Exercise 11 - Linear System Analysis

Once the system is modeled, it is of interest to analyze it. In particular, we will focus on:

- Stability,
- controllability, observability, and
- performance.

As a result from the modeling we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}x(t) &= f(x(t), v(t), t)\\ y(t) &= g(x(t), u(t), t),\end{aligned}$$

where $x(t)$ represents the states, $u(t)$ represents the inputs and $y(t)$ represents the outputs. Difficulties in analyzing this kind of systems arise because systems are

- Non-normalized (i.e., have non-normalized variables):
  - numerical problems,
  - comparisons hard to perform.
- Non-linear: difficult approaches (there is very little general theory on nonlinear systems).

In order to overcome this issues, the model is normalizes and linearized.

11.1 Normalization

Consider the scaling factors $x_{i,0}, u_{j,0}, y_{k,0}$ and the normalized variables $\bar{x}_i(t), \bar{u}_j(t), \bar{y}_k(t)$ such that

$$\begin{aligned}x_i(t) &= x_{i,0} \cdot \bar{x}_i(t) \quad \rightarrow \quad \bar{x}_i(t) = \frac{x_i(t)}{x_{i,0}}\\ u_j(t) &= u_{j,0} \cdot \bar{u}_j(t) \quad \rightarrow \quad \bar{u}_j(t) = \frac{u_j(t)}{u_{j,0}}\\ y_k(t) &= y_{k,0} \cdot \bar{y}_k(t) \quad \rightarrow \quad \bar{y}_k(t) = \frac{y_k(t)}{y_{k,0}}.\end{aligned}$$

The new normalized variables have no physical units and have roughly the order of magnitude 1. Since $x_{i,0}$ is a constant, there is no change in the dynamics (only scaled), i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}x_i(t) = x_{i,0} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\bar{x}_i(t)$$

We can write this in vector notation with

$$x = T\bar{x}, \quad T = \mathrm{diag}(x_{1,0}, \dots, x_{n,0}),$$

where T is the similarity transformation matrix. Similarly, we introduce

$$\begin{aligned} u &= T_u\bar{u}, \qquad T_u = \mathrm{diag}(u_{1,0}, \dots, y_{r,0}), \\ y &= T_y\bar{y}, \qquad T_y = \mathrm{diag}(y_{1,0}, \dots, u_{m,0}). \end{aligned}$$

From that we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\bar{x} &= T^{-1}f(T\bar{x}, T_u\bar{u}, t) = f_0(\bar{x}, \bar{u}, t) \\ \bar{y} &= T_y^{-1}g(T\bar{x}, T_u\bar{u}, t) = g_0(\bar{x}, \bar{u}, t). \end{aligned}$$

Remark. It can be shown that such a similarity transformation does not change characteristics of the system.

11.2 Linearization

After normalization the system is in the form

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\bar{x} &= T^{-1}f(T\bar{x}, T_u\bar{u}, t) = f_0(\bar{x}, \bar{u}, t) \\ \bar{y} &= T_y^{-1}g(T\bar{x}, T_u\bar{u}, t) = g_0(\bar{x}, \bar{u}, t). \end{aligned}$$

To simplify the notation we drop the $^-$ and $()_0$. Recall that $(x_\mathrm{e}, u_\mathrm{e})$ is an equilibrium point if

$$\begin{aligned} 0 &= f(x_\mathrm{e}, u_\mathrm{e}, t) \\ y_\mathrm{e} &= g(x_\mathrm{e}, u_\mathrm{e}, t). \end{aligned}$$

As the analysis of the nonlinear system is often too difficult, we consider the system in a neighbourhood of the equilibrium point. Mathematically, this translates in considering the Taylor expansion of the functions f and g around the equilibrium point and neglecting high order terms. Let $\delta x = x - x_\mathrm{e}$ and $\delta u = u - u_\mathrm{e}$. Then,

$$\delta\dot{x} = f(x_\mathrm{e} + \delta x, u_\mathrm{e} + \delta_\mathrm{e} u, t) = f(x_\mathrm{e}, u_\mathrm{e}, t) + \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x_\mathrm{e}, u_\mathrm{e}}}_{A} \delta x + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{x_\mathrm{e}, u_\mathrm{e}}}_{B} \delta u + \text{h.o.t.}.$$

By proceeding analogously for $g(\cdot)$ and neglecting higher order terms we get

$$\begin{aligned} \delta\dot{x} &= A\delta x + B\delta u \\ \delta y &= C\delta x + D\delta u, \end{aligned}$$

where $C = \left.\frac{\partial g}{\partial x}\right|_{x_\mathrm{e}, u_\mathrm{e}}$ and $D = \left.\frac{\partial g}{\partial u}\right|_{x_\mathrm{e}, u_\mathrm{e}}$.

Remark. Note that in general the matrices A, B, C, D are time-varying. However if f, g do not depend explicitly on t, then the linearized model will be time-invariant.

Remark. $\delta x, \delta u, \delta y$ describe a deviation from the equilibrium point. The (linearized) dynamics is given by $x = x_\mathrm{e} + \delta x, u = u_\mathrm{e} + \delta u, y = y_\mathrm{e} + \delta y$.

11.3 Solution to the Linear ODE

Assuming the system has been normalized and linearized and dropping the δ in the notation, the system is of the form

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du.\end{aligned}$$

Further, assume that the system is time-invariant, i.e., the matrices A, B, C, D do not depend explicitly on time. Then, the solution to the ODE reads

$$\begin{aligned}x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau \\ y(t) &= Cx(t) + Du(t),\end{aligned}$$

where

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \ldots = \sum_{k=0}^{\infty} \frac{A^kt^k}{k!}.$$

11.4 Stability of Linear Systems

For linear time-invariant systems of the form

$$\dot{x} = Ax$$

with solution

$$x = e^{At}x(0)$$

we consider the following three types of stability for the equilibrium point:

- Asymptotically stable, if $\lim_{t\to\infty} x(t) = 0$.
- Stable, if $\|x\| < \infty$.
- Unstable, if $\|x\| \to \infty$.

As it can be inferred, the stability properties of the equilibrium point $x = 0$ are characterized by the eigenvalues of the matrix A. In particular, the system is:

- Asymptotically stable if all eigenvalues of A have strictly negative real part.
- Stable if all eigenvalues of A have non-positive real part and the eigenvalues with zero real part are semisimple (that is, all eigenvalues with zero real part must have the same algebraic and geometric multiplicity).
- Unstable otherwise

Remark. If a linear system is asympotically stable, then also the nonlinear system is (locally) asymptotically stable. Similarly, if a linear system is unstable, also the nonlinear system is (locally) unstable. On the other hand, if a linear system is stable, no conclusions on the nonlinear system can be drawn.

Remark. It is important to emphasize that the stability properties are "of the equilibrium point". For linear system, we sometimes talk (with an abuse of notation) of the stability system to denote the stability of the origin.

11.5 Reachability and Observability

11.5.1 Reachability

Informally, a system is said to be reachable if there exists τ such that $x(\tau) = x_1$ for all $x_1 \in \mathbb{R}^n$ and for $x(0) = 0$, i.e., if any state can be reached in a finite time. For linear time-invariant systems this is true if the matrix

$$R = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

is full rank, i.e., if $\operatorname{rank}(R) = n$.

11.5.2 Observability

Informally, a system is said to be observable if given the sequence $y(t)$ up to some time τ the initial condition $x(0)$ can be reconstructed uniquely. For linear time-invariant systems this is true if the matrix

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is full rank, i.e., if $\operatorname{rank}(O) = n$.

11.6 Order Reduction

The above creteria provide a yes/no answer on controllability and observability. In this section, we investigate a quantitative creterium.

11.6.1 Gramian Matrices

The controllability and observability Gramian for linear time-invariant systems are

$$W_R = \int_0^\infty e^{A\tau} B B^\intercal e^{A^\intercal \tau} \, \mathrm{d}t,$$
$$W_O = \int_0^\infty e^{A^\intercal \tau} C^\intercal C e^{A\tau} \, \mathrm{d}t.$$

If the system is asymptotically stable, i.e., if all eigenvalues of A a strictly negative real part the Gramian matrices exist and are the solution to

$$A W_R + W_R A^\intercal = -B B^\intercal,$$
$$A^\intercal W_O + W_O A = -C^\intercal C.$$

11.6.2 Order Reduction

The idea is then to bring the Gramian in diagonal form and neglect the states with small weight. Assume the last ν states of the system are to be neglected. Write the system dynamics in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$
$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D u$$

where $x_1 \in \mathbb{R}^{n-\nu}$ and $x_2 \in \mathbb{R}^{\nu}$ and all matrices are of appropriate dimensions. There are two possible approaches:

Approach 1: Simple ignore the last ν states. Then,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}x_1 &= A_{11}x_1 + B_1u\\ y &= C_1x_1 + Du.\end{aligned}$$

This procedure gives good performances. However, the DC gain (i.e., $P(s=0)$) of the original system and of the reduced one are often different.

Approach 2 (single perturbation analysis): Ignore the dynamics of the last ν states, but not their contribution to the system. Then, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}x_2 = 0 \quad \Rightarrow \quad 0 = A_{21}x_1 + A_{22}x_2 + B_2u \quad \Rightarrow \quad x_2 = -A_{22}^{-1}(A_{21}x_1 + B_2u).$$

The dynamics of x_1 reads then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}x_1 &= A_{11}x_1 + A_{12}x_2 + B_1u = (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u\\ y &= C_1x_1 + C_2x_2 + Du = (C_1 - C_2A_{22}^{-1}A_{21})x_1 + (D - C_2A_{22}^{-1}B_2)u.\end{aligned}$$

This approach does no change the DC gain. Note that if the system is asymptotically stable, then the matrix A_{22} is guaranteed to be invertible.

11.7 Example

In order to optimize your spaghetti production, you want to reduce the order of a linear system with the matrices

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ -0.5 & -2 & -1 \end{pmatrix},\ b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ c = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},\ d = 0$$

from 3 to 2.

1. Give a sufficient condition for the existance of Gramian matrices.
2. Show that the Gramian matrices exist.
3. Explain the necessary steps to reduce the order of the system.
4. After point 3., you get the following balanced system

   $$\bar{A} = \begin{pmatrix} -1.9 & -0.06 & -0.08 \\ -0.06 & -0.04 & -1.15 \\ 0.08 & 1.15 & -0.06 \end{pmatrix},\ \bar{b} = \begin{pmatrix} 0.41 \\ 0.46 \\ 0.61 \end{pmatrix},\ \bar{c} = \begin{pmatrix} -0.41 & -0.46 & 0.61 \end{pmatrix},\ \bar{d} = 0.$$

   The observability and controllability gramians are given as

   $$W_{R,b} = W_{O,b} = \begin{pmatrix} 0.04 & 0 & 0 \\ 0 & 2.93 & 0 \\ 0 & 0 & 3.17 \end{pmatrix}.$$

   Which state would you eliminate in order to reduce the order of the system from 3 to 2? Why?
5. Give the system matrices of the reduced system.
6. Has the DC gain changed?
7. Show how you would overcome this issue.

Solution.

1. The condition is that the system is asymptotically stable.

2. We need to show that the condition is fulfilled. The fastest way to show this, is to prove that the matrix A is Hurwitz, i.e., all of its eigenvalues have negative real part. This can be investigated e.g. by looking at all the leading principal minors of H (the Hurwitz matrix of the characteristic polynomial of A): if they are all positive, the matrix is Hurwitz. It holds

$$\begin{aligned}\det(A-\lambda\mathbb{I}) &= \det\begin{pmatrix}-\lambda & 0 & 1\\ 1 & -1-\lambda & 0\\ -0.5 & -2 & -1-\lambda\end{pmatrix}\\ &= -\lambda\cdot(\lambda+1)^2 - 2 - \frac{1}{2}\cdot(\lambda+1)\\ &= \lambda^3+2\lambda^2+\frac{3}{2}\lambda+\frac{5}{2}=0.\end{aligned}$$

From here one has two choices:

- You compute the eigenvalues and you show that these have negative real part. Here

$$\begin{aligned}\lambda_1 &= -0.0488+1.1453i\\ \lambda_2 &= -0.0488-1.1453i\\ \lambda_3 &= -1.9023.\end{aligned}$$

- You compute the Hurwitz matrix of the characteristic polynomial of A. For a generic polynomial

$$p(x) = a_0z^3+a_1z^2+a_2z+a_3$$

this reads

$$\begin{pmatrix}a_1 & a_3 & 0\\ a_0 & a_2 & 0\\ 0 & a_1 & a_3\end{pmatrix}.$$

For our case we have

$$\begin{aligned}H &= \begin{pmatrix}2 & 2.5 & 0\\ 1 & 2 & 0\\ 0 & 2 & 2.5\end{pmatrix}\\ \det(2) &= 2>0\\ \det\begin{pmatrix}2 & 2.5\\ 1 & 2\end{pmatrix} &= 1.5>0\\ \det(H) &= 2.5\cdot 1.5=\frac{15}{4}>0.\end{aligned}$$

3. To reduce the order of the system one should:

- Calculate the Gramians for controllability and observability by solving the two discussed Lypunov equations.

- Find the coordinate transformation matrix T, which transforms the original system into a system whose controllability and observability Gramians are equal and diagonal.
- Do the coordination transformation to derive the balanced system $\{T^{-1}AT, T^{-1}b, cT, d\}$.

4. The first state has a two order smaller influence on the controllability and observability of the system. For this reason we can eliminate it.

5. The reduced system reads

$$\tilde{A} = \begin{pmatrix} -0.04 & -1.15 \\ 1.15 & -0.06 \end{pmatrix}, \ \tilde{b} = \begin{pmatrix} 0.46 \\ 0.61 \end{pmatrix}, \ \tilde{c} = \begin{pmatrix} -0.46 & 0.61 \end{pmatrix}, \ \tilde{d} = 0.$$

6. In steady state conditions (using $u = 1$), one can write

$$\dot{x} = Ax + b = 0,$$

which has the solution

$$x_{ss} = \begin{pmatrix} 0.4 \\ 0.4 \\ 0 \end{pmatrix}.$$

For this reason it holds

$$y_{ss} = x_1 = 0.4.$$

For the same input $u = 1$, one can compute

$$\begin{aligned} -0.04\tilde{x}_2 - 1.15\tilde{x}_3 &= -0.46 \\ 1.15\tilde{x}_2 - 0.06\tilde{x}_3 &= -0.61. \end{aligned}$$

This system has solution

$$\tilde{x}_{ss,red} = \begin{pmatrix} -0.51 \\ 0.42 \end{pmatrix}.$$

It follows

$$y_{ss,red} = -0.46\tilde{x}_2 + 0.61\tilde{x}_3 = 0.49,$$

which is higher than the one for the original system.

7. In order to solve this problem, you should use the singular perturbation method. In this method, the eliminated state is computed stationary and dynamically with the oter state variables. According to the balanced system matrix, the differential equation of the state which have to be eliminated is

$$\dot{\tilde{x}}_1 = -1.9\tilde{x}_1 - 0.06\tilde{x}_2 - 0.08\tilde{x}_3 + 0.41\tilde{u}$$

By setting the derivative to 0, one gets

$$\tilde{x}_1 = -0.03\tilde{x}_2 - 0.04\tilde{x}_3 + 0.22\tilde{u}$$