

Exercise 10 - Parameter Identification

10.1 Introduction

So far, we have investigated how to come up with a mathematical model of a system. This modeling was based on physics laws, whereby we assumed that some parameters of the system were known. Now, these parameters are to be identified by running experiments. The data collected experimentally may be used for the following purposes:

- To identify unknown system structures and system parameters;
- To validate the results of the system modeling and parameter identification.

It is important not to use the same data set for both purposes.

10.2 Least Squares Method

The least square is used to fit the parameters of a linear (in the parameters) and static model. Assume you have a model of the form

$$y(k) = h(u(k))^{\top} \cdot \pi + e(k), \quad (10.1)$$

where $y(k) \in \mathbb{R}$ is the output, $u(k) \in \mathbb{R}^m$ is the input, $\pi \in \mathbb{R}^q$ is the vector of q parameters, and $e(k)$ is the error.

Example 1. The model

$$y(k) = au(k)^2 + b + e(k)$$

is linear in the parameters. Let $\pi = [a \ b]^{\top}$, then the system can be brought in the form (10.1) with

$$y(k) = \underbrace{\begin{bmatrix} u(k)^2 & 1 \end{bmatrix}}_{h(u(k))^{\top}} \begin{bmatrix} a \\ b \end{bmatrix} + e(k).$$

Example 2. The model

$$y(k) = u(k)^a + au(k)$$

is nonlinear in the parameter.

Example 3. The model

$$y(k) = u(k)^a$$

is nonlinear in the parameter. However, if we write

$$\log(y(k)) = \log(u(k)^a) = a \log(u(k))$$

the model is linear and of the form of Equation (10.1).

The least square problem consists of finding the parameter vector π_{LS} that, given a data set, minimizes the error

$$\pi_{\text{LS}} = \arg \min_{\pi} \sum_{k=1}^n e(k)^2 = \arg \min_{\pi} \sum_{k=1}^n (y(k) - h(u(k))^{\top} \pi)^2$$

Now let

$$e = \begin{bmatrix} e(1) \\ \vdots \\ e(n) \end{bmatrix} \quad y = \begin{bmatrix} y(1) \\ \vdots \\ y(n) \end{bmatrix} \quad H = \begin{bmatrix} h(u(1))^\top \\ \vdots \\ h(u(n))^\top \end{bmatrix}$$

then

$$\pi_{\text{LS}} = \arg \min_{\pi} e^\top e = \arg \min_{\pi} (y - H\pi)^\top (y - H\pi).$$

The minimum can be obtained by setting the derivative w.r.t. π to zero¹

$$\frac{\partial}{\partial \pi} (y^\top y - y^\top H\pi - \pi^\top H^\top y + \pi^\top H^\top H\pi) = -2H^\top y + 2H^\top H\pi \stackrel{!}{=} 0$$

which gives

$$\pi_{\text{LS}} = (H^\top H)^{-1} H^\top y.$$

Remark. • The matrix $H^\dagger = (H^\top H)^{-1} H^\top$ is also called Moore-Penrose inverse of H .

- For the derivative in π , we used $\frac{\partial A \cdot x}{\partial x} = A^\top$, $\frac{\partial x^\top \cdot A}{\partial x} = A$.

Weighted Least Squares Method

Assume now the errors are weighted by some weights $w(k) > 0$. Then,

$$\pi_{\text{LS}} = \arg \min_{\pi} \sum_{k=1}^n w(k) e(k)^2 = \arg \min_{\pi} \sum_{k=1}^n w(k) (y(k) - h(u(k))^\top \pi)^2$$

Using the same notation as above and with $W = \text{diag}(w_1, \dots, w_n)$ we obtain

$$\pi_{\text{LS}} = \arg \min_{\pi} e^\top W e = \arg \min_{\pi} (y - H\pi)^\top W (y - H\pi).$$

Setting the derivative to 0 gives

$$\pi_{\text{LS}} = (H^\top W H)^{-1} H^\top W y.$$

Remark. In a probabilistic environment, it might be reasonable to set $w(k) = 1/\sigma(k)^2$, where $\sigma(k)^2$ is the standard deviation of measurement k .

10.3 Iterative Least Squares Methods

In the previous subsections, we presented a batch approach, i.e., an approach where data is first collected, organized, and then the solution is computed. However, this approach might be computationally demanding for real-time applications as at each time step the whole problem has to be solved again. Thus, we look for iterative solution the estimate is computed as function of the previous estimate and the latest measurement. That is,

$$\pi_{\text{LS}}(r+1) = f(\pi_{\text{LS}}(r), y(r+1))$$

¹Strictly speaking, it should be show that π_{LS} is a minimum. However, as the matrix of second derivatives $2H^\top H$ is positive semidefinite, π_{LS} a minimum.

initialized with $\pi_{\text{LS}}(0) = \mathbb{E}[\pi]$. Then, it can be shown that

$$\begin{aligned}\pi_{\text{LS}}(r+1) &= \pi_{\text{LS}}(r) + \frac{1}{1+c(r+1)}\Omega(r)h(u(r+1))(y(r+1) - h(u(r+1))^{\top}\pi_{\text{LS}}(r)) \\ \Omega(r+1) &= \Omega(r) - \frac{1}{1+c(r+1)}\Omega(r)h(u(r+1))h(u(r+1))^{\top}\Omega(r) \\ c(r+1) &= h(u(r+1))^{\top}\Omega(r)h(u(r+1)).\end{aligned}$$

Note that the recursion has to be initialized with some $\pi_{\text{LS}}(0)$ and some $\Omega(0)$.

Remark. There exist many extensions/variations for the recursive least squares method. Among them, we highlight:

- The least squares with exponential forgetting, where old data is weighted with some forgetting factor $\lambda \in (0, 1)$. That is, at time n the measurement $i < n$ has weight λ^{n-i} . Then,

$$\pi_{\text{LS},\lambda} = \arg \min_{\pi} \sum_{k=1}^n \lambda^{n-k} e(k)^2 = \arg \min_{\pi} \sum_{k=1}^n \lambda^{n-k} (y(k) - h(u(k))^{\top}\pi)^2$$

- The Kaczmarz's projection algorithm, which is more computationally efficient than the regular least squares algorithm.

10.4 Identification of Parameters in Dynamic Systems

So far, we assume that the model of the system was algebraic, i.e.,

$$y(k) = h(u(k))^{\top}u(k) + e(k).$$

How should we proceed for systems of the form

$$\pi_0 \frac{d}{dt}y(t) = h(u(t))\pi + e(t).$$

We distinguish between two cases:

- Taking measurements in steady state. That is, by considering

$$0 = h(u(t))\pi + e(t)$$

and doing least squares for all the available measurements. However, this does not allow to estimate π_0 as $\frac{d}{dt}y = 0$.

- Numerically finding the optimal parameters. That is,

$$\pi_0, \pi = \arg \min_{\pi_0, \pi} \sum_{k=1}^n (y(k) - \hat{y}(k))^2,$$

where \hat{y} is the output of the simulated system. In other words, we try to find the parameters that minimize the error between the output of the simulation and the measurements. Note that there exists no closed form equation of π_0 and π ; the problem has to be tackled numerically.

10.5 Example

Since your SpaghETH is going well, you want to optimize and control the production of your meals depending on how satisfied your customers are. In order to analyze this, you find a model to describe the satisfaction of your customers at any time t . This could help you standardizing your whole production process and designing a controller which can handle any situation.

The model is mathematically given as

$$\begin{aligned}\frac{dS(t)}{dt} &= T \cdot q_{\text{meal}} + D \cdot \alpha_{\text{meal}} - S(t) \\ \frac{T}{C} &= P.\end{aligned}$$

The first equation of the model relates the satisfaction $S(t)$ of the cooked meal to several factors. In fact, this depends on the percentage of waiting time T for the customers to receive their meal ($T = \frac{\text{waiting time}}{\text{total permanence time}}$), the quality of the meal q_{meal} , the digestibility factor for the meal D , the factor which describes how large the portion was α_{meal} and the satisfaction itself. The second equation, instead, defines the waiting time percentage T for receiving the meal, which is a function of the number of pots needed to cook the meals $P = \frac{\text{pots needed}}{\text{standard pots need}}$ and the fraction of customers $C = \frac{\text{customers}}{\text{capability}}$ eating. We are interested in finding the parameters T and α_{meal} with the Least Squares Method, using data from feedbacks from two different meals: pasta alla puttanesca and äplper magronen.

	C	D	S	P	q_{meal}
Pasta alla puttanesca (P)	0.5	3	4	1	1
Äplper Magronen (A)	1	$\sqrt{3}$	2	$\sqrt{3}$	$\sqrt{3}$

1. Write down the least squares problem for the steady state of the system.
2. Determine the parameters T and α_{meal} with the Least Squares Method.
3. By computing π_{LS} you realize that something does not work. You find out that the intern who should count the number of customers for äplper magronen was drunk and the number he reported is not reliable. How can you take this into account for the Least Squares Method? Provide a possible solution.
4. You realize that a better model for the waiting time is

$$\beta = (P \cdot C)^T.$$

Solve the Least Squares Problem (only for the model of the waiting time) for the following measurements. Note that $\beta \in \mathbb{R}$ is a known constant.

	P	C
Pasta alla puttanesca (P)	1	0.5
Älpler Magronen (A)	$\sqrt{3}$	1

Solution.

1. We consider the system in its equilibrium state. The relations become

$$S_i = T \cdot q_{\text{meal},i} + D_i \cdot \alpha_{\text{meal},i}$$

$$P_i = \frac{T}{C_i},$$

where $i \in \{P, A\}$. We can write the matrices for the Least Square problem. The parameter vector we look for reads

$$\pi = \begin{pmatrix} T \\ \alpha_{\text{meal}} \end{pmatrix}.$$

The weight matrix for the problem is the identity matrix, since we are not given special weights for the parameters, i.e.

$$W = \mathbb{I}_4.$$

Furthermore,

$$\tilde{y} = \begin{pmatrix} S_P \\ P_P \\ S_A \\ P_A \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \\ \sqrt{3} \end{pmatrix},$$

and

$$H = \begin{pmatrix} q_{\text{meal},P} & D_P \\ \frac{1}{C_P} & 0 \\ q_{\text{meal},A} & D_A \\ \frac{1}{C_A} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ \sqrt{3} & \sqrt{3} \\ 1 & 0 \end{pmatrix}.$$

2. Using the learned formula, we get

$$\begin{aligned}
\pi_{\text{LS}} &= (H^T \cdot W \cdot H)^{-1} \cdot H^T \cdot W \cdot \tilde{y} \\
&= (H^T \cdot H)^{-1} \cdot H^T \cdot \tilde{y} \\
&= \left[\begin{pmatrix} 1 & 2 & \sqrt{3} & 1 \\ 3 & 0 & \sqrt{3} & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 \\ \sqrt{3} & \sqrt{3} \\ 1 & 0 \end{pmatrix} \right]^{-1} \cdot \begin{pmatrix} 1 & 2 & \sqrt{3} & 1 \\ 3 & 0 & \sqrt{3} & 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ 2 \\ \sqrt{3} \end{pmatrix} \\
&= \begin{pmatrix} 9 & 6 \\ 6 & 12 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 6 + 3\sqrt{3} \\ 12 + 2\sqrt{3} \end{pmatrix} \\
&= \frac{1}{72} \begin{pmatrix} 12 & -6 \\ -6 & 9 \end{pmatrix} \cdot \begin{pmatrix} 6 + 3\sqrt{3} \\ 12 + 2\sqrt{3} \end{pmatrix} \\
&= \frac{1}{72} \cdot \begin{pmatrix} 72 + 36\sqrt{3} - 72 - 12\sqrt{3} \\ -36 - 18\sqrt{3} + 108 + 18\sqrt{3} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sqrt{3}}{3} \\ 1 \end{pmatrix}.
\end{aligned}$$

3. In order to take this into account we adjust the weight matrix to not trust the measurement for äpler magronen customers, i.e.

$$W_{\text{new}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.1 \end{pmatrix}.$$

We perform the same computation as before with the new weight matrix:

$$\begin{aligned}
\pi_{\text{LS,new}} &= (H^T \cdot W \cdot H)^{-1} \cdot H^T \cdot W \cdot \tilde{y} \\
&= (H^T \cdot H)^{-1} \cdot H^T \cdot \tilde{y} \\
&= \left[\begin{pmatrix} 1 & 2 & \sqrt{3} & 1 \\ 3 & 0 & \sqrt{3} & 0 \end{pmatrix} \cdot W_{\text{new}} \cdot \begin{pmatrix} 1 & 3 \\ \sqrt{3} & \sqrt{3} \\ 1 & 0 \end{pmatrix} \right]^{-1} \cdot \begin{pmatrix} 1 & 2 & \sqrt{3} & 1 \\ 3 & 0 & \sqrt{3} & 0 \end{pmatrix} \cdot W_{\text{new}} \cdot \begin{pmatrix} 4 \\ 1 \\ 2 \\ \sqrt{3} \end{pmatrix} \\
&= \dots \\
&= \begin{pmatrix} 0.3736 \\ 1.1019 \end{pmatrix}.
\end{aligned}$$

4. Since the model is not linear, we cannot directly apply the method we learned. One possible solution is to linearize the model, e.g. by computing:

$$\begin{aligned}
\ln(\beta) &= \ln((P \cdot C)^T) \\
\ln(\beta) &= T \cdot \ln(P \cdot C).
\end{aligned} \tag{10.2}$$

Applying the formalism, we have

$$\begin{aligned}
 \tilde{y} &= \begin{pmatrix} \ln(\beta) \\ \ln(\beta) \end{pmatrix}, \\
 H &= \begin{pmatrix} \ln(P_P \cdot C_P) \\ \ln(P_A \cdot C_A) \end{pmatrix} \\
 &= \begin{pmatrix} \ln(0.5) \\ \ln(\sqrt{3}) \end{pmatrix}, \\
 W &= \mathbb{I}_2.
 \end{aligned} \tag{10.3}$$

It holds

$$\begin{aligned}
 \pi_{\text{LS}} &= (H^T \cdot W \cdot H)^{-1} \cdot H^T \cdot W \cdot \tilde{y} \\
 &= \left(\begin{pmatrix} \ln(0.5) & \ln(\sqrt{3}) \end{pmatrix} \cdot \begin{pmatrix} \ln(0.5) \\ \ln(\sqrt{3}) \end{pmatrix} \right)^{-1} \cdot \begin{pmatrix} \ln(0.5) & \ln(\sqrt{3}) \end{pmatrix} \cdot \begin{pmatrix} \ln(\beta) \\ \ln(\beta) \end{pmatrix}.
 \end{aligned}$$