Exercise 4 - Hydraulic Systems

4.1 Hydraulic Systems

Hydraulic systems are in general described by the *Navier-Stokes equations* as you learned in Fluiddynamics. In order to simplify the modeling of such systems, it is in general worth to use simpler formulations. In general, typical elements which build up hydraulic systems are

- Ducts,
- compressible nodes,
- valves.

4.1.1 Hydraulic Ducts

A sketch for a water duct is depicted in Figure 1. Elements of the drawing are the height of inclination h, the top and bottom pressures $p_1(t), p_2(t)$, the length of the duct l, the velocity of the water flowing into the duct $v(t)$, the density of the flowing fluid ρ, and the cross-sectional area of the duct A. In general, we are interested in modeling the velocity

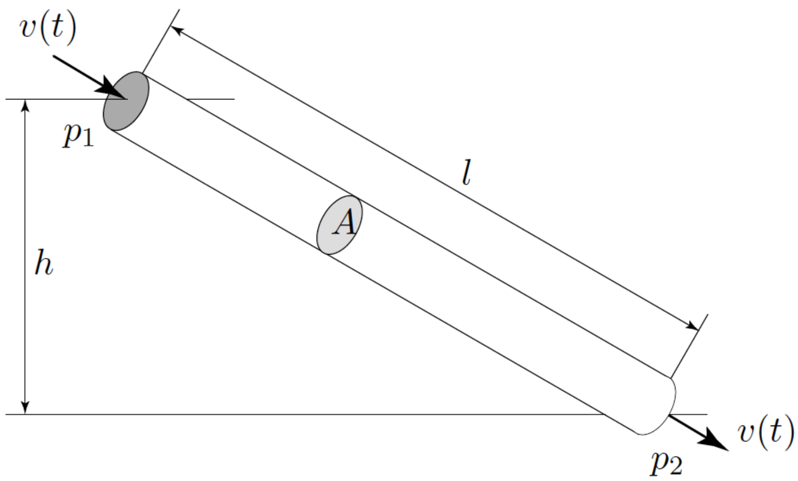

Figure 1: Sketch of a water duct.

of the water, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = f(p_1(t), p_2(t), v(t), h, \rho, A, l) \tag{4.1}$$

Applying the Newton's law to the given water duct along its longitudinal axis one gets

$$m \cdot \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \vec{F}_{\text{pressure}} + \vec{F}_{\text{gravity}} + \vec{F}_{\text{fric}} \tag{4.2}$$

The force caused by a pressure can be described with

$$\vec{F} = (p \cdot A) \cdot \vec{e}_{\text{long}}, \tag{4.3}$$

where A represents an area. Applying this with the constant area of the duct A, the pressure force reads

$$\vec{F}_{\text{pressure}} = (p_1(t) \cdot A - p_2(t) \cdot A) \cdot \vec{e}_{\text{long}}. \tag{4.4}$$

The angle of inclination of the duct follows the law

$$\sin(\alpha) = \frac{\mathrm{d}h}{\mathrm{d}l}$$

The mass of the fluid in the duct (water in this case) is given by

$$\begin{aligned} \rho \cdot V &= \rho \cdot A \cdot l \\ \Rightarrow \mathrm{d}m &= \rho \cdot A \cdot \mathrm{d}l. \end{aligned} \tag{4.5}$$

One can then use the definition of grAHitational force to compute the second term. This reads

$$\begin{aligned} \vec{F}_{\text{gravity}} &= \int_{\text{duct}} \vec{g} \cdot \mathrm{d}m \\ &= g \int_{\text{duct}} \begin{pmatrix} \sin(\alpha) \\ -\cos(\alpha) \end{pmatrix} \cdot \rho \cdot A \cdot \mathrm{d}l \\ &= \rho \cdot g \cdot A \cdot \left(\int_0^h \begin{pmatrix} \frac{\sin(\alpha)}{\sin(\alpha)} \\ -\frac{\cos(\alpha)}{\sin(\alpha)} \end{pmatrix} \mathrm{d}l \right) \\ &= h \cdot \rho \cdot g \cdot A \cdot \begin{pmatrix} 1 \\ -\frac{1}{\tan(\alpha)}. \end{pmatrix} \end{aligned}$$

The friction force depends on the shape factor $\frac{l}{d}$ and reads

$$\vec{F}_{\text{fric,x}}(t) = \frac{1}{2} \cdot \rho \cdot v(t)^2 \cdot \text{sign}(v(t)) \cdot \lambda(v(t)) \cdot \frac{Al}{d}, \tag{4.6}$$

where λ is to read from the *Moody-Diagram*. Plugging the found results in Equation 4.2 one gets the conservation law along the longitudinal axis of the duct

$$m\frac{\mathrm{d}}{\mathrm{d}t}v(t) = \rho \cdot l \cdot A \cdot \frac{\mathrm{d}}{\mathrm{d}t}v(t) = A \cdot (p_1(t) - p_2(t)) + A \cdot \rho \cdot g \cdot h - F_{\text{fric,x}}(t). \tag{4.7}$$

4.1.2 Compressibility

The **compressibility** is the property of a body (solid, liquid, gas,...) to deform under the effect of an applied pressure. Mathematically, it is defined as

$$\sigma_0 = \frac{1}{V_0}\frac{\mathrm{d}V}{\mathrm{d}P}, \tag{4.8}$$

where V_0 [m^3] is nominal volume, P [Pa] is the pressure and σ_0 [Pa^{-1}] is the compressibility. Although this is very small for liquid fluids, gas presence or elastic walls could increase its effects. In order to take this into account we model a lumped-parameter element, as depicted in Figure 2. The element causes a change in volume, which produces a change in pressure of the fluid. Mathematically, we define the change in volume with the conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = \overset{*}{V}_{\text{in}}(t) - \overset{*}{V}_{\text{out}}(t). \tag{4.9}$$

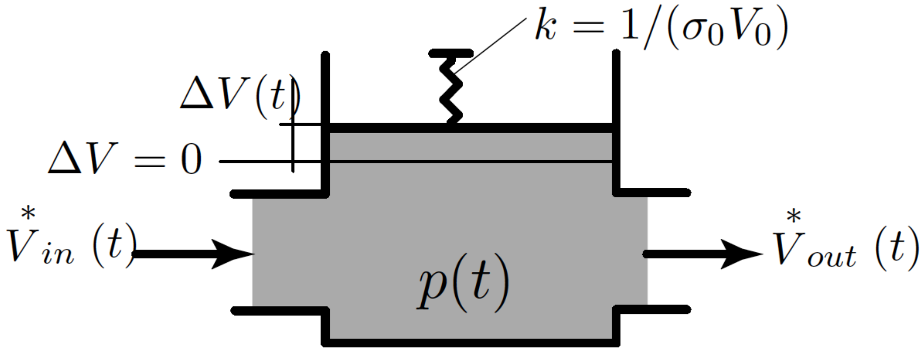

Figure 2: Sketch of a lumped-parameter element.

The elasticity effects can be described by defining the pressure with respect to time, i.e.

$$p(t) = \frac{1}{\sigma_0} \cdot \frac{\Delta V(t)}{V_0}, \qquad \Delta V(t) = V(t) - V_0, \tag{4.10}$$

where σ_0 and V_0 must be determined experimentally in each case.

4.2 Example

Your SpaghETH is growing every week more and although no particular production issues occur you are concerned about ecology. Since each tank of pasta you cook needs water and a correct salt seasoning for it to taste that delicious, you need a lot of salt and water, which are often wasted. For this reason, you open a research branch in your startup which decides to design a duct-hydraulic system to counteract the waste of water and salt. The tank where the pasta cooks has a duct, named *Tunnel*, connected to it. Inside of the *Tunnel* there are a salt source and a hydraulic element. The hydraulic element increases the water's pressure by Δp. Moreover, the water with increased pressure (and hence velocity) is then seasoned in a second duct, called *Seasoner*, compensating the loss of salt due to pasta's absorption. Note that only compressibility effects of the *Tunnel* should be taken into account. The pressure at the water's surface p_∞ is assumed to be known. Assume a circular tunnel, whose area reads $A_\mathrm{T} = \frac{d_\mathrm{T}^2 \cdot \pi}{4}$. Assume that the area of the water tank is A_W and is known. A sketch of the system with the relative parameters is shown in Figure 3.

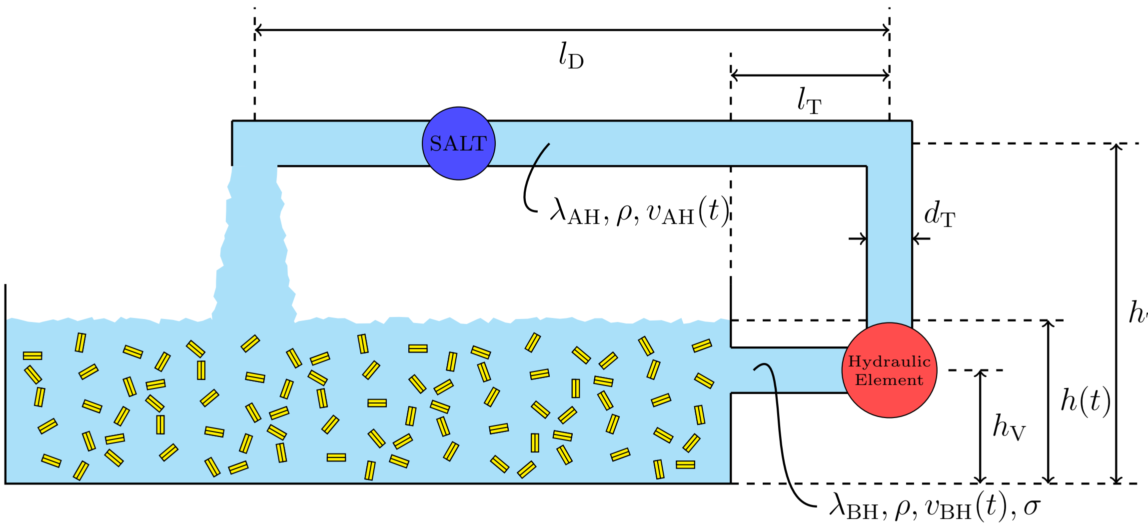

Figure 3: Sketch of the system.

1. List all the reservoirs and the relative state variables.
2. Find the pressure $p_1(t)$ at the beginning of the *Tunnel* as a function of the velocity in the *Tunnel* $v_\mathrm{BH}(t)$.
3. Formulate the differential equation for $v_\mathrm{BH}(t)$ as function of the pressure right before the hydraulic element $p_2(t)$.
4. Exploiting the compressibility of the *Tunnel*, find the pressure $p_2(t)$ explicitly.
5. Formulate the differential equation for $v_\mathrm{AH}(t)$.
6. Formulate the differential equation for $h(t)$.

Solution.

1. The reservoirs of the system and their relative state variables are:
   - The *Tunnel* kinetic energy. The state variable of this reservoir is the velocity of the water in the *Tunnel* $v_{\mathrm{BH}}(t)$.
   - The *Seasoner* kinetic energy. The state variable of this reservoir is the velocity of the water in the *Tunnel* $v_{\mathrm{AH}}(t)$.
   - The water mass in the tank. The state variable of this reservoir is the water's height $h(t)$.
   - The compressibility of the *Tunnel*. The state variable of this reservoir is the water volume $V(t)$.

2. **Bernoulli:**
   First, we need to use Bernoulli's law from the water's surface to the beginning of the *Tunnel*, in order to find the local pressure. This reads

$$\begin{aligned} 0^2 + \frac{p_\infty}{\rho} + g \cdot h(t) &= \frac{v_{\mathrm{BH}}(t)^2}{2} + \frac{p_1(t)}{\rho} + \rho \cdot g \cdot h_{\mathrm{V}} \\ p_1(t) &= p_\infty + g \cdot \rho \cdot (h(t) - h_{\mathrm{V}}) - \rho \cdot \frac{v_{\mathrm{BH}}(t)^2}{2}. \end{aligned} \tag{4.11}$$

3. **Tunnel:**
   The impulse equation for the *Tunnel* reads

$$\rho \cdot l_{\mathrm{T}} \cdot A_{\mathrm{T}} \cdot \frac{\mathrm{d}}{\mathrm{d}t} v_{\mathrm{BH}}(t) = A_{\mathrm{T}} \cdot (p_1(t) - p_2(t)) - F_{\mathrm{fric}}(t), \tag{4.12}$$

   where $F_{\mathrm{fric}}(t)$ is the friction force, which reads

$$F_{\mathrm{fric}}(t) = A_{\mathrm{T}} \cdot \lambda_{\mathrm{BH}} \cdot \frac{l_{\mathrm{T}} \cdot \rho}{2 \cdot d_{\mathrm{T}}} \cdot \mathrm{sign}(v_{\mathrm{BH}}(t)) \cdot v_{\mathrm{BH}}(t)^2. \tag{4.13}$$

   The pressure $p_2(t)$ is obtained from the compressibility of the *Tunnel*.

4. **Compressibility:**
   As learned in the lecture, the pressure before the valve $p_2(t)$ is computed as

$$\begin{aligned} p_2(t) &= \frac{1}{\sigma} \cdot \frac{\Delta V(t)}{V_0} + p_{\mathrm{stat}} \\ &= \frac{1}{\sigma} \cdot \frac{V(t) - V_0}{V_0} + \rho \cdot g \cdot (h(t) - h_{\mathrm{V}}) + p_\infty, \end{aligned} \tag{4.14}$$

   where

$$V_0 = \frac{l_{\mathrm{T}} \cdot \pi \cdot d_{\mathrm{T}}^2}{4}. \tag{4.15}$$

   The volume balance reads

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &= V^*_{\mathrm{in}} - V^*_{\mathrm{out}} \\ &= A_{\mathrm{T}} \cdot (v_{\mathrm{BH}}(t) - v_{\mathrm{AH}}(t)). \end{aligned} \tag{4.16}$$

5. **Seasoner:**
   The impulse equation for the *Seasoner* reads

$$\rho \cdot (l_{\mathrm{D}} + h_{\mathrm{T}} - h_{\mathrm{V}}) \cdot A_{\mathrm{T}} \cdot \frac{\mathrm{d}}{\mathrm{d}t} v_{\mathrm{AH}}(t) = A_{\mathrm{T}} \cdot (p_2(t) + \Delta p - p_\infty) + A_{\mathrm{T}} \cdot \rho \cdot g \cdot (h_{\mathrm{V}} - h_{\mathrm{T}}) - F_{\mathrm{fric}}(t), \tag{4.17}$$

   where $F_{\mathrm{fric}}(t)$ is the friction force, which reads

$$F_{\mathrm{fric}}(t) = A_{\mathrm{T}} \cdot \lambda_{\mathrm{AH}} \cdot \frac{(l_{\mathrm{D}} + h_{\mathrm{T}} - h_{\mathrm{V}}) \cdot \rho}{2 \cdot d_{\mathrm{T}}} \cdot \mathrm{sign}(v_{\mathrm{AH}}(t)) \cdot v_{\mathrm{AH}}(t)^2. \tag{4.18}$$

6. **Water Tank:**
   The water tank stores water's mass. Its state variable is the height of the water tank $h(t)$. The mass balance reads

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} m(t) &= \rho \cdot A_{\mathrm{W}} \cdot \frac{\mathrm{d}h(t)}{\mathrm{d}t} \\ &= \overset{*}{m}_{\mathrm{in}} - \overset{*}{m}_{\mathrm{out}} \\ &= \rho \cdot A_{\mathrm{T}} \cdot (v_{\mathrm{AH}}(t) - v_{\mathrm{BH}}(t)). \end{aligned} \tag{4.19}$$

   This leads to a relation for the change of the water height

$$\frac{\mathrm{d}}{\mathrm{d}t} h(t) = \frac{A_{\mathrm{T}}}{A_{\mathrm{W}}} \cdot (v_{\mathrm{AH}}(t) - v_{\mathrm{BH}}(t)) \,. \tag{4.20}$$