Liouville dynamical percolation

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Abstract

Dynamical percolation on the triangular lattice is a Markov process where the state space is the set of possible black/white colorings of the sites. Each site is associated with an independent Poisson clock, and the color of a site is flipped every time its clock rings. We consider the case where the rate of the Poisson clocks are defined in terms of a background $\sqrt{8/3}$-LQG surface. We prove that the process converges in the scaling limit to a process we call Liouville dynamical percolation (LDP). We prove ergodicity of both LDP and the Euclidean counterpart of the process. The latter process was studied by Garban, Pete and Schramm (2013).

Contents

1 Introduction

Consider a bounded simply connected domain $D \subset \mathbb{C}$ with smooth boundary, and let h be a zero boundary GFF in D. For $\eta \in (0,1)$ consider the rescaled triangular lattice $\eta\mathbb{T}$. At time $t = 0$ the sites of the rescaled triangular lattice are colored black or white, where the colors are chosen uniformly and independently at random. Let $\gamma = \sqrt{8/3}$ and $Q = 2/\gamma + \gamma/2$, and define $f : \mathbb{R} \to \mathbb{R}$ by $f(\alpha) := Q\alpha - \frac{1}{2}\alpha^2$. Then let α be such that $f(\alpha) = 3/4$. This gives $\alpha = \gamma/4 = 1/\sqrt{6}$. Let $\mu_{\alpha h}$ denote the α-LQG area measure associated with h. Each site is associated with an independent Poisson clock, such that the rate of the clock of node x is given by $\mu_{\alpha h}(B_\eta(x))\eta^d$. Let $\omega_\eta(t)$ denote the quad crossing element for the percolation at time t.

One of the main results of this note is the following theorem. We refer to [GPS13b] for notation and definitions not given above.

Theorem 1.1. *When $\eta \to 0$ the process $(\omega_\eta(t))_{t\geq 0}$ converges in law in $(\mathrm{Sk}, d_{\mathrm{Sk}})$ to a limiting stochastic process in the quad-crossing space $\mathscr{H}$ denoted by $(\omega_\infty(t))_{t\geq 0}$.*

We call the process $(\omega_\infty(t))_{t\geq 0}$ **Liouville dynamical percolation**.

2 Convergence of Gaussian multiplicative chaos measures

Let $\beta \in \mathbb{R}$, and let λ be a measure on $\mathbb{C}$. We define the β-energy of λ by

$$I_\beta(\lambda) = \iint \frac{1}{\|z-y\|^\beta} \lambda(dz)\lambda(dy).$$

Let $\alpha \geq 0$, let λ^n be a sequence of random measures in $\mathbb{C}$ converging in probability for the Prokhorov topology to a measure λ, and let $\nu^n_{h,\alpha}$ be the sequence of α-GMC of λ^n with respect to h. The objective of this section is to give a sufficient condition for $\nu^n_{h,\alpha}$ to converge to $\nu_{h,\alpha}$, the α-GMC of h with respect to λ. The result we need can be extracted from the results in [Ber15].

Lemma 2.1. *Take $\bar\beta > 0$ and assume that $\sup_n \mathbb{E}[I_{\bar\beta}(\lambda^n)] < \infty$. Then for all $\alpha^2 < \bar\beta$ and $\mathcal{O} \subseteq \mathbb{C}$ such that $\lambda(\mathcal{O}) < \infty$ and $\lambda(\partial\mathcal{O}) = 0$ a.s., the GMC measures considered above are well-defined and we have that $\nu^n_{h,\alpha}(\mathcal{O}) \to \nu_{h,\alpha}(\mathcal{O})$ in L^1.*

Proof. Fix $\alpha^2 < \bar\beta$. For simplicity we write $\nu^n_{h,\alpha}$ as ν^n. For some smooth approximation h_ϵ to h (e.g. the circle average approximation) we write $\nu^n_{h_\epsilon,\alpha}$ as ν^n_ϵ. By the triangle inequality, for $n, n' \in \mathbb{N}$,

$$\mathbb{E}|\nu^n(\mathcal{O}) - \nu^{n'}(\mathcal{O})| \leq \mathbb{E}|\nu^n(\mathcal{O}) - \nu^n_\varepsilon(\mathcal{O})| + \mathbb{E}|\nu^n_\varepsilon(\mathcal{O}) - \nu^{n'}_\varepsilon(\mathcal{O})| + \mathbb{E}|\nu^{n'}_\varepsilon(\mathcal{O}) - \nu^{n'}(\mathcal{O})|.$$

The second term on the right side converges to 0 as $n, n' \to \infty$ for any fixed ε, since $\sup_n \mathbb{E}[\lambda^n(\mathcal{O})^2] < \infty$. Therefore, to conclude it is sufficient to show that $\nu^n_\epsilon(\mathcal{O})$ converges to $\nu^n(\mathcal{O})$ uniformly in $n \in \mathbb{N}$, i.e., that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[|\nu^n_\epsilon(\mathcal{O}) - \nu^n(\mathcal{O})|\right] \to 0 \quad \text{ as } \epsilon \to 0.$$

This result follows from a close inspection of [Ber15]. When possible we try to use the same notations as in that paper. Write $J^n_\varepsilon = \nu^n_\varepsilon(\mathcal{O})$ to simplify notation. Thanks to the strategy of [Ber15, Chapter 4] we just need to show that $\mathbb{E}\left[(J^n_\epsilon)^2\right]$ and $\mathbb{E}\left[J^n_\epsilon J^n_\delta\right]$ converge as $\epsilon \to 0$, uniformly in n. Let us treat the first term, as the second term is treated in the same way. We have that $\mathbb{E}\left[(J^n_\epsilon)^2\right]$ is upper bounded by the sum of

$$\mathbb{E}\left[\iint_{|x-y|\leq\delta} e^{\gamma^2\mathbb{E}[h_\epsilon(x)h_\epsilon(y)]}\lambda^n(dx)\lambda^n(dy)\right] \tag{1}$$

and

$$\mathbb{E}\left[\iint_{|x-y|\geq\delta} e^{\gamma^2\mathbb{E}[h_\epsilon(x)h_\epsilon(y)]}\lambda^n(dx)\lambda^n(dy)\right], \tag{2}$$

Now [Ber15, equation (3.12)] shows that for all $n \in \mathbb{N}$ and with $\beta \in (0, \overline{\beta})$, (1) is smaller than a constant depending only on the correlation kernel of h times

$$\mathbb{E}\left[\iint_{|x-y|\leq\delta} |x-y|^{-\beta}\lambda^n(dx)\lambda^n(dy)\right] \leq \delta^{\bar{\beta}-\beta}\mathbb{E}\left[I_{\bar{\beta}}(\lambda^n)\right].$$

Observe that the right hand of this inequality converges to 0 as $\delta \to 0$, uniformly in n. Let us choose $\delta > 0$ such that (1) is smaller than $\eta/3$. Now, as in [Ber15, Lemma 4.1] when $|x-y| \geq \delta$, we have that $e^{\gamma^2\mathbb{E}[h_\epsilon(x)h_\epsilon(y)]}$ converges in the topology of uniform convergence to a function $F(x,y)$. Thus, uniformly in n, (2) converges to $\mathbb{E}[\iint_{|x-y|\geq\delta} F(x,y)\lambda(dx)\lambda(dy)] < \infty$. This finishes the proof. □

Recall the definition of ε-pivotal points in [GPS13b, Definition 2.14].

Lemma 2.2. *Let λ^ε_η be renormalized counting measure on the ε-pivotal points of critical site percolation on $\eta\mathbb{T}$ as defined in [GPS13a, GPS13b]. Then λ^ε_η satisfies the energy condition of Lemma 2.1 with $\overline{\beta} < 3/4$.*

Proof. This follows from the calculation of [GPS13a, page 43], where we multiply each term in the sum by $|z-w|^{-\overline{\beta}}$. The exponent of 2^k in the terms of the sum is $3/4 - \overline{\beta} + o(1)$. Since this is asymptotically positive, the last term will dominate the sum as before. We get the same bound as before, except for an additional factor of $r^{-\overline{\beta}}$. □

Combining Lemmas 2.2 and 2.3 we see that we may define the $1/\sqrt{6}$-GMC measure of h with respect to the limiting measure λ^ε of λ^ε_η. Denote the discrete and continuum $1/\sqrt{6}$-GMC measures by ν^ε_η and ν^ε, respectively.

Lemma 2.3. *Gaussian multiplicative chaos measures are always positive if the base measure is positive.*

Proof. Denote the GMC measure by ν and denote the base measure by λ. To conclude the proof we need to show that if $\lambda(D) > 0$ we have $\nu(D) > 0$ a.s. We assume throughout the proof that $\lambda(D) > 0$. Write $h = \sum_j \alpha_j \phi_j$ for an orthonormal basis $\{\phi_j\}$ and i.i.d. Gaussians α_j. Also define $h^n = \sum_{j>n} \alpha_j \phi_j$, and let ν^n denote the measure induced by h^n. Notice that if $\nu(D) = 0$ then $\nu^n(D) = 0$ for all n, since the measure ν^n equals the measure ν weighted by a smooth function. Since the α_j's are i.i.d., the Kolmogorov 0-1 law gives that $\mathbb{P}[\nu^n(D) = 0 \; \forall n \in \mathbb{N} \,|\, \lambda] \in \{0, 1\}$ λ-a.s. We see that the probability must be 0 a.s., since the expectation of the measure is positive. □

3 Dynamical percolation with cut-off

Let $\omega_\eta^\varepsilon(t)$ denote dynamical percolation on the triangular lattice rescaled by η, where we only update pivotal points which are ε-important. The following theorem is proved exactly as [GPS13b, Theorems 7.3 and 7.10], where we use Lemmas 2.1 and 2.2 to argue convergence of the pivotal measure. All notations used in the theorem are defined in [GPS13b].

Theorem 3.1. *Consider the setting of [GPS13b, Theorem 7.3], where PPP_T has intensity ν^ε instead of intensity λ^ε.*

- *One can define a càdlàg process $(\omega_\infty^\varepsilon(t))_{t\in[0,T]}$ with values in the quad-crossing space $\mathscr{H}$, which starts from $\omega_\infty \sim \mathbb{P}_\infty$, and which is determined from $(\omega_\infty, \mathrm{PPP}_T)$ in the exact same way as in [GPS13b, Theorem 7.3].*
- *The process $(\omega_\eta^\varepsilon(t))_{t\in[0,T]}$ converges in law in the Skorokhod space to the process $(\omega_\infty^\varepsilon(t))_{t\in[0,T]}$.*

4 Stability property of dynamical percolation

We will prove the following proposition. Combining with Theorem 3.1 it implies Theorem 1.1.

Proposition 4.1. *Let $T > 0$ and some instance h of the GFF be fixed. There exists a continuous function $\psi = \psi_{T,h} : [0, 1] \to [0, 1]$, with $\psi(0) = 0$ such that uniformly in $0 < \eta < \varepsilon$,*

$$\mathbb{E}\big[d_{\mathsf{Sk}_T}(\omega_\eta(\cdot), \omega_\eta^\varepsilon(\cdot))\big] \leq \psi(\varepsilon)\,.$$

The proof is similar to the proof of the Euclidean version [GPS13b, Proposition 8.1]. We will therefore omit many details in the proof, and point out only the places at which our argument differ from the one in [GPS13b].

The following lemma is needed for adapting the Euclidean arguments to our setting.

Lemma 4.2. *There exists a deterministic $\delta > 0$ and a random $C > 0$ such that for any $r \in (0,1)$ and $z \in D$ we have $\mu(B_r(z)) < C\alpha_4(r,1)r^\delta$. The same result holds for a mean zero free boundary GFF if we only require $\mu(B_r(z)) < Cr^\delta$.*

Proof. First we consider the case of the zero boundary GFF. Note that

$$\mathbb{P}[\mu(B_r(z)) > \alpha_4(r,1)r^\delta] \leq \mathbb{E}[\mu(B_r(z))^p]\alpha_4(r,1)^{-p}r^{-\delta p}$$
$$= r^{-\gamma^2 p^2/2+(2+\gamma^2/2)p-5p/4-\delta p+o(1)}.$$

If $\gamma^2 = 1/6$ then the exponent is larger than 2 for sufficiently small δ and $p = 5$. By a union bound and Borel-Cantelli we see that $\mu(B_r(z)) > \alpha_4(r,1)r^\delta$ finitely often, which implies the statement of the lemma for the zero boundary GFF.

For the free boundary GFF we proceed similarly. □

To prove the proposition, we will need to introduce some notations as well as some preliminary lemmas. Since the entire section is about discrete configurations $\omega_\eta \in \mathscr{H}$, we will often omit the subscript η and denote the percolation configurations simply by ω.

Definition 4.3 (Importance of a point)**.** *Given a percolation configuration $\omega = \omega_\eta \in \mathscr{H}$ and a site z, let $Z(z) = Z_\omega(z)$ denote the maximal radius r such that the four arm event holds from the hexagon of z to distance r away. This is also the maximum r for which changing the value of $\omega(z)$ will change the white connectivity in ω between two white points at distance r away from z, or will change the black connectivity between two black points at distance r away from z. The quantity $Z(z)$ will also be called the* **importance** *of z in ω.*

Definition 4.4. *Fix $T > 0$. We will denote by $X = X_{\eta,T}$ the random set of sites on $\eta\mathbb{T}$ which are updated along the dynamics $t \in [0,T] \mapsto \omega_\eta(t)$. Recall that this random subset of $\eta\mathbb{T}$ is independent of $\omega = \omega_\eta(t=0)$ and each site $z \in \eta\mathbb{T}$ is in X independently with probability $q_T := 1 - e^{-Tr_h(\eta)} \sim T\,\mu(B_\eta(z))/\alpha_4^\eta(\eta,1)$.*

Let $\Omega(\omega, X)$ denote the set of percolation configurations ω' such that $\omega'(x) = \omega(x)$ for all $x \notin X$. Finally, let $\mathcal{A}_4(z,r,r')$ denote the 4-arm event in the annulus $A(z,r,r')$.

Lemma 4.5. *Let $T > 0$ and the instance of h be fixed. Set $r_i := 2^i\,\eta$, $N := \lfloor \log_2(1/\eta) \rfloor$. Let $\mathcal{W}_z(i,j)$ denote the event that there is some $\omega' \in \Omega(\omega,X)$ satisfying $\mathcal{A}_4(z,r_i,r_j)$. Then for every pair of integers i,j satisfying $0 \leq i < j < N$ and every $z \in \mathbb{R}^2$,*

$$\mathbb{P}[\mathcal{W}_z(i,j)] \leq C_1\,\alpha_4(r_i,r_j)\,, \tag{3}$$

where $C_1 = C_1(T,h)$ is a constant that may depend only on T and h (note that here $\mathbb{P}$ includes the extra randomness in the choice of the subset X).

Proof. The proof proceeds exactly in the Euclidean case, except that we use the new definition $\mathbb{P}[x \in X]$. The reader is advised to also read the proof of [GPS13b, Lemma 8.4], since many steps are skipped here.

Define $A_n = A(z, r_n, r_{n+1})$. Note that conditioned on h the events $\{x_1 \in X\}$ and $\{x_2 \in X\}$ are independent, similarly as in the Euclidean case. In particular, defining $b_i^j := \sup_z \mathbb{P}\big[\mathcal{W}_z(i,j)\big]$, we get as before

$$\mathbb{P}\big[\mathcal{W}_z(i,j),\, \mathcal{D}\big] \le O(T) \sum_{n=i+1}^{j-2} \mu(A_n)\, \alpha_4^\eta(\eta,1)^{-1}\, b_1^{n-1}\, b_i^{n-1}\, b_{n+2}^j,$$

and further for some absolute constant C_2 and all i, j with $j > i$,

$$b_i^j \le C_2\, \alpha_4(r_i, r_j)\Big(1 + T\, C_1^3 \sum_{n=i+1}^{j-1} \frac{\mu(A_n)}{\alpha_4(r_n,1)}\Big). \tag{4}$$

Note that the latter bound is our variant of [GPS13b, Equation (8.3)]. As in [GPS13b] we show (3) by induction on j, and for a fixed j by induction on $j-i$. By Lemma 4.2 there is a random constant $C(h)$ depending on h and a deterministic constant $\delta > 0$ such that $\mu(A_n) < C(h)\alpha_4(r_n,1)r_n^\delta$. This implies that we can find a random constant $M = M(h,T) \in \mathbb{N}$ such that for $N - j \ge M$,

$$T\,(2\,C_2)^3 \sum_{n=i+1}^{j-1} \frac{\mu(A_n)}{\alpha_4(r_n,1)} \le 1\,.$$

Choosing $C_1 = 2C_2$ and insertion into (4) completes the proof by induction as in [GPS13b]. □

Set

$$Z^X(z) := \sup_{\omega' \in \Omega(\omega, X)} Z_{\omega'}(z)\,.$$

Lemma 4.6. *For every site z and every ε and r satisfying $2\,\eta < \varepsilon < 2^4\,\varepsilon < r \le 1$, we have*

$$\mathbb{P}\big[Z^X(z) \ge r,\, Z_\omega(z) \le \varepsilon\big] \le O_{T,h}(1)\alpha_4(\eta,r)\epsilon^\delta.$$

Proof. Assume ω' is such that $Z_{\omega'}(z) \ge r$, and let $x_1, \ldots, x_m$ be some enumeration of the sites in $B_\eta(z,\epsilon)$ where $\omega' \ne \omega$. For each $j = 0, 1, \ldots, m$, let ω_j denote the configuration that agrees with ω' on every site different from $x_{j+1}, x_{j+2}, \ldots, x_m$, and agrees with ω on $x_{j+1}, \ldots, x_m$. Then $\omega_m = \omega'$ and $Z_{\omega_0}(z) < \varepsilon$. Let $k_{\omega'}$ be the first j such that $Z_{\omega_j}(z) > r$.

Let $\widehat{X}$ be the set of sites $x \in B_\epsilon(z)$ such that $x = x_{k_{\omega'}}$ for some ω' satisfying $Z_{\omega'}(z) \ge r$. Proceeding as in the proof of [GPS13b, Lemma 8.5] we see that

$$\mathbb{P}\big[Z^X(z) \ge r,\, Z(z) \le \varepsilon,\, x \in \widehat{X}\big] \le O_T(1)\, \alpha_4(\eta, r^x)\, \mu(B_\eta(x))\, \alpha_4(r,1)^{-1}\,.$$

Since $\widehat{X}$ is non-empty if $Z^X(z) \geq r$ and $Z(z) \leq \varepsilon$ both occur,

$$\begin{aligned}\mathbb{P}\big[Z^X(z) \geq r,\, Z(z) \leq \varepsilon\big] &\leq \sum_{x \in B_\epsilon(z)} \mathbb{P}\big[Z^X(z) \geq r,\, Z(z) \leq \varepsilon,\, x \in \widehat{X}\big] \\ &\leq O_T(1) \sum_{n=0}^{\log_2(\epsilon/\eta)} \alpha_4(\eta, r_n)\mu(A_n)\alpha_4(r,1)^{-1}.\end{aligned}$$

The lemma now follows by Lemma 4.2, and by using $\alpha_4(\eta,1)\alpha_4(r,1)^{-1} \asymp \alpha_4(\eta, r)$. □

Next we state a similar result to Lemma 4.6 which will be needed in a later work. Set

$$Z_X(z) := \inf_{\omega' \in \Omega(\omega, X)} Z_{\omega'}(z)\,.$$

The proof of the following lemma is omitted, since it is identical to the proof of Lemma 4.6.

Lemma 4.7. *For every site z and every ε and r satisfying $2\,\eta < \varepsilon < 2^4\,\varepsilon < r \leq 1$, we have*

$$\mathbb{P}\big[Z_X(z) \leq \epsilon,\, Z_\omega(z) \geq r\big] \leq O_{T,h}(1)\alpha_4(\eta, r)\epsilon^\delta.$$

For any quad $Q \in \mathscr{Q}$, if $r > 0$ is smaller than the minimal distance from $\partial_1 Q$ to $\partial_3 Q$, we will say that Q is r**-almost crossed** by $\omega = \omega_\eta \in \mathscr{H}$, if there is an open path in the r-neighborhood of Q that comes within distance r of each of the two arcs $\partial_1 Q$ and $\partial_3 Q$.

The following proposition and lemma are proved exactly as [GPS13b, Lemma 8.5 and Proposition 8.6], and the proofs are therefore omitted. Proposition 4.1 follows from Lemma 4.9 exactly as in [GPS13b]. In particular, we observe from the proof that to deduce Proposition 4.1 from the lemma it is sufficient with a variant of the lemma for which the considered probability converges 0 as $\epsilon \to 0$.

Proposition 4.8. *Let T, h, and X be as above, and fix some quad $Q \in \mathscr{Q}$. Let $r > 0$ be smaller than the minimal distance between $\partial_1 Q$ and $\partial_3 Q$, and suppose that $0 < \eta < 2\,\eta < \varepsilon < 2^5\,\varepsilon < r \leq 1$. Then the probability that there are some $\omega', \omega'' \in \Omega(\omega, X)$ such that (a) Q is crossed by ω', (b) Q is not r-almost crossed by ω'', and (c) $\omega'(z) = \omega''(z)$ for every site z satisfying $Z_\omega(z) \geq \varepsilon$ is at most*

$$O_{T,Q,h}(\varepsilon^\delta)\,\alpha_4(r,1)^{-1}.$$

See [GPS13a] for the notation $\emptyset_k(\cdot)$ used in the following lemma.

Lemma 4.9. *Let $k \in \mathbb{N}$ and $T > 0$ be fixed and suppose that $0 < \eta < 2\eta < \varepsilon < 2^{-k-20}$. Then, the probability that there are some $\omega', \omega'' \in \Omega(\omega, X)$ such that* *(a)* $\omega' \notin \text{Ø}_k(\omega'')$*;*

(b) $\omega'' \notin \text{Ø}_k(\omega')$*;*

(c) $\omega'(z) = \omega''(z)$ *for every site* z *satisfying* $Z_\omega(z) \geq \varepsilon$*;*

is at most

$$O_{T,k,h}(\varepsilon^\delta).$$

We say that a pivotal point is ε-pivotal if the symmetric difference of the loop collection before and after flipping the color has at least three loops enclosing regions of LQG area at least ε. We say that a pivotal is ρ-important if the following holds (see [GPS13b, Definition 2.14]): It belongs to a square in $\rho\mathbb{Z}^2$, and there are four arms from the pivotal to the outer boundary of the width ρ annulus around this square. Note that the cut-off we consider above is defined in terms of ρ-importance.

5 Size of the spectral sample for multiple quad crossings

Let $\mathcal{Q}$ be a collection of finitely many quads, and for $R > 1$ let $f = f_{R\mathcal{Q}}$ be the indicator function describing whether all the quads of $\mathcal{Q}$ have been crossed. Throughout this section we do *not* rescale the triangular lattice; to be consistent with [GPS10] we instead rescale the quads by R. For any set V let $\mathcal{A}_\square(V, \mathcal{Q})$ denote the event that V is pivotal for $\mathcal{Q}$. Let $\mathcal{Q}^o \subset \mathbb{C}$ denote the union of the complementary components V of the quad boundaries which are such that $\mathbb{P}[\mathcal{A}_\square(V, \mathcal{Q})] > 0$ for sufficiently large R. We assume throughout this and the next section that $\mathcal{Q}^o$ has finitely many connected components and the the boundaries of the quads are piecewise smooth.

Theorem 5.1.

$$\lim_{s\to\infty} \inf_{R>1} \mathbb{P}\left[|\mathscr{S}_f| \in [s^{-1}R^2\alpha_4(R), sR^2\alpha_4(R)) \cup \{0\}\right] = 1.$$

Proof. The theorem follows from Proposition 5.3 and Theorem 5.2 by the exactly the same argument as in the proof of [GPS10, Theorem 7.4]. □

Theorem 5.2. *Let* $U \subset \mathcal{Q}^o$ *be open, and let* $U' \subset \overline{U'} \subset U$*. Then, for some constants* $\bar{r} = \bar{r}(U', U, \mathcal{Q}) > 0$ *and* $q(U', U, \mathcal{Q}) > 0$*, for any* $r \in [\bar{r}, R\mathrm{diam}(U)]$*,*

$$\mathbb{P}\big[0 < |\mathscr{S}_f \cap RU| \leq r^2\,\alpha_4(r),\ \mathscr{S}_f \cap RU \subset RU'\big] \leq q(U', U, \mathcal{Q})\,\frac{R^2\,\alpha_4(R)^2}{r^2\,\alpha_4(r)^2}\,. \qquad (5)$$

Proof. The theorem follows from Propositions 5.8 and 5.4, and from [GPS10, Proposition 6.1]. See the proof of [GPS10, Theorem 7.1] for a similar argument. □

Proposition 5.3. *Given any $\delta > 0$ we can find an open set $U \subset \overline{U} \subset \mathcal{Q}^o$, such that $\mathbb{P}[\mathscr{S}_f \subset RU] > 1 - \delta$.*

Proof. Let $\alpha = \bigcup_{Q \in \mathcal{Q}} \partial Q \subset \mathbb{C}$ be the union of the quad boundaries. Given $s > 0$ let $g : \Omega \to \{-1, 1\}$ be measurable with respect to the σ-algebra $\mathcal{F}_s$ of quad crossing information at distance $> s$ from α, such that

$$g(\omega) = \begin{cases} -1 & \text{if} \quad \mathbb{P}[f = -1 \,|\, \mathcal{F}_s] > 1/2, \\ 1 & \text{otherwise.} \end{cases}$$

By [SS11, Theorem 1.5], given any $\epsilon > 0$ and quad Q it holds for all s and η sufficient small that $\mathbb{P}[\epsilon < \mathbb{P}[Q(\omega) \,|\, \mathcal{F}_s] < 1 - \epsilon] < \epsilon$. Therefore, for sufficiently small s and η,

$$\mathbb{P}[\epsilon < \mathbb{P}[f = -1 \,|\, \mathcal{F}_s] < 1 - \epsilon] < \epsilon.$$

It follows that for sufficiently small s and η,

$$\begin{aligned} \mathbb{P}[f \neq g] \leq \mathbb{P}[f \neq g; \epsilon < \mathbb{P}[f = -1 \,|\, \mathcal{F}_s] < 1 - \epsilon] \\ + \mathbb{P}[f = 1; \mathbb{P}[f = -1 \,|\, \mathcal{F}_s] > 1 - \epsilon] \\ + \mathbb{P}[f = -1; \mathbb{P}[f = 1 \,|\, \mathcal{F}_s] > 1 - \epsilon] < 3\epsilon, \end{aligned}$$

which implies that with $\| \cdot \|$ denoting the L^2 norm we have $\|f - g\| < 10\sqrt{\epsilon}$. From [GPS10, equation (2.7)] we get further that with tv denoting total variation distance,

$$\text{tv}(\mathscr{S}_f, \mathscr{S}_g) \leq \sum_{S \subset \mathcal{I}} |\widehat{f}(S)^2 - \widehat{g}(S)^2| \leq \|f - g\| \|f + g\| < 20\sqrt{\varepsilon}.$$

The spectral sample of $\mathscr{S}_g$ has distance at least s from α, so we see that the proposition holds with U' instead of U if we let U' be the points contained of $\mathbb{C}$ which have distance at least s from the quad boundaries. We have that $\mathscr{S} \cap (U' \setminus \mathcal{Q}^o) = \emptyset$ for all sufficiently large R, and we obtain the proposition by defining $U = U' \cap \mathcal{Q}^o$. □

5.1 Few squares intersecting the spectral sample

The following is our version of [GPS10, Proposition 4.2] for the case of multiple quads. It may be deduced from the lemmas below just as in [GPS10].

Proposition 5.4. *Consider a collection $\mathcal{Q}$ of finitely many quads, and let $\mathscr{S}$ be the spectral sample of f. Let $U' \subset U \subset \mathcal{Q}^o$, let R denote the diameter of U, let $a \in (0,1)$, and suppose that the distance from U' to the complement of U is at least $a\,R$. Let $\mathcal{S}(r,k)$ be the collection of all sets $S \subseteq \mathcal{I}$ such that $\big|(S \cap U)_r\big| = k$ and $S \cap (U \setminus U') = \emptyset$. Then for $g(k) := 2^{\vartheta \log_2^2(k+2)}$, with $\vartheta > 0$ large enough, and $\gamma_r(R) := (R/r)^2 \alpha_4(r,R)^2$, we have*

$$\forall k, r \in \mathbb{N}_+ \qquad \mathbb{P}\big[\mathscr{S} \in \mathcal{S}(r,k)\big] \le c_a\, g(k)\, \gamma_r(R)\,,$$

where c_a is a constant that depends only on a and $\mathcal{Q}$.

The following is proved in [GPS10, Section 4.2]. See [GPS10] for the definition of an annulus structure and compatibility.

Lemma 5.5 ([GPS10])**.** *Let $\mathcal{Q}, U', U, R, a$, and $S(r,k)$ be as in Proposition 5.4, where we assume $\mathcal{Q}$ is a single quad. It is possible to construct a set $\mathfrak{U}$ of annulus structures $\mathcal{A}$ such that for each $S \in S(r,k)$ there is an annulus structure $\mathcal{A} \in \mathfrak{U}$ compatible with S, and such that*

$$\sum_{\mathcal{A} \in \mathfrak{U}} h(\mathcal{A})^2 \le c_a g(k) \gamma_r(R).$$

Lemma 5.6. *Lemma 5.5 also holds when $\mathcal{Q}$ is a collection of finitely many quads.*

Proof. The set $\mathcal{Q}^o$ consists of $m < \infty$ connected components of in-radius greater than aR. Let $S'(r,k)$ be the set of S such that each of the m components intersect at most k r-squares, and such that $S \cap (U \setminus U') = \emptyset$. Then $S(r,k) \subset S'(r,k)$. Let $S_j(r,k)$ be the collection of sets $S \in \mathcal{I}$ such that $|(S \cap U \cap Q_j)_r| \le k$ and $S \cap Q_j \cap (U \setminus U') = \emptyset$. Suppose $\mathfrak{U}_j$ is a set of annulus structures such that each annulus is contained in Q_j, and such that $\mathfrak{U}_j$ has the property that each $S \in S_j(r,k)$ is compatible with some $A \in \mathfrak{U}_j$. Note that $\mathfrak{U}_j$ contains the empty annulus structure. Then let $\mathfrak{U}$ be the set of annulus structures which can be obtained by taking a union of one annulus structure from each of the sets $\mathfrak{U}_j$, except that we do not include the empty annulus structure in $\mathfrak{U}$. Note that for each $S \in S(r,k)$ we can find $A \in \mathfrak{U}$ which is compatible with S. We have upon increasing ϑ if necessary and using $\gamma_r(R) = (r/R)^{1/2+o(1)} < 1$,

$$\begin{aligned}\sum_{\mathcal{A} \in \mathfrak{U}} h(\mathcal{A})^2 = \prod_{j=1}^m \sum_{\mathcal{A} \in \mathfrak{U}_j} h(\mathcal{A}_j)^2 - 1 \le c_a \prod_{j=1}^m \left(1 + \sum_{1 \le i \le k} g(i) \gamma_r(R)\right) - 1 \\ \le c_{a,m} g(k) \gamma_r(R).\end{aligned}$$

□

Lemma 5.7. *For any annulus structure $\mathcal{A}$ with $\bigcup \mathcal{A} \subset \mathcal{Q}^o$,*

$$\mathbb{P}\big[\mathscr{S} \text{ is compatible with } \mathcal{A}\big] \leq h(\mathcal{A})^2.$$

Proof. The proof is identical to the proof of [GPS10, Lemma 4.3]. Note in particular that both in our and their setting the following holds for any set S which is compatible with $\mathcal{A}$ on the event that some annulus in $\mathcal{A}$ does not have a 4-arm event: $\mathbb{E}[f\chi_S \,|\, \omega|_{\mathcal{A}}] = 0$. □

5.2 Partial independence in the spectral sample

Proposition 5.8. *Let Q be a collection of finitely many quads, and let U be an open set whose closure is contained in $\mathcal{Q}^o$. For $R > 0$, let $\mathscr{S} := \mathscr{S}_{f_{R\mathcal{Q}}}$ be the spectral sample of $f_{R\mathcal{Q}}$, the ± 1 indicator function for the crossing event in $R\mathcal{Q}$. Then, there is a constant $\bar{r} = \bar{r}(U, \mathcal{Q})$ such that for any box $B \subset RU$ of radius $r \in [\bar{r}, R\mathrm{diam}(U)]$ and any set W with $W \cap B = \emptyset$, we have*

$$\mathbb{P}\big[\mathscr{S}_{f_{R\mathcal{Q}}} \cap B' \cap \mathcal{Z} \neq \emptyset \,\big|\, \mathscr{S}_{f_{R\mathcal{Q}}} \cap B \neq \emptyset,\, \mathscr{S}_{f_{R\mathcal{Q}}} \cap W = \emptyset\big] \geq a(U, \mathcal{Q})\,,$$

where B' is concentric with B and has radius $r/3$, the random set $\mathcal{Z}$ contains each element of $\mathcal{I}$ independently with probability $1/(\alpha_4(r)r^2)$, and $a(U, \mathcal{Q}) > 0$ is a constant that depends only on U and $\mathcal{Q}$.

Proof. The proof is identical to the proof of [GPS10, Proposition 5.11]. Propositions 5.11 and 5.12 below give the required first and second moment estimates. □

Let $B, W \subset \mathcal{I}$ be disjoint. Define $\Lambda_B = \Lambda_{f,B}$ as the event that B is pivotal for f. More precisely, Λ_B is the set of $\omega \in \Omega$ such that there is some $\omega' \in \Omega$ that agrees with ω on B^c while $f(\omega) \neq f(\omega')$. Also define $\lambda_{B,W} = \Lambda(B, W) := \mathbb{P}\big[\Lambda_B \,\big|\, \mathcal{F}_{W^c}\big]$.

The result of the following lemma is derived in [GPS10, Section 5.3, equation (5.10)].

Lemma 5.9. *Let f be a monotone function with values in $\{-1, 1\}$. Then the following two inequalities are equivalent for any constant $c_1 > 0$*

$$\mathbb{P}\big[\omega', \omega'' \in A_\square(x, \mathcal{Q})\big] \geq c_1\, \mathbb{P}\big[\omega', \omega'' \in A_4(x, B)\big]\, \mathbb{P}\big[\omega', \omega'' \in A_\square(B, \mathcal{Q})\big],$$

$$\mathbb{P}\big[x \in \mathscr{S},\ \mathscr{S} \cap W = \emptyset\big] \geq c_1\, \mathbb{E}\big[\lambda_{B,W}^2\big]\, \alpha_4(r).$$

Lemma 5.10. *Let V be a connected component of $\mathcal{Q}^o$, and assume $V' \Subset V$. We can find quads $q_1, \ldots, q_4$, and for each $Q \in \mathcal{Q}$ we can find collections of quads $A(Q)$ and $B(Q)$, such that the following hold.*

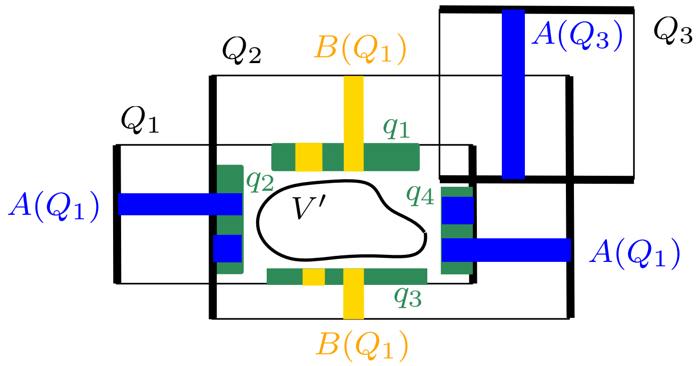

Figure 1: Illustration of the quads defined in Lemma 5.10. The bold boundary arcs of the quads Q_1, Q_2, Q_3 indicate the boundary arcs which are connected on the event that the quads have an open crossing.

- *The quads $q_1, \dots, q_4$ are bounded away from V' and each other, and they are contained in V. One of the open boundaries of q_1 (resp. q_3) is equal to a boundary arc of V, while the other boundary arcs of q_1 (resp. q_3) are in the interior of V. The same property holds for q_2 and q_4, but with closed instead of open. The quads $q_1, \dots, q_4$ are in counterclockwise order around ∂V.*

- *For each quad $Q \in \mathcal{Q}$ one of the following properties (i) or (ii) holds.*

  (i) $A(Q)$ consists of a single quad q, which is such that the open boundary arcs of q are contained in each of the open boundary arcs of Q.

  (ii) $A(Q)$ consists of two quads q'_a, q''_a, which are such that one open boundary arc of q'_a is contained in an open boundary arc of Q and the other open boundary arc of q'_a is contained in the closed boundary arc of q_2 which does not intersect ∂V. The same property holds for q''_a, except that the open boundary arcs intersect the other open boundary arc of Q and a closed boundary arc of q_4, respectively. $B(Q)$ consists of two quads q'_b, q''_b which satisfy the exact same properties, except that we consider the closed boundary arcs of Q and the quads q_1 and q_3.

- *$q_1 \cup q_3 \cup (\cup_{Q \in \mathcal{Q}} B(Q))$ and $q_2 \cup q_4 \cup (\cup_{Q \in \mathcal{Q}} A(Q))$ are disjoint.*

- *$\cup_{B \in \mathcal{Q}} B(Q) \neq \emptyset$.*

Observe that V is pivotal for $\mathcal{Q}$ if all the quads in $\{q_2, q_4\} \cup A(Q)$ have open crossings, and none of the quads in $\{q_1, q_3\} \cup B(Q)$ have open crossings.

Proof. Choose R large and consider a coloring such that there is a pivotal point $x \in V$ for the event that quads $\mathcal{Q}$ are crossed. Let x be closed. We may assume x is bounded away from ∂V. For quads for which there is an open crossing define $A(Q)$ as in (i) by using the open crossing. For the remaining quads $Q \in \mathcal{Q}$ the vertex x is pivotal. Define $A(Q)$ and $B(Q)$ as in (ii) using the 4 arms from x to the quad boundaries. The third bullet point is satisfied if we let the quads be contained in the interior of the hexagons which define the crossings; by this definition we see that appropriate quads $q_1, \ldots, q_4$ satisfying the first bullet point can be found. Note that the fourth bullet point holds since x is pivotal and closed. □

The following is our first moment estimate. Compare with [GPS10, Proposition 5.2].

Proposition 5.11. *Consider the setup of Proposition 5.8. There is a constant* $c_1 > 0$ *(depending on* U *and* $\mathcal{Q}$*) such that for any* $x \in B' \cap \mathcal{I}$,

$$\mathbb{P}\big[x \in \mathscr{S},\ \mathscr{S} \cap W = \emptyset\big] \geq c_1\, \mathbb{E}\big[\lambda_{B,W}^2\big]\, \alpha_4(r). \qquad (6)$$

Proof. It is sufficient to prove the first inequality of Lemma 5.9. We may assume $x \in V$ for V some fixed component of $\mathcal{Q}$.

Let $\widehat{B}$ be a macroscopic square concentric with B of side length $d/2$, where $d = \mathrm{dist}(V', V^c)$. Let r be the side length of $\widehat{B}$ divided by 8. Let $\widetilde{\mathbb{P}}$ denote the probability measure where ω', ω'' are independent both on W and outside $B(x, r)$, and let $E = \{\mathcal{Z}', \mathcal{Z}'', \widetilde{\mathcal{Z}}\}$ be as in [GPS10], except that $\widetilde{\mathcal{Z}}$ is only the subset of the event in [GPS10] which considers the square $B(x, 8r)$. Let E' be the event that the quads $A(Q), q_2, q_4$ have open crossings, and that $B(Q), q_1, q_3$ have closed crossings. Let E'' be the event that there is an open crossing from $\partial B(x, 6r)$ in $L_0 \cup V \setminus B(x, 8r)$ to $\partial q_2 \cap \widehat{B}$, that there is a similar crossing with L_4 and q_4, and that there are similar closed crossings. We have

$$\begin{aligned}
&\mathbb{P}[\omega', \omega'' \in \mathscr{A}_4(x, B)]\mathbb{P}[\omega', \omega'' \in \mathscr{A}_\square(B, \mathcal{Q})] \\
&\quad \leq \mathbb{P}[\omega', \omega'' \in \mathscr{A}_4(x, B)]\mathbb{P}[\omega', \omega'' \in \mathscr{A}_4(B, \widehat{B})] \qquad (a) \\
&\quad \preceq \widetilde{\mathbb{P}}[\omega', \omega'' \in \mathscr{A}_4(x, \widehat{B})] \qquad (b) \\
&\quad \preceq \widetilde{\mathbb{P}}[\omega', \omega'' \in \mathscr{A}_4(x, \widehat{B}) \cap E] \qquad (c) \\
&\quad \preceq \widetilde{\mathbb{P}}[\omega', \omega'' \in \mathscr{A}_4(x, \widehat{B}) \cap E \cap E' \cap E''] \qquad (d) \\
&\quad \leq \widetilde{\mathbb{P}}[\omega', \omega'' \in \mathscr{A}_\square(B, \mathcal{Q})] \qquad (e) \\
&\quad \preceq \mathbb{P}[\omega', \omega'' \in \mathscr{A}_\square(B, \mathcal{Q})] \qquad (f)
\end{aligned}$$

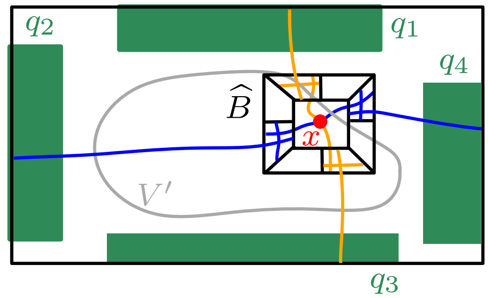

Figure 2: Illustration of the events E and E'' in the proof of Proposition 5.11.

Here (a) is immediate by inclusion of events; (b) is [GPS10, Proposition 5.6]; (c) follows from the inequality at the very end of [GPS10, proof Lemma 5.7] combined with [GPS10, equation (5.19)]; (d) follows by using the Russo-Seymour-Welsh theorem, the FKG inequality, and compactness; (e) is immediate by inclusion of events; and (f) follows from monotonicity similarly as in [GPS10, equation (5.12)]. □

The following is our second moment estimate. Compare with [GPS10, Proposition 5.3].

Proposition 5.12. *Let $\mathscr{S}$ be the spectral sample of $f = f_{R\mathcal{Q}}$, where $\mathcal{Q}$ is a collection of finitely many quads. Let $z \in \mathcal{Q}$ and $r > 0$. Set $B := B(z, r)$ and $B' := B(z, r/3)$. Suppose that $B(z, r/2) \subset \mathcal{Q}^o$ and that B and W are disjoint. Then for every $x, y \in B' \cap \mathcal{I}$ we have*

$$\mathbb{P}\big[x, y \in \mathscr{S},\ \mathscr{S} \cap W = \emptyset\big] \le c_2\, \mathbb{E}\big[\lambda_{B,W}^2\big]\, \alpha_4(|x-y|)\, \alpha_4(r)\,,$$

where $c_2 < \infty$ is an absolute constant.

Proof. The proof is identical to the proof in [GPS10]. Note that [GPS10, Lemmas 2.1 and 2.2], which are used in the proof, hold for the spectral sample of general real-valued functions f of the percolation configuration. For an arbitrary set $A \subset \mathcal{I}$ we let Λ_A be the event that A is pivotal for our quad crossing event. One key geometric argument in proof which still holds in our setting is that if we condition on ω restricted to the complement of $W \cup \{x, y\}$ and if flipping ω_x affects $f(\omega)$, then we must have a four arm event from x to distance $|x-y|/4$, and four arms in an annulus with outer boundary ∂B and inner boundary defined by a box centered at $(x+y)/2$ with radius $2|x-y|$. □

6 Mixing of Liouville dynamical percolation

The main result is the following.

Proposition 6.1. *Consider quantum dynamical percolation* $(\omega(t))_{t\geq 0}$*. Let* k *be a natural number, and let* $Q_1,\dots,Q_k$ *be quads. For some* $j\leq k$ *and any* $t\geq 0$ *let* $A(t)$ *be the event that* $Q_1,\dots,Q_j$ *are all crossed at time* t*, and that* $Q_{j+1},\dots,Q_k$ *are not crossed at time* t*. Then for any event* B *measurable with respect to* $\omega(0)$ *we have* $\mathbb{P}[A(t)B]\to\mathbb{P}[A(t)]\mathbb{P}[B]$*.*

Proposition 6.2. *Proposition 6.1 holds if* $j=k$*.*

Proof. Let f be the function with value in $\{-1,1\}$ which says whether the quads $\mathcal{Q}$ are all crossed. Let $\mathscr{S}$ be the spectral sample for this event, which is defined such that $\mathbb{P}[\mathscr{S}=A]=\widehat{f}(A)^2$. Let $a(\mathscr{S})=\sum_{x\in\mathscr{S}}a(x)$ with $a(x)$ the renormalized $1/\sqrt{6}$-LQG area of $B_\eta(x)$. Note that $a(\mathscr{S})$ is defined exactly as the quantum pivotal measure, but with the spectral sample instead of pivotal points. Then if we fix h, let $\mathbb{E}_{\mathscr{S}}$ denote the expectation when averaging over the spectral sample, let $\mathbb{E}_p$ denote the expectation when averaging over the randomness of the dynamical percolation, and assume $a(x)t<1/2$ for all x,

$$
\begin{aligned}
\mathbb{E}_p[f(\omega^\eta(0))f(\omega^\eta(t))]-\mathbb{E}_p[\omega^\eta(0)]^2 &= \sum_{S\neq\emptyset}\widehat{f}(S)^2\mathbb{E}_p[\chi_S(\omega^\eta(0))\chi_S(\omega^\eta(t))]\\
&=\sum_{S\neq\emptyset}\widehat{f}(S)^2\prod_{x\in S}\mathbb{E}[\omega^\eta(0)(x)\omega^\eta(t)(x)]\preceq\sum_{S\neq\emptyset}\widehat{f}(S)^2\prod_{x\in S}(1-a(x)t) \qquad (7)\\
&\preceq\sum_{S\neq\emptyset}\widehat{f}(S)^2\exp(-a(S)t)=\mathbb{E}_{\mathscr{S}}[\exp(-a(\mathscr{S})t)\mathbf{1}_{\mathscr{S}\neq\emptyset}],
\end{aligned}
$$

where the implicit constant in $\preceq$ is universal. This gives

$$
\mathbb{E}[f(\omega^\eta(0))f(\omega^\eta(t))\,|\,h]-\mathbb{E}[\omega^\eta(0)\,|\,h]^2\preceq\mathbb{E}[\exp(-a(\mathscr{S})t)\mathbf{1}_{\mathscr{S}\neq\emptyset}\,|\,h], \qquad (8)
$$

where we average over h and the dynamical percolation on the left side, and we average over h and the spectral sample on the right side. Sending $\eta\to 0$ (along some arbitrary subsequence) the left side converges to $\mathbb{E}[f(\omega(0))f(\omega(t))\,|\,h]-\mathbb{E}[\omega(0)\,|\,h]^2$, so to conclude it is sufficient to show that $\lim_{t\to\infty}\limsup_{\eta\to 0}\mathbb{E}[\exp(-a_\eta(\mathscr{S})t)\mathbf{1}_{\mathscr{S}\neq\emptyset}\,|\,h]=0$ a.s. We consider some subsequence along which the spectral sample converges to a limiting measure λ; such a subsequence exists since the expectation of the total measure remains bounded. We may consider a coupling such that the convergence is almost sure. On the event E that $\mathscr{S}\neq\emptyset$ for arbitrarily small η we see that the limiting measure λ is non-trivial by Theorem 5.1. We also have that $a=\nu_\eta$ converges to LQG ν on λ, so for η along this subsequence $\lim_{\eta\to 0}\mathbb{E}[\exp(-a_\eta(\mathscr{S})t)\mathbf{1}_{\mathscr{S}\neq\emptyset}\,|\,h]=\mathbb{E}[\exp(-\nu(\mathbb{C})t)\mathbf{1}_{\lambda(\mathbb{C})>0}\,|\,h]$. Since $\nu(\mathbb{C})>0$ a.s. on the event $\lambda(\mathbb{C})>0$ by Lemma 2.3, we have $\lim_{t\to\infty}\mathbb{E}[\exp(-\nu(\mathbb{C})t)\mathbf{1}_{\lambda(\mathbb{C})>0}\,|\,h]=0$ a.s., which concludes the proof. □

Proposition 6.3. *Let $f : \Omega \to [-1,1]$, and assume that $\mathbb{E}[f(\omega^\eta)f(\omega^\eta(t))] \to \mathbb{E}[f(\omega^\eta)]^2$ as $t \to \infty$. Then, for any $g : \Omega \to [-1,1]$,*

$$\mathbb{E}[f(\omega^\eta)g(\omega^\eta(t))] \to \mathbb{E}[f(\omega^\eta)]\mathbb{E}[g(\omega^\eta)].$$

Proof. Note that

$$\hat{f}(\emptyset) = \mathbb{E}[f].$$

This means

$$\mathbb{E}[f^2] - \mathbb{E}[f]^2 = \sum_{S \neq \emptyset} \hat{f}(S)^2$$

By [GS12, equation (IV.3)],

$$\mathrm{Cov}[f(\omega(0)), f(\omega(t))] = \mathbb{E}[f(\omega(0))f(\omega(t))] - \mathbb{E}[f]^2 = \sum_{S \neq \emptyset} \hat{f}(S)^2 e^{-|S|t}.$$

By the exact same calculation we get

$$\mathrm{Cov}[f(\omega(0)), g(\omega(t))] = \sum_{S \neq \emptyset} \hat{f}(S) g(S) e^{-|S|t}.$$

In particular, $\mathrm{Cov}[f(\omega(0)), g(\omega(t))]$ is also an inner product. By Cauchy-Schwarz for this inner product,

$$\mathrm{Cov}[f(\omega(0)), g(\omega(t))] \leq \mathrm{Cov}[f(\omega(0)), f(\omega(t))]^{\frac{1}{2}} \mathrm{Cov}[g(\omega(0)), g(\omega(t))]^{\frac{1}{2}}. \quad (9)$$

Under the assumptions of the lemma, we see that the right side converges to 0, which concludes the proof. □

Proof of Proposition 6.1. By Proposition 6.3 it is sufficient to prove that $A(t)$ decorrelates. This follows from inclusion-exclusion and a proof by induction on k, where for each fixed k we do induction on j. □

Throughout the paper we assumed that h is a zero boundary GFF. We end the paper with a short discussion about other fields. In particular, we will discuss fields h that can be written on the form $h = h_0 + g_0 + g_1$, where h_0 is a zero boundary or mean zero free boundary GFF, g is a continuous function which is bounded from above by a possibly random constant, and g_0 is the sum of finitely many logarithmic singularities.

- Convergence of quantum pivotal measure in probability holds for general fields, since adding the function g_0+g_1 to the field has the effect of reweighing the measure by $e^{\alpha(g_0+g_1)}$ or $e^{\gamma(g_0+g_1)}$, and since the free boundary GFF and the zero boundary GFF are absolutely continuous bounded away from ∂D. Convergence in L^1 holds on the event that $g_1 = 0$ and that g_0 is bounded by a deterministic constant.

- Theorem 1.1 holds for general fields. For any $M > 0$, if we consider the process at distance at least M^{-1} from the logarithmic singularities and ∂D, and on the event that $\|g_0\|_{L^\infty} < M$ and that there are at most M logarithmic singularities each with weight at most M, we may define the process exactly as before. Sending $M \to \infty$, we obtain the theorem in the general case.

- The proof of Proposition 6.1 carries through exactly as before for the case of general fields.

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