Exercise 7 - Fluiddynamic Systems

7.1 Valves

Flows of fluids between reservoirs are characterized by **valves**. The inputs of these elements are the pressure of the upstreams/downstreams. We consider the upstream pressure p_{in} and the downstream pressure p_{out}. The two modeling approaches are:

- Incompressible fluids;
- Compressible fluids.

7.1.1 Valves with Incompressible Fluids

First, some assumptions should be made:

- The friction effects in the flow is modeled accounting it with experimentally determined correction factors.
- Inertial effects in the flow are neglected. Note that the mass of fluid around the valve is very small if compared to the mass stored in the receiving reservoirs.
- We consider an insulated system.
- All flow-phenomena are *zero dimensional*, i.e., spatial effects are neglected.

Incompressible fluids (constant density) are, e.g., liquids and fluids at low **Mach** numbers, where the Mach number is defined as

$$M = \frac{u}{c},$$

with u local flow velocity and c speed of sound the medium.

Remark. As a rule of thumb, we can consider a fluid to be incompressible if $M < 0.3$ and the flow is quasi-steady and isothermal.

The fluid mass flow $\overset{*}{m}$ can be then modeled using *Bernoull's law*. This reads

$$\overset{*}{m}(t) = c_{\text{d}} \cdot A \cdot \sqrt{2\rho} \cdot \sqrt{p_{\text{in}}(t) - p_{\text{out}}(t)},$$

where c_{d} is the discharge coefficient (factor which takes into account flow restrictions, friction and other losses), A is the open area of the valve, ρ is the density of the fluid, p_{in} and p_{out} are the upstream and downstream pressures of the valve.

7.1.2 Valves with Compressible Fluids

The key concept to model compressible fluids through valves is the **Isenthalpic throttle**.

Isenthalpic Process

A fluid circulates in a tube with

- No moving wall (no work exerted by pressure forces).
- No heat exchange.
- $\rightarrow \mathrm{d}H = 0$, no enthalpy variation in the system.

By referring to Figure 1, one can divide the model in two distinct regions. The first part is

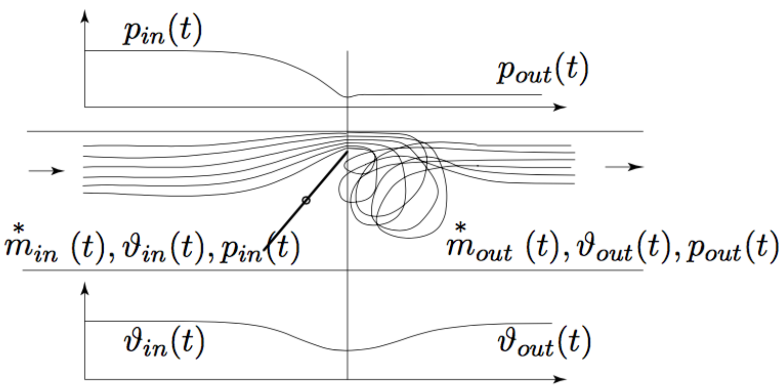

Figure 1: Two regions for the model.

the acceleration part, i.e. pressure decrease, up to the narrowest point. The flow remains laminar during this phase. All the potential energy stored in the flow (using the flow pressure as level variable) is converted isentropically (with no losses) into kinetic energy. The second part is the deceleration part, where the flow becomes turbulent. In this phase the kinetic energy is dissipated into thermal energy and no pressure recuperation occurs. With this modeling approach we can conclude that

- The pressure in the narrowest part of the valve is $\approx$ to the downstream pressure.
- The temperature of the flow before and after the valve is $\approx$ the same.

Using the first law of thermodynamics and the properties of isentropic expansions for a perfect gas, the equation for the fluid mass flow reads

$$\overset{*}{m}(t) = c_{\mathrm{d}} \cdot A(t) \cdot \frac{p_{\mathrm{in}}(t)}{\sqrt{R \cdot \vartheta_{\mathrm{in}}}} \cdot \Psi(p_{\mathrm{in}}(t), p_{\mathrm{out}}(t)),$$

where

$$\Psi(p_{\mathrm{in}}(t), p_{\mathrm{out}}(t)) = \begin{cases} \sqrt{\kappa \cdot \left(\frac{2}{\kappa+1}\right)^{\frac{\kappa+1}{\kappa-1}}} & \text{for } p_{\mathrm{out}}(t) < p_{\mathrm{cr}}(t), \\ \left(\frac{p_{\mathrm{out}}(t)}{p_{\mathrm{in}}(t)}\right)^{\frac{1}{\kappa}} \cdot \sqrt{\frac{2\kappa}{\kappa-1} \cdot \left[1 - \left(\frac{p_{\mathrm{out}}(t)}{p_{\mathrm{in}}(t)}\right)^{\frac{\kappa-1}{\kappa}}\right]} & \text{for } p_{\mathrm{out}}(t) \geq p_{\mathrm{cr}}(t), \end{cases}$$

and

$$p_{\text{cr}} = \left(\frac{2}{\kappa+1}\right)^{\frac{\kappa}{\kappa-1}} \cdot p_{\text{in}}(t).$$

At critic pressure p_{cr}, the flow reaches in its narrowest part sonic conditions. For air and similar gases a simplification can be taken into account. This reads

$$\Psi(p_{\text{in}}(t), p_{\text{out}}(t)) = \begin{cases} \frac{1}{\sqrt{2}} & \text{for } p_{\text{out}} < 0.5 p_{\text{in}}, \\ \sqrt{\frac{2p_{\text{out}}}{p_{\text{in}}} \cdot \left[1 - \frac{p_{\text{out}}}{p_{\text{in}}}\right]} & \text{for } p_{\text{out}} \geq 0.5 p_{\text{in}}. \end{cases}$$

7.2 Example

As SpaghETH is rapidly growing, you decide to buy a building where pasta, risotto, and sauces can be easily stored. Due to the large orders, your chief technical manager suggests to study a way to efficiently move around the heavy boxes. You opt for a spring-fluiddynamic system, consisting of a chamber, two valves, a piston, and a box, as depicted in Figure 2.

The chamber, modeled as a cylinder of diameter D, is connected to a reservoir of air with constant pressure $p_\mathrm{r} = 10\,\mathrm{bar}$ and constant temperature ϑ_r through a valve of opening area $A_1(t)$ and discharge coefficient $c_{\mathrm{d},1}$. Moreover, the chamber is connected to the ambient, where the pressure and temperature are $p_\infty = 1\,\mathrm{bar}$ and ϑ_∞, through second valve with opening $A_2(t)$ and discharge coefficient $c_{\mathrm{d},2}$. Experiments have shown that the pressure in the chamber changes dynamically and is (on average) approximately 1.5 bar. To model the valves you may assume constant $\kappa = 1.4$ and constant specific heats c_v and c_p. You use simplified models. The walls of the chamber have the constant temperature ϑ_∞ and the heat transfer coefficient for the internal wall is α. No heat is transferred to the piston. The piston and the box have mass m_p and mass m_b, respectively. They move frictionless on a flat surface. The spring has the constant k and is unstretched for $x = 0$.

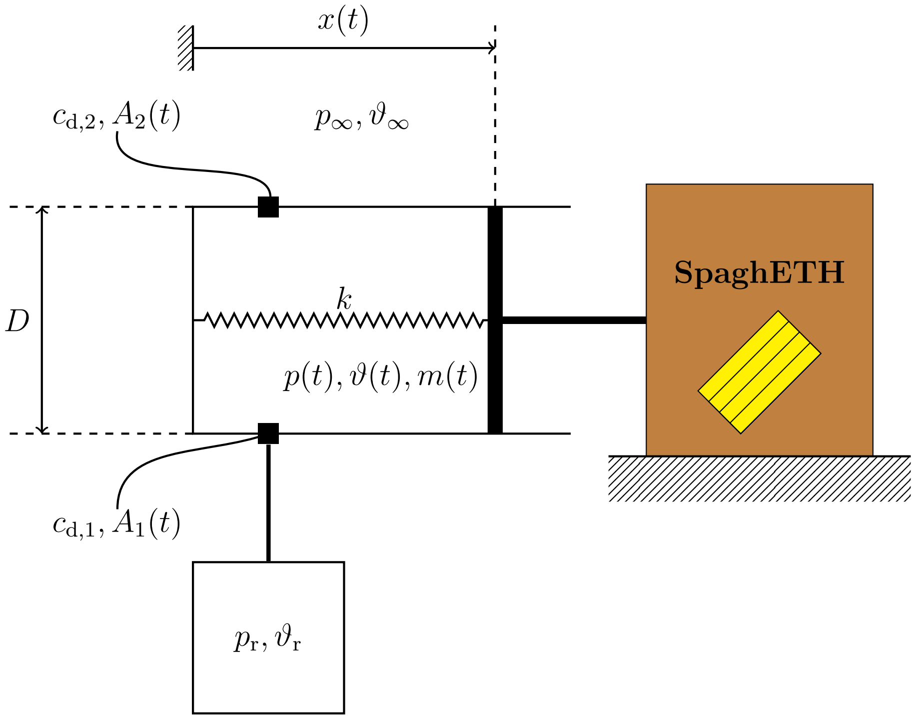

Figure 2: Sketch of the system.

1. What are the input(s) and output(s) to the system.
2. List the reservoirs with the level variables.
3. Draw a causality diagram of the system.
4. In what conditions will the valves operate?
5. Formulate the relations describing each block.

Solution.

1. The inputs are the opening surfaces of the two valves. The output is the position of the piston (or of the box).
2. The reservoirs of the system are:
   - Mass of air in the chamber with level variable $m(t)$;
   - Internal energy in the chamber with level variable $\vartheta(t)$;
   - Kinetic energy of the piston and of the box with level variable $\dot{x}(t)$;
   - Potential energy in the spring with level variable $x(t)$.
3. The causality diagram is sketched in Figure 3.

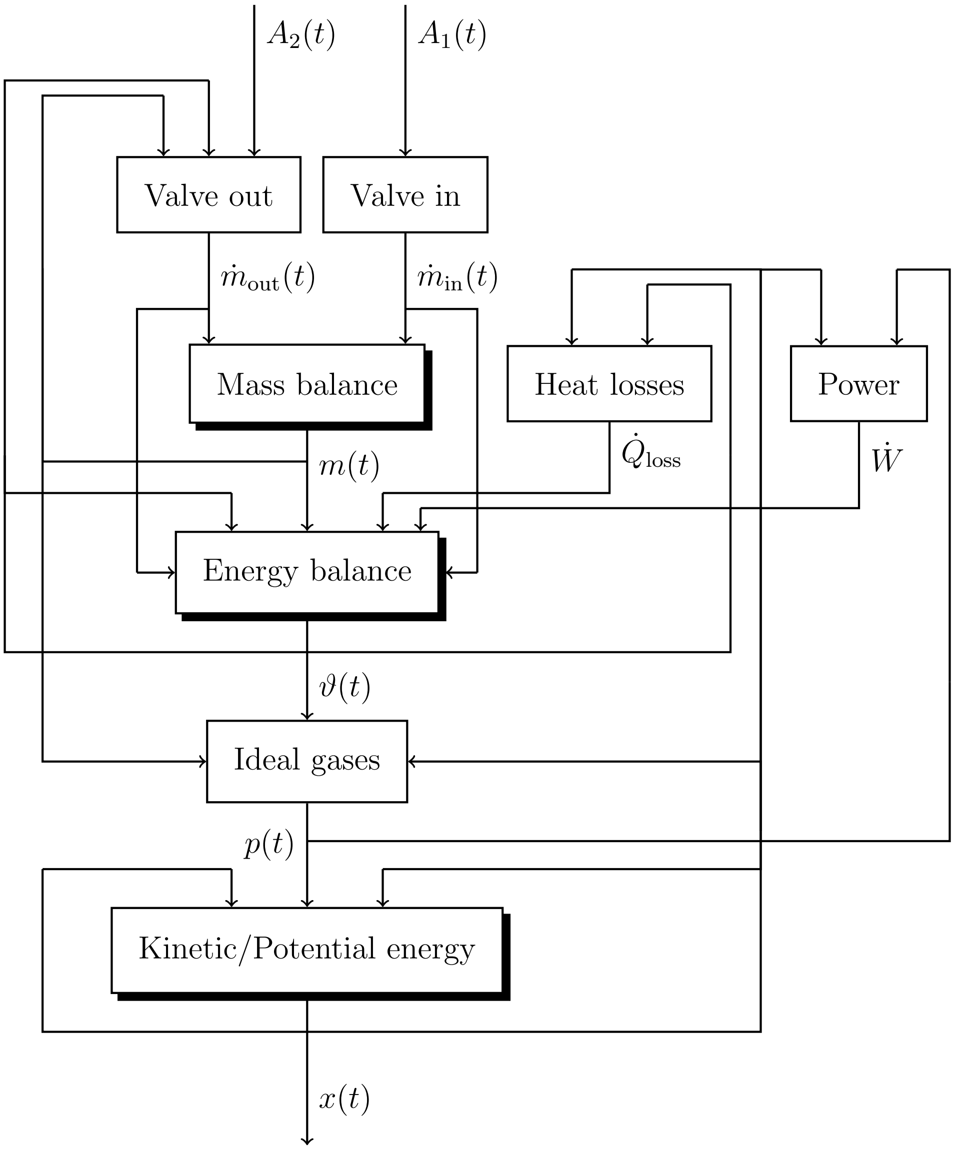

Figure 3: Causality diagram of the system.

4. For the first valve we have

$$p_{\text{cr}} = \left(\frac{2}{\kappa+1}\right)^{\frac{\kappa}{\kappa-1}} \cdot 10\,\text{bar} \approx 5.3\,\text{bar} > p_{\text{out}}(t).$$

Hence, the valve operates in "sonic conditions". For the second valve we have

$$p_{\text{cr}} = \left(\frac{2}{\kappa+1}\right)^{\frac{\kappa}{\kappa-1}} \cdot 1.5\,\text{bar} \approx 0.8\,\text{bar} < p_{\text{out}}(t).$$

Hence, the valve operates in "normal conditions".

5. The relations for the valves are

$$\dot{m}_{\text{in}}(t) = c_{\text{d},1} \cdot A_1(t) \cdot \frac{p_{\text{r}}}{\sqrt{R \cdot \vartheta_{\text{r}}}} \cdot \sqrt{\kappa \cdot \left(\frac{2}{\kappa+1}\right)^{\frac{\kappa+1}{\kappa-1}}}$$

$$\dot{m}_{\text{out}}(t) = c_{\text{d},2} \cdot A_2(t) \cdot \frac{p(t)}{\sqrt{R \cdot \vartheta(t)}} \cdot \left(\frac{p_\infty}{p(t)}\right)^{\frac{1}{\kappa}} \cdot \sqrt{\frac{2\kappa}{\kappa-1} \cdot \left[1 - \left(\frac{p_\infty}{p(t)}\right)^{\frac{\kappa-1}{\kappa}}\right]}$$

The mass balance yields then

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t) = \dot{m}_{\text{in}} - \dot{m}_{\text{out}}.$$

The energy balance reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(c_{\text{v}} \cdot m(t) \cdot \vartheta(t)\right) = \dot{m}_{\text{in}}(t) \cdot c_{\text{p}} \cdot \vartheta_{\text{r}} - \dot{m}_{\text{out}}(t) \cdot c_{\text{p}} \cdot \vartheta(t) - \dot{W} - \dot{Q}_{\text{loss}},$$

where

$$\dot{Q}_{\text{loss}} = \alpha \cdot (S + \pi D \cdot x(t)) \cdot (\vartheta(t) - \vartheta_\infty),$$
$$\dot{W} = (p(t) - p_\infty) \cdot S \cdot \dot{x}(t).$$

Note that the pressure can then be computed by using the ideal gas relation

$$p(t) = \frac{1}{V} \cdot m(t) \cdot R \cdot \vartheta(t).$$

where $V(t) = S \cdot x(t)$. The position of the piston obeys to

$$(m_{\text{p}} + m_{\text{b}}) \cdot \frac{\mathrm{d}^2}{\mathrm{d}t^2} x(t) = (p(t) - p_\infty) \cdot S - k \cdot x(t).$$