

Exercise 3 - Lagrange Formalism

3.1 The Lagrange Formalism

The Lagrange formalism is a mechanical method that allows to derive the equations of motion for a mechanical system. The big advantage of the Lagrange formalism over the Newton formulation, i.e. over $F = m \cdot \ddot{x}$, lies in the analysis of the constraints. In fact, constraints can be included much more easily in the Lagrange formalism.

3.1.1 Generalized Coordinates

Generalized coordinates are coordinates that can be used to describe a system uniquely. Usually, they are stored in a vector \vec{q} . Moreover, constraints reduce the system's degrees of freedom and generalized coordinates can be adapted in order to describe them.

3.1.2 Holonomic vs. Non-holonomic Constraints

The constraints acting on a mechanical system can be divided into two groups. **Non-holonomic** constraints and **holonomic** constraints. The most general of the two constraints' types are non-holonomic constraints, which can be described by

$$f(\vec{q}, \dot{\vec{q}}, t) = 0,$$

where \vec{q} is the generalized-coordinates vector. When no dependency on the general velocities (state-variable vector derived) is present, the constraint is holonomic. This type of constraints can be once more subdivided into two subcategories: scleronomic constraints are described with

$$f(\vec{q}) = 0,$$

and rheonomic constraints with

$$f(\vec{q}, t) = 0.$$

The reason why holonomic constraints are important is that they allow us reducing the number of variables used to describe a system. This is done by removing dependent variables. Let's assume to have a constraint acting on a system described by n general coordinates q_i :

$$f(q_1, \dots, q_n, t) = 0.$$

Then, only if a constraint is holonomic, we can write

$$q_1 = f(q_2, \dots, q_n). \tag{3.1}$$

Remark. If interested, recall the *implicit function theorem*.

In general, it holds

$$p = n - r,$$

where p is the number of degrees of freedom of a system, n the number of generalized coordinates used to describe the system and r the number of holonomic constraints acting on the system. Moreover, if a non-holonomic constraint can be integrated (is integrable), then it will become holonomic and eliminates degrees of freedom as well.

Integrability

Every holonomic constraint f can be written in the *Pfaffian Form*

$$\begin{aligned} \frac{df}{dt} &= \sum_{i=1}^n \frac{\partial f(\vec{q}, t)}{\partial q_i} \dot{q}_i + \frac{\partial f(\vec{q}, t)}{\partial t} = 0 \\ &= \frac{df}{dt} = \sum_{i=1}^n a_i(\vec{q}, t) \dot{q}_i + b(\vec{q}, t) = 0 \end{aligned} \quad (3.2)$$

3.1.3 The Lagrange Method

Given a mechanical system, the method reads

- (I) Identify the generalized-coordinates vector \vec{q} .
- (II) Is the system holonomic or non holonomic? Are non-holonomic constraints integrable?
- (III) Define the total kinetic and the potential energy of the system. Recall that the kinetic energy expressed with respect to the center of mass of the system reads

$$T = \underbrace{\frac{1}{2} m \vec{v}_s^T \vec{v}_s}_{\text{translation}} + \underbrace{\frac{1}{2} \vec{\Omega}^T \Theta \vec{\Omega}}_{\text{rotation}}.$$

Recall that the potential energy for a mechanical system generally reads

$$U = mgh + \underbrace{\frac{1}{2} k_{\text{lin}} \Delta x^2}_{\text{linear spring}} + \underbrace{\frac{1}{2} k_{\text{rot}} \Delta \varphi^2}_{\text{torsional spring}}. \quad (3.3)$$

- (IV) Define the Lagrange function

$$\mathcal{L}(\vec{q}, \dot{\vec{q}}) = T(\vec{q}, \dot{\vec{q}}) - U(\vec{q}),$$

where T is the total kinetic energy of the system and U is the total potential energy of the system.

- (V) If there are forces and torques acting on the system, contrary to the holonomic constraints these should be taken into account. In order to do this, we want to compute the generalized forces Q_k . Suppose to have a force \vec{F} acting on point A of the system. The velocity of point A can always be written as

$$\vec{v}_A = J_A \dot{\vec{q}} + \vec{v}_A,$$

where J_A is the translational Jacobian matrix of the translation in Q , \vec{q} is the generalized-coordinates vector and \vec{v}_A is an offset term. The resulting general force can be written as

$$\vec{Q}_A = J_A^T \vec{F}.$$

Suppose to have a torque \vec{M} acting on the body. The angular velocity of the system $\vec{\Omega}$ can always be written as

$$\vec{\Omega} = J_R \dot{\vec{q}} + \vec{v}_E,$$

where J_R is the rotational Jacobian matrix and \vec{v}_E is an offset term. The resulting general force can be written as

$$\vec{Q}_R = J_R^T \vec{M}.$$

(VI) Write the Lagrange formalism for each generalized coordinate q_k . For holonomic systems this reads

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}}{\partial q_k} = Q_k, \quad k = 1, \dots, n, \quad (3.4)$$

where Q_k are the generalized forces of the system.
These k relations build the system's equations of motion.

3.2 Example

Due to the large demand, your SpaghETH shall increase the production. You decide to automate the blending to the pasta in the water by carefully optimizing its operation. To do that, you start by formulating a model of the system. A sketch is shown in Figure 1.

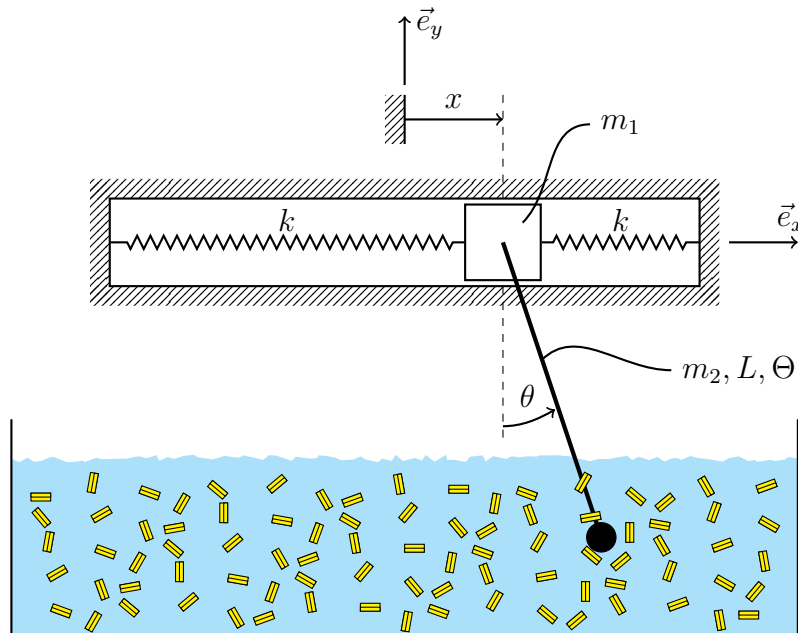


Figure 1: Sketch of the system.

The blender is modeled as a bar of mass m_2 , length L , and moment of inertia (w.r.t. center of mass) $\Theta = \frac{1}{12}m_2L^2$. The blender is attached to a point mass m_1 . In order to deal with possible vibrations that might occur in the system, the mass is attached to two springs with spring constant k . Assume the springs are unstretched at $x = 0$. Gravitational effects are to be considered; friction and aerodynamic losses are neglected.

1. How many degrees of freedom does the system have? List at least two possible choices of generalized coordinates.
2. Is the system holonomic?
3. Is the system conservative?
4. Determine the kinetic energy of the system.
5. Determine the potential energy of the system.
6. Determine the equations of motion of the system.
7. Describe qualitatively how your answer would change if
 - (a) an external force F acts horizontally on the mass m_1 .
 - (b) the system is brought in an electric field.

Solution.

1. The system has two degrees of freedom. Possible choices of generalized coordinates are (x, θ) and (x, ϕ) , where ϕ denotes the angle between the bar the horizontal bar. Note that (x, x_{bar}) and (x, y_{bar}) are not valid choices.
2. Since there are no constraints, the system is holonomic.
3. All forces acting on the system have a potential (or do no work) and therefore the system is conservative.
4. Let $\vec{r}_1(t)$ be the position vector of the mass m and $\vec{r}_2(t)$ the position of the center of mass of the bar. In order to compute the kinetic energy, we first compute the velocities of the mass m and of the bar. In an inertial frame we get

$$\vec{r}_1(t) = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}, \quad \vec{r}_2(t) = \begin{bmatrix} x + \frac{L}{2} \sin(\theta) \\ -\frac{L}{2} \cos(\theta) \\ 0 \end{bmatrix}.$$

The velocities are then

$$\dot{\vec{r}}_1(t) = \begin{bmatrix} \dot{x} \\ 0 \\ 0 \end{bmatrix}, \quad \dot{\vec{r}}_2(t) = \begin{bmatrix} \dot{x} + \frac{L}{2} \dot{\theta} \cos(\theta) \\ \frac{L}{2} \dot{\theta} \sin(\theta) \\ 0 \end{bmatrix}.$$

The angular velocity of the bar is given by

$$\vec{\omega} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}.$$

Thus, the kinetic energy of the system is

$$\begin{aligned} T &= \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2 + \frac{1}{2} \Theta |\vec{\omega}|^2 \\ &= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left(\left(\dot{x} + \frac{L}{2} \dot{\theta} \cos(\theta) \right)^2 + \frac{L^2}{4} \dot{\theta}^2 \sin^2(\theta) \right) + \frac{1}{2} \frac{1}{12} m_2 L^2 \dot{\theta}^2 \\ &= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left(\dot{x}^2 + \frac{L^2}{4} \dot{\theta}^2 + \dot{x} L \dot{\theta} \cos(\theta) \right) + \frac{1}{2} \frac{1}{12} m_2 L^2 \dot{\theta}^2. \end{aligned}$$

Remark. Alternatively, the velocity of the center of mass of the bar can be computed with

$$\dot{\vec{r}}_2 = \dot{\vec{r}}_1 + \vec{\omega} \times \vec{r}_{12} = \begin{bmatrix} \dot{x} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \times \begin{bmatrix} \frac{L}{2} \sin(\theta) \\ -\frac{L}{2} \cos(\theta) \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{x} + \frac{L}{2} \dot{\theta} \cos(\theta) \\ \frac{L}{2} \dot{\theta} \sin(\theta) \\ 0 \end{bmatrix}.$$

5. The potential energy of the system is

$$U = \frac{1}{2} k x^2 + \frac{1}{2} k x^2 - m_2 g \frac{L}{2} \cos(\theta) = k x^2 - m_2 g \frac{L}{2} \cos(\theta).$$

6. The Lagrangian is defined as

$$\begin{aligned}\mathcal{L} &= T - V \\ &= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x}^2 + \frac{L^2}{4}\dot{\theta}^2 + \dot{x}L\dot{\theta}\cos(\theta)\right) + \frac{1}{2}\frac{1}{12}m_2L^2\dot{\theta}^2 - kx^2 + m_2g\frac{L}{2}\cos(\theta).\end{aligned}$$

The derivatives are

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\dot{x}} &= m_1\dot{x} + m_2\dot{x} + \frac{1}{2}m_2L\dot{\theta}\cos(\theta), \\ \frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= m_2\frac{L^2}{4}\dot{\theta} + \frac{1}{2}m_2L\dot{x}\cos(\theta) + \frac{1}{12}m_2L^2\dot{\theta}, \\ \frac{\partial\mathcal{L}}{\partial x} &= -2kx, \\ \frac{\partial\mathcal{L}}{\partial\theta} &= -\frac{1}{2}m_2\dot{x}L\dot{\theta}\sin(\theta) - m_2g\frac{L}{2}\sin(\theta).\end{aligned}$$

The equation of motion for the generalized coordinate x is

$$\begin{aligned}\frac{d}{dt}\left(m_1\dot{x} + m_2\dot{x} + \frac{1}{2}m_2L\dot{\theta}\cos(\theta)\right) - (-2kx) &= 0 \\ \Rightarrow (m_1 + m_2)\ddot{x} + \frac{1}{2}m_2L\left(\ddot{\theta}\cos(\theta) - \dot{\theta}^2\sin(\theta)\right) + 2kx &= 0.\end{aligned}$$

The equation of motion for the generalized coordinate θ is

$$\begin{aligned}\frac{d}{dt}\left(m_2\frac{L^2}{4}\dot{\theta} + \frac{1}{2}m_2L\dot{x}\cos(\theta) + \frac{1}{12}m_2L^2\dot{\theta}\right) - \left(-\frac{1}{2}m_2\dot{x}L\dot{\theta}\sin(\theta) - m_2g\frac{L}{2}\sin(\theta)\right) &= 0 \\ \Rightarrow m_2\left(\frac{L^2}{4} + \frac{L^2}{12}\right)\ddot{\theta} + \frac{1}{2}m_2L\left(\ddot{x}\cos(\theta) - \dot{x}\dot{\theta}\sin(\theta)\right) + \frac{1}{2}m_2\dot{x}L\dot{\theta}\sin(\theta) + m_2g\frac{L}{2}\sin(\theta) &= 0.\end{aligned}$$

7. (a) The system would not be conservative anymore. The force F has to be considered as generalized force Q_1 acting on the generalized coordinate x . In particular, we have $Q_1 = F$.
- (b) The system is still conservative. The electric potential can be included in U .