Exercise 12 - Zero Dynamics

12.1 Zero Dynamics

In order to analyze the effect of zeros on systems, we consider the zero dynamics. It is known that non-minimumphase zeros limit the achievable bandwidth on any closed-loop system. In order to give more precise informations of this behaviours, we start by analyzing the case of a SISO (single input single output) system. In the SISO case, the transfer function matrix of a system

$$\begin{aligned}\dot{x}(t) &= A \cdot x(t) + B \cdot u(t)\\ y(t) &= C \cdot x(t) + D \cdot u(t),\end{aligned}$$

is given as

$$P(s) = C \cdot (s \cdot \mathbb{I} - A)^{-1} \cdot B + D.$$

This function can be rewritten in the form

$$\begin{aligned}P(s) &= \frac{Y(s)}{U(s)}\\ &= k \cdot \frac{s^{n-r} + b_{n-r-1} \cdot s^{n-r-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_2 s^2 + a_1 s + a_0},\end{aligned}$$

where

- n is the highest power of s,
- k is the input gain,
- r is the relative degree:
  - This is the difference between the highest power of s in the denominator and the highest power of s in the numerator.

After choosing this structure for rewriting the problem, it is easy to show that the **state-space representation** reads

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}x(t) &= \begin{pmatrix} 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{pmatrix} \cdot x(t) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ k \end{pmatrix} \cdot u(t)\\ y(t) &= \begin{pmatrix} b_0 & \cdots & b_{n-r-1} & 1 & 0 & \cdots & 0 \end{pmatrix} \cdot x(t)\end{aligned}$$

Remark.

- This form is called *controller canonical form with gain* and has the minimum number of parameters.
- Remember that infinitely other possible choices of coordinate systems for this transfer function exist.

The relative degree r is the number of differentiations needed in order to have the input $u(t)$ explicitly appearing in the output $y^{(r)}(t)$. In fact, we have that

$$\begin{aligned}
y(t) &= Cx(t),\\
\dot{y}(t) &= C\dot{x}(t) = CAx(t) + CBu(t) = CAx(t),\\
\ddot{y}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\dot{y}(t) = CA^2x(t) + CABu(t) = cA^2x(t),\\
&\vdots\\
y^{(r)}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}y^{(r-1)}(t) = CA^rx(t) + CA^{r-1}Bu(t) = CA^rx(t) + ku(t),\ r \leq n.
\end{aligned}$$

The zero dynamics of a system correspond to its behaviour for the special

- non-zero inputs $u^*(t)$,
- and initial conditions x^*

for which its outout $y(t) = 0$ for a finite interval (we do not consider $x^* = u^* = 0$ as a possible solution). If we want to do reference tracking, i.e.

$$e = y_{\text{ref}} - y = 0,$$

and $y_{\text{ref}} = 0$, then $y(t) = 0$ should be zero for all times (meaning that its derivatives should be zero as well). In order to simplify calculations, we introduce the coordinate transformation $z = \Phi^{-1} \cdot x$, from which follows:

$$\begin{aligned}
z_1(t) &= y(t) = Cx(t) = (b_0x_1 + b_1x_2 + \ldots + b_{n-r-1}x_{n-r} + x_{n-r+1}),\\
z_2(t) &= \dot{y}(t) = CAx(t) = (b_0x_2 + b_1x_3 + \ldots + b_{n-r-1}x_{n-r+1} + x_{n-r+2}),\\
&\vdots\\
z_r(t) &= y^{(r-1)}(t) = CA^{r-1}x(t) = (b_0x_r + b_1x_{r+1} + \ldots + b_{n-r-1}x_{n-1} + x_n),\ r \leq n
\end{aligned}$$

The next $n - r$ coordinates should be chosen such that Φ is regular and such that their derivatives do not depend on the input u. This requirement is satisfied by

$$\begin{aligned}
z_{r+1} &= x_1\\
z_{r+2} &= x_2\\
&\vdots\\
z_n &= x_{n-r}.
\end{aligned}$$

Then, the whole problem can be rewritten with the matrices:

$$z = \begin{pmatrix} \epsilon \\ \eta \end{pmatrix},\ \epsilon = \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix},\ \eta = \begin{pmatrix} z_{r+1} \\ \vdots \\ z_n \end{pmatrix}$$

and the dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\epsilon\\ \eta\end{pmatrix} = \left(\begin{array}{ccccc|ccccc} 0 & 1 & 0 & \dots & 0 & 0 & \dots & \dots & \dots & 0\\ 0 & 0 & 1 & \dots & 0 & 0 & \dots & \dots & \dots & 0\\ 0 & \dots & \dots & 0 & 1 & 0 & \dots & \dots & \dots & 0\\ - & - & r^\intercal & - & - & - & - & s^\intercal & - & -\\ \hline 0 & \dots & \dots & \dots & 0 & 0 & 1 & 0 & \dots & 0\\ 0 & \dots & \dots & \dots & 0 & 0 & 0 & 1 & \dots & 0\\ 0 & \dots & \dots & \dots & 0 & 0 & 0 & \dots & 0 & 1\\ - & - & p^\intercal & - & - & - & - & q^\intercal & - & - \end{array}\right) \cdot \begin{pmatrix}\epsilon\\ \eta\end{pmatrix} + \left(\begin{array}{c} 0\\ \vdots\\ 0\\ k\\ \hline 0\\ \vdots\\ \vdots\\ 0\end{array}\right)\cdot u(t),$$

with

$$\begin{aligned} y &= \epsilon_1\\ q^\intercal &= \begin{pmatrix}-b_0 & -b_1 & \dots & -b_{n-r-2} & -b_{n-r-1}\end{pmatrix}\\ p^\intercal &= \begin{pmatrix}1 & 0 & \dots & 0\end{pmatrix}. \end{aligned}$$

We want to have a vanishing output. For this to happen, it is necessary to choose

$$\epsilon^*(0) = 0, \quad u^*(t) = -\frac{1}{k}s^\intercal \cdot \eta^*(t).$$

If the system is initialized and controlled according to these parameters, the output $y(t)$ and the state variable $\epsilon(t)$ will be zero for all times $t > 0$. The trajectories of the state variables $\eta(t)$ are governed by the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta^*(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ 0 & \dots & \dots & 0 & 1\\ - & - & q^\intercal & - & - \end{pmatrix}\cdot\eta^*(t) = Q\cdot\eta^*(t), \quad \eta^*(0) = \eta_0^*.$$

These equations define the *zero dynamics* of the system. If Q is asymptotically stable (i.e. all eigenvalues with negative real part), then the system is minimum phase. As soon as there is a zero with positive real part:

- The system is non-minimum phase,
- the system has its zero dynamics unstable, and
- its internal states η can diverge without $y(t)$ being affected.

In these cases, $u(t)$ may not be chosen such that the output $y(t)$ is almost zero before the states η associated with the zero dynamics are almost zero. Of course this situation must be avoided: we want that the bandwidth of che closed-loop system to be substantially smaller than the slowest non-minimumphase zero.

12.2 Nonlinear Systems

12.2.1 First and Second Order Systems

First-order time invariant nonlinear systems can be generally described with

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = f(x(t), u(t), t), \;\; x(t_0) = x_0 \neq 0,$$

where $f : \mathbb{R} \to \mathbb{R}$ is some nonlinear function. These are the easiest to be analyzed: Setting $u = 0$ we can separate the problem as

$$\int \frac{\mathrm{d}x}{f(x)} = \int \mathrm{d}t = t + c,$$

and solve it explicitly. The equilibrium point x_e is defined as

$$f(x_e, t) = 0, \quad \forall t.$$

The equilibrium point x_e is *uniformly Lyapunov stable (ULS)*, if for each scalar $R > 0$, there is a scalar $r(R) > 0$, such that if the initial condition x_0 satisfies

$$\|x_0 - x_e\| < r,$$

the corresponding solution will satisfy the condition

$$\|x(t) - x_e\| < R$$

for all times t greater than t_0. The same point is *asymptotically stable* if it is ULS and attractive, i.e.

$$\lim_{t\to\infty} x(t) = x_e.$$

An equilibrium point is *exponentially asymptotically stable* if there exist constant scalars $a, b > 0$ such that

$$\|x(t) - x_e\| \leq a \cdot e^{-b(t-t_0)} \cdot \|x_0 - x_e\|.$$

Nonlinear systems can have infinitely many isolated equilibrium points. These can

- have a finite region of attraction,
- be non-exponentially asymptotically stable and
- be unstable (escaping to infinity in finite time).

Second-order time invariant nonlinear systems can be generally described with

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = f(x_1(t), x_2(t)), \;\; x_1(0) = x_{1,0}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = f(x_1(t), x_2(t)), \;\; x_2(0) = x_{2,0}.$$

Generally, we look at one equilibrium point and we write the first term of the Taylor expansion around it:

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta x(t) = A \cdot \delta x(t), \;\; A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Excluding the case where matrix A has two eigenvalues with 0 real part, the local behaviour of the nonlinear system and its linearized version are geometrically equivalent.

12.2.2 Lyapunov Principle - General Systems

1. The Lyapunov Principle is valid for all finite-order systems: as long as the linearized system has no eigenvalues on the imaginary axis.
2. The local stability properties of an arbitrary-order nonlinear system are fully understood once the eigenvalues of the linearization are known.
3. Particularly, if the linearization of a nonlinear system around an isolated equilibrium point x_e is asymptotically stable (or unstable), then this equilibrium is an asymptotically stable (or unstable) equilibrium of the nonlinear system as well. We can but not say that this holds also for the concept of *stable system* ($\text{Re}(\lambda) = 0$).

If we are interested in non-local results or in the case of stable systems, we should use the Lyapunov's direct method. A scalar function $\alpha(p)$ with $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function if $\alpha(0) = 0$ and $\alpha(p) \geq \alpha(q) \forall p > q$. A function $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a candidate global Lyapunov function if

- the function is strictly positive, i.e., $V(x,t) > 0 \,\forall\, x \neq 0, \forall\, t$ and $V(0) = 0$ and
- there are two nondecreasing functions α and β that satisfy the inequalities

$$\beta(\|x\|) \leq V(x,t) \leq \alpha(\|x\|)$$

Remark. If these conditions are not met, only local assumptions can be made.

Theorem 1. The system

$$\dot{x}(t) = f(x(t), t), \quad x(t_0) = x_0 \neq 0,$$

is globally/locally stable in the sense of Lyapunov if there is a global/local Lyapunov function candidate $V(x,t)$ for which the following inequality holds $\forall x \neq 0$ and $\forall t$:

$$\dot{V}(x(t), t) = \frac{\partial V(x,t)}{\partial t} + \frac{\partial V(x,t)}{\partial x} f(x(t), t) \leq 0$$

Theorem 2. The same system is globally/locally asymptotically stable if there is a global/local Lyapunov function candidate $V(x,t)$ such that $-\dot{V}(x(t),t)$ satisfies all conditions of a global/local Lyapunov function candidate.

Remark. In general it is difficult to find suitable functions. A good way to approach the problem is to use physical laws (Lyapunov functions can be seen as generalized energy functions).

For linear systems one can find the Lyapunov function

$$V(x(t)) = x(t)^\intercal \cdot P \cdot x(t), \quad P = P^\intercal > 0,$$

where P is the solution of the Lyapunov equation

$$PA + A^\intercal P = -Q.$$

For arbitrary $Q = Q^\intercal > 0$, a solution to this equation exists if and only if A is a Hurwitz matrix.

Remark. Lyapunov theorems provide sufficient but not necessary conditions.

12.3 Example

Since you are at the end of the semester and your engineering team goes on holiday, you decide to work with the financial team of SpaghETH to build predictions for the future. After rigorous studies in the customers' behavior you come up with a dynamical model for the revenue for the costumers' interest in SpathETH, denoted by x_1, and the revenue's, denoted by x_2. Note that x_1 and x_2 denotes deviations from a reference configuration. The nonlinear system is described by the following differential equations:

$$\begin{aligned}\dot{x}_1 &= x_1 x_2^2 \\ \dot{x}_2 &= x_1^2 x_2 + 2x_2^3 - 6x_2.\end{aligned}$$

1. Linearize the system around the equilibrium $x_{1e} = x_{2e} = 0$ and find the matrix A.
2. What can you conclude about the stability properties of the origin?
3. Evaluate the stability of the origin using the Lyapunov function $V = \frac{1}{2}(x_1^2 + x_2^2)$ and find an estimate of the region of attraction about the equilibrium point. Explain your results.

Solution.

1. The linearization matrix A reads

$$\begin{aligned} A &= \left.\begin{pmatrix} \frac{\partial}{\partial x_1}(x_1 x_2^2) & \frac{\partial}{\partial x_2}(x_1 x_2^2) \\ \frac{\partial}{\partial x_1}(x_1^2 x_2 + 2x_2^3 - 6x_2) & \frac{\partial}{\partial x_2}(x_1^2 x_2 + 2x_2^3 - 6x_2) \end{pmatrix}\right|_{(0,0)} \\ &= \left.\begin{pmatrix} x_2^2 & 2x_1 x_2 \\ 2x_1 x_2 & x_1^2 + 6x_2^2 - 6 \end{pmatrix}\right|_{(0,0)} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & -6 \end{pmatrix}. \end{aligned}$$

2. The eigenvalues of matrix A are $\lambda_1 = 0$ and $\lambda_2 = -6$. Using the Lyapunov principle, we cannot evaluate the stability of the origin of the nonlinear system, since the linearized one is just stable around the equilibrium.

3. The total time derivative of the Lyapunov function reads

$$\begin{aligned} \dot{V} &= x_1 \dot{x}_1 + x_2 \dot{x}_2 \\ &= x_1 \cdot (x_1 x_2^2) + x_2 \cdot (x_1^2 x_2 + 2x_2^3 - 6x_2) \\ &= x_1^2 x_2^2 + x_2^2 x_1^2 + 2x_2^4 - 6x_2^2 \\ &= 2x_2^2 \cdot (x_1^2 + x_2^2 - 3). \end{aligned}$$

   In order for $\dot{V}$ to be negative semi-definite, it must hold $x_1^2 + x_2^2 \leq 3$. Hence, we can conclude that the origin in locally stable. A (possibly conservative) estimate of the region of attraction is $\{x \in \mathbb{R}^2 | x_1^2 + x_2^3 \leq 3\}$. Note that the actual region of attraction might be much larger.