdot x_2(t) + \frac{1}{2} \cdot u(t)\\ y(t) &= 4 \cdot x_1(t) + \frac{7}{3} \cdot x_2(t).\end{aligned} \tag{6.60}$$

a) Solve the LQR problem for the criterion

$$J(x(t), u(t)) = \int_0^\infty 7 \cdot x_1(t)^2 + 3 \cdot x_2(t)^2 + \frac{1}{4} \cdot u(t)^2 \mathrm{d}t \tag{6.61}$$

and find the state feedback controller K using the direct method.

b) Solve a) using the Hamiltonian method.

c) Find the eigenvalues of the closed-loop system with the LQ regulator K.

d) Does the new criterion

$$J_{\text{new}} = \int_0^\infty 70 \cdot x_1^2 + 30 \cdot x_2^2 + \frac{10}{4} \cdot u^2 \mathrm{d}t \tag{6.62}$$

affect the solution for K?

Solution.

a) Using quadratic forms, one can identify $\tilde{Q}$ and $\tilde{R}$ to be (using the null matrix for N)

$$Q = \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix} \tag{6.63}$$

and

$$R = \frac{1}{4}. \tag{6.64}$$

The state-space description of the system can be re-written in standard form as

$$\begin{aligned} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} &= \underbrace{\begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix}}_{A} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}}_{B} u(t) \\ y(t) &= \underbrace{\begin{pmatrix} 4 & \frac{7}{3} \end{pmatrix}}_{C} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \underbrace{0}_{D} u(t). \end{aligned} \tag{6.65}$$

In order to find the controller K, one has to compute the symmetric, positive definite solution of the Riccati equation related to this problem. First, one has to look at the form that this solution should have. Here $B \in \mathbb{R}^{2\times 1}$. This means that since $\Phi = \Phi^\intercal$ we are dealing with $\Phi \in \mathbb{R}^{2\times 2}$ of the form

$$\Phi = \begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_2 & \varphi_3 \end{pmatrix}. \tag{6.66}$$

With the Riccati equation, it holds

$$\begin{aligned} \Phi \cdot (B \cdot R^{-1} \cdot B^\intercal) \cdot \Phi - \Phi \cdot (A - BR^{-1}N^\intercal) - (A - BR^{-1}N^\intercal)^\intercal \Phi - Q = 0 \\ \Phi \cdot B \cdot R^{-1} \cdot B^\intercal \cdot \Phi - \Phi \cdot A - A^\intercal \cdot \Phi - Q = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \Phi \cdot \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \cdot 4 \cdot \begin{pmatrix} 0 & \frac{1}{2} \end{pmatrix} \cdot \Phi - \Phi \cdot \begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix} - \begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix} \cdot \Phi - \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 2\varphi_2 \\ 2\varphi_3 \end{pmatrix} \cdot \begin{pmatrix} \frac{\varphi_2}{2} & \frac{\varphi_3}{2} \end{pmatrix} - \begin{pmatrix} 3\varphi_2 & 3\varphi_1 - 2\varphi_2 \\ 3\varphi_3 & 3\varphi_2 - 2\varphi_3 \end{pmatrix} - \begin{pmatrix} 3\varphi_2 & 3\varphi_3 \\ 3\varphi_1 - 2\varphi_2 & 3\varphi_2 - 2\varphi_3 \end{pmatrix} - \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} \varphi_2^2 & \varphi_2\varphi_3 \\ \varphi_2\varphi_3 & \varphi_3^2 \end{pmatrix} - \begin{pmatrix} 6\varphi_2 + 7 & 3\varphi_1 - 2\varphi_2 + 3\varphi_3 \\ 3\varphi_1 - 2\varphi_2 + 3\varphi_3 & 6\varphi_2 - 4\varphi_3 + 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{6.67}$$

Hence, one gets 3 equations (two elements are equal because of symmetry):

$$\begin{aligned} \varphi_2^2 - 6 \cdot \varphi_2 - 7 = 0 \qquad (I) \\ \varphi_2 \cdot \varphi_3 - 3 \cdot \varphi_1 + 2 \cdot \varphi_2 - 3 \cdot \varphi_3 = 0 \qquad (II) \\ \varphi_3^2 - 6 \cdot \varphi_2 + 4 \cdot \varphi_3 - 3 = 0 \qquad (III). \end{aligned} \tag{6.68}$$

Sylvester's Criterion: An Hermitian (here symmetric) matrix $M \in \mathbb{C}^{m\times m}$ is positive definite if and only if all the *upper-left* $i \times i$ *submatrices (leading minors)*, $i \in 1, \dots, m$ has positive determinant. Applying this to Φ one gets the conditions:

(a) $\varphi_1 > 0$.

(b) $\varphi_1\varphi_3 - \varphi_2^2 > 0$.

From the Equation (I), one gets

$$\begin{aligned}\varphi_2 &= \frac{6 \pm \sqrt{64}}{2} \\ &= \{-1, 7\}.\end{aligned} \tag{6.69}$$

Since we cannot discard a specific value, we pursue with $\varphi_{2,1} = -1$ and $\varphi_{2,2} = 7$:

Case $\varphi_{2,1} = -1$:

Plugging this into the Equation (III), one gets

$$\begin{aligned}\varphi_3^2 + 6 + 4\varphi_3 - 3 &= 0 \\ \varphi_3^2 + 4\varphi_3 + 3 &= 0,\end{aligned} \tag{6.70}$$

and

$$\begin{aligned}\varphi_3 &= \frac{-4 \pm \sqrt{4}}{2} \\ &= \{-3, -1\}.\end{aligned} \tag{6.71}$$

In order for these two values to fulfill the second Sylvester condition, it should hold $\varphi_1 < 0$, which violates the first condition. For this reason this is not a possible choice.

Case $\varphi_{2,1} = 7$:

Plugging this into the Equation (III), one gets

$$\begin{aligned}\varphi_3^2 - 42 + 4\varphi_3 - 3 &= 0 \\ \varphi_3^2 + 4\varphi_3 - 45 &= 0,\end{aligned} \tag{6.72}$$

and

$$\begin{aligned}\varphi_3 &= \frac{-4 \pm \sqrt{196}}{2} \\ &= \{-9, 5\}.\end{aligned} \tag{6.73}$$

$\varphi_3 = 5$ is the only value which does not violate the two Sylverster's conditions. Plugging the values into Equation (II) one gets

$$\begin{aligned}35 - 3\varphi_1 + 14 - 15 &= 0 \\ \varphi_1 &= \frac{34}{3}.\end{aligned} \tag{6.74}$$

The solution of the Riccati equation hence is

$$\Phi = \begin{pmatrix} \frac{34}{3} & 7 \\ 7 & 5 \end{pmatrix}. \tag{6.75}$$

The controller K can be computed as

$$\begin{aligned} K &= R^{-1}B^\intercal\Phi \\ &= 4\cdot\begin{pmatrix}0 & \frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}\frac{34}{3} & 7\\ 7 & 5\end{pmatrix} \\ &= \begin{pmatrix}14 & 10\end{pmatrix}. \end{aligned} \tag{6.76}$$

b) Before applying the Hamiltonian method, one need to check

- (A,B) stabilizable (all unstable modes are reachable). The reachability matrix for this pair

$$\begin{aligned} \mathcal{R} &= \begin{pmatrix}B & AB\end{pmatrix} \\ &= \begin{pmatrix}0 & \frac{3}{2}\\ \frac{1}{2} & -1\end{pmatrix} \end{aligned} \tag{6.77}$$

has full rank, hence the system is reachable.

- (A,Q) detectable. The observability matrix for the pair

$$\begin{aligned} \mathcal{O} &= \begin{pmatrix}Q\\ QA\end{pmatrix} \\ &= \begin{pmatrix}7 & 0\\ 0 & 3\\ 0 & 21\\ 9 & -6\end{pmatrix} \end{aligned} \tag{6.78}$$

has full column rank, hence the system is observable.

In order to use the Hamiltonian method, one needs to build the Hamiltonian matrix

$$H = \begin{pmatrix}\tilde{A} & \tilde{R}\\ -\tilde{Q} & -\tilde{A}^\intercal\end{pmatrix}, \tag{6.79}$$

where

$$\begin{aligned} \tilde{A} &= A - BR^{-1}N^\intercal \\ \tilde{R} &= -BR^{-1}B^\intercal \\ \tilde{Q} &= Q - NR^{-1}N^\intercal. \end{aligned} \tag{6.80}$$

For this specific example ($N=0$), one has

$$\begin{aligned} \tilde{A} &= A. \\ \tilde{R} &= -\begin{pmatrix}0\\ \frac{1}{2}\end{pmatrix}\frac{1}{\frac{1}{4}}\begin{pmatrix}0 & \frac{1}{2}\end{pmatrix} \\ &= \begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}. \\ \tilde{Q} &= Q. \end{aligned} \tag{6.81}$$

Therefore, the Hamiltonian is

$$H = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 3 & -2 & 0 & -1 \\ -7 & 0 & 0 & -3 \\ 0 & -3 & -3 & 2 \end{pmatrix}. \tag{6.82}$$

In order to have the eigenvalues, one computes

$$\begin{aligned}
\det(H - \lambda \mathbb{I}) &= \det \begin{pmatrix} -\lambda & 3 & 0 & 0 \\ 3 & -2-\lambda & 0 & -1 \\ -7 & 0 & -\lambda & -3 \\ 0 & -3 & -3 & 2-\lambda \end{pmatrix} \\
&= -\lambda \det \begin{pmatrix} -2-\lambda & 0 & -1 \\ 0 & -\lambda & -3 \\ -3 & -3 & 2-\lambda \end{pmatrix} - 3\det \begin{pmatrix} 3 & 0 & -1 \\ -7 & -\lambda & -3 \\ 0 & -3 & 2-\lambda \end{pmatrix} \\
&= \lambda \left[(2+\lambda) \det \begin{pmatrix} -\lambda & -3 \\ -3 & 2-\lambda \end{pmatrix} + \det \begin{pmatrix} 0 & -\lambda \\ -3 & -3 \end{pmatrix} \right] \\
&\quad - \left[9 \det \begin{pmatrix} -\lambda & -3 \\ -3 & 2-\lambda \end{pmatrix} - 3\det \begin{pmatrix} -7 & -\lambda \\ 0 & -3 \end{pmatrix} \right] \\
&= (\lambda^2 + 2\lambda - 9) \det \begin{pmatrix} -\lambda & -3 \\ -3 & 2-\lambda \end{pmatrix} - 3\lambda^2 + 63 \\
&= (\lambda^2 - 9 - 2\lambda)(\lambda^2 - 9 + 2\lambda) - 3\lambda^2 + 63 \\
&= \lambda^4 - 25\lambda^2 + 144.
\end{aligned} \tag{6.83}$$

Therefore, the eigenvalues are

$$\lambda_{1,2} = \pm 3, \quad \lambda_{3,4} = \pm 4. \tag{6.84}$$

Since we only care about stable eigenvalues (in LHP), we compute the eigenvectors for $\lambda_2 = -3$ and $\lambda_4 = -4$.
It holds:

- $E_{\lambda_2} = E_{-3}$: from $(H - \lambda_2 \mathbb{I}) \cdot x = 0$ one gets the linear system of equations

$$\left(\begin{array}{cccc|c} 3 & 3 & 0 & 0 & 0 \\ 3 & 1 & 0 & -1 & 0 \\ -7 & 0 & 3 & -3 & 0 \\ 0 & -3 & -3 & 5 & 0 \end{array}\right).$$

  Using the first row as reference and subtracting the correct multiples of it from the other rows, one gets the form

$$\left(\begin{array}{cccc|c} 3 & 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & 7 & 3 & -3 & 0 \\ 0 & -3 & -3 & 5 & 0 \end{array}\right).$$

  Using the second row as reference and subtracting the correct multiples of it from the other rows, one gets the form

$$\left(\begin{array}{cccc|c} 3 & 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & 3 & -\frac{13}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Since one has one zero row, one can introduce a free parameter. Let $x_4 = s$, then $x_3 = \frac{13}{6}s$, $x_2 = -\frac{s}{2}$, $x_1 = \frac{s}{2}$, $s \in \mathbb{R}$. This defines the first eigenspace, which is (multiplying everything by 6)

$$E_{-3} = \left\{ \begin{pmatrix} 3 \\ -3 \\ 13 \\ 6 \end{pmatrix} \right\}. \tag{6.85}$$

- $E_{\lambda_4} = E_{-4}$: from $(H - \lambda_4 \mathbb{I}) \cdot x = 0$ one gets the linear system of equations

$$\left(\begin{array}{cccc|c} 4 & 3 & 0 & 0 & 0 \\ 3 & 2 & 0 & -1 & 0 \\ -7 & 0 & 4 & -3 & 0 \\ 0 & -3 & -3 & 6 & 0 \end{array}\right).$$

Using the first row as reference and subtracting the correct multiples of it from the other rows, one gets the form

$$\left(\begin{array}{cccc|c} 4 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -4 & 0 \\ 0 & 21 & 16 & -12 & 0 \\ 0 & -3 & -3 & 6 & 0 \end{array}\right).$$

Using the second row as reference and subtracting the correct multiples of it from the other rows, one gets the form

$$\left(\begin{array}{cccc|c} 4 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -4 & 0 \\ 0 & 0 & 16 & -96 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Since one has one zero row, one can introduce a free parameter. Let $x_4 = s$, then $x_3 = 6s$, $x_2 = -4s$, $x_1 = 3s$, $s \in \mathbb{R}$. This defines the second eigenspace, which is (multiplying everything by 6)

$$E_{-4} = \left\{ \begin{pmatrix} 3 \\ -4 \\ 6 \\ 1 \end{pmatrix} \right\}. \tag{6.86}$$

Stacking the eigenvectors one gets

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ -4 & -3 \\ 6 & 13 \\ 1 & 6 \end{pmatrix}. \tag{6.87}$$

It holds

$$\begin{aligned}\Phi &= X_2 X_1^{-1}\\ &= \begin{pmatrix} 6 & 13 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ -4 & -3 \end{pmatrix}^{-1}\\ &= \frac{1}{3} \begin{pmatrix} 6 & 13 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} -3 & -3 \\ 4 & 3 \end{pmatrix}\\ &= \begin{pmatrix} \frac{34}{3} & 7 \\ 7 & 5 \end{pmatrix},\end{aligned} \tag{6.88}$$

which confirms the result of a).

c) The closed-loop matrix to analyse is

$$\begin{aligned} A - B \cdot K &= \begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 14 & 10 \end{pmatrix}\\ &= \begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 7 & 5 \end{pmatrix}\\ &= \begin{pmatrix} 0 & 3 \\ -4 & -7 \end{pmatrix}. \end{aligned} \tag{6.89}$$

The eigenvalues of the closed loop system are given by

$$\begin{aligned} \det((A - B \cdot K) - \lambda \cdot \mathbb{I}) &= 0\\ \lambda^2 + 7\lambda + 12 &= 0 \end{aligned} \tag{6.90}$$

from which it follows: $\lambda_1 = -3$ and $\lambda_2 = -4$.

d) No. Since it holds $J_{\text{new}} = 10 \cdot J$, K remains the same.

Example 53. You design with MATLAB a LQ Regulator:

```
1  A = [1 0 0 0; 1 1 0 0; 1 1 1 0;  0 0 1 1];
2  B = [1 1 1 1; 0 1 0 2]';
3  C = [0 0 0 1; 0 0 1 1; 0 1 1 1];
4
5  nx = size(A,1); Number of state variables of the plant, in Script: n
6  nu = size(B,2); Number of input variables of the plant, in Script: m
7  ny = size(C,1); Number of output variables of the plant, in Script: p
8
9  q = 1;
10 r = 1;
11 Q = q*eye(###);
12 R = r*eye(###);
13
14 K = lqr(A,B,Q,R);
```

Fill the following rows:

```
11  : ### =
12  : ### =
```

Solution. The matrix Q is a weight for the states and the matrix R is a weight for the inputs. The correct filling is

```
11  : ### = nx
12  : ### = nu
```

7 State Estimation

In the previous chapters, we assumed that all the state variables of a given systen system were available at each time. In real systems, however, this is not the case: one knows just the output $y(t)$ and the input $u(t)$. Hence, one has to figure out how to get the actual state $x(t)$. The idea is to use an **observer** to get an estimate of $x(t)$, also called $\hat{x}(t)$. A whole course about estimation if offered at IDSC in the master by Prof. D'Andrea: *Recursive Estimation*.

7.1 Preliminary Definitions

Definition 23. A system is said to be *observable* if for any time $T > 0$ it is possible to determine the state of the system $x(T) \in \mathbb{R}^n$ through the measurements of $u(t)$ and $y(t)$, with $t \in [0, T]$.

A pair (A, C) is observable if and only if $\text{rank}(\mathcal{O}) = n = \dim(A)$, where

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} \tag{7.1}$$

is the observability matrix.

Definition 24. A pair (A, C) is detectable if all the unobservable modes are stable.

Definition 25. A linear system is **reachable** if for any $x_0, x_\text{f} \in \mathbb{R}^n$, there exists a $T > 0$ and $u(t) : [0, T] \to \mathbb{R}$ such that the corresponding solution satisfies $x(0) = x_0$ and $x(T) = x_\text{f} \in \mathbb{R}^n$.

Reachability test: A system (A, B) is reachable if and only if $\text{rank}(\mathcal{R}) = n$, where $x \in \mathbb{R}^n$ and

$$\mathcal{R} = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}. \tag{7.2}$$

Remark. Note that for continuous linear time invariant systems, controllability is the same as reachability. In general, reachability implies controllability, but not the converse.

7.2 Problem Definition

Given a linear, time invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{7.3}$$

at each time instant t construct an estimate of the state $\hat{x}(t)$ by only measuring the system's inputs and outputs, such that

$$\lim_{t \to \infty} (x(t) - \hat{x}(t)) = 0. \tag{7.4}$$

7.3 The Luenberger Observer

In order to do this, one creates a numerical copy of the system (an observer)

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t)\\ \hat{y}(t) &= C\hat{x}(t) + Du(t),\end{aligned} \tag{7.5}$$

and one observes the dynamics of the estimation error

$$\hat{e} = x(t) - \hat{x}(t). \tag{7.6}$$

It holds

$$\begin{aligned}\dot{\hat{e}}(t) &= \dot{x}(t) - \dot{\hat{x}}(t)\\ &= Ax(t) + Bu(t) - A\hat{x}(t) - Bu(t)\\ &= A\hat{e}(t).\end{aligned} \tag{7.7}$$

If matrix A has all the eigenvalues in the left half-plane, the error $\hat{e}(t)$ will converge to zero, resulting in a correct state estimation. But is this what we want? Essentially, our error is converging to zero because the states of the two systems are designed to converge to zero. In particular, we are not using the output as an information. How can we solve the problem even for unstable systems? For the following, consider the structure reported in Figure 54. Let's add feedback from the measured output by considering the observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu + L(y(t) - \hat{y}(t)). \tag{7.8}$$

It holds

$$\begin{aligned}\dot{\hat{e}}(t) &= \dot{x}(t) - \dot{\hat{x}}(t)\\ &= Ax(t) + Bu(t) - A\hat{x}(t) - Bu(t) - L(y(t) - \hat{y}(t))\\ &= Ax(t) - A\hat{x}(t) - L(Cx(t) - C\hat{x}(t))\\ &= (A - LC)\hat{e}(t).\end{aligned} \tag{7.9}$$

With this new equation, one can choose a matrix L such that the matrix $A - LC$ has eigenvalues with negative real parts and hence such that the error $\hat{e}(t)$ will converge to 0. This observer is known as the Luenberger observer.

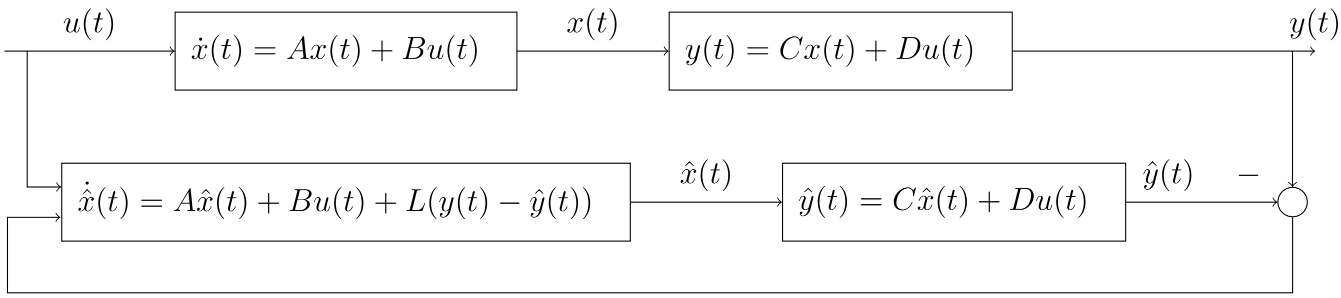

Figure 54: Luenberger Observer

7.3.1 Duality of Estimation and Control

The structure of Equation 7.9 is similar to the one of state feedback we introduce in the previous chapter. Considering the state feedback dynamic equation

$$\dot{x}(t) = (A - BK)x(t), \tag{7.10}$$

one can find in Equation 7.9 some analogies. In particular, it holds

$$\dot{\hat{e}}(t) = (\underbrace{A^\intercal}_{\tilde{A}} - \underbrace{C^\intercal}_{\tilde{B}}\underbrace{L^\intercal}_{\tilde{K}})^\intercal \hat{e}(t). \tag{7.11}$$

Since the problem has the same form, one can use the same methodology to solve it. One recalls that pole placement is allowed if and only if the system is reachable, i.e. if $\text{rank}(\mathcal{R}) = n = \dim(A)$, where $\mathcal{R}$ is the reachability matrix defined in Equation 7.2 By using the analogies, one can define similarly

$$\begin{aligned}\tilde{\mathcal{R}} &= \begin{pmatrix}\tilde{B} & \tilde{A}\tilde{B} & \dots & \tilde{A}^{n-1}\tilde{B}\end{pmatrix}\\ &= \begin{pmatrix}C^\intercal & A^\intercal C^\intercal & \dots & (A^\intercal)^{n-1}C^\intercal\end{pmatrix}\\ &= \begin{pmatrix}C\\ CA\\ \vdots\\ CA^{n-1}\end{pmatrix}^\intercal\\ &= \mathcal{O}^\intercal,\end{aligned} \tag{7.12}$$

where $\mathcal{O}$ is the observability matrix defined in Equation 7.1. Using the well known rule

$$\text{rank}(\mathcal{O}) = \text{rank}(\mathcal{O}^\intercal), \tag{7.13}$$

one can impose $\text{rank}(\mathcal{O})$ to be n in order for observer pole placement to be feasible. Starting from the Ackermann formula for state feedback

$$K = \begin{pmatrix}0 & \dots & 1\end{pmatrix}\mathcal{R}^{-1}p^*_{\text{cl}}(A), \tag{7.14}$$

one can write

$$\begin{aligned}\tilde{K} &= L^\intercal\\ &= \begin{pmatrix}0 & \dots & 0 & 1\end{pmatrix}\tilde{\mathcal{R}}^{-1}p^*_{\text{cl}}(A)\\ \Rightarrow L &= p^*_{\text{cl}}(A)\mathcal{O}^{-1}\begin{pmatrix}0 & \dots & 0 & 1\end{pmatrix}^\intercal.\end{aligned} \tag{7.15}$$

7.3.2 Putting Things Together

Considering Figure 55, one can define the augmented state

$$\tilde{x}(t) = \begin{pmatrix}x(t)\\ \hat{e}(t)\end{pmatrix}. \tag{7.16}$$

The dynamics read

$$\begin{aligned}\dot{\tilde{x}}(t) &= \begin{pmatrix}\dot{x}(t)\\ \dot{\hat{e}}(t)\end{pmatrix}\\ &= \underbrace{\begin{pmatrix}A - BK & BK\\ 0 & A - LC\end{pmatrix}}_{A_{\text{cl}}}\begin{pmatrix}x(t)\\ \hat{e}(t)\end{pmatrix}\end{aligned} \tag{7.17}$$

Since A_{cl} is upper triangular, it holds

$$\sigma(A_{\text{cl}}) = \sigma(A - BK) \cup \sigma(A - LC). \tag{7.18}$$

This is known as the separation principle. Intuitively, this means that control and estimation do not interact with each other, hence can designed independently.

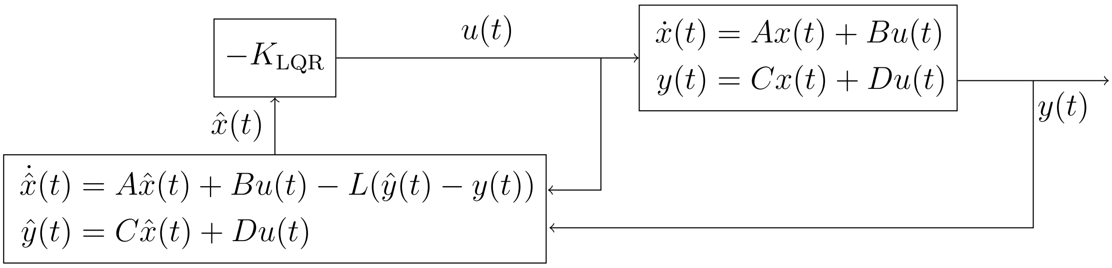

Figure 55: Observer Problem: Closed loop system.

In general one follows this procedure:

1. Design the controller first. Find K to place the poles of $A - BK$ where you desire in the LHP.
2. Design the observer. Find L to place the poles of $A - LC$ in the LHP. As a rule of thumb, make the observer 10 times faster than the controller.

7.4 Linear Quadratic Gaussian (LQG) Control

LQR relies on the assumption that the states are known. How can one integrate the defined estimation procedure in the LQR framework? Is the optimality defined for the LQR method affected by this? In the following, we recall the LQR problem definition and its solution.

7.4.1 LQR Problem Definition

With LQR one wants to find a stabilizing input $u_{\text{LQR}}(t),\ t \in [0, \infty]$ such that

$$u_{\text{LQR}}(t) = \underset{u}{\operatorname{argmin}} \int_0^\infty u(t)^\intercal R u(t) + x(t)^\intercal Q x(t) + 2x(t)^\intercal N u(t) \mathrm{d}t \tag{7.19}$$

satisfying

- the dynamics
$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0 \tag{7.20}$$
and
$$z(t) = Ex(t) + Fu(t), \tag{7.21}$$
with $u(t) \in \mathbb{R}^{m\times 1}$, $z \in \mathbb{R}^{k\times 1}$ and $x \in \mathbb{R}^{n\times 1}$.
- $Q > 0$, $R > 0$, $Q = Q^\intercal$, $R = R^\intercal$, with
$$\begin{aligned} R &= F^\intercal \bar{Q} F + \rho \bar{R}, \quad \rho \in \mathbb{R}^+ \\ Q &= E^\intercal \bar{Q} E \\ N &= E^\intercal \bar{Q} F. \end{aligned} \tag{7.22}$$

7.4.2 LQR Problem Solution

If

1. The system (A, B) is stabilizable and
2. the pair $(\tilde{A}, \tilde{Q}) = (A - BR^{-1}N^\intercal, Q - NR^{-1}N^\intercal)$ is detectable,

then

$$u_{\text{LQR}}(t) = -\underbrace{R^{-1}(N + PB)^\intercal}_{K_{\text{LQR}}} x(t), \tag{7.23}$$

where P is the real, symmetric, positive definite solution of the algebraic Riccati equation (ARE)

$$\underbrace{(A - BR^{-1}N^\intercal)^\intercal}_{\bar{A}^\intercal} P + P \underbrace{(A - BR^{-1}N^\intercal)}_{\bar{A}} + P \underbrace{(-BR^{-1}B^\intercal)}_{\bar{R}} P + \underbrace{(Q - NR^{-1}N^\intercal)}_{\bar{Q}} = 0 \tag{7.24}$$

7.4.3 Simplified Case

It turns out that choosing $N = 0$ results in nice robustness properties. By writing P_∞ instead of P (we are solving the finit horizon), one can simplify the ARE as

$$A^\intercal P_\infty + P_\infty A - P_\infty BR^{-1}B^\intercal P_\infty + Q = 0. \tag{7.25}$$

7.4.4 Stady-state Kalman Filter

The observer problem, i.e. find L in

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t)) \tag{7.26}$$

such that $(A - LC)$ is stable, shows duality with the control problem, i.e.

$$C^\intercal \to B, \quad A^\intercal \to A, \quad L^\intercal \to K. \tag{7.27}$$

Thank to this duality, one can solve the estimation problem by solving the control one. The algebraic Riccati equation for estimation is

$$AP_\infty + P_\infty A^\intercal - P_\infty C^\intercal R^{-1} C P_\infty + Q = 0. \tag{7.28}$$

The matrix L can be found with

$$L^\intercal = R^{-1}CP_\infty \Rightarrow L = P_\infty C^\intercal R^{-1}. \tag{7.29}$$

The duality exists also for the technical conditions for the ARE:

$$\begin{aligned} &(A^\intercal, C^\intercal) \text{ stabilizable } \leftrightarrow (A, C) \text{ detectable .} \\ &(A^\intercal, Q) \text{ detectable } \leftrightarrow (A, Q) \text{ stabilizable .} \end{aligned} \tag{7.30}$$

Deterministic Interpretation

It is worth mentioning that the Kalman filter's theory is not that short. Among others, stochastic interpretation, recursive formulatiom, finite horizon and discrete time implementation represent important topics to be discussed. However, for the aim of this course, we use the deterministic interpretation of Kalman filters. Considering the effect of disturbances/noises on the plant

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + w(t),\ x(0) = x_0 \\ y(t) &= Cx(t) + Du(t) + n(t), \end{aligned} \tag{7.31}$$

where $w(t)$ represents the process noise and $n(t)$ the measurement noise. The Kalman Filter can be interpreted deterministically as minimizing an uncertainty measure

$$\|x_0\|_2^2 + \int_0^T \|w\|_2^2 + \|n\|_2^2 \mathrm{d}t, \tag{7.32}$$

i.e. estimating the last energy/most likely initial condition, disturbance and measurement noise that justify the measurements.

7.4.5 Summary

The linear quadratic gaussian (LQG) regulator is the union of a LQR controller and a Kalman Filter. One can see such a closed loop system in Figure 56. The closed loop is

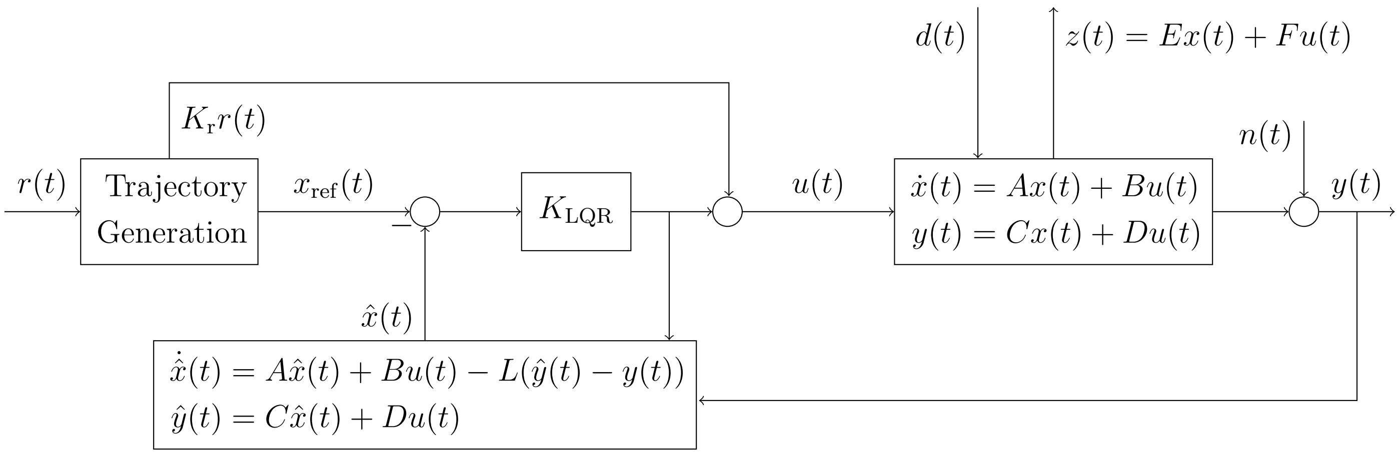

Figure 56: LQG Problem: Closed loop system.

stable if and only if K is a stabilizing state feedback gain and L is a stabilizing estimation gain.

7.5 Examples

Example 54. You are given the following system matrices:

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -4 & 0 \\ 3 & -2 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{7.33}$$

a) ☐ The system is fully observable and fully controllable.
☐ The system is fully observable but not controllable.
☐ The system is not observable but fully controllable.
☐ The system not observable and not controllable.

b) ☐ The eigenvalue corresponding to the non observable state is: 1
☐ The eigenvalue corresponding to the non observable state is: -2
☐ The eigenvalue corresponding to the non observable state is: -4
☐ All the states are observable.

c) ☐ The eigenvalue corresponding to the non reachable state is: 1
☐ The eigenvalue corresponding to the non reachable state is: -2
☐ The eigenvalue corresponding to the non reachable state is: -4
☐ All the states are reachable.

d) ☐ The system is detectable and stabilizable.
☐ The system is detectable but not stabilizable.
☐ The system is not detectable but stabilizable.
☐ The system is not detectable and not stabilizable.

Solution. You are given the following system matrices:

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -4 & 0 \\ 3 & -2 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{7.34}$$

a) ☐ The system is fully observable and fully controllable.
☐ The system is fully observable but not controllable.
☐ The system is not observable but fully controllable.
☑ The system not observable and not controllable.

b) ☐ The eigenvalue corresponding to the non observable state is: 1
☑ The eigenvalue corresponding to the non observable state is: -2
☐ The eigenvalue corresponding to the non observable state is: -4
☐ All the states are observable.

c) ☐ The eigenvalue corresponding to the non reachable state is: 1
☐ The eigenvalue corresponding to the non reachable state is: -2
☑ The eigenvalue corresponding to the non reachable state is: -4
☐ All the states are reachable.

d) ☑ The system is detectable and stabilizable.
☐ The system is detectable but not stabilizable.
☐ The system is not detectable but stabilizable.
☐ The system is not detectable and not stabilizable.

- The observability and controllability of a system can be determined by analyzing the rank of the controllability and observability matrices respectively. After a short calculation you will see that both have only rank 2 which means that neither controllability nor observability of the system is given.
- To determine the eigenvalue corresponding to a possible non observable state, a PBH test for detectability needs to be done with every eigenvalue of the system.

$$\text{rank}\begin{pmatrix} 1 \cdot I - A \\ C \end{pmatrix} = 3 \qquad \text{rank}\begin{pmatrix} -2 \cdot I - A \\ C \end{pmatrix} = 2 \qquad \text{rank}\begin{pmatrix} -4 \cdot I - A \\ C \end{pmatrix} = 3 \tag{7.35}$$

  Thus, the non observable state corresponds to the eigenvalue -2.
- To determine the eigenvalue corresponding to a possible non reachable state, a PBH test for stabilizability needs to be done with every eigenvalue of the system.

$$\text{rank}[1 \cdot I - A \quad B] = 3 \qquad \text{rank}[-2 \cdot I - A \quad B] = 3 \qquad \text{rank}[-4 \cdot I - A \quad B] = 2 \tag{7.36}$$

  Thus, the non reachable state corresponds to the eigenvalue -4.
- As the unstable state with eigenvalue 1 has full rank for both tests, the overall system is detectable and stabilizable.

Example 55. The dynamics of a system are given as

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= u(t)\\ y(t) &= x_1(t).\end{aligned} \tag{7.37}$$

You want to design a state observer. The observer should use the measurements for $y(t)$ and $u(t)$ in order to estimate the state variables $\hat{x}(t) \sim x(t)$.

(a) Which dimension should the observer matrix L have?

(b) Compute the observer matrix L for $R = 1$ and $Q = BB^\intercal$.

(c) You have already computed a state feedback matrix $K = \begin{pmatrix}1 & 1\end{pmatrix}$ for the system above. What is the complete transfer function of the controller $C(s)$?

Solution.

(a) Since $C \in \mathbb{R}^{1\times 2}$ matrix and $L \cdot C$ has the same dimensions of $A \in \mathbb{R}^{2\times 2}$, L is a 2×1 matrix, i.e.

$$L = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}. \tag{7.38}$$

(b) First of all, let's read from 7.37 the system matrices:

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ B &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ C &= \begin{pmatrix} 1 & 0 \end{pmatrix} \\ D &= 0. \end{aligned} \tag{7.39}$$

Plugging these matrices into the algebraic Riccati equation and using the unknown matrix

$$\Psi = \begin{pmatrix} \psi_1 & \psi_2 \\ \psi_2 & \psi_3 \end{pmatrix}, \tag{7.40}$$

one gets:

$$\begin{gathered} \frac{1}{R} \cdot \Psi \cdot C^T \cdot C \cdot \Psi - \Psi \cdot A^T - A \cdot \Psi - B \cdot B^T = 0 \\ \Psi \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \Psi - \Psi \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \Psi - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} \psi_1^2 & \psi_1 \cdot \psi_2 \\ \psi_1 \cdot \psi_2 & \psi_2^2 \end{pmatrix} - \begin{pmatrix} 2\psi_2 & \psi_3 \\ \psi_3 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{gathered} \tag{7.41}$$

The matrix Ψ is symmetric and positive definite and with these informations we can compute its elements:

- From the last term of the equation one gets

$$\psi_2^2 = 1 \Rightarrow \psi_2 = \pm 1. \tag{7.42}$$

- By plugging this into the first equation one gets $\psi_1 = \pm\sqrt{2}$. Because the positive definite condition, one gets $\psi_1 = \sqrt{2}$, $\psi_2 = 1$.
- Because of the form of C we don't care about ψ_3.

From these calculations it follows

$$\begin{aligned} L^T &= \frac{1}{R} \cdot C \cdot \Psi \\ &= \frac{1}{1} \cdot \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & 1 \\ 1 & * \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{2} & 1 \end{pmatrix}, \end{aligned} \tag{7.43}$$

and so

$$L = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}. \tag{7.44}$$

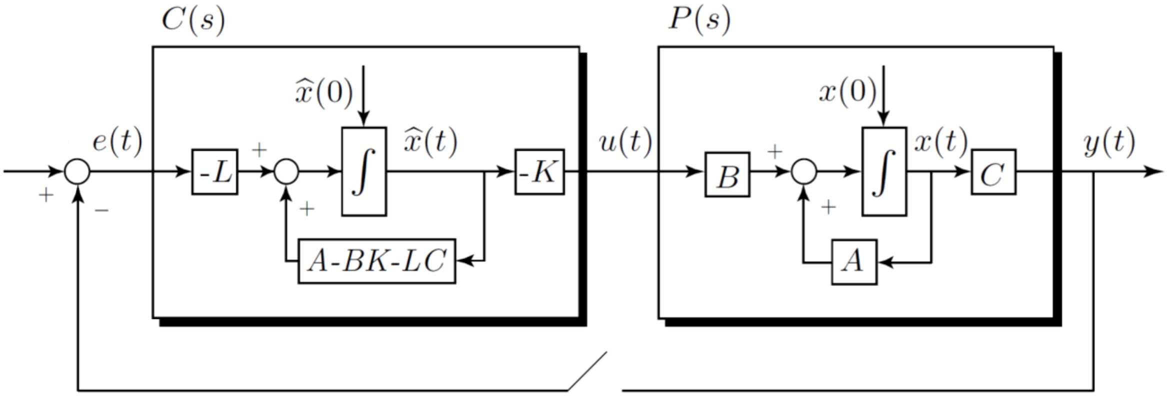

Figure 57: Structure of LQG controller.

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(c) By looking at Figure 57, one can write the transfer function of the feedback controller as

$$\hat{C}(s) = K \cdot (s \cdot \mathbb{I} - (A - B \cdot K - L \cdot C))^{-1} \cdot L. \tag{7.45}$$

By plugging in the found matrices one gets

$$\begin{aligned}(A - B \cdot K - L \cdot C) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \end{pmatrix} - \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} \sqrt{2} & 0 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\sqrt{2} & 1 \\ -2 & -1 \end{pmatrix}.\end{aligned} \tag{7.46}$$

It follows

$$\begin{aligned}(s \cdot \mathbb{I} - (A - B \cdot K - L \cdot C))^{-1} &= \begin{pmatrix} s + \sqrt{2} & -1 \\ 2 & s + 1 \end{pmatrix}^{-1} \\ &= \frac{1}{(s + \sqrt{2}) \cdot (s + 1) + 2} \cdot \begin{pmatrix} s + 1 & 1 \\ -2 & s + \sqrt{2} \end{pmatrix}.\end{aligned} \tag{7.47}$$

By plugging this into the formula one gets

$$\begin{aligned}\hat{C}(s) &= K \cdot (s \cdot \mathbb{I} - (A - B \cdot K - L \cdot C))^{-1} \cdot L \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \cdot \frac{1}{(s + \sqrt{2}) \cdot (s + 1) + 2} \cdot \begin{pmatrix} s + 1 & 1 \\ -2 & s + \sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \\ &= \frac{1}{(s + \sqrt{2}) \cdot (s + 1) + 2} \cdot \begin{pmatrix} s - 1 & s + 1 + \sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \\ &= \frac{1}{(s + \sqrt{2}) \cdot (s + 1) + 2} \cdot (\sqrt{2}s - \sqrt{2} + s + 1 + \sqrt{2}) \\ &= \frac{(\sqrt{2} + 1)s + 1}{(s + \sqrt{2}) \cdot (s + 1) + 2}.\end{aligned} \tag{7.48}$$

Example 56. Design a Luenberger Observer for the following system:

$$A = \begin{pmatrix} -2 & 1 \\ 0 & -4 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{7.49}$$

with the desired observer poles $(-4, -4)$.

a) With the direct method, i.e. by imposing the poles by hand.

b) Using the Ackermann formula.

Solution.

a) The observer gain is defined as $L = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$.
The evolution in the estimation error for a Luenberger observer can be written as:

$$\dot{\hat{e}} = (A - LC)\hat{e} \tag{7.50}$$

By plugging in the matrices one gets:

$$A - LC = \begin{pmatrix} -2 & 1 \\ 0 & -4 \end{pmatrix} - \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} -2 - l_1 & 1 \\ -l_2 & -4 \end{pmatrix} \tag{7.51}$$

The characteristic polynomial of this system can be written as:

$$\det(\lambda \mathbb{I} - A + LC) = \lambda^2 + (6 + l_1)\lambda + 4l_1 + l_2 + 8 \tag{7.52}$$

As the observer poles are required to be $\begin{pmatrix} -4 \\ -4 \end{pmatrix}$, the characteristic polynomial is once more:

$$(\lambda + 4)^2 = \lambda^2 + 8\lambda + 16 \tag{7.53}$$

We can thus deduce that $l_1 = 2$ and $l_2 = 0$ by comparing the coefficients of the two polynomials.
The Luenberger observer can therefore be written as:

$$\begin{aligned} \dot{\hat{x}}(t) &= (A - LC)\hat{x}(t) + Bu(t) + Ly(t) \\ \dot{\hat{x}}(t) &= \begin{pmatrix} -4 & 1 \\ 0 & -4 \end{pmatrix} \hat{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) + \begin{pmatrix} 2 \\ 0 \end{pmatrix} y(t) \end{aligned} \tag{7.54}$$

b) To apply the Ackermann formula, the inverse of the observability matrix of the system needs to be known:

$$\mathcal{O} = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \Rightarrow \mathcal{O}^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \tag{7.55}$$

The desired observer poles are at $\begin{pmatrix} -4 \\ -4 \end{pmatrix}$ the characteristic polynomial is therefor written as:

$$p_{cl}^*(s) = s^2 + 8s + 16 \Rightarrow p_{cl}^*(A) = A^2 + 8A + 16\mathbb{I} \tag{7.56}$$

$$p_{cl}^*(A) = \begin{pmatrix} -2 & 1 \\ 0 & -4 \end{pmatrix}^2 + 8 \begin{pmatrix} -2 & 1 \\ 0 & -4 \end{pmatrix} + 16 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 0 & 0 \end{pmatrix} \tag{7.57}$$

The observer gain can now be derived using the Ackermann formula:

$$L = p_{cl}^*(A)\mathcal{O}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7.58}$$

$$L = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{7.59}$$

Example 57. You are working for your semester thesis at a project which includes a water reservoir. Your task is to determine the disturbance $d(t)$ that acts on the reservoir. Figure 58 shows the situation. The only state of the system is the water volume $x(t) = V(t)$. The volume flows in the reservoir $V_{\text{in}}^*(t)$ are the known system input $u(t)$ and the unknown disturbance $d(t) > 0$. The volume flow of the system is assumed to be only dependend on the water volume, i.e.

$$V_{\text{out}}^*(t) = -\beta \cdot x(t). \tag{7.60}$$

The system output $y(t)$ is the water level $h(t)$. The model of this reservoir reads

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= -\beta \cdot x(t) + u(t) + d(t), \\ y(t) &= \frac{1}{\alpha} \cdot x(t), \ \alpha > 0, \ \beta > 0. \end{aligned} \tag{7.61}$$

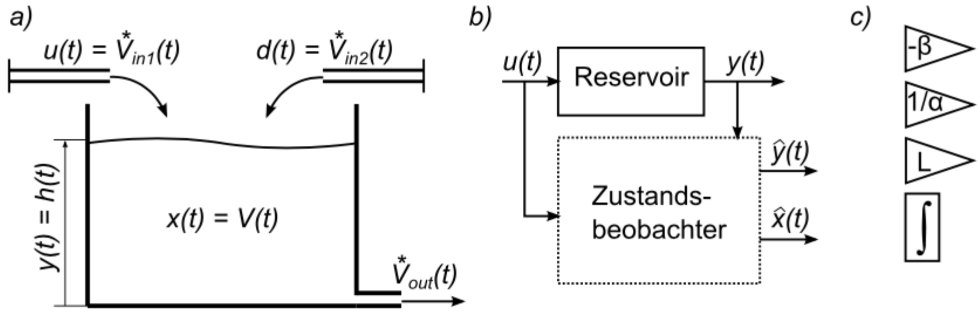


Figure 58: a) Drawing of the reservoir; b) Inputs and Outputs of the observer; c) Blocks for signal flow diagram.

The goal is to determine $d(t)$. Your supervisor has already tried to solve the model equations for $d(t)$: he couldn't determine the change in volume $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$ with enough precision. Hence, you want to solve this problem with a state observer.

(a) Draw the signal flow diagram of such a state observer. Use the blocks of Figure 58c).

(b) The state feedback matrix L is in this case some scalar value. Which value can L be, in order to get an asymptotically stable state observer?

(c) Introduce a new signal $\hat{d}(t)$ in the state observer. This should approximate the real disturbance $d(t)$.

(d) Find the state space description of the observer with inputs $u(t)$ and $y(t)$ and output $\hat{d}(t)$.

Solution.

(a) The signal flow diagram can be seen in Figure 59.

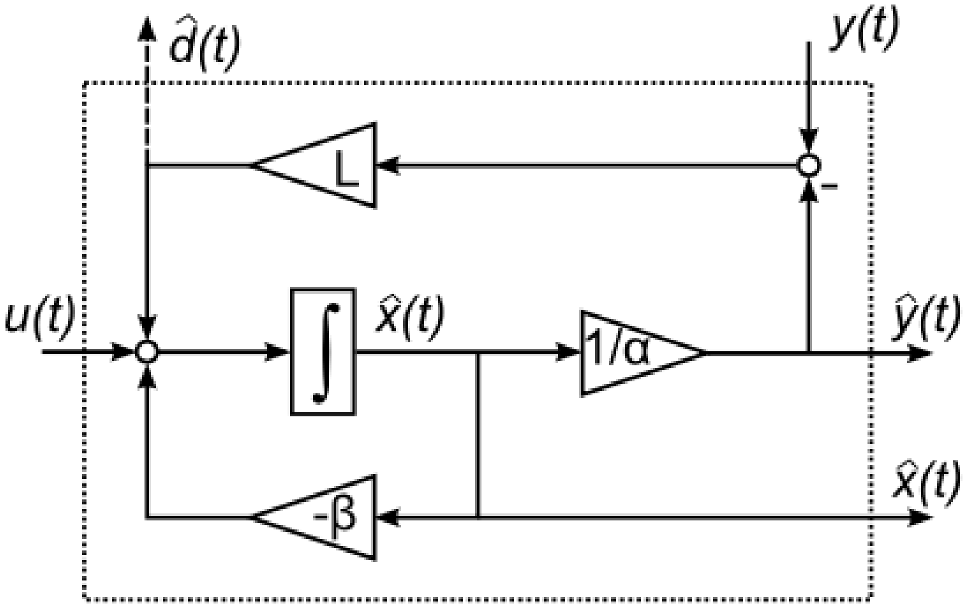


Figure 59: Signal flow diagram of the state observer.

.

(b) The stability of the ovserver depends on the eigenvalues of $A - L \cdot C$. In this case, since $A - L \cdot C$ is a scalar,

$$A - L \cdot C < 0 \tag{7.62}$$

should hold. This leads to

$$L > \frac{A}{C}. \tag{7.63}$$

With the given informations it follows

$$L > -\frac{\beta}{\alpha}. \tag{7.64}$$

(c) The dashed line in Figure 59 represents the new output $\hat{d}(t)$. The integrator in Figure 59 has now 3 inputs. The arrow from downwards from the reservoir is $V^*_{\text{out}}(t)$, the arrow from left is the input flow $u(t) = V^*_{\text{in1}}(t)$. If we simulate the system without the dashed arrow, there is a deviation between the measured $y(t)$ and the simulated $\hat{y}(t)$. This results from the extra inflow $d(t) = V^*_{\text{in}}(t)$, which is not considered in the simulation.

(d) The new state-space description reads

$$\frac{\mathrm{d}\hat{x}(t)}{\mathrm{d}t} = \left(-\beta - \frac{L}{\alpha}\right) \cdot \hat{x}(t) + (1) \cdot u(t) + (L) \cdot y(t) \tag{7.65}$$

$$\hat{d}(t) = \left(-\frac{L}{\alpha}\right) \cdot \hat{x}(t) + (0) \cdot u(t) + (L) \cdot y(t). \tag{7.66}$$

8 $\mathcal{H}_\infty$ Control

The big disadvantage of LQR/LQG is that one cannot directly impose frequency domain specifications to the control loop. A solution to this problem is given by the $\mathcal{H}_\infty$ control formulation.

8.1 Problem Formulation

In $\mathcal{H}_\infty$ control we consider the closed-loop system representation reported in Figure 60. By referring to Figure 60, one can define

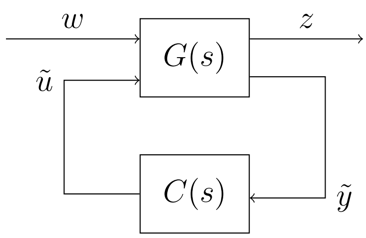


Figure 60: General system for $\mathcal{H}_\infty$ control.

- $G(s)$ is called the *extended system* and is real, rational and proper.
- $C(s)$ is the controller and is real, rational and proper.
- $w(t) \in \mathbb{C}^{m_1 \times 1}$ is called *exogenous input*, and contains at least the reference signal $r(t)$ and possibly other exogenous signals, such as a noise model $n(t)$.
- $z(t) \in \mathbb{C}^{p_1 \times 1}$ is called the *performance output* and is a virtual output signal only used for design.
- $\tilde{u}(t) \in \mathbb{C}^{m_2 \times 1}$ is the control input, computed by the controller $C(s)$.
- $\tilde{y}(t) \in \mathbb{C}^{p_2 \times 1}$ is the measured output, available to the controller $C(s)$.

Remark. As a side node, *exogenous* means: caused or produced by factors external to a model.

With $\mathcal{H}_\infty$ control we are interested in finding the controller $C(s)$ that stabilizes internally and externally the closed loop system and minimizes

$$
\begin{aligned}
\|T_{zw}(s)\|_\infty &= \sup_{w\neq 0} \frac{\|z\|_2}{\|w\|_2} \\
&= \sup_{w\neq 0} \bar{\sigma}\left(T_{zw}(j\omega)\right) \\
&:= \gamma_{\min},
\end{aligned}
\tag{8.1}
$$

where $T_{zw}(s)$ is the transfer function which relates signals $z(t)$ and $w(t)$. Intuitively, this is equivalent to

- Minimize the energy ($\|\cdot\|_2$ norm) gain of the closed-loop system.

- Have a chance to incorporate constraints on regulated variables in frequency space, e.g. the tracking error $E(s)$.

One can hence state the aim of $\mathcal{H}_\infty$ control to be:

Optimal $\mathcal{H}_\infty$ Control: Find all admissible controllers $C(s)$ such that $\|T_{zw}(j\omega)\|_\infty$ is minimized.

Differently to what we observed in $\mathcal{H}_2$ control, the optimal $\mathcal{H}_\infty$ controllers are not *unique*. Moreover, the process of finding an optimal controller is complicated, numerically and theoretically. This said, in practice is often not necessary to design an optimal controller: often it is sufficient to find controllers which are close to optimality, but *easier* to compute, i.e. *suboptimal controllers*.

Suboptimal $\mathcal{H}_\infty$ Control: Given $\gamma > 0$, find all admissible controllers $C(s)$ such that $\|T_{zw}(j\omega)\|_\infty < \gamma$.

8.2 Mixed Sensitivity Approach

8.2.1 Transfer Functions Recap

By considering the standard MIMO control system structure with 0 disturbance depicted in Figure 61 and defining the signals of interest to be $E(s)$, $U(s)$ and $Y(s)$, one can derive them as

$$\begin{aligned} E(s) &= R(s) - Y(s) \\ &= R(s) - N(s) - P(s)C(s)E(s) \\ \Rightarrow E(s) &= \left(\mathbb{I} + P(s)C(s)\right)^{-1}\left(R(s) - N(s)\right) \\ &= S(s)(R(s) - N(s)) \end{aligned} \tag{8.2}$$

$$\begin{aligned} U(s) &= C(s)E(s) \\ &= C(s)S(s)(R(s) - N(s)), \end{aligned} \tag{8.3}$$

$$\begin{aligned} Y(s) &= N(s) + P(s)C(s)E(s) \\ &= N(s) + P(s)C(s)S(s)(R(s) - N(s)) \\ &= N(s) + T(s)(R(s) - N(s)) \\ &= (\mathbb{I} - T(s))N(s) + T(s)R(s) \\ &= S(s)N(s) + T(s)R(s), \end{aligned} \tag{8.4}$$

which can be rewritten in matrix form as

$$\begin{pmatrix} E(s) \\ U(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} S(s) & -S(s) \\ C(s)S(s) & -C(s)S(s) \\ T(s) & S(s) \end{pmatrix} \begin{pmatrix} R(s) \\ N(s) \end{pmatrix}. \tag{8.5}$$

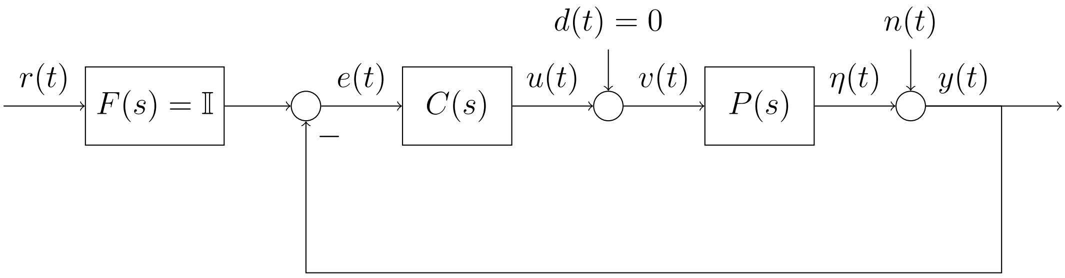


Figure 61: Standard feedback control system structure.

8.2.2 How to ensure Robustness?

As previously mentioned, $\mathcal{H}_\infty$ control approach allows to introduce specifications in the frequency domain. In particular, we always focused ourselves in the analysis of the useful system transfer functions $S(s)$ (sensitivity function) and $T(s)$ (complementary sensitivity function). As a general reminder, recall that a small sensitivity $S(s)$ corresponds to disturbance rejection and a small complementary sensitivity function $T(s)$ corresponds to noise attenuation and robustness on modeling errors. We now recall the bound we defined in previous classes

$$\|W_1(j\omega)S(j\omega)\| + \|W_2(j\omega)T(j\omega)\| < 1, \tag{8.6}$$

where $W_1(s)$ and $W_2(s)$ are the robust weighting functions for the sensitivity and the complementary sensitivity, respectively. In order to ensure robustness, we would like to minimize the left term in Equation 8.6. Since there exist no controller which is able to solve this problem directly, one can write the two conditions separately, i.e.

$$\begin{aligned} &\|W_1(s)S(s)\|_\infty < 1 \text{ nominal performance,} \\ &\|W_2(s)T(s)\|_\infty < 1 \text{ robust stability.} \end{aligned} \tag{8.7}$$

8.2.3 How to use this in $\mathcal{H}_\infty$ Control?

Once the weighting functions are designed, one needs to augment the original plant in order to let the approach meet the $\mathcal{H}_\infty$ problem definition. In particular, the general form of such an augmentation with $\tilde{u}(t) = u(t)$, $\tilde{y}(t) = e(t)$ and $w(t) = r(t)$ can be seen in Figure 62. One can note that the signals resulting from the weighting are three and are $z_e(t), z_u(t), z_y(t)$. The error $e(t)$ is fed through the weighting function $W_e(s)$ ($W_1(s)$ in our previous considerations). Since the transfer function from the reference signal $r(t)$ to the error $e(t)$ is known to be the sensitivity function $S(s)$ (refer to Equation 8.5), one can write

$$Z_e(s) = W_e(s)S(s)R(s). \tag{8.8}$$

The measured output $y(t)$ is fed through the weighting function $W_y(s)$ ($W_2(s)$ in our previous considerations). Since the transfer function from the reference $r(t)$ to the output $y(t)$ is known to be the complementary sensitivity function $T(s)$ (refer to Equation 8.5), one can write

$$Z_y(s) = W_y(s)T(s)R(s). \tag{8.9}$$

The clever reader will notice that a third weighting function is present. The input $u(t)$ is fed through the weighting function $W_u(s)$. Since the transfer function from the reference

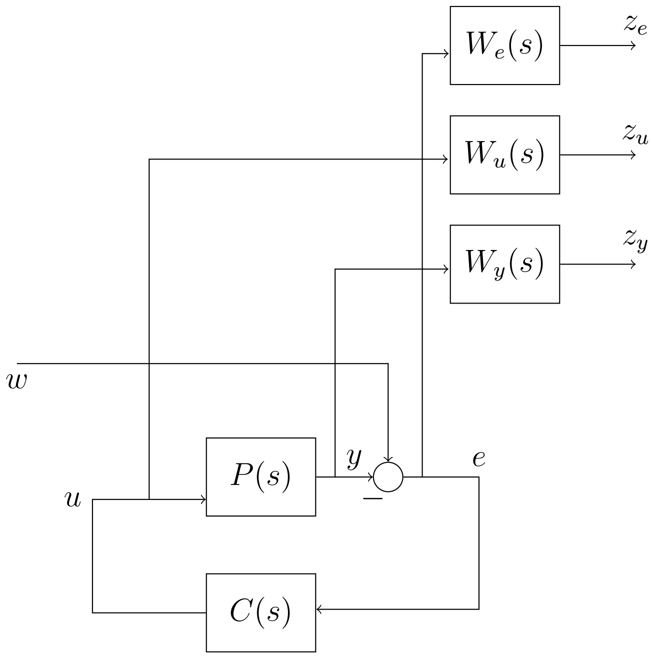


Figure 62: General extended system structure.

$r(t)$ to the input $u(t)$ is known to be $C(s)S(s)$ (refer to Equation 8.5), one can write

$$Z_u(s) = W_u(s)C(s)S(s)R(s). \tag{8.10}$$

Summarizing, one can write

$$\underbrace{\begin{pmatrix} Z_e(s) \\ Z_u(s) \\ Z_y(s) \end{pmatrix}}_{Z(s)} = \underbrace{\begin{pmatrix} W_e(s)S(s) \\ W_u(s)C(s)S(s) \\ W_y(s)T(s) \end{pmatrix}}_{T_{zw}(s)} \underbrace{R(s)}_{W(s)}. \tag{8.11}$$

8.3 Finding $T_{zw}(s)$

8.3.1 General Form

By looking at Figure 60, one can write the system into standard form, which reads

$$\begin{aligned} \begin{pmatrix} Z(s) \\ \tilde{Y}(s) \end{pmatrix} &= G(s) \begin{pmatrix} W(s) \\ \tilde{U}(s) \end{pmatrix} \\ &= \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix} \begin{pmatrix} W(s) \\ \tilde{U}(s) \end{pmatrix}. \end{aligned} \tag{8.12}$$

Furthermore, it holds

$$\begin{aligned} \tilde{U}(s) &= C(s)\tilde{Y}(s) \\ &= C(s)\left(G_{21}(s)W(s) + G_{22}(s)\tilde{U}(s)\right), \end{aligned} \tag{8.13}$$

from which it follows

$$\tilde{U}(s) = \left(\mathbb{I} - C(s)G_{22}(s)\right)^{-1} C(s)G_{21}(s)W(s). \tag{8.14}$$

Combining Equation 8.12 and Equation 8.14 results in

$$\begin{aligned} Z(s) &= G_{11}(s)W(s) + G_{12}(s)\tilde{U}(s) \\ &= G_{11}(s)W(s) + G_{12}(s)\left(\mathbb{I} - C(s)G_{22}(s)\right)^{-1} C(s)G_{21}(s)W(s) \\ &= \underbrace{\left(G_{11}(s) + G_{12}(s)\left(\mathbb{I} - C(s)G_{22}(s)\right)^{-1} C(s)G_{21}(s)\right)}_{T_{zw}(s)} W(s). \end{aligned} \tag{8.15}$$

The infinity norm of $T_{zw}(s)$ is per definition

$$\|T_{zw}(j\omega)\|_\infty = \max_\omega \left(\max_i \sigma_i\left(T_{zw}(j\omega)\right)\right), \tag{8.16}$$

which corresponds to the maximum magnitude of its frequency response. Minimizing the infinity means in practice, minimizing this maximum singular value, i.e. minimizing the worst-case amplification from $w(t)$ to $z(t)$ at any frequency.

8.3.2 Applying Mixed Sensitivity Approach

If the plant is augmented using the mixed sensitivity approach, it holds

$$T_{zw}(s) = \begin{pmatrix} W_e(s)S(s) \\ W_u(s)C(s)S(s) \\ W_y(s)T(s) \end{pmatrix}. \tag{8.17}$$

Let's define $\hat{S}(s)$, $\hat{T}(s)$ and $\hat{R}(s)$ to be acceptable upper bounds for the sensitivity $S(s)$, the complementary sensitivity $T(s)$ and the transfer function $r \to u$, $C(s)S(s)$. By setting

$$\begin{aligned} W_e(s) &= \hat{S}(s)^{-1} \\ W_u(s) &= \hat{R}(s)^{-1} \\ W_y(s) &= \hat{T}(s)^{-1}, \end{aligned} \tag{8.18}$$

one can write the control problem as:

Suboptimal $\mathcal{H}_\infty$ Control: Find $C(s)$ such that for sufficiently small $\gamma \in \mathbb{R}_+$ it holds.

$$\| \begin{pmatrix} W_e(s)S(s) \\ W_u(s)C(s)S(s) \\ W_y(s)T(s) \end{pmatrix} \|_\infty \leq \gamma. \tag{8.19}$$

Remark. Note that

$$\bar{\sigma}\left(T_{zw}(s)\right) = \bar{\sigma} \to \underline{\sigma}\left(T_{zw}(s)^{-1}\right) = \frac{1}{\bar{\sigma}}. \tag{8.20}$$

One can then in general define the generalized optimization problem related to this control problem to be

$$\min_{\|T_{zw}(s)\|_\infty \leq \gamma} \gamma. \tag{8.21}$$

One can hase essentially two cases:

- If the solution of the optimization problem results into $\gamma^* \leq 1$, then the imposed specifications are filfilled and satisfied by $C(s)$.
- If the solution of the optimization problem results into $\gamma^* > 1$, the conditions are not satisfied and one should use relaxed weights.

What is done in practice is an iterative procedure, which solves the optimization problem by restricting the weighting functions.

8.4 Implementation

By fixing a γ^*, one can solve the optimization problem. The augmented plant $G(s)$ has to be represented in state space form, i.e. we need to express the weighting functions' dynamics. In general, one can always write

$$\begin{aligned} P(s) &= C(s\mathbb{I} - A)^{-1}B + D, \\ W_e(s) &= C_e(s\mathbb{I} - A_e)^{-1}B_e + D_e, \\ W_u(s) &= C_u(s\mathbb{I} - A_u)^{-1}B_u + D_u, \\ W_y(s) &= C_y(s\mathbb{I} - A_y)^{-1}B_y + D_y. \end{aligned} \tag{8.22}$$

8.4.1 State Space Representation

The extended system can be re-written as a dynamic system with $A_{\text{ext}}, B_{\text{ext}}, C_{\text{ext}}, D_{\text{ext}}$ as

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{x}_e(t) \\ \dot{x}_u(t) \\ \dot{x}_y(t) \end{pmatrix} &= A_{\text{ext}} \begin{pmatrix} x(t) \\ x_e(t) \\ x_u(t) \\ x_y(t) \end{pmatrix} + B_{\text{ext}} \begin{pmatrix} w(t) \\ u(t) \end{pmatrix} \\ \begin{pmatrix} z_e(t) \\ z_u(t) \\ z_y(t) \\ e(t) \end{pmatrix} &= C_{\text{ext}} \begin{pmatrix} x(t) \\ x_e(t) \\ x_u(t) \\ x_y(t) \end{pmatrix} + D_{\text{ext}} \begin{pmatrix} w(t) \\ u(t) \end{pmatrix}. \end{aligned} \tag{8.23}$$

By using Figure 62, one can write

$$\begin{aligned} \dot{x}_e(t) &= A_e x_e(t) + B_e u_e(t) \\ &= A_e x_e(t) + B_e(r(t) - y(t)) \\ &= A_e x_e(t) + B_e(w(t) - Cx(t) - Du(t)) \\ &= A_e x_e(t) + B_e w(t) - B_e C x(t) - B_e D u(t), \\ z_e(t) &= y_e(t) \\ &= C_e x_e(t) + D_e u_e(t) \\ &= C_e x_e(t) + D_e(w(t) - Cx(t) - Du(t)) \\ &= C_e x_e(t) - D_e C x(t) + D_e w(t) - D_e D u(t), \end{aligned} \tag{8.24}$$

and

$$\begin{aligned} \dot{x}_y(t) &= A_y x_y(t) + B_y u_y(t) \\ &= A_y x_y(t) + B_y y(t) \\ &= A_y x_y(t) + B_y(Cx(t) + Du(t)) \\ &= A_y x_y(t) + B_y C x(t) + B_y D u(t), \\ z_y(t) &= y_y(t) \\ &= C_y x_y(t) + D_y u_y(t) \\ &= C_y x_y(t) + D_y(Cx(t) + Du(t)) \\ &= C_y x_y(t) + D_y C x(t) + D_y D u(t), \end{aligned} \tag{8.25}$$

and

$$\begin{aligned}
\dot{x}_u(t) &= A_u x_u(t) + B_u u_u(t) \\
&= A_u x_u(t) + B_u u(t), \\
z_u(t) &= y_u(t) \\
&= C_u x_u(t) + D_u u_u(t) \\
&= C_u x_u(t) + D_u u(t).
\end{aligned} \tag{8.26}$$

Combining these results with Equation 8.23, one gets

$$A_{\text{ext}} = \begin{pmatrix} A & 0 & 0 & 0 \\ -B_e C & A_e & 0 & 0 \\ 0 & 0 & A_u & 0 \\ B_y C & 0 & 0 & A_y \end{pmatrix}, \quad B_{\text{ext}} = \left(\begin{array}{c|c} 0 & B \\ B_e & -B_e D \\ 0 & B_u \\ 0 & B_y D \end{array}\right) = \begin{pmatrix} B_{\text{ext},w} & B_{\text{ext},u} \end{pmatrix},$$
$$C_{\text{ext}} = \left(\begin{array}{cccc} -D_e C & C_e & 0 & 0 \\ 0 & 0 & C_u & 0 \\ D_y C & 0 & 0 & C_y \\ \hline -C & 0 & 0 & 0 \end{array}\right) = \begin{pmatrix} C_{\text{ext},z} \\ C_{\text{ext},y} \end{pmatrix}, \quad D_{\text{ext}} = \left(\begin{array}{c|c} D_e & -D_e D \\ 0 & D_u \\ 0 & D_y D \\ \hline \mathbb{I} & -D \end{array}\right) = \begin{pmatrix} D_{\text{ext},zw} & D_{\text{ext},zu} \\ D_{\text{ext},yw} & D_{\text{ext},yu} \end{pmatrix}. \tag{8.27}$$

Compactly, one can write

$$\left(\begin{array}{c|cc} A_{\text{ext}} & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right) = \left(\begin{array}{c|cc} A_{\text{ext}} & B_{\text{ext},w} & B_{\text{ext},u} \\ \hline C_{\text{ext},z} & D_{\text{ext},zw} & D_{\text{ext},zu} \\ C_{\text{ext},y} & D_{\text{ext},yw} & D_{\text{ext},yu} \end{array}\right) \tag{8.28}$$

8.4.2 $\mathcal{H}_\infty$ Solution

Once that one has the extended plant $G(s)$ and the state space description of the system, one can solve the optimization problem.

Simplified Case

Assuming

- $C_{\text{ext},z}^\intercal D_{\text{ext},zu} = 0$,
- $B_{\text{ext},w} D_{\text{ext},yw}^\intercal = 0$, i.e. process noise and sensor noise are uncorrelated,
- $D_{\text{ext},zu}^\intercal D_{\text{ext},zu} = \mathbb{I}$,
- $D_{\text{ext},yw}^\intercal D_{\text{ext},yw} = \mathbb{I}$,

find a controller $C(s)$ such that $\|T_{zw}\|_\infty < \gamma$ for $\gamma > 0$. It turns out that by simplifying the problem, the solution has similarities with the one of LQG (state feedback). The procedure to solve this problem is:

Kochrezept $\mathcal{H}_\infty$ Control

A controller $C(s)$ which satisfies the objective exists if and only if the conditions contained in the different steps are fulfilled.

I) Fix a large value for γ.

II) Find the quatratic, real matrix $X_\infty \geq 0$ which solves the algebraic Riccati equation

$$A_{\text{ext}}^\mathsf{T} X_\infty + X_\infty A + C_{\text{ext},z}^\mathsf{T} C_{\text{ext},z} + X_\infty \left(\frac{1}{\gamma^2} B_{\text{ext},w} B_{\text{ext},w}^\mathsf{T} - B_{\text{ext},u} B_{\text{ext},u}^\mathsf{T} \right) X_\infty \tag{8.29}$$

and such that it is stabilizing, i.e.

$$\text{Re}\left(\lambda_i \left(A_{\text{ext}} + \left(\frac{1}{\gamma^2} B_{\text{ext},w} B_{\text{ext},w}^\mathsf{T} - B_{\text{ext},u} B_{\text{ext},u}^\mathsf{T} \right) X_\infty \right) \right) < 0 \quad \forall i, \tag{8.30}$$

where $\lambda_i(\cdot)$ denotes the $i-$th eigenvalue.

III) Find the quadratic, real matrix $Y_\infty \geq 0$ which solves the algebraic Riccati equation

$$A_{\text{ext}} Y_\infty + Y_\infty A_{\text{ext}}^\mathsf{T} + B_{\text{ext},w} B_{\text{ext},w}^\mathsf{T} + Y_\infty \left(\frac{1}{\gamma^2} C_{\text{ext},z}^\mathsf{T} C_{\text{ext},z} - C_{\text{ext},y}^\mathsf{T} C_{\text{ext},y} \right) Y_\infty = 0, \tag{8.31}$$

and it is stabilizing, i.e.

$$\text{Re}\left(\lambda_i (A_{\text{ext}} + Y_\infty \left(\frac{1}{\gamma^2} C_{\text{ext},z}^\mathsf{T} C_{\text{ext},z} - C_{\text{ext},y}^\mathsf{T} C_{\text{ext},y} \right) \right) \right) < 0 \quad \forall i, \tag{8.32}$$

where $\lambda_i(\cdot)$ denotes the $i-$th eigenvalue.

IV) It must holds

$$\gamma \mathbb{I} - Y_\infty X_\infty > 0, \tag{8.33}$$

or, equivalently,

$$\max_i |\lambda_i(X_\infty Y_\infty)| = \rho(X_\infty Y_\infty) \leq \gamma^2, \tag{8.34}$$

where ρ denotes the spectral radius.

V) Reduce γ until no solution is found.

VI) If the resulting minimal $\gamma^* > 1$, the feasibility conditions we intoduced in the previous chapter are no more valid. In order to make the problem feasible, one need to relax the weights $W_e(s), W_u(s), W_y(s)$. If the resulting minimal $\gamma^* \leq 1$, the result is acceptable. One can use the matrices X_∞ and Y_∞ to calclulate the $\mathcal{H}_\infty$ control dynamics. Considering the extended state

$$\hat{x}(t) = \begin{pmatrix} x(t) \\ x_e(t) \\ x_u(t) \\ x_y(t) \end{pmatrix}, \tag{8.35}$$

one can write the controller dynamics to be

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) &= A_\infty\hat{x}(t) - ZLy(t)\\ u(t) &= F\hat{x}(t),\end{aligned} \tag{8.36}$$

where

$$\begin{aligned}A_\infty &= A_{\text{ext}} + \left(\frac{1}{\gamma^2}B_{\text{ext},w}B_{\text{ext},w}^\mathsf{T}\right)X_\infty + B_{\text{ext},u}F + ZLC_{\text{ext},y}\\ F &= -B_{\text{ext},u}^\mathsf{T}X_\infty\\ L &= -Y_\infty C_{\text{ext},y}^\mathsf{T}\\ Z &= \left(\mathbb{I} - \frac{1}{\gamma^2}Y_\infty X_\infty\right)^{-1}.\end{aligned} \tag{8.37}$$

Remark. In order to solve this kind of problems, a popular strategy is bisection. Let γ^* be the optimal solution. By maintaining lower and upper bounds $\gamma_- < \gamma^* < \gamma_+$ one uses the following procedure:

1. Initialize $\gamma_- = 0$ and $\gamma_+ = \alpha$, where α is the $\mathcal{H}_\infty$ norm of the $\mathcal{H}_2$ optimal design (LQG). Let K_+ be the optimal LQG controller.
2. Let
$$\gamma \leftarrow \frac{\gamma_- + \gamma_+}{2}. \tag{8.38}$$
Check if a controller exists such that $\|T_{zw}\|_\infty < \gamma$. If yes, set $\gamma_+ = \gamma$ and K_+ to the controller just designed. If not, set $\gamma_- \leftarrow \gamma$.
3. Repear from step 2. until
$$\gamma_+ - \gamma_- < \varepsilon, \tag{8.39}$$
where ε is a user-defined threshold.
4. Return K_+.

8.4.3 Feasibility Conditions

Conditions for the feasibility of the problem are

(a) The controllability of the extended plant $G(s)$ must be verified: if there are non controllable states, one needs to make sure that these states remain bounded. The pair (A_{ext}, B_2) must be **stabilizable**.

(b) The extended plant $G(s)$ must be fully observable: if there are not observable states, one needs to make sure that these are stable. The pair $(C_{\text{ext},y}, A_{\text{ext}})$ must be detectable.

(c) The four matrices
$$D_{\text{ext,zu}}, \quad D_{\text{ext,yw}}, \quad \begin{pmatrix} A_{\text{ext}} - \gamma\omega\mathbb{I} & B_{\text{ext},u}\\ C_{\text{ext},z} & D_{\text{ext,zu}}\end{pmatrix}, \quad \begin{pmatrix} A_{\text{ext}} - \gamma\omega\mathbb{I} & B_{\text{ext},w}\\ C_{\text{ext},y} & D_{\text{ext,yw}}\end{pmatrix} \tag{8.40}$$
must have full rank $\forall\omega$.

(d) $\bar{\sigma}(D_{\text{ext,zw}}) < \gamma$.

Example 58. Given an extended system of a SISO plant with performance output

$$\begin{pmatrix} Z_e(s) \\ Z_u(s) \\ Z_y(s) \end{pmatrix} = \underbrace{\begin{pmatrix} W_e(s)S(s) \\ W_u(s)C(s)S(s) \\ W_y(s)T(s) \end{pmatrix}}_{=T_{zw}} R(s) \tag{8.41}$$

where

- $W_e(s)$, $W_u(s)$ and $W_y(s)$ are weights for the corresponding sensitivities,
- $S(s)$ is the sensitivity and $T(s)$ is the complementary sensitivity,
- $R(s)$ is the reference signal to the feedback loop.

The state space representation of the extended system is given as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 \tilde{u}(t) \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} \tilde{u}(t) \\ \tilde{y}(t) &= C_2 x(t) + D_{21} w(t) + D_{22} \tilde{u}(t) \end{aligned} \tag{8.42}$$

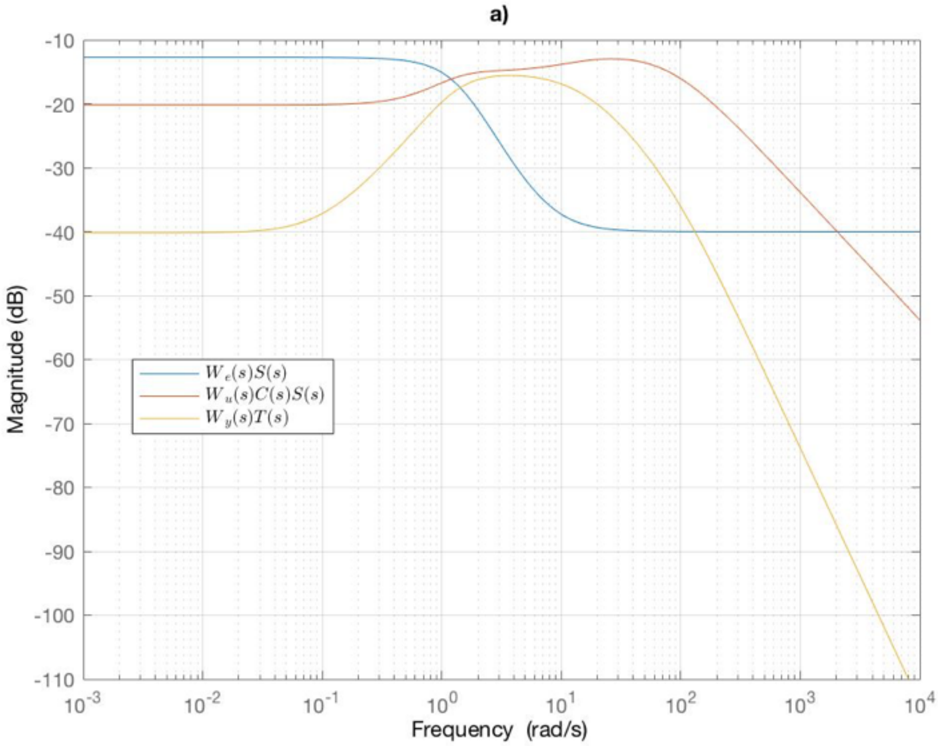
Note: The system matrix A and state vector $x(t)$ correspond to the extended system state and not only to the plant's one.

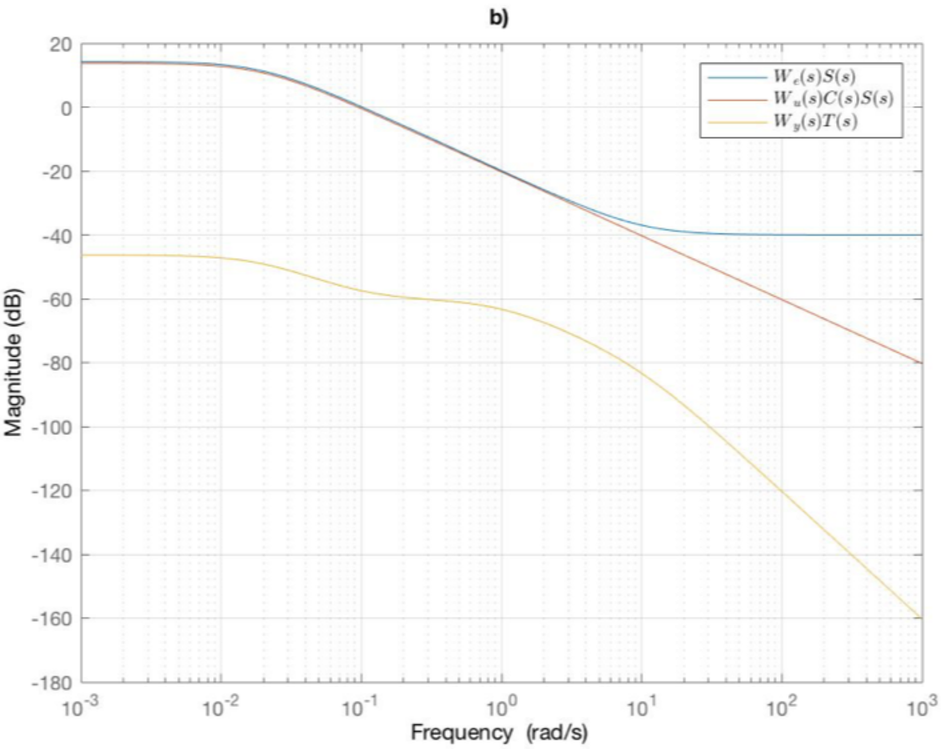
Assume that an $\mathcal{H}_\infty$-controller $C(s)$ was found so that

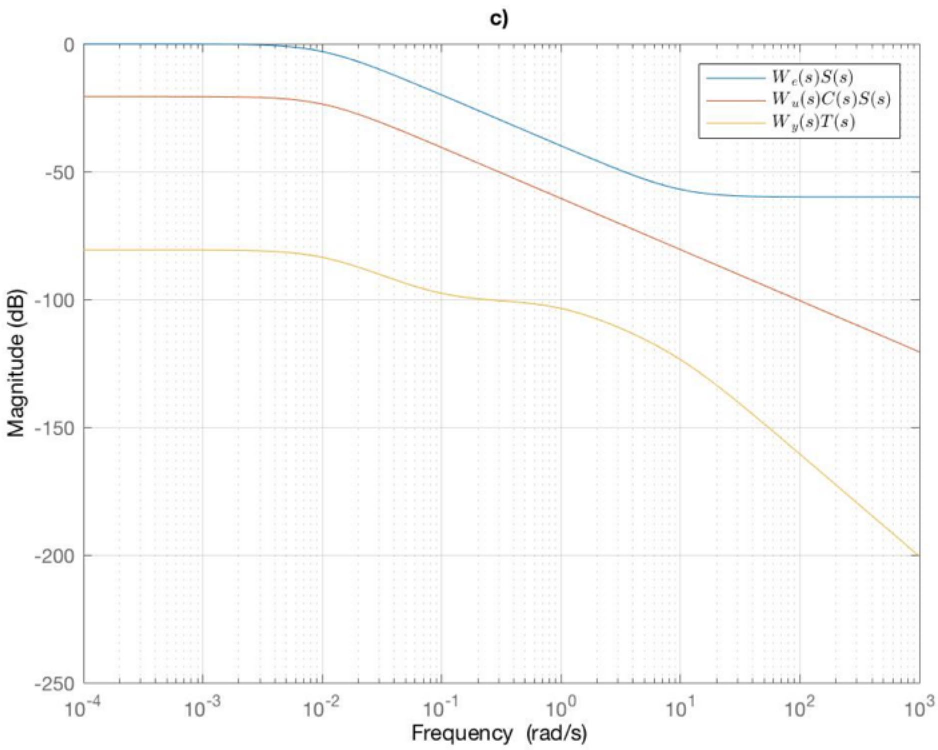
$$\left\| \begin{pmatrix} W_e(s)S(s) \\ W_u(s)C(s)S(s) \\ W_y(s)T(s) \end{pmatrix} \right\|_\infty \leq 1 \tag{8.43}$$

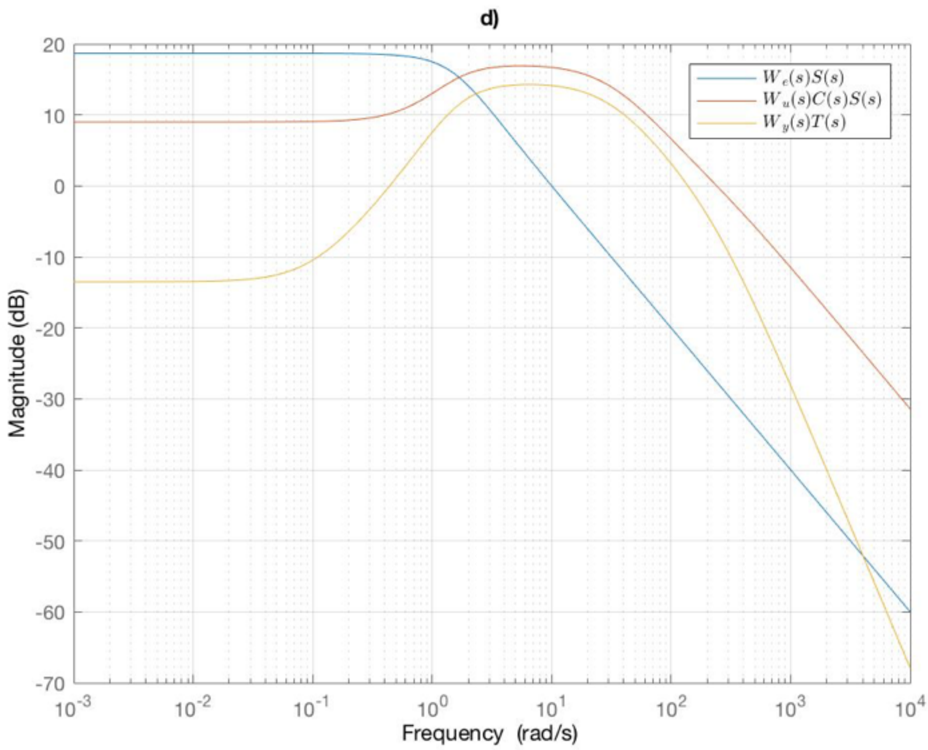
holds.

a) Which of the following magnitude plots are possible for the given extended system?






c)
0
-50
-100
-150
-200
-250
Magnitude (dB)
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/s)
$W_e(s)S(s)$
$W_u(s)C(s)S(s)$
$W_y(s)T(s)$


d)
20
10
0
-10
-20
-30
-40
-50
-60
-70
Magnitude (dB)
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
Frequency (rad/s)
$W_e(s)S(s)$
$W_u(s)C(s)S(s)$
$W_y(s)T(s)$

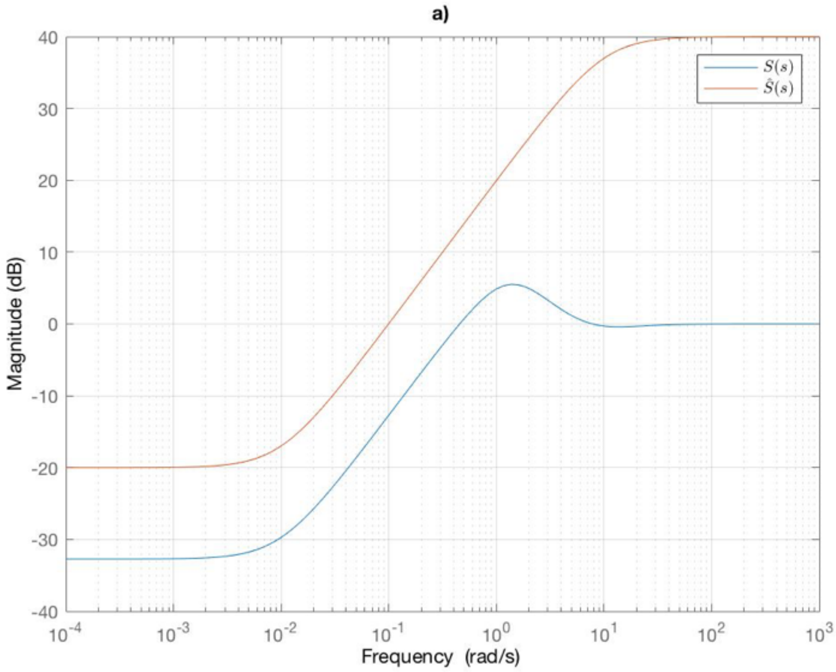
☐ a)

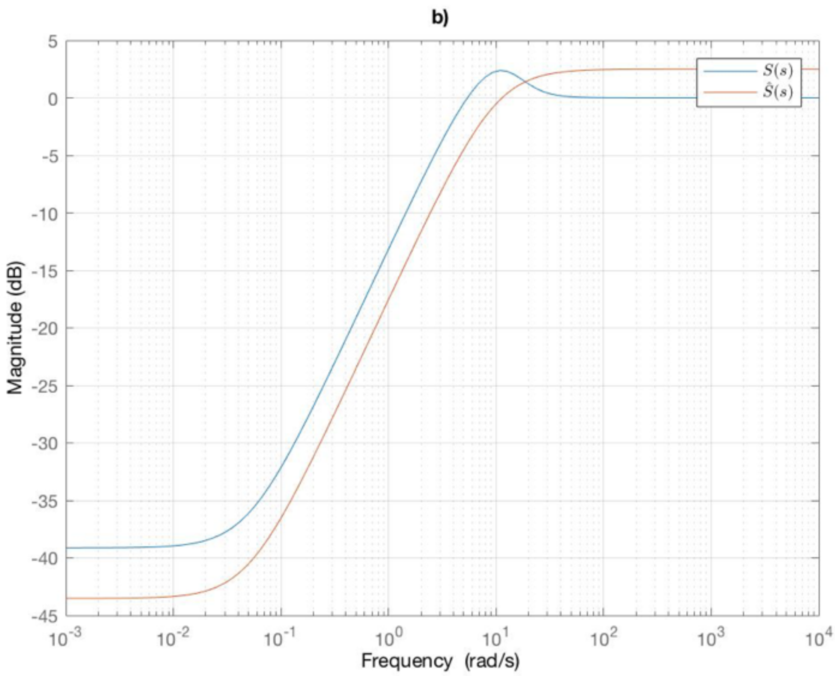
☐ b)

☐ c)

☐ d)

b) Let $W_e(s) = \hat{S}(s)^{-1}$ where $\hat{S}(s)$ is a designer defined upper boundary for $S(s)$. Which magnitude plot of $S(s)$ and $\hat{S}(s)$ could correspond to the the given system?





☐ a)

☐ b)

Given that all matrices in (8.42) are the same as the one derived in the lecture. One can show that the eigenvalues of A are the eigenvalues of all separately considered subsystems. Assume for the next two subtasks that the not considered conditions (summarized on slide 18, lecture 11) for well-posedness of the $\mathcal{H}_\infty$-problem hold.

c) Given stabilizability to the extended system. There exists a solution to the problem if all eigenvalues of the weight system matrices have negative real part.

- ☐ True.
- ☐ False.

d) Assume for the given extended system that $\bar{\sigma}(D_{11}) = 1.15$. There exists a solution.

- ☐ True.
- ☐ False.

Solution.

a) ☑ a)

☐ b)

☑ c)

☐ d)

From the lecture we know that if

$$\left\| \begin{pmatrix} W_e(s)S(s) \\ W_u(s)C(s)S(s) \\ W_y(s)T(s) \end{pmatrix} \right\|_\infty \leq 1 \tag{8.44}$$

holds, then each individual inequality holds as well. Therefore the Bode magnitude plots of $W_e(s)S(s)$, $W_u(s)C(s)S(s)$ and $W_y(s)T(s)$ must not exceed the $0dB$-line. We see that only the plots of **a)** and **c)** satisfy this condition.

b) ☑ a)

☐ b)

Since $\left\|\hat{S}(s)^{-1}S(s)\right\|_\infty \leq 1$, the magnitude plot of $S(s)$ has to be always below the one of $\hat{S}(s)$. This means, that the $\mathcal{H}_\infty$-controller is exact the solution of the minimization problem which leads to a $S(s)$ satisfying the above condition.

c) ☑ True.

☐ False.

From the lecture we know that a sufficient condition for well-posedness of the problem is $[A, B_2]$ stabilizable and $[A, C_2]$ detectable. Since $C_2 = \begin{pmatrix} -C_s & 0 & 0 & 0 \end{pmatrix}$, where C_s corresponds to the plant's LTI-representation, only the states of the plant can be observed. However, the states of the plant are not influenced by the remaining one. In other words, there is no possibility to observe the weight's states. Therefore the poles of the weights has to be stable in order to have well-posedness.

d) ☐ True.

☑ False.

From the lecture we know that a sufficient condition for well-posedness of the problem is $\bar{\sigma}(D_{11}) = \gamma$, where in our case $\gamma = 1$. Therefore the problem is not well-posed.

9 Elements of Nonlinear Control

9.1 Equilibrium Point and Linearization

A nonlinear system can be written as

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= g(x(t), u(t)), \end{aligned} \tag{9.1}$$

where $f(\cdot)$ and $g(\cdot)$ are nonlinear functions. Recall that $(x_\mathrm{e}, u_\mathrm{e})$ represents an **equilibrium point** if and only if

$$\begin{aligned} 0 &= f(x_\mathrm{e}, u_\mathrm{e}, t) \\ y_\mathrm{e} &= g(x_\mathrm{e}, u_\mathrm{e}, t). \end{aligned} \tag{9.2}$$

As the analysis of the nonlinear system is often difficult, we previously considered such a system in a neighbourhood of its equilibrium points. Mathematically, this translates into considering the Taylor expansion of the functions $f(\cdot)$ and $g(\cdot)$ around the equilibrium points of the system and neglecting high order terms. Let $\delta x = x - x_\mathrm{e}$ and $\delta u = u - u_\mathrm{e}$. It holds then

$$\begin{aligned} \delta\dot{x} &= f(x_\mathrm{e} + \delta x, u_\mathrm{e} + \delta u, t) \\ &= \left.\frac{\partial f}{\partial x}\right|_{x_\mathrm{e}, u_\mathrm{e}} \delta x + \left.\frac{\partial f}{\partial u}\right|_{x_\mathrm{e}, u_\mathrm{e}} \delta u + \text{ high order terms} \\ &= A\delta x + B\delta u + \text{ high order terms.} \end{aligned} \tag{9.3}$$

By proceeding analogously for $g(\cdot)$ and neglecting high order terms, one gets

$$\begin{aligned} \delta\dot{x} &= A\delta x + B\delta u \\ \delta y &= C\delta x + D\delta u, \end{aligned} \tag{9.4}$$

where $C = \left.\frac{\partial g}{\partial x}\right|_{x_\mathrm{e}, u_\mathrm{e}}$ and $D = \left.\frac{\partial g}{\partial u}\right|_{x_\mathrm{e}, u_\mathrm{e}}$.

Remark.

- Note that in general, matrices A, B, C, D are time-varying. However, if $f(\cdot), g(\cdot)$ do not depend explicitly on time t, the linearized model will be time-invariant.
- $\delta x, \delta u, \delta y$ describe a deviation from the equilibrium point. The linearized dynamics are given by

$$\begin{aligned} x &= x_\mathrm{e} + \delta x \\ y &= y_\mathrm{e} + \delta y \\ u &= u_\mathrm{e} + \delta u. \end{aligned} \tag{9.5}$$

9.2 Nominal Stability

During the course Control Systems I, you learned about different stability concepts. Moreover, you have learned the differences between internal and external stability: let's recall them here. Consider a generic nonlinear system defined by the dynamics

$$\dot{x}(t) = f(x(t)), \quad t \in \mathbb{R}, \quad x(t) \in \mathbb{R}^n, \quad f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n. \tag{9.6}$$

9.2.1 Internal/Lyapunov Stability

Internal stability, also called Lyapunov stability, characterises the stability of the trajectories of a dynamic system subject to a perturbation near the to equilibrium. Let now $\hat{x} \in \mathbb{R}^n$ be an equilibrium of system (9.6).

Definition 26. An equilibrium $\hat{x} \in \mathbb{R}^n$ is said to be *Lyapunov* stable if

$$\forall \varepsilon > 0, \quad \exists \delta > 0 \quad \text{s.t. } \|x(0) - \hat{x}\| < \delta \Rightarrow \|x(t) - \hat{x}\| < \varepsilon. \tag{9.7}$$

In words, an equilibrium point is said to be *Lyapunov* stable if for any bounded initial condition and zero input, the state remains bounded.

Definition 27. An equilibrium $\hat{x} \in \mathbb{R}^n$ is said to be *asymptotically stable* in $\Omega \subseteq \mathbb{R}^n$ if it is Lyapunov stable and attractive, i.e. if

$$\lim_{t\to\infty} (x(t) - \hat{x}) = 0, \quad \forall x(0) \in \Omega. \tag{9.8}$$

In words, an equilibrium is said to be asymptotically stable if, for any bounded initial condition and zero input, the state converges to the equilibrium.

Definition 28. An equilibrium $\hat{x} \in \mathbb{R}^n$ is said to be *unstable* if it is not stable.

Remark. Note that stability is a property of the equilibrium and not of the system in general.

9.2.2 External/BIBO Stability

External stability, also called BIBO stability (Bounded Input-Bounded Output), characterises the stability of a dynamic system which for bounded inputs gives back bounded outputs.

Definition 29. A signal $s(t)$ is said to be bounded, if there exists a finite value $B > 0$ such that the signal magnitude never exceeds B, that is

$$|s(t)| \leq B \quad \forall t \in \mathbb{R}. \tag{9.9}$$

Definition 30. A system is said to be *BIBO-stable* if

$$\|u(t)\| \leq \varepsilon \quad \forall t \geq 0, \text{ and } x(0) = 0 \Rightarrow \|y(t)\| < \delta \quad \forall t \geq 0, \quad \varepsilon, \delta \in \mathbb{R}. \tag{9.10}$$

In words, for any bounded input, the output remains bounded.

9.2.3 Stability for LTI Systems

Above, we focused on general nonlinear system. However, in Control Systems I you learned that the output $y(t)$ for a LTI system of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{9.11}$$

can be written as

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}(\tau) + Du(t). \tag{9.12}$$

The transfer function relating input to output is a rational function

$$P(s) = C(s\mathbb{I} - A)^{-1}B + D = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} + d. \tag{9.13}$$

Furthermore, it holds:

- The zeros of the numerator of Equation (9.13) are the *zeros* of the system, i.e. the values s_i which fulfill
$$P(s_i) = 0. \tag{9.14}$$
- The zeros of the denominator of Equation (9.13) are the *poles* of the system, i.e. the values s_i which fulfill $\det(s_i\mathbb{I} - A) = 0$, or, in other words, the eigenvalues of A.

One can show, that the following Theorem holds:

Theorem 13. The equilibrium $\hat{x} = 0$ of a linear time invariant system is stable if and only if the following two conditions are met:

1. For all $\lambda \in \sigma(A)$, $\quad \text{Re}(\lambda) \leq 0$.
2. The algebraic and geometric multiplicity of all $\lambda \in \sigma(A)$ such that $\text{Re}(\lambda) = 0$ are equal.

Remark. For linear systems, the stability of an equilibrium point does not depend on the point itself. For nonlinear systems, it does.

9.3 Local Stability

Let $x = x_\mathrm{e}$ be an equilibrium for the autonomous nonlinear system

$$\dot{x}(t) = f(x(t)), \tag{9.15}$$

where $f : \mathcal{D} \to \mathbb{R}^n$ is a continuously differentiable function and $\mathcal{D}$ is a neighborhood of x_e. Let

$$A = \frac{\partial f}{\partial x}(x)\Big|_{x=x_\mathrm{e}}. \tag{9.16}$$

Then:

1. x_e is aymptotically stable if $\text{Re}(\lambda_i) < 0$ for all eigenvalues of A.
2. x_e is unstable if $\text{Re}(\lambda_i) > 0$ for one or more of the eigenvalues of A.

It holds:

- In linear systems, local stability $\Leftrightarrow$ global stability.
- In nonlinear systems, this is not true.

9.3.1 Region of Attraction

Definition 31. A function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is said to be Lipschitz on Ω if for $K \geq 0$ it holds

$$\frac{\|f(x) - f(y)\|}{\|x - y\|} \leq K, \quad \forall x, y \in \Omega. \tag{9.17}$$

Definition 32. Let x_e be an asymptotically stable equilibrium point of the system $\dot{x}(t) = f(x(t))$, where $f(\cdot)$ is a locally Lipschitz function defined over a domain $\mathcal{D} \subset \mathbb{R}^n$ and x_e is contained in $\mathcal{D}$. The **region of attraction** (also known as region of asymptotic stability, domain of attraction) is the set of all points $x_0 \in \mathcal{D}$ such that the solution of

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \tag{9.18}$$

is defined for all $t \geq 0$ and converges to x_e as $t \to \infty$. Note that x_e is said to be **globally asymptotically stable** if the region of attraction is the whole space $\mathbb{R}^n$.

9.4 Lyapunov Stability

9.4.1 Lyapunov Principle - General Systems

1. The Lyapunov Principle is valid for all finite-order systems: as long as the linearized system has no eigenvalues on the imaginary axis.

2. The local stability properties of an arbitrary-order nonlinear system are fully understood once the eigenvalues of the linearization are known.

3. Particularly, if the linearization of a nonlinear system around an isolated equilibrium point x_e is asymptotically stable (or unstable), then this equilibrium is an asymptotically stable (or unstable) equilibrium of the nonlinear system as well. We can but not say that this holds also for the concept of *stable system* ($\text{Re}(\lambda) = 0$).

If we are interested in non-local results or in the case of stable systems, we should use the Lyapunov's direct method.
A scalar function $\alpha(p)$ with $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function if $\alpha(0) = 0$ and $\alpha(p) \geq \alpha(q) \forall p > q$. A function $V : \mathbb{R}^{n+1} \to \mathbb{R}$ is a candidate global Lyapunov function if

- The function is strictly positive, i.e., $V(x,t) > 0 \ \forall x \neq 0, \forall t$ and $V(0) = 0$ and

- there are two nondecreasing functions α and β which satisfy the inequalities

$$\beta(\|x\|) \leq V(x,t) \leq \alpha(\|x\|) \tag{9.19}$$

Remark. If these conditions are not met, only local assumptions can be made.

Theorem 14. The system

$$\dot{x}(t) = f(x(t), t), \quad x(t_0) = x_0 \neq 0, \tag{9.20}$$

is globally/locally stable in the sense of Lyapunov if there is a global/local Lyapunov function candidate $V(x,t)$ for which the following inequality holds $\forall x(t) \neq 0$ and $\forall t$:

$$\dot{V}(x(t), t) = \frac{\partial V(x,t)}{\partial t} + \frac{\partial V(x,t)}{\partial x} f(x(t), t) \leq 0 \tag{9.21}$$

Theorem 15. The same system is globally/locally asymptotically stable if there is a global/local Lyapunov function candidate $V(x,t)$ such that $-\dot{V}(x(t),t)$ satisfies all conditions of a global/local Lyapunov function candidate.

Remark. In general it is difficult to find suitable functions. A good way to approach the problem is to use physical laws (Lyapunov functions can be seen as generalized energy functions).

For linear systems one can find the Lyapunov function

$$V(x(t)) = x(t)^\intercal \cdot P \cdot x(t), \quad P = P^\intercal > 0, \tag{9.22}$$

where P is the solution of the Lyapunov equation

$$PA + A^\intercal P = -Q. \tag{9.23}$$

For arbitrary $Q = Q^\intercal > 0$, a solution to this equation exists if and only if A is a Hurwitz matrix.

Remark. Lyapunov theorems provide sufficient but not necessary conditions!

Example 59. Consider the nonlinear system described by the following differential equations:

$$\begin{aligned} \dot{x}_1 &= x_1 x_2^2 \\ \dot{x}_2 &= x_1^2 x_2 + 2x_2^3 - 6x_2. \end{aligned} \tag{9.24}$$

a) Linearize the system around the equilibrium $x_{1,e} = x_{2,e} = 0$ and find matrix A.

b) Can you say something about the stability of the nonlinear system?

c) Evaluate the stability of the nonlinear system using the Lyapunov function $V = \frac{1}{2}(x_1^2 + x_2^2)$ and find the region of attraction about the equilibrium point.

Solution.

1. The linearization matrix A reads

$$\begin{aligned} A &= \begin{pmatrix} \frac{\partial}{\partial x_1}(x_1 x_2^2) & \frac{\partial}{\partial x_2}(x_1 x_2^2) \\ \frac{\partial}{\partial x_1}(x_1^2 x_2 + 2x_2^3 - 6x_2) & \frac{\partial}{\partial x_2}(x_1^2 x_2 + 2x_2^3 - 6x_2) \end{pmatrix} \Bigg|_{(0,0)} \\ &= \begin{pmatrix} x_2^2 & 2x_1 x_2 \\ 2x_1 x_2 & x_1^2 + 6x_2^2 - 6 \end{pmatrix} \Bigg|_{(0,0)} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & -6 \end{pmatrix}. \end{aligned} \tag{9.25}$$

2. The eigenvalues of matrix A are $\lambda_1 = 0$ and $\lambda_2 = -6$. Using the Lyapunov principle, we cannot evaluate the stability of the nonlinear system, since the linearized one is just stable around the equilibrium.

3. The derivative of the Lyapunov function reads

$$\begin{aligned} \dot{V} &= x_1 \dot{x}_1 + x_2 \dot{x}_2 \\ &= x_1 \cdot (x_1 x_2^2) + x_2 \cdot (x_1^2 x_2 + 2x_2^3 - 6x_2) \\ &= x_1^2 x_2^2 + x_2^2 x_1^2 + 2x_2^4 - 6x_2^2 \\ &= 2x_2^2 \cdot (x_1^2 + x_2^2 - 3). \end{aligned} \tag{9.26}$$

In order for $\dot{V}$ to be negative definite, it must hold $x_1^2 + x_2^2 < 3$.

9.5 Gain Scheduling

As for most systems stability is guaranteed in some neighborhood of the equilibrium point, we are limited when we design a stabilizing controller. A first method to overcome this problem could be to stabilize the system around each equilibrium point and to design a local controller to get stability. The procedure can be defined as

I) Given the nonlinear system
$$\dot{x}(t) = f(x(t), u(t)), \tag{9.27}$$
choose n equilibrium points $(x_{\mathrm{e,i}}, u_{\mathrm{e,i}})$, $i = 1, \ldots, n$.

II) For each of these equilibria, linearize the system and design a local control law
$$u_{\mathrm{l}}(x(t)) = u_{\mathrm{l,e}} - K(x(t) - x_{\mathrm{l,e}}) \tag{9.28}$$
for the linearization.

III) The global control law consists of:
- Choosing the correct control law, as a function of the state: $i = \sigma(x)$.
- Use that control law: $u(x) = u_{\sigma(x)}(x)$.

9.6 Feedback Linearization

9.6.1 Input-State Feedback Linearization

Input-state feedback linearization is the ability to use feedback to convert a nonlinear state equation into a linear state equation by canceling nonlinearities. This requires the nonlinear state equation to have the structure
$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\beta^{-1}(x(t))\left[u(t) - \alpha(x(t))\right],\\ y(t) &= h(x(t)).\end{aligned} \tag{9.29}$$
where
- The pair (A, B) is controllable.
- $$\begin{aligned}\alpha &: \mathbb{R}^n \to \mathbb{R}^p\\ \beta &: \mathbb{R}^n \to \mathbb{R}^{p\times p}\end{aligned} \tag{9.30}$$
  are defined on the domain $\mathcal{D}_x \subset \mathbb{R}^n$, which contains the origin.
- The matrix $\beta(x(t))$ is assumed to be invertible $\forall x \in \mathcal{D}_x$.

If the system is in the form presented in Equation 9.29, one can linearize it using the feedback law
$$u(t) = \alpha(x(t)) + \beta(x(t))v(t). \tag{9.31}$$
Remark. The form presented in Equation 9.31 has a specific meaning. In fact, it holds
$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\beta^{-1}(x(t))\left[u(t) - \alpha(x(t))\right]\\ &= Ax(t) + B\beta^{-1}(x(t))\left[\alpha(x(t)) + \beta(x(t))v(t) - \alpha(x(t))\right]\\ &= Ax(t) + Bv(t),\end{aligned} \tag{9.32}$$
where $v(t)$ can be chosen with respect to the design constraints. This allows to linearize the dynamics of the system.

9.6.2 Input-State Linearizability

Let $z(x(t)) = T(x(t))$ be a change of variables (also known as bijection). If both T and T^{-1} are continuously differentiable, we call it a **diffeomorphism**. A nonlinear system

$$\dot{x}(t) = f(x(t)) + \Gamma(x(t))u(t), \tag{9.33}$$

where

$$f : \mathcal{D}_x \to \mathbb{R}^n \tag{9.34}$$

and

$$\Gamma : \mathcal{D}_x \to \mathbb{R}^{p \times p} \tag{9.35}$$

are sufficiently smooth on a domain $\mathcal{D}_x \subset \mathbb{R}^n$, is said to be **input-state linearizable** if there exists a diffeomorphism

$$T : \mathcal{D}_x \subset \mathbb{R}^n \tag{9.36}$$

such that

$$\mathcal{D}_z = T(\mathcal{D}_x) \tag{9.37}$$

contains the origin and the change of variables $z(x(t)) = T(x(t))$ transforms the system into the form

$$\dot{z}(x(t)) = Az(x(t)) + B\beta^{-1}(x(t)) \left[u(t) - \alpha(x(t))\right], \tag{9.38}$$

with (A, B) controllable and $\beta(x(t))$ invertible for all $x \in \mathcal{D}_x$.

Conditions for Linearizability - General Case

But when is this the case? In general, holds

$$\begin{aligned} \dot{z}(x(t)) &= \frac{\partial T}{\partial x}\dot{x}(t) \\ &= \frac{\partial T}{\partial x}\left[f(x(t)) + \Gamma(x(t))u(t)\right]. \end{aligned} \tag{9.39}$$

On the other hand, one can also write

$$\begin{aligned} \dot{z}(t) &= Az(t) + B\beta^{-1}(x(t))\left[u(t) - \alpha(x(t))\right] \\ &= AT(x(t)) + B\beta^{-1}(x(t))\left[u(t) - \alpha(x(t))\right]. \end{aligned} \tag{9.40}$$

Using Equation 9.39 and Equation 9.40, one can write the general equality which must hold for all $x(t)$ and $u(t)$ in the domain of interest:

$$\frac{\partial T}{\partial x}\left[f(x(t)) + \Gamma(x(t))u(t)\right] = AT(x(t)) + B\beta^{-1}(x(t))\left[u(t) - \alpha(x(t))\right]. \tag{9.41}$$

If one sets $u(t) = 0$, one can split the equation into two:

$$\begin{aligned} \frac{\partial T}{\partial x}f(x(t)) &= AT(x(t)) - B\beta^{-1}(x(t))\alpha(x(t)) \\ \frac{\partial T}{\partial x}\Gamma(x(t)) &= B\beta^{-1}(x(t)). \end{aligned} \tag{9.42}$$

Each correct transformation $T(\cdot)$ must satisfy the partial differential equations given in Equation 9.42.
Having $T(x(t))$ which fulfills these partial differential equations is a necessary and sufficient conditions that a transformation from the form in Equation 9.33 to the form in Equation 9.38 exists.

Conditions for Linearizability - Single Input

With a single input ($p = 1$), one can define a linear transformation $\xi(x(t)) = Mz(x(t))$ with M invertible and write

$$\dot{\xi} = MAM^{-1}\xi + MB\beta^{-1}(x(t))\left[u(t) - \alpha(x(t))\right]. \tag{9.43}$$

We choose M such that the controller canonical form can be written as

$$\left(\begin{array}{c|c} A_c + B_c\gamma^\intercal & B_c \\ \hline C_c & D_c \end{array}\right) = \left(\begin{array}{cccccc|c} 0 & 1 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \\ 0 & \dots & \dots & 0 & 1 & 0 & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 & 0 \\ -\gamma_0 & -\gamma_1 & \dots & \dots & --\gamma_{n-2} & -\gamma_{n-1} & 1 \\ \hline c_0 & \dots & c_m & 0 & \dots & 0 & 0 \end{array}\right). \tag{9.44}$$

This means

$$MAM^{-1} = A_c + B_c\gamma^\intercal \tag{9.45}$$

and

$$MB = B_c. \tag{9.46}$$

The term

$$B_c\gamma^\intercal\xi = B_c\gamma^\intercal MT(x(t)) \tag{9.47}$$

is included into the nonlinearity

$$B_c\beta^{-1}(x(t))\alpha(x(t)), \tag{9.48}$$

which allows to reformulate the partial differential equations as

$$A_cT(x(t)) - B_c\beta^{-1}(x(t))\alpha(x(t)) = \begin{pmatrix} T_2(x) \\ T_3(x) \\ \vdots \\ T_{n-1}(x) \\ T_n(x) \end{pmatrix} \tag{9.49}$$

and

$$B_c\beta^{-1}(x(t)) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{\beta(x(t))} \end{pmatrix}. \tag{9.50}$$

Finally, one can write

$$\begin{aligned} \frac{\partial T_1}{\partial x(t)} f(x(t)) &= T_2(x(t)) \\ \frac{\partial T_2}{\partial x(t)} f(x(t)) &= T_3(x(t)) \\ &\vdots \\ \frac{\partial T_{n-1}}{\partial x(t)} f(x(t)) &= T_n(x(t)) \\ \frac{\partial T_n}{\partial x} f(x(t)) &= -\frac{\alpha(x(t))}{\beta(x(t))} \end{aligned} \tag{9.51}$$

and

$$\begin{aligned}\frac{\partial T_1}{\partial x}\gamma(x(t)) &= 0\\ \frac{\partial T_2}{\partial x}\gamma(x(t)) &= 0\\ &\vdots\\ \frac{\partial T_{n-1}}{\partial x}\gamma(x(t)) &= 0\\ \frac{\partial T_n}{\partial x}\gamma(x(t)) &= \frac{1}{\beta(x(t))}\end{aligned} \tag{9.52}$$

9.7 Examples

Example 60.

a) Consider the continuous-time system

$$\dot{x}(t) = 0.5x(t), \quad x(t) \in \mathbb{R}, \tag{9.53}$$

and the test function

$$V(x) = 2x. \tag{9.54}$$

Which of the following statements is true?

- ☐ $V(x)$ is a Lyapunov function for this system and therefore the system is asymptotically stable.
- ☐ $V(x)$ is not a Lyapunov function for this system and therefore the system is not stable.
- ☐ $V(x)$ is not a Lyapunov function for this system. Furthrermore, given this information, we cannot conclude anything about the stability of the system.

b) Which of the following functions are positive definite

- ☐ $V(x) = x_1(t)^2 + x_2(t)^2$.
- ☐ $V(x) = x_1(t)^2$.
- ☐ $V(x) = (x_1(t) + x_2(t))^2$.
- ☐ $V(x) = -x_1(t)^2 - (3x_1(t) + 2x_2(t))^2$.
- ☐ $V(x) = x_1(t)x_2(t) + x_2(t)^2$.
- ☐ $V(x) = x_1(t)^2 + \frac{2x_2(t)^2}{1+x_2(t)^2}$.

c) You are given the nonlinear system

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)x_2(t)^2\\ \dot{x}_2(t) &= x_1(t)^2x_2(t) + 2x_2(t)^3 - 6x_2(t).\end{aligned} \tag{9.55}$$

Evaluate the stability of the origin using the Lyapunov function

$$\frac{1}{2}(x_1^2(t) + x_2(t)^2). \tag{9.56}$$

- ☐ The largest region of attraction of the system is $\{x(t) \in \mathbb{R}^2 | x_1(t)^2 + x_2(t)^2 \leq 3\}$.
- ☐ The largest region of attraction of the system is $\{x(t) \in \mathbb{R}^2 | x_1(t)^2 + x_2(t)^2 \leq \sqrt{3}\}$.
- ☐ The largest region of attraction of the system is $\{x(t) \in \mathbb{R}^2 | x_1(t)^2 + x_2(t)^2 \leq 2\}$.
- ☐ The largest region of attraction of the system is $\{x(t) \in \mathbb{R}^2 | x_1(t)^2 + x_2(t)^2 \leq \sqrt{2}\}$.
- ☐ None of the above.

Solution.

a)

- ☐ $V(x)$ is a Lyapunov function for this system and therefore the system is asymptotically stable.
- ☐ $V(x)$ is not a Lyapunov function for this system and therefore the system is not stable.
- ☑ $V(x)$ is not a Lyapunov function for this system. Furthermore, given this information, we cannot conclude anything about the stability of the system.

Solution: The test function is not a Lyapunov function. One can verify this by observing that:

- $V(x)$ is not a positive definite function or
-
$$\begin{aligned}\dot{V}(x) &= \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}x(t)}{\mathrm{d}t} \\ &= \frac{\partial V}{\partial x}\dot{x}(t) \\ &= 2\cdot 0.5\cdot x \\ &= x\end{aligned} \tag{9.57}$$

  is not a negative definite function.

Since $V(x)$ is not a Lyapunov function, we cannot conclude anything about the stability of the system. Moreover, we know that the system is unstable only from the positive eigenvalue $\lambda_1 = 0.5$, and not from $V(x)$.

b)

- ☑ $V_1(x(t)) = x_1(t)^2 + x_2(t)^2$.
- ☐ $V_2(x(t)) = x_1(t)^2$.
- ☐ $V_3(x(t)) = (x_1(t) + x_2(t))^2$.
- ☐ $V_4(x(t)) = -x_1(t)^2 - (3x_1(t) + 2x_2(t))^2$.
- ☐ $V_5(x(t)) = x_1(t)x_2(t) + x_2(t)^2$.

Solution:

- $V_1(x(t)) > 0\ \forall x(t) \neq 0$ and $V_1(x(t)) = 0$ if $x = 0$.
- $V_2(x(t)) > 0\ \forall x(t) \neq 0$ and $V_2(x(t)) = 0$ if $x_1 = 0$. This still holds for any $x_2(t) \neq 0$, which makes $V_2(x(t))$ positive semi-definite.
- $V_3(x(t)) \geq 0\ \forall x(t)$, but can be 0 as soon as $x_1(t) = -x_2(t)$.
- $V_4(x(t)) < 0\ \forall x(t) \neq 0$.
- As soon as $x_1(t)x_2(t) < x_2(t)^2$, $V_5(x(t)) < 0$.

c) ☐ The largest region of attraction of the system is $\{x(t) \in \mathbb{R}^2 | x_1(t)^2 + x_2(t)^2 \leq 3\}$.

- ☐ The largest region of attraction of the system is $\{x(t) \in \mathbb{R}^2 | x_1(t)^2 + x_2(t)^2 \leq \sqrt{3}\}$.
- ☐ The largest region of attraction of the system is $\{x(t) \in \mathbb{R}^2 | x_1(t)^2 + x_2(t)^2 \leq 2\}$.
- ☐ The largest region of attraction of the system is $\{x(t) \in \mathbb{R}^2 | x_1(t)^2 + x_2(t)^2 \leq \sqrt{2}\}$.
- ☑ None of the above.

Solution: It holds

$$\begin{aligned}
\dot{V}(x_1(t), x_2(t)) &= \frac{\partial V}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial V}{\partial x_2}\frac{\partial x_2}{\partial t} \\
&= x_1(t)\left(x_1(t)x_2(t)^2\right) + x_2(t)\left(x_1(t)^2 x_2(t) + 2x_2(t)^3 - 6x_2(t)\right) \\
&= 2x_1(t)^2 x_2(t)^2 + 2x_2(t)^4 - 6x_2(t)^2 \\
&= 2x_2(t)^2\left(x_1(t)^2 + x_2(t)^2 - 3\right).
\end{aligned} \tag{9.58}$$

In order to find the region of attraction for which the system is asymptotically stable, $\dot{V}(x)$ must be negative definite. This is the case if

$$\{x(t) \in \mathbb{R}^2 | x_1(t)^2 + x_2(t)^2 < 3\}. \tag{9.59}$$

This ensures that the region of attraction for the origin is at least the one presented in Equation 9.59. However, the choice of another Lyapunov function could result in a larger region of attraction. This explains why none of the first four answers is correct.

Example 61. You are given the system

$$\dot{x}(t) = f(x(t)) + gu(t), \tag{9.60}$$

with

$$f(x(t)) = \begin{pmatrix} x_2(t) \\ -a\sin(x_1(t)) - b(x_1(t) - x_3(t)) \\ x_4(t) \\ c(x_1(t) - x_3(t)) \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 0 \\ 0 \\ d \end{pmatrix} \tag{9.61}$$

where a, b, c and d are positive constants. We want to find a diffeomorphismus such that $T_1(x(t))$ fulfills:

$$\frac{\partial T_i}{\partial x} g = 0, \quad i = 1, 2, 3; \; \frac{\partial T_4}{\partial x} g \neq 0. \tag{9.62}$$

The system has clearly an equilibrium point at $x = 0$. From the first condition

$$\frac{\partial T_1}{\partial x} g = 0, \tag{9.63}$$

one knows that

$$\frac{\partial T_1}{\partial x_4} d = 0. \tag{9.64}$$

This means that one must choose $T_1(x(t))$ independent of $x_4(t)$. Using this, one can write

$$T_2(x(t)) = \frac{\partial T_1}{\partial x_1} x_2(t) + \frac{\partial T_1}{\partial x_2} \left(-a\sin(x_1(t)) - b(x_1(t) - x_3(t)\right) + \frac{\partial T_1}{\partial x_3} x_4(t). \tag{9.65}$$

From the second condition

$$\frac{\partial T_2}{\partial x} g = 0, \tag{9.66}$$

one knows that

$$\frac{\partial T_1}{\partial x_4} = 0. \tag{9.67}$$

This implies

$$\frac{\partial T_2}{\partial x_4} = 0 \Rightarrow \frac{\partial T_1}{\partial x_3} = 0. \tag{9.68}$$

$T_1(x(t))$ needs to be independent of $x_3(t)$ and hence

$$T_2(x(t)) = \frac{\partial T_1}{\partial x_1} x_2(t) + \frac{\partial T_1}{\partial x_2} \left(-a\sin(x_1(t)) - b(x_1(t) - x_3(t)\right), \tag{9.69}$$

and

$$T_3(x(t)) = \frac{\partial T_2}{\partial x_1} x_2(t) + \frac{\partial T_2}{\partial x_2} \left(-a\sin(x_1(t)) - b(x_1(t) - x_3(t)\right) + \frac{\partial T_2}{\partial x_3} x_4(t). \tag{9.70}$$

From the third condition

$$\frac{\partial T_3}{\partial x} g = 0, \tag{9.71}$$

one knows that

$$\frac{\partial T_3}{\partial x_4} = 0. \tag{9.72}$$

This implies

$$\frac{\partial T_3}{\partial x_4} = 0 \Rightarrow \frac{\partial T_2}{\partial x_3} = 0 \Rightarrow \frac{\partial T_1}{\partial x_2} = 0. \tag{9.73}$$

$T_1(x(t))$ needs to be independent of $x_2(t)$ and hence

$$T_4(x(t)) = \frac{\partial T_3}{\partial x_1} x_2(t) + \frac{\partial T_3}{\partial x_2} \left(-a\sin(x_1(t)) - b(x_1(t) - x_3(t)\right) + \frac{\partial T_3}{\partial x_3} x_4(t). \tag{9.74}$$

The last condition

$$\frac{\partial T_4}{\partial x} g \neq 0 \tag{9.75}$$

is satisfied if

$$\frac{\partial T_3}{\partial x_3} = 0 \Rightarrow \frac{\partial T_2}{\partial x_2} = 0 \Rightarrow \frac{\partial T_1}{\partial x_1} = 0. \tag{9.76}$$

With $T_1(x(t)) = x_1(t)$, one can write

$$\begin{aligned}
z_1(x(t)) &= T_1(x(t)) = x_1(t) \\
z_2(x(t)) &= T_2(x(t)) = x_2(t) \\
z_3(x(t)) &= T_3(x(t)) = -a\sin(x_1(t)) - b(x_1(t) - x_3(t)) \\
z_4(x(t)) &= T_4(x(t)) = -a x_2(t)\cos(x_1(t)) - b(x_2(t) - x_4(t)).
\end{aligned} \tag{9.77}$$

Example 62. Your SpaghETH startup, which cooks pasta on the polyterrasse everyday, is growing every week more and although no particular production issues occur you are concerned about ecology. Since each tank of pasta you cook needs water and a correct salt seasoning for it to taste that delicious, you need a lot of salt and water, which are often wasted. For this reason, you open a research branch in your startup which decides to design a duct-hydraulic system to counteract the waste of water and salt. The idea is to use a two water tank system, which helps you seasoning the water and changing it, without substituting the whole pot. The dynamics of the system are given by

$$\begin{aligned}\dot{x}_1(t) &= 1 + u(t) - \sqrt{1 + x_1(t)}\\ \dot{x}_2(t) &= \sqrt{1 + x_1(t)} - \sqrt{1 + x_2(t)}\\ y &= x_2(t).\end{aligned} \tag{9.78}$$

a) Linearize the nonlinear system around the equilibrium

$$\begin{pmatrix} x_{1,\text{eq}}(t) & x_{2,\text{eq}}(t) & u_{\text{eq}}(t) \end{pmatrix} = \begin{pmatrix} 3 & 3 & 1 \end{pmatrix}. \tag{9.79}$$

b) Determine the coordinate transformation such that the system can be written in the form

$$\begin{aligned}\dot{z}_1(t) &= z_2(t)\\ \dot{z}_2(t) &= \alpha(z) + \beta(z)u(t)\\ y(t) &= z_1(t).\end{aligned} \tag{9.80}$$

c) Find a feedback control law by exactly linearizing the system.

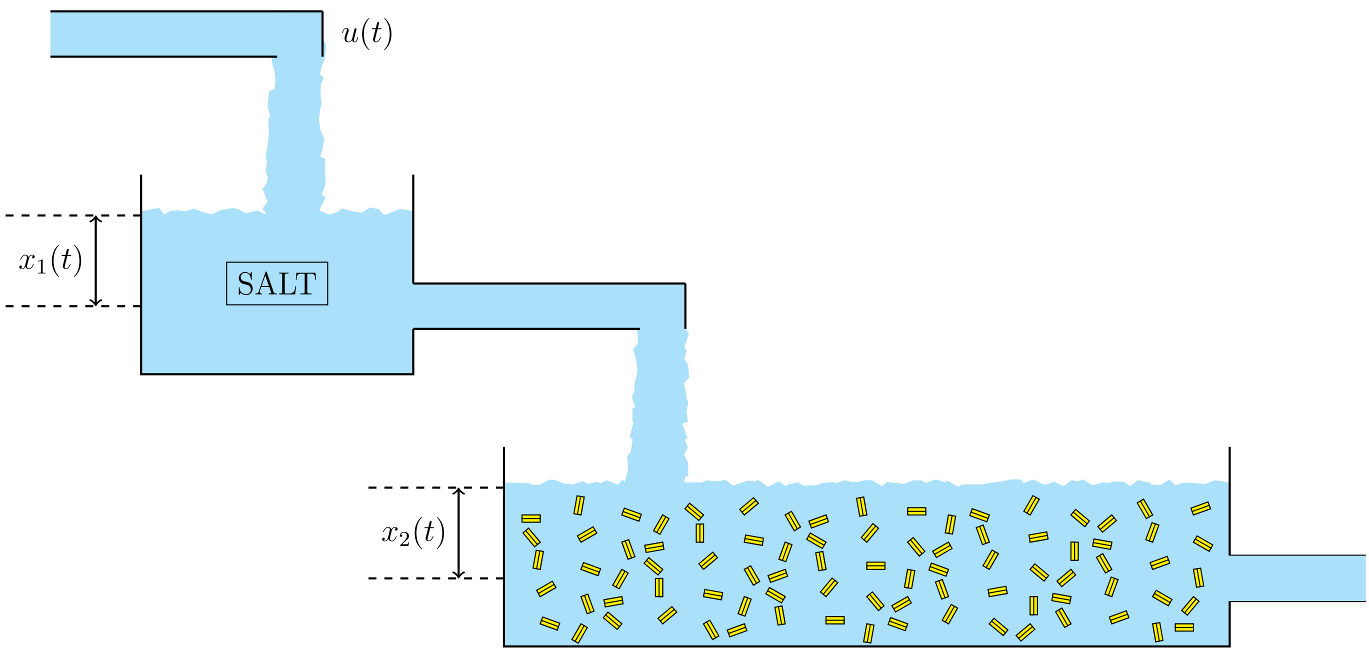


Figure 63: Sketch of the system.

Solution.

a) It holds

$$\begin{aligned} A &= \begin{pmatrix} -\frac{1}{2\sqrt{1+x_1(t)}} & 0 \\ \frac{1}{2\sqrt{1+x_1(t)}} & -\frac{1}{2\sqrt{1+x_2(t)}} \end{pmatrix}\Bigg|_{x_{1,\mathrm{eq}}(t)=x_{2,\mathrm{eq}}(t)=3} \\ &= \begin{pmatrix} -\frac{1}{4} & 0 \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}, \\ B &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ C &= \begin{pmatrix} 0 & 1 \end{pmatrix}, \\ D &= 0. \end{aligned} \tag{9.81}$$

b) By choosing the states

$$z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix}, \tag{9.82}$$

one gets

$$\begin{aligned} \dot{z}_1(t) &= z_2(t) \\ \dot{z}_2(t) &= \frac{\partial}{\partial t}\dot{y}(t) \\ &= \frac{\partial}{\partial t}\dot{x}_2(t) \\ &= \frac{1}{2\sqrt{1+x_1(t)}}\dot{x}_1(t) - \frac{1}{2\sqrt{1+x_2(t)}}\dot{x}_2(t) \\ &= \frac{1}{2\sqrt{1+x_1(t)}}\left(1+u(t)-\sqrt{1+x_1(t)}\right) - \frac{1}{2\sqrt{1+x_2(t)}}\left(\sqrt{1+x_1(t)}-\sqrt{1+x_2(t)}\right) \\ &= \frac{1}{2}\left(\frac{1}{\sqrt{1+x_1(t)}} - \frac{\sqrt{1+x_1(t)}}{\sqrt{1+x_2(t)}}\right) + \frac{u(t)}{2\sqrt{1+x_1(t)}}. \end{aligned} \tag{9.83}$$

Furthermore, we know

$$\begin{aligned} z_1(t) &= y(t) = x_2(t) \\ z_2(t) &= \dot{y}(t) = \dot{x}_2(t) \\ &= \sqrt{1+x_1(t)} - \sqrt{1+x_2(t)}, \end{aligned} \tag{9.84}$$

from which it follows

$$\begin{aligned} x_2(t) &= z_1(t) \\ \sqrt{1+x_1(t)} &= z_2(t) + \sqrt{1+z_1(t)} \end{aligned} \tag{9.85}$$

Plugging Equation 9.85 into Equation results in

$$\begin{aligned} \dot{z}_1(t) &= z_2(t) \\ \dot{z}_2(t) &= \frac{1}{2}\left(\frac{1}{z_2(t)+\sqrt{1+z_1(t)}} - \frac{z_2(t)+\sqrt{1+z_1(t)}}{\sqrt{1+z_1(t)}}\right) + \frac{1}{2}\frac{u(t)}{z_2(t)+\sqrt{1+z_1(t)}} \\ &= \alpha(z(t)) + \beta(z(t))u(t). \end{aligned} \tag{9.86}$$

c) With the form obtained in Equation 9.86, one can write

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v(t), \tag{9.87}$$

where

$$\begin{aligned} u(t) &= \frac{1}{\beta(z(t))} \left(v(t) - \alpha(z(t)) \right) \\ &= 2(z_2(t) + \sqrt{1 + z_1(t)}) \left(v(t) - \frac{1}{2} \left(\frac{1}{z_2(t) + \sqrt{1 + z_1(t)}} - \frac{z_2(t) + \sqrt{1 + z_1(t)}}{\sqrt{1 + z_1(t)}} \right) \right). \end{aligned} \tag{9.88}$$

Example 63. You are given the system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} a\sin(x_2(t)) \\ -x_1(t)^2 + u(t) \end{pmatrix}. \tag{9.89}$$

Use the linearizability conditions for SIMO systems to find the transformation (diffeomorphism) $z(x(t)) = T(x(t))$.

Solution. We identify the system to be of the form

$$\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix} = \begin{pmatrix}a\sin(x_2(t))\\ -x_1(t)^2\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}u(t) = f(x(t)) + \gamma u(t). \tag{9.90}$$

From this we deduce

$$\gamma = \begin{pmatrix}0\\ 1\end{pmatrix}. \tag{9.91}$$

The first condition on $T(x(t))$ implies

$$\begin{aligned}\frac{\partial T_1}{\partial x}\gamma(x(t)) &= 0\\ \begin{pmatrix}\frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2}\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} &= 0\\ \frac{\partial T_1}{\partial x_2} &= 0.\end{aligned} \tag{9.92}$$

From the very definition and the result of Equation 9.92, one gets

$$\begin{aligned}T_2(x(t)) &= \begin{pmatrix}\frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2}\end{pmatrix} f(x(t))\\ &= \frac{\partial T_1}{\partial x_1}a\sin(x_2(t)) - \underbrace{\frac{\partial T_1}{\partial x_2}}_{=0}x_1(t)^2\\ &= \frac{\partial T_1}{\partial x_1}a\sin(x_2(t)).\end{aligned} \tag{9.93}$$

The second condition on $T(x(t))$ implies

$$\begin{aligned}\frac{\partial T_2}{\partial x}\gamma(x(t)) &= \frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x_1}a\sin(x_2(t))\right)\begin{pmatrix}0\\ 1\end{pmatrix}\\ &= \begin{pmatrix}\frac{\partial}{\partial x_1}\left(\frac{\partial T_1}{\partial x_1}a\sin(x_2(t))\right) & \frac{\partial}{\partial x_2}\left(\frac{\partial T_1}{\partial x_1}a\sin(x_2(t))\right)\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}\\ &= \frac{\partial}{\partial x_2}\left(\frac{\partial T_1}{\partial x_1}a\sin(x_2(t))\right)\\ T_1 \textit{ is independent of } x_2 \qquad &= \frac{\partial T_1}{\partial x_1}a\cos(x_2(t)) \neq 0.\end{aligned} \tag{9.94}$$

In order for Equation 9.94 to hold, $\cos(x_2(t)) \neq 0$ and $\frac{\partial T_1}{\partial x_1} \neq 0$. Choosing $z_1(x(t)) = T_1(x(t)) = x_1(t)$, results in the diffeomorphism

$$\begin{aligned}z_1(x(t)) &= x_1(t)\\ z_2(x(t)) &= a\sin(x_2(t)) = \dot{x}_1(t).\end{aligned} \tag{9.95}$$

Note that this is not the only possible choice. Choosing

$$z_1(x(t)) = x_1(t) + x_1(t)^3 \tag{9.96}$$

would result in

$$\begin{aligned}z_2(x(t)) &= \dot{x}_1(t) + 3x_1(t)^2\dot{x}_1(t)\\ &= a\sin(x_2(t)) + 3x_1(t)^2 a\sin(x_2(t)).\end{aligned} \tag{9.97}$$

A Linear Algebra

A.1 Matrix-Inversion

$$A^{-1} = \frac{1}{\det(A)} \cdot \mathrm{adj}(A), \qquad \{\mathrm{adj}(A)\}_{ij} = (-1)^{i+j} \cdot \det(A_{ij}) \tag{A.1}$$

Special Cases:

- $n = 2$:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{a \cdot d - b \cdot c} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{A.2}$$

- $n = 3$:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} e \cdot i - f \cdot h & c \cdot h - b \cdot i & b \cdot f - c \cdot e \\ f \cdot g - d \cdot i & a \cdot i - c \cdot g & c \cdot d - a \cdot f \\ d \cdot h - e \cdot g & b \cdot g - a \cdot h & a \cdot e - b \cdot d \end{bmatrix} \tag{A.3}$$

- $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

A.2 Differentiation with Matrices

$$\frac{\mathrm{d}}{\mathrm{d}x} A \cdot x = A^T$$

$$\frac{\mathrm{d}}{\mathrm{d}x} x^T \cdot A \cdot x = (A^T + A) \cdot x$$

A.3 Matrix Inversion Lemma

$$[M + v \cdot v^T]^{-1} = M^{-1} - \frac{1}{1 + v^T \cdot M^{-1} \cdot v} \cdot M^{-1} \cdot v \cdot v^T \cdot M^{-1}$$

B Rules

B.1 Trigo

$\alpha[°]$	0	30	45	60	90	120	180
α[rad]	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
$\sin(\alpha)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	0
$\cos(\alpha)$	1	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1
$\tan(\alpha)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\pm\infty$	$-\sqrt{3}$	0
$\cot(\alpha)$	$\pm\infty$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$\pm\infty$

B.2 Euler-Forms

$$e^{ix} = \cos(x) + i \cdot \sin(x)$$
$$a + i \cdot b = |a + i \cdot b| \cdot e^{i \cdot \angle(a+i\cdot b)}$$
$$\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$$
$$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$$

B.3 Derivatives

$$(\log_a |x|)' = (\log_a e)\frac{1}{x} = \frac{1}{x \ln a}$$
$$(a^{cx})' = (c \ln a) a^{cx}$$
$$(\tan x)' = \frac{1}{\cos^2 x} = 1 + \tan^2 x$$
$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}$$
$$(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}$$
$$(\arctan x)' = \frac{1}{1 + x^2}$$

B.4 Logarithms

$$\ln |y| \cdot C = \ln |y^C|$$
$$-\ln |r| = \ln |r^{-1}|$$
$$\ln(1) = \log(1) = 0$$

B.5 Magnitude and Phase

In the Bode diagram are magnitude and phase separate from each other. Magnitude and phase for complex fractions are:

$$\left|\frac{a+i\cdot b}{c+i\cdot d}\right| = \sqrt{\frac{a^2+b^2}{c^2+d^2}}$$

$$\angle\left(\frac{a+i\cdot b}{c+i\cdot d}\right) = \arg\left\{\frac{a+i\cdot b}{c+i\cdot d}\right\} = \arctan\left(\frac{bc-ad}{ac+bd}\right)$$

$$\arg\{a+i\cdot b\} = \arctan\left(\frac{b}{a}\right)$$

$$\arg\{(a+i\cdot b)^c\} = c\cdot\arg\{a+i\cdot b\}$$

$$\arg\left\{\frac{c}{(a+i\cdot b)}\right\} = \arg\{c\} - \arg\{a+i\cdot b\}$$

B.6 dB-Scale

Typically the unit for the magnitude is dB:

$$|\Sigma(j\omega)|_{\text{dB}} = 20\cdot\log_{10}|\Sigma(j\omega)|$$

$$|\Sigma(j\omega)| = 10^{\frac{|\Sigma(j\omega)|_{\text{dB}}}{20}}$$

$$\left.\frac{1}{X}\right|_{\text{dB}} = -X|_{\text{dB}}$$

$$(X\cdot Y)|_{\text{dB}} = X|_{\text{dB}} + Y|_{\text{dB}}$$

Value	0.001	0.01	0.1	0.5	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	2	10	100
dB	-60	-40	-20	≈ -6	≈ -3	0	≈ 3	≈ 6	20	40

C MATLAB

C.1 General Commands

Command	Description
`A(i,j)`	Element of `A` in position `i` (row) and `j` (column)
`abs(X)`	Magnitude of all elements of `X`
`angle(X)`	Phase of all elements of `X`
`X'`	Complex conjugate and transpose of `X`
`X.'`	Transpose, not complex conjugate of `X`
`conj(X)`	Complex conjugate of all elements of `X`
`real(X)`	Realteil von allen Einträge von `X`
`imag(X)`	Imaginary part of all elements of `X`
`eig(A)`	Eigenvalues of `A`
`[V,D]=eig(A)`	Eigenvalues `D` (diagonal elements), eigenvectors `V` (column vectors)
`s=svd(A)`	singular values of `A`
`[U,Sigma,V]=svd(A)`	Singular Values Decomposition of `A`
`rank(A)`	Rank of `A`
`det(A)`	Determinant of `A`
`inv(A)`	Inverse of `A`
`diag([a1,...,an])`	Diagonalmatrix with `a1,...,an` as diagonal elements
`zeros(x,y)`	Zero matrix of dimension `x`×`y`
`zeros(x)`	Zero matrix of dimension `x`×`x`
`eye(x,y)`	Identity matrix of dimension `x`×`y`
`eye(x)`	Identity matrix of dimension `x`×`x`
`ones(x,y)`	One-Matrix (all elements = 1) of dimension `x`×`y`
`ones(x)`	One-Matrix (all elements = 1) of dimension `x`×`x`
`max(A)`	Largest element in vector `A` (`A` Matrix: Max in column vectors)
`min(A)`	Smallest element in vector `A` (`A` Matrix: Max in column vectors)
`sum(A)`	Sum of elements of `A` (`A` Matrix: Sum row pro row)
`dim=size(A)`	Dimension of `A` (`size=[#rows #columns]`)
`dim=size(A,a)`	`a=1`: `dim=#rows`, `a=2`: `dim=#columns`, sonst `dim=1`
`t=a:i:b`	`t=[a,a+i,a+2i,...,b-i,b]` (row vector)
`y=linspace(a,b)`	row vector with 100 "linear-spaced" points in range `[a,b]`
`y=linspace(a,b,n)`	row vector with `n` "linear-spaced" points in range `[a,b]`
`y=logspace(a,b)`	row vector with 50 "logarithmically-spaced" points in range `[10^a,10^b]`
`y=logspace(a,b,n)`	row vectors with `n` "logarithmically-spaced" points in range `[10^a,10^b]`
`I=find(A)`	`I`: Index of non zero elements of `A`
`disp(A)`	Print on screen of `A` (String: `'name'`)

C.2 Control Systems Commands

Befehl	Beschreibung
`sys=ss(A,B,C,D)`	State-Space M. with `A,B,C,D` in time domain
`sys=ss(A,B,C,D,Ts)`	State-Space M. with `A,B,C,D` and sampling `Ts` (discrete-time)
`sys=zpk(Z,P,K)`	State-Space M. with zeros `Z`, poles `P` and gain `K`
`sys=zpk(Z,P,K,Ts)`	State-Space M. with zeros `Z`, poles `P`, gain `K` and sampling `Ts`
`sys=tf([bm ...b0],[an ...a0])`	Transfer function with `bn` in numerator and `an` in denom.
`P=tf(sys)`	Transfer function of `sys`
`P.iodelay=...`	Insers to `P` delay.
`pole(sys)`	Poles of System
`zero(sys)`	Zeros og System
`[z,p,k]=zpkdata(sys)`	`z`: Zeros, `p`: Poles, `k`: static gain
`ctrb(sys)` or `ctrb(A,b)`	Controllability Matrix
`obsv(sys)` or `obsv(A,c)`	Observability Matrix
`series(sys1,sys2)`	series of `sys1` and `sys2`
`feedback(sys1,sys2)`	`sys1` with `sys2` as (negative) Feedback
`[Gm,Pm,Wgm,Wpm]=margin(sys)`	`Gm`: gain margin, `Pm`: phase margin, `Wpm`: crossover freq.
`[y,t]=step(sys,Tend)`	`y`: step response von `sys` until `T`, `t`: time
`[y,t]=impulse(sys,Tend)`	`y`: impulse response of `sys` until `Tend`, `t`: time
`y=lsim(sys,u,t)`	Simulation of `sys` with input `u` for the time`t`
`sim('Simulink model',Tend)`	Simulation of `Simulink Model'` until `Tend`
`p0=dcgain(sys)`	static gain ($P(0)$)
`K=lqr(A,B,Q,R)`	Gain Matrix `K` (solution of the LQR-Problem)
`[X,L,K]=care(A,B,Q)`	`X`: solution of the Riccati equation, `G`: Gain matrix
`Paug=augw(G,W1,W3,W2)`	Space State M. for $\mathcal{H}_\infty$
`[K,Cl,gamma]=hinfsyn(Paug)`	$\mathcal{H}_\infty$: `K`: Controller
`fr=evalfr(sys,f)`	`sys` evaluated in `f` ($s = f$)
`sysd=c2d(sys,Ts,method)`	Discretization of `sys` with `method` with Sampling Time `Ts`

C.3 Plot and Diagrams

Befehl	Beschreibung
`nyquist(sys)`	Nyquist diagram of the system `sys`
`nyquist(sys,{a,b})`	Nyquis diagramm in interval `[a,b]` of the system `sys`
`bode(sys)`	Bode diagram of the system `sys`
`bode(sys,{a,b})`	Bode diagram in intervall `[a,b]` of the system `sys`
`bodemag(sys)`	Bode diagram (just magnitude) of the system `sys`
`bodemag(sys,{a,b})`	Bode diagram (just magnitude) in interval `[a,b]` of the system. `sy`
`rlocus(sys)`	Root Locus diagram
`impulse(sys)`	Impulse Response of the system `sys`
`step(sys)`	Step response of the system `sys`
`pzmap(sys)`	Poles and zeros mapping of the system `sys`
`svd(sys)`	Singular values dynamics of the dystem `sys`
`plot(X,Y)`	Plot of `Y` as function of `X`
`plot(X,Y,...,Xn,Yn)`	Plot of `Yn` as function of `Xn` (for all `n`)
`stem(X,Y)`	Discrete plot of `Y` as function of `X`
`stem(X,Y,...,Xn,Yn)`	Discrete plot of `Yn` as function of `Xn` (for all `n`)
`xlabel('name')`	Name of the `x`-Axis
`ylabel('name')`	Name of the `y`-Axis
`title('name')`	Title of the plot
`xlim([a b])`	Range for the `x`-Axis (Plot between `a` and `b`)
`ylim([a b])`	Range for the `y`-Axis (Plot between `a` and `b`)
`grid on`	Grid
`title('name')`	Title of the plot
`legend('name1',...,'name')`	Legend
`subplot(m,n,p)`	Grid `m`×`n`, Plot in Position `p`
`semilogx(X,Y)`	Logarithmitic Plot with `y`-Axis linear

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