Exercise 1 - Introduction

1.1   Aim of the Course

The aim of this course is to come up to *mathematical representations* or *models* for given systems. We distinguish in particular three purposes:

- Analysis and understanding,

- simulation,

- control.

1.2   Preliminary Definitions

Let's define some terms which will be often used along the course and which lay the foundation for system modeling.

Definition 1. A **dynamic system** is a non static system which evolves with respect to time. The time evolution can be caused by

- input signals,

- external perturbations,

- naturally.

Examples of time evolutions are multiple:

- Trajectory changes (position, velocity, and acceleration).

- Physical properties changes (temperature, pressure, volume, etc.).

One can subdivide mathematical models of dynamic systems in two major classes:

- parametric models

- non-parametric models.

Definition 2. A **parametric model** describes a system through its parameters.

Example 1. Mechanical systems are described with the general differential equation

$$m\ddot{y}(t) + d\dot{y}(t) + ky(t) = F(t),$$

where the parameters are: mass m, viscous damping coefficient d, and spring constant k.

Definition 3. A **non-parametric model** describes the system through the knowledge of its *actual* response to an actual given input.

Example 2. Knowing the impulse response of a damped mechanical oscillator.

In this course, we will consider only parametric models. We can further distinguish three different types of models:

- **Black-box models**: derived from experiments only.

- **Grey-box models**: starting from a model, one uses experiments to identify model parameters and validate the model.

- **White-box models**: no experiments needed, since one knows each part of the system.

1.3   How to Build a Model

When modeling a system, two big groups of actors are important to be defined:

Definition 4. The **reservoirs** are accumulative elements. Examples of reservoirs are various energies, mass, informations. Moreover, two important aspects should be underlined:

- Only systems including one or more reservoirs exhibit dynamic behaviour and

- for each reservoir an associated *level* variable can be defined, being a function of the reservoir's content (**state variable**).

Definition 5. Flows describe the flowing of heat, mass, etc. between the reservoirs.

In order to be able to model a system, a specific methodology should be used.

Reservoir-based Approach

For a given system:

(I) Define the system boundaries:

- Inputs: what can you control?
- Outputs: what can you measure?

(II) Identify the relevant reservoirs and the corresponding state variables.

(III) Formulate the differential equations which decribe the relevant reservoirs. These are usually conservation laws and can be generally expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\text{reservoir content} \right) = \sum \text{inflows} - \sum \text{outflows}. \tag{1.1}$$

(IV) Formulate the algebraic relations which express the flows between the reservoirs. These should be expressed as a function of the state variables.

(V) Solve algebraic equations and simplify the resulting mathematical relations as much as possible.

(VI) Identify the unknown system parameters using experiments.

(VII) Validate the model with experiments different from the ones used to identify the parameters.

Remark. With relevant dynamics one refers to the variables whose time constant is of the same of magnitude as the one to the sytem. Too quick dynamics translate to algebraic relations, whereas too slow dynamics lead to constant variables.

Causality Diagram

A causality diagram is a graphical representation of the system equations. Here, we distinguish between algebraic and dynamic blocks, as shown in Figure 1. The blocks are connected through arrows, which represent variables.

Figure 1: Block types.

Remark. In general, there are mulitple ways to draw the causality diagram of a system.

Example 3. For a general mass-spring mechanical system with mass m and spring constant k, one can write the Newton's law as

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = u - F_{\text{spring}},$$

where

$$F_{\text{spring}} = kx$$

is the spring force. In this case, one can observe two types of relations: the Newton lows contains derivatives in time and hence dynamics. The spring force is an algebraic relation ($y = f(x)$). The causality diagram hence is as shown in Figure 2.
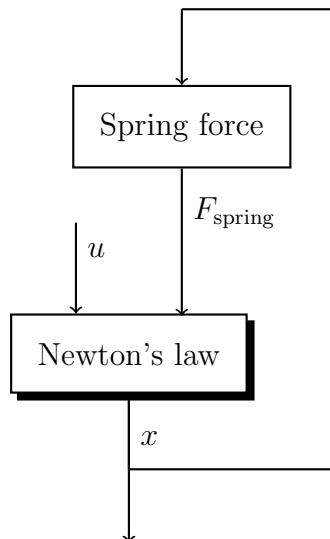

Figure 2: Causality diagram of the system.

1.4   Example

In recent years, the market of food trucks has seen a tremendous increase. Willing to get into the market, you decide to found the company SpaghETH. You decide to focus on serving fresh pasta with different sauces everyday. In order not to run into delays during your activity, you aim to have hot water right after parking and therefore you decide to warm up the water while driving. To find the optimal way to do that, you start by formulating a model of the system.
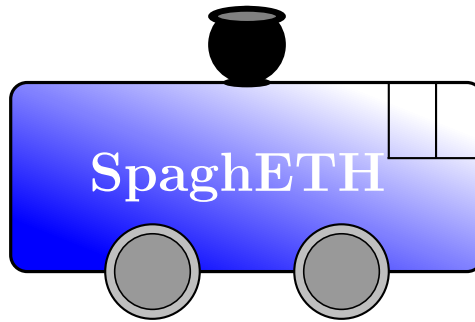

Figure 3: Sketch of the system.

The truck is modeled as a point mass of mass m. The propulsive power acting on the truck is given by

$$P_{\mathrm{p}}(\theta) = P_{\max} \cdot (1 - \exp(-c_1\theta)),$$

where $\theta(t)$ is the (normalized) angular position of the pedal and c_1 and $P_{\max}$ are known constants. The truck is also subject to the aerodynamic drag force

$$F_{\mathrm{drag}} = \frac{1}{2}c_2 v^2,$$

where c_2 is a known constant. Clearly, your truck is equipped with a pot of known area A and volume V. Your cooker allows you to set the heat flow given to the water. The heat trasfer coefficient α between the water surface and the air is a known function velocity of the truck. The ambient temperature T_∞ as well as the specific heat c and the density ρ of water are known. You may assume no water evaporates. Your truck is equipped with one sensor measuring the velocity and a temperature sensor.

1. Determine the inputs and the outputs of the system.

2. List the reservoir(s) and the corresponding level variable(s).

3. Draw a causality diagram of the system.

4. Formulate the differential/algebraic equations needed to describe the system.

5. Is the system linear or nonlinear? Explain.

Solution.

1. The inputs are the position of the pedal θ and the heat trasfer to the water $\dot{Q}$. The outputs are the temperature of the water in the pot and the velocity of the truck. Furthermore, assume that the mass of the truck is much larger than the mass of the driver and of the equipment.

2. The system has two reservoirs:

   - the kinetic energy of the truck E_{kin}, whose level variable is the velocity v of the truck;

   - the thermal energy of the water in the pot U, whose level variable is the temperature of the water in the pot.

   Note that, since no vapor leaves the water, the mass of the water in the system is not a reservoir.
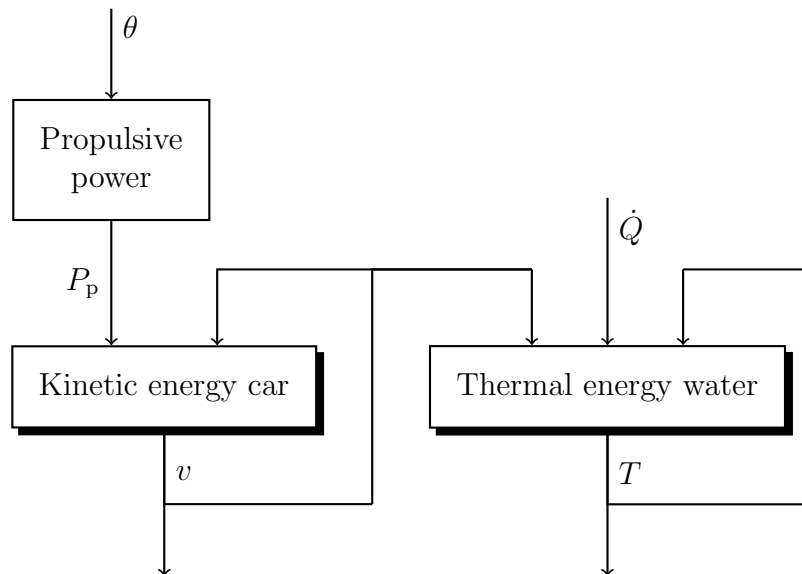
3. The causality diagram is shown in Figure 4.

Figure 4: Causality diagram of the system.

4. The algebraic equation that relates the propulsive power and the position of the pedal reads
$$P_{\text{p}}(\theta) = P_{\text{max}} \cdot (1 - \exp(-c_1\theta)).$$

   The differential equation for the kinetic energy of the car

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\text{kin}} = P_{\text{in}} - P_{\text{out}}$$
$$= P_{\text{p}} - F_{\text{drag}} \cdot v$$

   leads to

$$mv \cdot \frac{\mathrm{d}}{\mathrm{d}t}v = P_{\text{p}} - \frac{1}{2}c_2 v^3,$$

where we used $\frac{d}{dt}v^2 = 2v\frac{d}{dt}v$. The heat balance of the pot

$$\frac{d}{dt}U = P_{\text{in}} - P_{\text{out}}$$
$$= \dot{Q} - \dot{Q}_{\text{loss}}$$

gives

$$\rho V c \cdot \frac{d}{dt}T = \dot{Q} - \alpha(v)A \cdot (T - T_\infty).$$

5. The system is nonlinear. Examples of nonlinearities are the relation between the pedal angular position and the propulsive power and the dynamics of the car.