Analysis III

December 22, 2015.

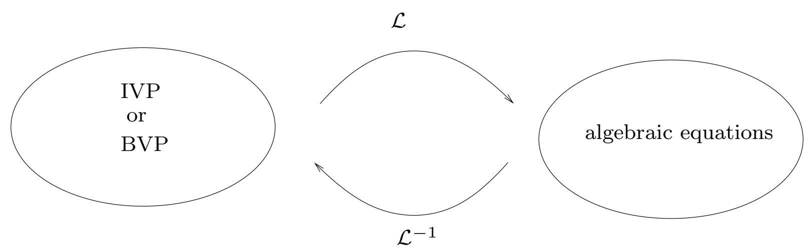
Contents

CHAPTER 1

The Laplace Transform

1.1. Introduction

The *Laplace transform* is one of the most powerful methods in *operational calculus*, a series of methods that transform differential problems into algebraic problems.



Typically these methods are integral transforms of which the Laplace transform and the *Fourier transform* are examples that we will see.

There are several advantages in using integral transforms in general, and the Laplace transform in particular:

(1) Often algebraic equations are easier to solve;
(2) The data of an IVP or a BVP are all encoded in the algebra equation and the solution is hence found at once without the need of finding the general solution and then the particular solution;
(3) It is particularly useful to solve differential equations with input that are not continuous, for example with short impulses. The way to deal with these issues is to introduce the *Heaviside function* and the *Dirac delta* (see § 1.6).

1.2. Definitions and examples

Definition 1.1. Let $f : [0, \infty] \to \mathbb{R}$ be a function. Its *Laplace transform* is defined as

$$\boxed{\mathcal{L}f(s) := \int_0^\infty e^{-st} f(t)\,dt\,.} \tag{1.1}$$

Notation. (1) There are several ways of denoting the Laplace transform. Among the most common are $\mathcal{L}(f)$ (and hence $\mathcal{L}(f)(s)$ when applied explicitly to a variable s) or $F(s)$.

(2) It is customary to denote by t the variable for the original function f (e.g. $f(t)$ and $g(t)$) and by s the variable of the Laplace transform (e.g. $\mathcal{L}(f)(s)$ or $F(s)$ and $\mathcal{L}(g)(s)$ or $G(s)$).

DEFINITION 1.2. The function f in (1.1) is the *inverse Laplace transform* of the function $F(s)$ and is often denoted by $f := \mathcal{L}^{-1}(F)$.

REMARK 1.3. The inverse Laplace transform is "essentially" uniquely defined, that is

$$F_1 = F_2 \Rightarrow f_1 = f_2\,.$$

"Essentially" here means that if two functions (both defined on $\mathbb{R}_{\geq 0}$ have the same Laplace transform, then they cannot differ on an interval of positive length, but they might differ at single points.

EXAMPLE 1.4. We want to compute $\mathcal{L}(f_n)$, where $f_n(t) = t^n$, for $n \geq 0$.

$\boxed{n=0}$ If $s > 0$, then

$$\mathcal{L}(f_0)(s) = \int_0^\infty e^{-st}\,dt = \lim_{T\to\infty}\left(\int_0^T e^{-st}\,dt\right) = \lim_{T\to\infty}\left(-\frac{1}{s}(e^{-st}-1)\right) = \frac{1}{s}.$$

$\boxed{n \geq 1}$

$$\begin{aligned}
\mathcal{L}(f_n)(s) &= \int_0^\infty e^{-st}t^n\,dt = \lim_{T\to\infty}\left(\int_0^T e^{-st}t^n\,dt\right)\\
&= \lim_{T\to\infty}\left(t^n\left(-\frac{1}{n}\right)e^{-st}\Big|_0^T + \frac{1}{s}\int_0^T nt^{n-1}e^{-st}\,dt\right)\\
&= \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}\,dt = \frac{n}{s}\mathcal{L}(f_{n-1}(s)\\
&= \frac{n(n-1)}{s^2}\mathcal{L}(f_{n-2})(s) = \cdots = \frac{n(n-1)\cdots 2\cdot 1}{s^n}\mathcal{L}(f_0)(s) = \frac{n!}{s^{n+1}}.
\end{aligned}$$

EXAMPLE 1.5. We compute the Laplace transform of $f(t) := e^{at}$, where $a \in \mathbb{R}$.

$$\begin{aligned}
\mathcal{L}(e^{at})(s) &= \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt\\
&= \begin{cases} \infty & \text{if } s = a\\ \frac{-1}{s-a}\lim_{T\to\infty}(e^{-(s-a)T}-1) & \text{if } s \neq a \end{cases}\\
&= \begin{cases} \infty & \text{if } s \leq a\\ \frac{1}{s-a} & \text{if } s > a\,. \end{cases}
\end{aligned}$$

REMARK 1.6. (1) The Laplace transform $\mathcal{L}(f)$ is defined exactly for the values of s for which the integral exists. For example if

(a) f is piecewise continuous[1] and
(b) there exist $M, k \in \mathbb{R}$ with $M, k > 0$ such that

$$|f(t)| \leq Me^{kt} \tag{1.2}$$

for all $t \geq 0$, then $\mathcal{L}(f)(s)$ exists on the interval (k, ∞). In fact, if (1.2) holds, then if $s > k$

$$|\mathcal{L}(f)(s)| \leq \int_0^\infty e^{-st}|f(t)|\,dt \leq M\int_0^\infty e^{-(s-k)t}\,dt < \frac{1}{s-k} < \infty\,.$$

Note that the piecewise continuity gives the integrability on finite intervals and the condition (1.2) in (b) insures the finiteness of $\mathcal{L}(f)$. The inequality (1.2) is called *growth restriction*.

(2) Let us assume that f is a "good" function, that is a function for which the Laplace transform exists. Then:
(a) $\lim_{s\to\infty}\mathcal{L}(f)(s) = 0$.
(b) $s\mathcal{L}(f)(s)$ is bounded as $s \to \infty$.
(c) $\mathcal{L}(f)$ is continuous for all $s \in \mathbb{R}$ such that $\alpha \leq s \leq \beta$, where $k < \alpha$.
(d)

$$\int_\alpha^\beta \left(\int_0^\infty e^{-st} f(t)\,dt\right) ds = \int_0^\infty \left(\int_\alpha^\beta e^{-st} f(t)\,ds\right) ds\,.$$

(e)

$$\frac{d}{ds}\left(\int_0^\infty e^{-st} f(t)\,dt\right) = -\int_0^\infty te^{-st} f(t)\,dt = -\mathcal{L}(tf(t))\,.$$

1.3. First properties and applications

1.3.1. Linearity.

Property 1. *Let f and g be functions for which the Laplace transform exist. Then for all $\alpha, \beta \in \mathbb{R}$*

$$\boxed{\mathcal{L}(\alpha f + \beta g) = \alpha\mathcal{L}(f) + \beta\mathcal{L}(g)}\,.$$

Moreover

$$\boxed{\mathcal{L}^{-1}(\alpha F + \beta G) = \alpha\mathcal{L}^{-1}(F) + \beta\mathcal{L}^{-1}(G)\,.}$$

Example 1.7. Compute $\mathcal{L}(\cosh at)$ for $s > |a|$.

$$\begin{aligned}\mathcal{L}(\cosh at) =& \mathcal{L}\left(\frac{e^{at}+e^{-at}}{2}\right) = \frac{1}{2}\mathcal{L}(e^{at}) + \frac{1}{2}\mathcal{L}(e^{-at})\\ =& \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a} = \frac{s}{s^2-a^2}\,,\end{aligned}$$

[1] f is piecewise continuous on $I \subset \mathbb{R}$ if there exists a subdivision of I into finitely many intervals on which f is continuous and has finite limits at both endpoints of each interval.

where we have applied Example 1.5 to $f(x) = e^{at}$ and to $f(x) = e^{-at}$. $\square$

1.3.2. Shifting Theorem (s-shifting).

PROPERTY 2. *Let us assume that the function f has a Laplace transform $\mathcal{L}(f)(s)$ for all $s > k$ for some k. Then*

$$\boxed{\mathcal{L}(e^{at}f(t))(s) = F(s-a)\,.} \tag{1.3}$$

VERIFICATION.

$$\mathcal{L}(e^{at}f(t))(s) = \int_0^\infty e^{-(s-a)t} f(t)\,dt = F(s-a)\,. \qquad \square$$

EXAMPLE 1.8. Assume that we have shown (see Example 1.9) that

$$\boxed{\mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2}} \quad \text{and} \quad \boxed{\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2}}\,. \tag{1.4}$$

We use these formulas to find the inverse Laplace transform of $F(s) := \frac{2s-3}{s^2+2s+26}$. It follows in fact from the Shifting Theorem that

$$\mathcal{L}(e^{at}\cos\omega t) = \frac{s-a}{(s-a)^2+\omega^2} \text{ and } \mathcal{L}(e^{at}\sin\omega t) = \frac{\omega}{(s-a)^2+\omega^2}\,.$$

Since we can write

$$F(s) = \frac{2s-3}{s^2+2s+26} = \frac{2(s+1)-5}{(s+1)^2+5^2} = \frac{2(s+1)}{(s+1)^2+5^2} - \frac{5}{(s+1)^2+5^2}\,,$$

then

$$\begin{aligned} f(t) = \mathcal{L}^{-1}(F) =& 2\mathcal{L}^{-1}\left(\frac{s+1}{s^2+2s+26}\right) - 5\mathcal{L}^{-1}\left(\frac{1}{s^2+2s+26}\right) \\ =& 2e^{-t}\cos 5t - e^{-t}\sin 5t = e^{-t}(2\cos 5t - \sin 5t)\,. \end{aligned}$$

Note that f is a function that describes damped vibrations.

1.3.3. Differentiation.

PROPERTY 3. *Let $f, f', \ldots, f^{(n-1)}$ be continuous functions for all $t \geq 0$ and let us assume that they satisfy the growth condition* (1.2). *Assume that $f^{(n)}$ is piecewise continuous on every finite subinterval in* $[0,\infty)$. *Then*

$$\boxed{\mathcal{L}(f^{(n)})(s) = s^n\mathcal{L}(f) - \sum_{j=0}^{n-1} s^{n-1-j} f^{(j)}(0), \text{ for every } n \geq 1} \tag{1.5}$$

For $n = 1$ and $n = 2$ (1.5) *reads as follows:*

$$\boxed{\begin{aligned} &\mathcal{L}(f')(s) = s\mathcal{L}(f) - f(0) \\ &\mathcal{L}(f'')(s) = s^2\mathcal{L}(f) - sf(0) - f'(0) \end{aligned}} \tag{1.6}$$

VERIFICATION. We assume that f and f' are piecewise continuous and f satisfies the growth restriction (1.2). To verify the first of the two equations in (1.6), observe that

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)\, dt = \lim_{T\to\infty} \left(e^{-st} f(t)|_0^T + s \int_0^T e^{-st} f(t)\, dt \right) \tag{1.7}$$

If (1.2) is satisfied, that is if for some $k, M > 0$, f satisfies the inequality $|f(t)| \leq Me^{kt}$, then

$$\left| e^{-sT} f(T) \right| \leq Me^{-(s-k)T} \to 0 \quad \text{as } T \to \infty \quad \text{for } s > k\,.$$

Then (1.7) becomes

$$\mathcal{L}(f') = \lim_{T\to\infty} e^{-sT} f(T) - f(0) + s \int_0^\infty e^{-st} f(t)\, dt = -f(0) + s\mathcal{L}(f)\,.$$

To verify the second of the equations in (1.6), we use the first equation applied first to f' then again to f. Then

$$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0) = s(s\mathcal{L}(f) - f(0)) - f'(0) = s^2\mathcal{L}(f) - sf(0) - f'(0)\,.$$

The formula (1.5) for any n can be proven by induction. □

EXAMPLE 1.9. We compute now $\mathcal{L}(\cos \omega t)$ and $\mathcal{L}(\sin \omega t)$. To this purpose, set $f(t) = \cos \omega t$. Then

$$f'(t) = -\omega \sin \omega t \quad \text{and hence} \quad f''(t) = -\omega^2 \cos \omega t\,.$$

From

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

and since $f(0) = 1$ and $f'(0) = 0$, we obtain

$$-\omega^2\mathcal{L}(\cos \omega t) = s^2\mathcal{L}(\cos \omega t) - s\,,$$

that is

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}\,.$$

The computation of $\mathcal{L}(\sin \omega t)$ can be performed analogously. □

EXAMPLE 1.10 (First application to solving an ODE). We want to find $y = y(t)$ that satisfies the differential equation

$$y'' - y = t\,,$$

with initial values

$$y(0) = 1 \quad \text{and} \quad y'(0) = 1\,.$$

Applying the Laplace transform to the equation $y'' - y = t$, and using the linearity property and Example 1.4, we obtain

$$\begin{aligned} \mathcal{L}(y'' - y) &= \mathcal{L}(t) \iff \\ s^2\mathcal{L}(y) - sy(0) - y'(0) - \mathcal{L}(y) &= \frac{1}{s^2} \iff \\ (s^2-1)\mathcal{L}(y) &= \frac{1}{s^2} + s + 1 \iff \\ \mathcal{L}(y) &= \frac{1}{s^2-1}\left(\frac{1}{s^2} + s + 1\right) . \end{aligned}$$

Using (1.4), Example 1.4 and Example 1.5, it follows that

$$\begin{aligned} \mathcal{L}(y) =& \frac{1}{s^2(s^2-1)} + \frac{1}{s-1} = \frac{1}{s^2-1} - \frac{1}{s^2} + \frac{1}{s-1} \\ =& \mathcal{L}(\sinh t) - \mathcal{L}(t) + \mathcal{L}(e^t) \\ =& \mathcal{L}(\sinh t - t + e^t) \, , \end{aligned}$$

from which we obtain $y(t) = \sinh t - t + e^t$.

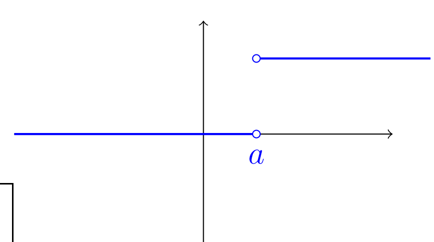
1.4. The Heaviside function and t-shifting

We want to consider now the vibrations of a mass m on an elastic spring and we denote by $y(t)$ the displacement. We want to consider the case in which there is a damping force and an external force $r(t)$, so that the differential equation satisfied by the displacement is

$$my'' + cy' + ky = r(t) \, , \tag{1.8}$$

where c and k are respectively the damping and the spring constants, and where $r(t)$ is the external force.

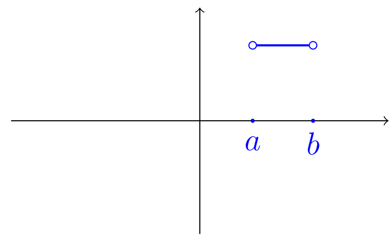
The interesting case we want to discuss is when the external force is applied only on one interval of time. To do this, we need to introduce the *Heaviside function* (or *unit step function*), so defined:



$$\boxed{\text{If } a \geq 0, \quad u(t-a) := \begin{cases} 1 & \text{if } t > a \\ 0 & \text{if } t < a \, . \end{cases}}$$

In the following examples we see how we can use the Heaviside function to express some discontinuous functions.

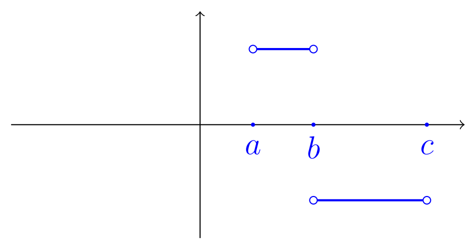
Example 1.11. The function



$$f(t) := \begin{cases} 1 & a < t < b \\ 0 & t < a \text{ or } t > b \end{cases}$$

can be written as $f(t) = u(t-a) - u(t-b)$.

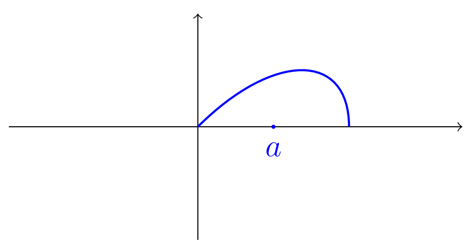
Example 1.12. The function



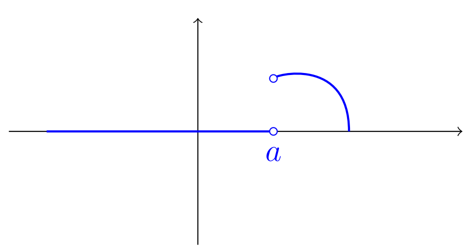
$$f(t) := \begin{cases} 1 & a < t < b \\ -1 & b < t < c \\ 0 & t < a \text{ or } t > c \end{cases}$$

can be written as $f(t) = u(t-a) - 2u(t-b) + u(t-c)$.

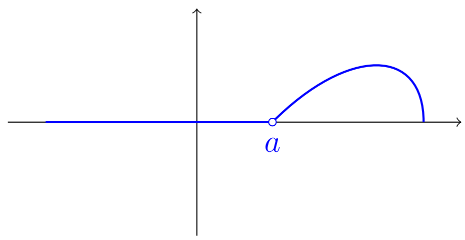
Example 1.13. If $f(t)$ has graph



then $u(t-a)f(t)$ has graph



but $u(t-a)f(t-a)$ has graph



1.4.1. Second shifting theorem (t-shifting).

Property 4. *Let f be a function with Laplace transform $\mathcal{L}(f)$ and let u be the Heaviside function. Then*

$$\boxed{\mathcal{L}(u(t-a)f(t-a)) = e^{-as}\mathcal{L}(f)\,,} \tag{1.9}$$

and

$$\boxed{u(t-a)f(t-a) = \mathcal{L}^{-1}(e^{-as}\mathcal{L}(f))\,,}$$

Verification. We have

$$\mathcal{L}(u(t-a)f(t-a))(s) = \int_0^\infty e^{-st}u(t-a)f(t-a)\,dt\,.$$

Setting $\overline{t} := t-a$, we obtain

$$\begin{aligned}\mathcal{L}(u(t-a)f(t-a))(s) &= \int_0^\infty e^{-st}u(t-a)f(t-a)\,dt\\ &= \int_{-a}^\infty e^{s(\overline{t}+a)}u(\overline{t})f(\overline{t})d\,\overline{t}\\ &= \int_0^\infty e^{-s\overline{t}}e^{-sa}f(\overline{t})d\,\overline{t}\\ &= e^{-sa}\mathcal{L}(f)\,. \quad \square\end{aligned}$$

Example 1.14. We now look at the mass-spring system in (1.8) with damping constant $c = 3$ and external force the square wave $u(t-1) - u(t-2)$. That is we want to solve the initial value problem

$$y'' + 3y' + 2y = u(t-1) - u(t-2)\,, \quad y(0) = 0\,, \quad y'(0) = 0\,. \tag{1.10}$$

Solution. We apply the Laplace transform to the left hand side of the ODE in (1.10). Using the convention $Y = \mathcal{L}(y)$ and using (1.6) we obtain

$$\begin{aligned}&\mathcal{L}(y'' + 3y' + 2y)\\ &= s^2Y - sy(0) - y'(0) + 3sY - 3y(0) + 2Y\\ &= s^2Y + 3sY + 2Y\,.\end{aligned}$$

Applying now the Laplace transform to the right hand side of (1.10) and using the t-shifting, we obtain

$$\mathcal{L}(u(t-1)-u(t-2)) = \frac{1}{s}(e^{-s}-e^{-2s})\,.$$

Setting the two equations above equal to each other, and solving for Y, we obtain, with the help of (1.9) and of Examples 1.4 and 1.5,

$$\begin{aligned}
Y(s) &= \frac{1}{(s^2+3s+2)s}(e^{-s}-e^{-2s}) \\
&= \frac{1}{s(s+1)(s+2)}(e^{-s}-e^{-2s}) \\
&= \left(\frac{1}{2s}-\frac{1}{s+1}+\frac{1}{2(s+2)}\right)(e^{-s}-e^{-2s}) \\
&= \mathcal{L}\left(\frac{1}{2}-e^{-t}+\frac{1}{2}e^{-2t}\right)(e^{-s}-e^{-2s}) \\
&= \mathcal{L}\left(\frac{1}{2}-e^{-t}+\frac{1}{2}e^{-2t}\right)e^{-s}-\mathcal{L}\left(\frac{1}{2}-e^{-t}+\frac{1}{2}e^{-2t}\right)e^{-2s} \\
&= \mathcal{L}\left(\left(\frac{1}{2}-e^{-(t-1)}+\frac{1}{2}e^{-2(t-1)}\right)u(t-1)\right)-\mathcal{L}\left(\left(\frac{1}{2}-e^{-(t-2)}+\frac{1}{2}e^{-2(t-2)}\right)u(t-2)\right) \\
&= \mathcal{L}\left(\left(\frac{1}{2}-e^{-(t-1)}+\frac{1}{2}e^{-2(t-1)}\right)u(t-1)-\left(\frac{1}{2}-e^{-(t-2)}+\frac{1}{2}e^{-2(t-2)}\right)u(t-2)\right)\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
y(t) &= \left(\frac{1}{2}-e^{-(t-1)}+\frac{1}{2}e^{-2(t-1)}\right)u(t-1)-\left(\frac{1}{2}-e^{-(t-2)}+\frac{1}{2}e^{-2(t-2)}\right)u(t-2) \\
&= \begin{cases} 0 & t<1 \\ \frac{1}{2}-e^{-(t-1)}+\frac{1}{2}e^{-2(t-1)} & 1<t<2 \\ e^{-(t-2)}-e^{-(t-1)}+\frac{1}{2}e^{-2(t-1)}-\frac{1}{2}e^{-2(t-2)} & t>2\,. \end{cases}
\end{aligned}$$

1.5. Integration

Property 5. *Let f be a piecewise continuous function for $t \geq 0$ that satisfies the growth condition* (1.2)*. Then for $s>k$, $s>0$, and $t>0$*

$$\boxed{\mathcal{L}\left(\int_0^t f(x)\,dx\right) = \frac{1}{s}F(s)\,.} \tag{1.11}$$

Verification. Let $g(t) := \int_0^t f(x)\,dx$ and let us verify first that if f satisfies the growth condition, so does g. In fact,

$$|g(t)| \leq \int_0^t |f(x)|\,dx \leq M\int_0^t e^{kx}\,dx = \frac{M}{k}(e^{kt}-1) \leq \frac{M}{k}e^{kt}\,.$$

Moreover $g'(t) = f(t)$ and $g(0) = 0$, hence from (1.6) we have

$$\mathcal{L}(g') = s\mathcal{L}(g) - g(0) = s\mathcal{L}(g)\,,$$

which implies that

$$\mathcal{L}\left(\int_0^t f(x)\,dx\right) = \mathcal{L}(g(t)) = \frac{1}{s}\mathcal{L}(g') = \frac{1}{s}\mathcal{L}(f)\,. \quad \square$$

EXAMPLE 1.15. Compute $\mathcal{L}^{-1}\left(\frac{1}{s(s^2+\omega^2)}\right)$ and $\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+\omega^2)}\right)$.

SOLUTION. We know that $\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2}$. Since ω is a constant, by linearity of the Laplace transform we obtain

$$\mathcal{L}\left(\frac{\sin\omega t}{\omega}\right) = \frac{1}{s^2+\omega^2}\,,$$

and multiplying by $\frac{1}{s}$

$$\frac{1}{s}\mathcal{L}\left(\frac{\sin\omega t}{\omega}\right) = \frac{1}{s}\left(\frac{1}{s^2+\omega^2}\right)\,.$$

Taking the inverse Laplace transform on both sides, we obtain

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{s}\left(\frac{1}{s^2+\omega^2}\right)\right) &= \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}\left(\frac{\sin\omega t}{\omega}\right)\right)\\
&= \int_0^t \frac{\sin\omega x}{\omega}\,dx\\
&= \frac{1}{\omega^2}(-\cos\omega x)\big|_0^t\\
&= \frac{1}{\omega^2}(1-\cos\omega t)\,.
\end{aligned}$$

To compute $\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+\omega^2)}\right)$ we iterate the above calculation. We know that

$$\mathcal{L}\left(\frac{1}{\omega^2}(1-\cos\omega t)\right) = \frac{1}{\omega^2}\left(\frac{1}{s} - \frac{s}{s^2+\omega^2}\right) = \frac{1}{s(s^2+\omega^2)}\,,$$

from which it follows that

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+\omega^2)}\right) &= \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}\left(\frac{1}{\omega^2}(1-\cos\omega t)\right)\right)\\
&= \frac{1}{\omega^2}\int_0^t (1-\cos\omega x)\,dx\\
&= \frac{1}{\omega^2}\left(t - \frac{1}{\omega}\sin\omega t\right)\,.
\end{aligned}$$

EXAMPLE 1.16 ([**?**, Example 6, page 232]). Solve

$$y'' + y = 2t\,, \quad y\left(\frac{\pi}{4}\right) = \pi/2\,, \quad y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}\,. \tag{1.12}$$

SOLUTION. We set $\overline{t} := t - \pi/4$, from which $t = \overline{t} + \pi/4$. With respect to this new variable we have

$$y(t) = \overline{y}(\overline{t})\,, \quad y'(t) = \overline{y}'(\overline{t})\,, \quad y''(t) = \overline{y}''(\overline{t})\,,$$

and hence (1.12) becomes

$$\overline{y}'' + \overline{y} = 2\left(\overline{t} + \frac{\pi}{4}\right)\,, \quad \overline{y}(0) = \frac{\pi}{2}\,, \quad \overline{y}'(0) = 2 - \sqrt{2}\,.$$

Applying the Laplace transform to the ODE we obtain

$$s^2\mathcal{L}(\overline{y}) - s\overline{y}(0) - \overline{y}'(0) + \mathcal{L}(\overline{y}) = 2\mathcal{L}(\overline{t}) + \frac{\pi}{2}\mathcal{L}(1)\,,$$

from which it follows that

$$(s^2+1)\overline{Y} - \frac{\pi}{2}s - 2 + \sqrt{2} = \frac{2}{s^2} + \frac{\pi}{2s}\,.$$

Solving for $\overline{Y}$, and using the result of both computations in Example 1.15, we obtain

$$\begin{aligned}
\overline{Y} &= \frac{2}{s^2(s^2+1)} + \frac{\pi}{2}\frac{1}{s(s^2+1)} + \frac{2-\sqrt{2}}{s^2+1} + \frac{\pi}{2}\frac{s}{s^2+1} \\
&= 2\mathcal{L}(\overline{t} - \sin\overline{t}) + \frac{\pi}{2}\mathcal{L}(1 - \cos\overline{t}) + (2-\sqrt{2})\mathcal{L}(\sin\overline{t}) + \frac{\pi}{2}\mathcal{L}(\cos\overline{t}) \\
&= \mathcal{L}(2\overline{t} + \frac{\pi}{2} - \sqrt{2}\sin\overline{t})\,.
\end{aligned}$$

Hence

$$\overline{y}(\overline{t}) = 2\overline{t} + \frac{\pi}{2} - \sqrt{2}\sin\overline{t}\,,$$

and substituting back to obtain the original variable t we obtain

$$\begin{aligned}
y(t) &= 2\left(t - \frac{\pi}{4}\right) + \frac{\pi}{2} - \sqrt{2}\sin\left(t - \frac{\pi}{4}\right) \\
&= 2t - \sqrt{2}\frac{1}{\sqrt{2}}(\sin t - \cos t) \\
&= 2t - \sin t + \cos t\,.
\end{aligned}$$

1.6. Dirac's delta function

We are going to look for solutions of the differential equation $my''+cy'+ky = r(t)$, where $r(t)$ is a force over a very short interval of time, almost instantaneously. Phenomena of this impulsive nature are common and can be dealt with using Dirac's delta.

We saw already how to deal with forces applied on an interval $[a, a+k]$ and we assume now that the magnitude of the ofrce is $1/k$, so that the impulse of the force is one. So if

$$f_k(t-a) := \begin{cases} 1/k & a \leq t \leq a+k \\ 0 & \text{otherwise}, \end{cases}$$

then

$$I_k := \int_0^\infty f_k(t-a)\,dt = \int_a^{a+k} \frac{1}{k}\,dt = 1\,.$$

DEFINITION 1.17. The *Dirac delta* is the limit

$$\boxed{\delta(t-a) := \lim_{k\to a} f_k(t) = \begin{cases} \infty & t = a \\ 0 & t \neq a\,, \end{cases}}$$

and

$$\boxed{\int_0^\infty \delta(t-a)\,dt = 1\,.} \tag{1.13}$$

REMARK 1.18. The Dirac delta function is not a function, but a *generalised function* or a *distribution.*

The following properties are important and should be shown using the definition of distribution:

PROPERTY 6. (1) *(Sifting property)*

$$\boxed{\int_0^\infty g(t)\delta(t-a)\,dt = g(a)\,,} \tag{1.14}$$

(2) $\boxed{\mathcal{L}(\delta(t-a)) = e^{-as}\,.}$

VERIFICATION. (1) This is consistent with (1.13) in the case in which $g(t) \equiv 1$.

(2) We use here properties of the Dirac delta and of the Laplace transform. Because the Laplace transform has good convergence properties, we can write

$$\begin{aligned}
\mathcal{L}(\delta(t-a)) &= \mathcal{L}(\lim_{k\to 0} f_k(t-a)) \\
&= \lim_{k\to 0} \mathcal{L}(f_k(t-a)) \\
&= \lim_{k\to 0} \frac{1}{k}\mathcal{L}(u(t-a) - u(t-(a+k))) \\
&= \lim_{k\to 0} \frac{1}{ks}(e^{-as} - e^{-(a+k)s}) \\
&= \lim_{k\to 0} e^{-as}\frac{1-e^{-ks}}{ks} \\
&= e^{-as} \lim_{k\to 0} \frac{se^{-ks}}{s} \\
&= e^{-as}\,.
\end{aligned}$$

□

Example 1.19. (Hammerblow response of a mass-spring system) We want to solve the IVP

$$y'' + 3y' + 2y = \delta(t-1)\,, \quad y(0) = 0\,, \quad y'(0) = 0\,.$$

Solution. We apply the Laplace transform to both sides of the ODE

$$\mathcal{L}(y'' + 3y' + 2y) = \mathcal{L}(\delta(t-1))$$

and, using the calculation in Example 1.14 we obtain that

$$s^2Y + 3sY + 2Y = e^{-s}\,,$$

or, equivalently,

$$Y = \frac{1}{(s+1)(s+2)}e^{-s} = \left(\frac{1}{s+1} - \frac{1}{s+2}\right)e^{-s}\,. \tag{1.15}$$

We apply now the second Shifting theorem (Property 4) with $\mathcal{L}(e^{-at}) = \frac{1}{s+a}$ and $a = 1, 2$, that is

$$\tfrac{1}{s+1}e^{-s} = \mathcal{L}(e^{-t})e^{-s} = \mathcal{L}(u(t-1)e^{-(t-1)}) \tag{1.16}$$

$$\tfrac{1}{s+2}e^{-s} = \mathcal{L}(e^{-2t})e^{-s} = \mathcal{L}(u(t-1)e^{-2(t-1)})\,. \tag{1.17}$$

Plugging in the result of the two equations in (1.16) into (1.15) we obtain

$$Y = \mathcal{L}(u(t-1)(e^{-(t-1)} - e^{-2(t-1)}))\,,$$

that is

$$y = u(t-1)(e^{-(t-1)} - e^{-2(t-1)})$$

□

1.7. Convolution and integral equations

While the Laplace transform is linear and satisfies the property $\mathcal{L}(f+g) = \mathcal{L}(f)+\mathcal{L}(g)$, the same cannot be said about the product of two functions. In other words $\mathcal{L}(f\cdot g) \neq \mathcal{L}(f)\mathcal{L}(g)$. However one can define a different "product", called *convolution* and denoted by $*$, that has the desired property that $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$.

Definition 1.20. The *convolution* $f * g$ of two functions f and g is defined as

$$f * g(t) := \int_0^t f(t')g(t-t')\,dt'\,.$$

Properties. *Let f, g and h be fucntions. Then:*

(1) $f * g = g * f$;
(2) $f * (g + h) = f * g + f * h$;
(3) $f * (g * h) = (f * g) * h$;
(4) $f * 0 = 0 * f = 0$;
(5) $f * 1 \neq f$;
(6) $f * f$ *is not always non-negative.*

The verification of these properties is straightforward and is similar to the verification of the Property 7 that we will do later. An example that illustrates (5) is $f(t) = t$, since $f * 1 = \frac{1}{2}t^2$. On the other hand, taking $f(t) = \sin t$, shows that $f * f = \frac{1}{2}\cos t + \frac{1}{2}\sin t$ can take also negative values.

Property 7. *If f, g are two functions, then*

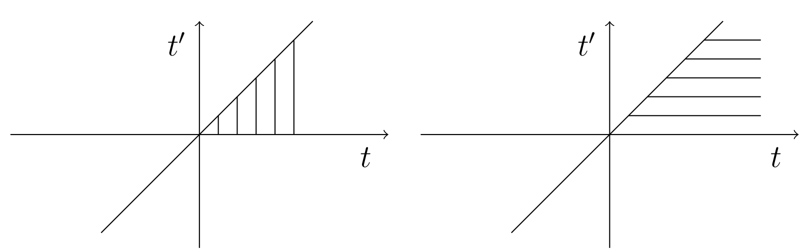
$$\boxed{\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)}\,.$$

Verification. By definition we have

$$\begin{aligned}\mathcal{L}(f*g)(s) &= \int_0^\infty e^{-st}(f*g)(t)\,dt \\ &= \int_0^\infty e^{-st}\left(\int_0^t f(t')g(t-t')\,dt'\right)dt\end{aligned} \tag{1.18}$$

Since

$$\{(t,t') : 0 < t' < t,\, 0 < t < \infty\} = \{(t,t') : t' < t < \infty,\, 0 < t' < \infty\}\,,$$



we can exchange the order on integration in the right hand side of (1.18) to obtain

$$\begin{aligned}
\mathcal{L}(f*g)(s) &= \int_0^\infty e^{-st}\left(\int_0^t f(t')g(t-t')\,dt'\right)\,dt \\
&= \int_0^\infty \left(\int_{t'}^\infty e^{-st}g(t-t')\,dt\right) f(t')\,dt' \\
&\overset{\alpha:=t-t'}{=} \int_0^\infty \left(\int_0^\infty e^{-s(\alpha+t')}g(\alpha)\,d\alpha\right) f(t')\,dt' \\
&= \int_0^\infty e^{-st'}\left(\int_0^\infty e^{-s\alpha}g(\alpha)\,d\alpha\right) f(t')\,dt' \\
&= \left(\int_0^\infty e^{-st'}f(t')\,dt'\right)\left(\int_0^\infty e^{-s\alpha}g(\alpha)\,\alpha\right) \\
&= \mathcal{L}(f)\mathcal{L}(g)\,.
\end{aligned}$$

□

Example 1.21. This is Example 1.14 revisited. In other words, we will solve again the IVP in (1.10)

$$y'' + 3y' + 2y = u(t-1) - u(t-2)\,, \quad y(0) = 0\,, \quad y'(0) = 0\,,$$

but this time we will use the convolution. By applying the Laplace transform on both sides of the differential equation we obtain

$$(s^2 + 3s + 2)Y = \mathcal{L}(u(t-1) - u(t-2))\,,$$

and, solving for Y,

$$\begin{aligned}
Y &= \frac{1}{s^2+3s+2}\mathcal{L}(u(t-1) - u(t-2)) \\
&= \left(\frac{1}{s+1} - \frac{1}{s+2}\right)\mathcal{L}(u(t-1) - u(t-2)) \\
&= \mathcal{L}(e^{-t} - e^{-2t})\mathcal{L}(u(t-1) - u(t-2)) \\
&= \mathcal{L}((e^{-t} - e^{-2t}) * (u(t-1) - u(t-2)))\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
y(t) &= (e^{-t} - e^{-2t}) * (u(t-1) - u(t-2)) \\
&= (u(t-1) - u(t-2)) * (e^{-t} - e^{-2t}) \\
&= \int_0^t (u(t'-1) - u(t'-2))(e^{-(t-t')} - e^{-2(t-t')})\,dt'
\end{aligned} \tag{1.19}$$

Since

$$\int e^{-a(t-t')}\,dt' = \frac{1}{a}e^{-a(t-t')} + C\,,$$

then

$$\int (e^{-(t-t')} - e^{-2(t-t')})\, dt' = e^{-(t-t')} - \frac{1}{2}e^{-2(t-t')} + C\,.$$

If follows from (1.19) that, if $1 < t < 2$,

$$\begin{aligned} y(t) &= \int_1^t (u(t'-1) - u(t'-2))(e^{-(t-t')} - e^{-2(t-t')})\, dt' \\ &= \int_1^t (e^{-(t-t')} - e^{-2(t-t')})\, dt' \\ &= (e^{-(t-t')} - \frac{1}{2}e^{-2(t-t')})\Big|_1^t \\ &= \frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\,. \end{aligned}$$

On the other hand, if $t > 2$, (1.19) becomes

$$\begin{aligned} y(t) &= \int_1^2 (u(t'-1) - u(t'-2))(e^{-(t-t')} - e^{-2(t-t')})\, dt' \\ &= \int_1^2 (e^{-(t-t')} - e^{-2(t-t')})\, dt' \\ &= (e^{-(t-t')} - \frac{1}{2}e^{-2(t-t')})\Big|_1^2 \\ &= e^{-(t-2)} - \frac{1}{2}e^{-2(t-2)} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\,, \end{aligned}$$

which is consistent with the result obtained in Exercise 1.14.

EXAMPLE 1.22. Solve the integral equation

$$y(t) - \int_0^t y(t')\sin(t-t')\, dt' = t\,.$$

SOLUTION. Since $\int_0^t y(t')\sin(t-t') = y(t) * \sin t$, by applying the Laplace transform to the equation

$$y - y * \sin t = t$$

we obtain

$$Y - Y\frac{1}{s^2+1} = \frac{1}{s^2}\,.$$

Solving for Y we get

$$Y = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4} = \mathcal{L}\left(t + \frac{1}{6}t^3\right)\,,$$

from which we gather that $y(t) = t + \frac{1}{6}t^3$. □

1.8. Last properties: differentiation and integration of transforms

We saw already how the Laplace transform of the derivative or of the integral of a function is related to the Laplace transform of the function (Properties 3 and 4). Here we see the effect of taking the derivative or the integral of the Laplace transform.

PROPERTY 8. *Let f be a piecewise continuous function that satisfies the growth condition* (1.2). *Then*

(1) $\mathcal{L}'(f) = -\mathcal{L}(tf(t))$.

(2) *If in addition* $\lim_{t\to 0^+} \frac{f(t)}{t}$ *exists, then* $\int_s^\infty \mathcal{L}(f)(s')\,ds' = \mathcal{L}\left(\frac{f(t)}{t}\right)$.

VERIFICATION. (1) The continuity properties of the Laplace transform allow us to write

$$\frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty -te^{-st} f(t)\,dt\,.$$

The right hand side of the above equality equals $-\mathcal{L}(tf(t))$, from which the first assertion follows.

(2) The same continuity properties of the Laplace transform used in (1) allow us to write

$$\int_s^\infty \left(\int_0^\infty e^{-ts'} f(t)\,dt\right) ds' = \int_0^\infty \left(\int_s^\infty e^{-ts'}\,ds'\right) f(t)\,dt\,. \tag{1.20}$$

The right hand side of (1.20) equals

$$\int_0^\infty \left(-\frac{1}{t}e^{-ts'}\bigg|_s^\infty\right) f(t)\,dt = \int_0^\infty e^{-st}\frac{1}{t} f(t)\,dt = \mathcal{L}\left(\frac{1}{t}f(t)\right)\,,$$

which shows the second assertion. □

REMARK 1.23. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is piecewise continuous, satisfies the growth condition (1.2) with constants M and k and has the property that $f(t) = 0$ if $t < 0$. Then the Laplace transform of f exists for all complex numbers $z \in \mathbb{C}$ with real part $\Re z > k$. Considering the Laplace transform as a function of a complex variable has for example the advantage that there exists a formula for the inverse Laplace transform. However the knowledge needed to use this formula is beyond the scope of this course and is therefore not treated here.

CHAPTER 2

Fourier Analysis

Fourier series are an instrument to decompose periodic phenomena, such as the behaviour of a rotating part in a machine, into simpler functions such as sines and cosines. We will see that Fourier integrals give a way of extending a similar decomposition to the case of non-periodic phenomena.

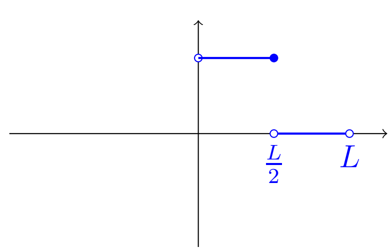
2.1. Fundamentals

DEFINITION 2.1. A function $f(x)$ is called *periodic* if it is defined for "most" $x \in \mathbb{R}$ and if there is a positive real number $p \in \mathbb{R}$, $p > 0$ such that $f(x) = f(x+p)$ for all x.

EXAMPLE 2.2. $f(x) = \sin x$ and $f(x) = \cos x$ are periodic of period $p = 2\pi$.

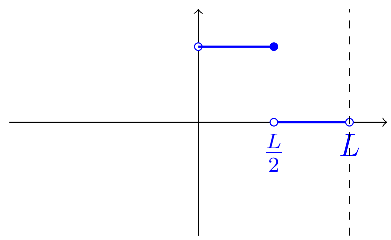
REMARK 2.3. The period p is not necessarily the smallest real number for which $f(x) = f(x+p)$. In fact, if p is a period, then any multiple of p is also a period. This is important in the following examples:

EXAMPLE 2.4.
(1) $f(x) = \sin nx$ and $f(x) = \cos nx$ are periodic of period $2\pi/n$ but also periodic of period $p = 2\pi$: we will consider them as periodic of period 2π.
(2) $\sin\left(\frac{n\pi}{L}x\right)$ and $\cos\left(\frac{n\pi}{L}x\right)$ are periodic of period $2L/n$ but we will consider them in the following as functions of period $2L$.
(3) $f(x) = \tan x$ is periodic of period π. This is the typical example of a function that is defined for "most" x, as it is defined for all $x \neq \pm(2k+1)\frac{\pi}{2}$, for $k = 0, 1, 2, \ldots$.
(4) The function $f : (0, L) \to \mathbb{R}$ defined as

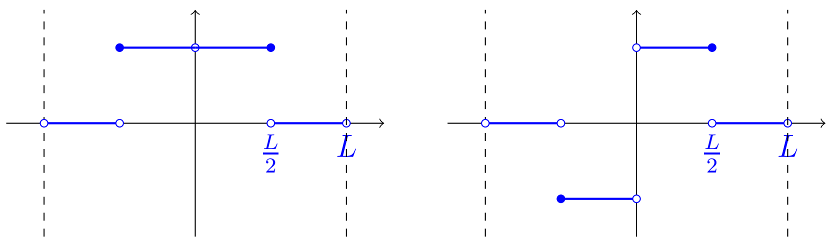


$$f(x) := \begin{cases} 1 & x \in \left(0, \frac{L}{2}\right] \\ 0 & x \in \left(\frac{L}{2}, L\right) \end{cases} .$$

can be extended to the whole real axis as a periodic function. There are at least three ways to do this, one as a function of period L



and two as a function of period $2L$



We will see that the possibility of extending a function defined on a bounded interval to be periodic is very important.

(5) If f, g are functions periodic of period p, and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is periodic of period p.

The functions $\sin\left(\frac{n\pi}{L}x\right)$ and $\cos\left(\frac{n\pi}{L}x\right)$ are particularly important periodic functions and the collection

$$\left\{\sin\left(\frac{n\pi}{L}x\right), \cos\left(\frac{n\pi}{L}x\right) : n \in \mathbb{Z}, n \geq 0\right\}$$

is called *trigonometric system*.

Properties (Orthogonality of the trigonometric system). *Let n, m be non-negative integers. Then*

(1)

$$\boxed{\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \\ 2L & n = m = 0\,. \end{cases}}$$

(2)

$$\boxed{\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\,dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0\,. \end{cases}}$$

(3)

$$\boxed{\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right)\, dx = 0, \ \textit{for every } n, m\,.}$$

VERIFICATION. To simplify the calculation we consider the case in which $2L = 2\pi$. Observe first of all that if $n = m = 0$, then

$$\int_0^{2\pi} \cos nx \cos mx \, dx = \int_0^{2\pi} dx = 2\pi\,.$$

Using the following trigonometric formulas

$$\begin{aligned}\cos nx \cos mx &= \frac{1}{2}\cos(n+m)x + \frac{1}{2}\cos(n-m)x\\ \sin nx \sin mx &= \frac{1}{2}\cos(n-m)x - \frac{1}{2}\cos(n+m)x\\ \sin nx \cos mx &= \frac{1}{2}\sin(n+m)x + \frac{1}{2}\sin(n-m)x\,,\end{aligned}$$

we obtain

$$\begin{aligned}\int_0^{2\pi} \cos nx \cos mx \, dx &= \frac{1}{2}\int_0^{2\pi} \cos(n+m)x\, dx + \frac{1}{2}\int_0^{2\pi} \cos(n-m)x\, dx\\ &= \begin{cases} \frac{1}{2}0 + \frac{1}{2}0 = 0 & n \neq m\\ \pi & n = m \neq 0 \end{cases},\end{aligned}$$

$$\begin{aligned}\int_0^{2\pi} \sin nx \sin mx \, dx &= \frac{1}{2}\int_0^{2\pi} \cos(n-m)x\, dx + \frac{1}{2}\int_0^{2\pi} \cos(n+m)x\, dx\\ &= \begin{cases} \frac{1}{2}0 + \frac{1}{2}0 = 0 & n \neq m\\ \pi & n = m \neq 0 \end{cases},\end{aligned}$$

and finally

$$\begin{aligned}\int_0^{2\pi} \cos nx \sin mx \, dx &= \frac{1}{2}\int_0^{2\pi} \sin(n+m)x\, dx + \frac{1}{2}\int_0^{2\pi} \sin(n-m)x\, dx\\ &= \frac{1}{2}0 + \frac{1}{2}0 = 0\,.\end{aligned}$$

□

DEFINITION 2.5. (1) A *trigonometric polynomial* is a function of the kind

$$a_0 + \sum_{n=1}^{N}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)\,,$$

while

(2) a *trigonometric series* is a function of the kind

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right) ,$$

where the a_n, b_n are constants called the *coefficients*.

Since the functions in the trigonometric system are periodic of period $2L$ and since a trigonometric series is the sum of functions in the trigonometric system, if the series converges it will converge to a periodic function of period $2L$. Suppose this is the case and let $f(x)$ be a function whose sum is a trigonometric series, that is

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right) . \tag{2.1}$$

The natural question is what is the relation between $f(x)$ and the coefficients a_n, b_n. By multiplying both sides of (2.1) by $\cos\left(\frac{m\pi}{L}x\right)$ and integrating the result over the interval $[-L, L]$, we obtain

$$\begin{aligned}
&\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)\,dx \\
&= a_0\int_{-L}^{L}\cos\left(\frac{m\pi}{L}x\right)\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-L}^{L}\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,dx\right. \\
&\qquad\qquad\left. + b_n \underbrace{\int_{-L}^{L}\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,dx}_{=0}\right).
\end{aligned}$$

Using the orthogonality properties of the trigonometric system (in particular (1) and (3) for m fixed and n varying between 1 and ∞) we obtain

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)\,dx = \begin{cases} a_0 2L & m = 0 \\ a_m L & m > 0\,, \end{cases}$$

from which it follows that

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx\,, \\ a_m &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)\,dx\,,\ \text{if } m > 0\,. \end{aligned}} \tag{2.2}$$

Likewise, multiplying both sides of (2.1) by $\sin\left(\frac{m\pi}{L}x\right)$ and integrating the result over the interval $[-L, L]$, we obtain

$$\begin{aligned}
&\int_{-L}^{L} f(x)\sin\left(\frac{m\pi}{L}x\right)\,dx \\
&= a_0 \underbrace{\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\,dx}_{=0} + \sum_{n=1}^{\infty}\left(a_n \underbrace{\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\,dx}_{=0}\right. \\
&\qquad\qquad\qquad\qquad \left. + b_n \int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\,dx\right).
\end{aligned}$$

Again using the orthogonality properties of the trigonometric system we deduce

$$\boxed{b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi}{L}x\right)\,dx\,,\ \text{if } m > 0\,.} \tag{2.3}$$

Example 2.6. We compute the Fourier series of the function

$$f(x) := \begin{cases} -k & -2 < x < 0 \\ k & 0 < x < 2\,, \end{cases}$$

extended by periodicity so as to have $f(x) = f(x+4)$. In this case $2L = 4$, that is $L = 2$. It follows that

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}(-k(0-(-2)) + k(2-0) = 0\,.$$

and

$$\begin{aligned}
a_m &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)\,dx \\
&= \frac{k}{2}\left(-\int_{-2}^{0}\cos\left(\frac{m\pi}{2}x\right)\,dx + \int_{0}^{2}\cos\left(\frac{m\pi}{2}x\right)\right)\,dx \\
&= \frac{k}{\cancel{2}}\frac{\cancel{2}}{m\pi}\left(-\sin\left(\frac{m\pi}{2}x\right)\Big|_{-2}^{0} + \sin\left(\frac{m\pi}{2}x\right)\Big|_{0}^{2}\right) \\
&= \frac{k}{m\pi}\left(\sin\left(\frac{m\pi}{\cancel{2}}(-\cancel{2})\right) + \sin\left(\frac{m\pi}{\cancel{2}}\cancel{2}\right)\right) \\
&= 0\,.
\end{aligned}$$

Analogously

$$
\begin{aligned}
b_m &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi}{L}x\right)\,dx \\
&= \frac{k}{2}\left(-\int_{-2}^{0}\sin\left(\frac{m\pi}{2}x\right)\,dx + \int_{0}^{2}\sin\left(\frac{m\pi}{2}x\right)\right)\,dx \\
&= \frac{k}{\cancel{2}}\frac{\cancel{2}}{m\pi}\left(\cos\left(\frac{m\pi}{2}x\right)\Big|_{-2}^{0} - \cos\left(\frac{m\pi}{2}x\right)\Big|_{0}^{2}\right) \\
&= \frac{k}{m\pi}(1-\cos(-m\pi)-\cos(m\pi)+1) \\
&= \frac{2k}{m\pi}(1-\cos m\pi) \\
&= \begin{cases} \frac{4k}{m\pi} & m = 1,3,5,\dots \\ 0 & m = 2,4,6,\dots\,. \end{cases}
\end{aligned}
$$

Thus $f(x)$ can be represented by a Fourier series as follows

$$
f(x) = \frac{4k}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\left(\frac{(2n+1)\pi}{2}x\right)\,.
$$

(See Fig. 257 on page 481 of Kreyszig's book.) □

REMARK 2.7. In order to be able to find a formula for the Fourier coefficients, we assumed that the function f could be represented as a Fourier series. This means not only that the Fourier series of the given function converges, but also that the sum of the Fourier series of f at the point x equals the value of the function f at x.

If f is periodic of period $2L$ and piecewise continuous and if f has left and right derivative at each point in a given interval of length $2L$, then the Fourier series of f with the coefficients determined above converges to $f(x)$ in all points where f is continuous and it converges to $\frac{1}{2}(f(x_0^-)+f(x_0^+))$ in the points of discontinuity. Here we use the notation

$$
f(x_0^{\pm}) = \lim_{x\to x_0^{\pm}} f(x) = \lim_{\epsilon\to 0} f(x_0 \pm \epsilon)\,.
$$

In particular this means that if f is continuous and its Fourier series converges, then the sum of the Fourier series at x equals $f(x)$.

2.2. Even and odd functions, half-range expansion

DEFINITION 2.8. A function f is

(1) *even* if $f(x) = f(-x)$
(2) *odd* if $f(x) = -f(-x)$

for all x in the domain.

REMARK 2.9. The graph of an even function (such as $\cos x$) is symmetric with respect to the y-axis, while the graph of an odd function (such as $\sin x$) is symmetric with respect to the origin.

PROPERTIES. (1) *The product of two even functions or of two odd functions is even, while the product of an odd function with an even one is odd.*
(2) *If g is even, then $\int_{-L}^{L} g(x)\,dx = 2\int_0^L g(x)\,dx$.*
(3) *If g is odd, then $\int_{-L}^{L} g(x)\,dx = 0$.*

2.2.1. Simplified form of the Fourier series for even and for odd functions.

THEOREM 2.10. *Let f be a function of period $2L$ representable by its Fourier series.*
(1) *If f is even then*

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)\,,$$

where

$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^L f(x)\,dx \\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\,dx\,,\ \ \text{if } n>0\,. \end{aligned}$$

(2) *If f is odd then*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)\,,$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx\,.$$

EXAMPLE 2.11. We will find the Fourier series of the function $f(x)$ defined as $f(x) = x+\pi$, for $-\pi < x < \pi$ and extended to be 2π periodic $f(x) = f(x+2\pi)$. To do so, we write $f(x) = f_1(x) + f_2(x)$, where $f_1(x) = x$ and $f_2(x) = \pi$. Because of the linearity of the integrals, the Fourier coefficients of f are the sum of the Fourier coefficients of f_1 and of f_2. The coefficients b_n of f_2 vanish since f_2 is even: since also if $n \geq 1$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \pi\cos\left(\frac{n\pi}{\pi}x\right)\,dx = \int_0^{\pi}\cos(nx)\,dx = \frac{1}{n}\sin(nx)\big|_0^{\pi} = 0\,,$$

the coefficients of f_2 are all zero, except for $a_0 = \pi$; that is $f_2(x) = \pi$ is its own Fourier series.

Since f_1 is odd, the only nonzero coefficients are the b_n. From (2.3), and using integration by parts, we have

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi x\sin nx\,dx \\ &= \frac{2}{\pi}\left(-\frac{x}{n}\cos nx\Big|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\,dx\right) \\ &= \frac{2}{\pi n}\left(-\pi\cos n\pi + \frac{1}{n}\sin nx\Big|_0^\pi\right) \\ &= -\frac{2}{n}\cos n\pi \\ &= (-1)^{n-1}\frac{2}{n}, \end{aligned}$$

from which it follows that

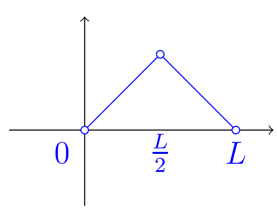
$$f(x) = \pi + 2\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}\sin nx\,.$$

(See Fig. 266 on page 493 of Kreyszig's book.)

2.2.2. The half-range expansion. We saw in Example 2.4(4) that a function f defined on the interval $(0, L)$ can be extended periodically with period L and in this case its Fourier series will contain both coefficients a_n and b_n. Sometimes it is however convenient to extend the function f to be either even or odd, and in this case the period will be $2L$. This is particularly convenient in connection with PDEs, as we will see in Chapter 3.

Example 2.12. Compute the half-range expansion of

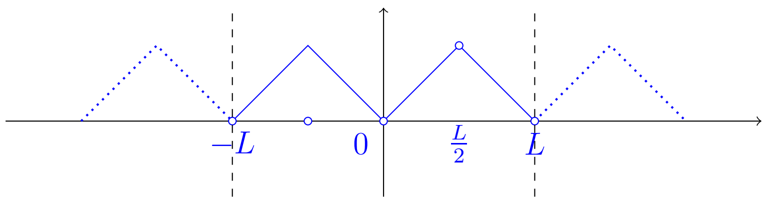
$$f(x) = \begin{cases} \frac{2k}{L}x & x\in\left[0, \frac{L}{2}\right] \\ \frac{2k}{L}(L-x) & x\in\left[\frac{L}{2}, L\right] \end{cases}.$$



We are going to compute the extension first as an *even* periodic function and then as an odd periodic function.

Extension as an even periodic function: We assume that f is as above and extend it to be an even function by setting $f(x) = f(-x)$ for $x \in [-L, 0]$. Then set

$f(x+2L) = f(x)$ for all $x \in \mathbb{R}$.



By Theorem 2.10(1) we have that

$$
\begin{aligned}
b_n &= 0 \\
a_0 &= \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^{L/2} \frac{2k}{L}x\,dx + \frac{1}{L}\int_{L/2}^L \frac{2k}{L}(L-x)\,dx \\
&= \frac{2k}{L^2}\frac{1}{2}\frac{L^2}{4} + \frac{2k}{L^2}\frac{L^2}{2} - \frac{2k}{L^2}\frac{1}{2}\frac{3L^2}{4} = \frac{1}{2}k\,, \\
a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\,dx \\
&= \frac{2}{L}\int_0^{L/2} \frac{2k}{L}x\cos\left(\frac{n\pi}{L}x\right)\,dx + \frac{2}{L}\int_{L/2}^L \frac{2k}{L}(L-x)\cos\left(\frac{n\pi}{L}x\right)\,dx = (*)
\end{aligned}
$$

Since

$$
\begin{aligned}
\int x\cos\left(\frac{n\pi}{L}x\right)\,dx &= \frac{L}{n\pi}x\sin\left(\frac{n\pi}{L}x\right) - \frac{L}{n\pi}\int \sin\left(\frac{n\pi}{L}x\right)\,dx \\
&= \frac{L}{n\pi}x\sin\left(\frac{n\pi}{L}x\right) + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right) + C\,,
\end{aligned}
$$

then

$$
\begin{aligned}
(*) =& \frac{4k}{L^2}\left(\frac{L}{n\pi}x\sin\left(\frac{n\pi}{L}x\right)+\frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right)\right)\Bigg|_0^{L/2}+\frac{4k}{L}\int_{L/2}^{L}\cos\left(\frac{n\pi}{L}x\right)\,dx\\
&-\frac{4k}{L^2}\left(\frac{L}{n\pi}x\sin\left(\frac{n\pi}{L}x\right)+\frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right)\right)\Bigg|_{L/2}^{L}\\
=&\frac{2k}{n\pi}\sin\left(n\frac{\pi}{2}\right)+\frac{4k}{n^2\pi^2}\cos\left(n\frac{\pi}{2}\right)-\frac{4k}{n^2\pi^2}\\
&+\frac{4k}{n\pi}\sin n\pi-\frac{4k}{n\pi}\sin\left(n\frac{\pi}{2}\right)\\
&-\frac{4k}{L^2}\left(\frac{L^2}{n\pi}\sin n\pi-\frac{L^2}{2n\pi}\sin\left(n\frac{\pi}{2}\right)\right)+\frac{L^2}{n^2\pi^2}\left(\cos n\pi-\cos\left(n\frac{\pi}{2}\right)\right)\\
=&\underbrace{\left(\frac{2k}{n\pi}-\frac{4k}{n\pi}+\frac{2k}{n\pi}\right)}_{=0}\sin\left(n\frac{\pi}{2}\right)+\underbrace{\left(\frac{4k}{n^2\pi^2}+\frac{4k}{n^2\pi^2}\right)}_{=\frac{8k}{n^2\pi^2}}\cos\left(n\frac{\pi}{2}\right)\\
&+\underbrace{\left(\frac{4k}{n\pi}-\frac{4k}{n\pi}\right)}_{=0}\sin n\pi-\frac{4k}{n^2\pi^2}\cos n\pi-\frac{4k}{n^2\pi^2}\\
=&\frac{8k}{n^2\pi^2}\cos\left(n\frac{\pi}{2}\right)-\frac{4k}{n^2\pi^2}\cos n\pi-\frac{4k}{n^2\pi^2}\\
=&\frac{4k}{n^2\pi^2}\left(2\cos\left(n\frac{\pi}{2}\right)-\cos n\pi-1\right).
\end{aligned}
$$

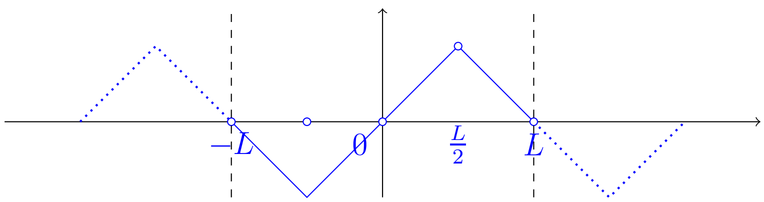
Since $\cos n\pi=(-1)^n$ and $\cos\left(n\frac{\pi}{2}\right)=\begin{cases}0 & n=2j+1\\ (-1)^j & n=2j\end{cases}$, then

$$
\begin{aligned}
a_n &= \left(2\cos\left(n\frac{\pi}{2}\right)-\cos n\pi-1\right)\\
&=\begin{cases}\frac{4k}{n^2\pi^2}\left(2\cdot 0-(-1)^{2j+1}-1\right) & n=2j+1\\ \frac{4k}{n^2\pi^2}\left(2(-1)^j-(-1)^{2j}-1\right) & n=2j\end{cases}\\
&=\begin{cases}\frac{4k}{n^2\pi^2}\left((-1)^{2j}-1\right) & n=2j+1\\ \frac{4k}{n^2\pi^2}\left(2(-1)^j-2\right) & n=2j\end{cases}\\
&=\begin{cases}0 & n=2j+1\\ 0 & n=2j,\text{with } j \text{ even, that is } j=2m\\ \frac{4k}{n^2\pi^2}(-4)=-\frac{16k}{n^2\pi^2} & n=2j,\text{with } j \text{ odd, that is } j=2m+1\,,\end{cases}
\end{aligned}
$$

so that

$$\begin{aligned} f(x) =& \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\left(\frac{2\pi}{L}x\right) + \frac{1}{6^2}\cos\left(\frac{6\pi}{L}x\right) + \dots\right) \\ =& \frac{k}{2} - \frac{16k}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2(2m+1))^2}\cos\left(\frac{2(2m+1)}{L}x\right). \end{aligned}$$

Extension as an odd periodic function: With an analogous calculation, after having set $f(x) = -f(-x)$ for $x \in [-L, 0]$ and then $f(x+2L) = f(x)$ for all $x \in \mathbb{R}$,



we obtain that

$$a_n = 0 \text{ for all } n \text{ and } b_n = \frac{8k}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} o & n \text{ even} \\ \frac{8k(-1)^j}{n^2\pi^2} & n = 2j+1\,, \end{cases}$$

so that

$$f(x) = \frac{8k}{\pi^2}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^2}\sin\left(\frac{(2j+1)\pi}{L}x\right). \tag{2.4}$$

□

2.3. Complex Fourier series

We saw that the Fourier series of a periodic function f of period $2L$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi}{L}x\right) + b_n\sin\left(\frac{n\pi}{L}x\right)\right].$$

We also know that

$$e^{it} = \cos t + i\sin t \tag{2.5}$$

or, equivalently,

$$\cos t = \frac{e^{it}+e^{-it}}{2} \qquad \text{and} \qquad \sin t = \frac{e^{it}-e^{-it}}{2i}\,,$$

where these last formulas can be obtained from (2.5), by replacing t with $-t$ and using that $\cos t = \cos(-t)$ and that $\sin t = -\sin(-t)$. It follows that, by setting

$\frac{n\pi}{L}x =: t_n$ for $n > 0$ for simplicity,

$$\begin{aligned}
a_n \cos t_n + b_n \sin t_n &= a_n \frac{e^{it_n} + e^{-it_n}}{2} + b_n \frac{e^{it_n} - e^{-it_n}}{2i} \\
&= e^{it_n} \underbrace{\frac{1}{2}(a_n - ib_n)}_{=:c_n} + e^{-it_n} \underbrace{\frac{1}{2}(a_n + ib_n)}_{=:k_n} \\
&= c_n e^{it_n} + k_n e^{-it_n} .
\end{aligned}$$

We can hence set $c_0 := a_0$: moreover, if we compute

$$\begin{aligned}
c_n =& \frac{1}{2}(a_n - ib_n) \\
=& \frac{1}{2L}\int_{-L}^{L} f(x)\left(\cos\left(\frac{n\pi}{L}x\right) - i\sin\left(\frac{n\pi}{L}x\right)\right)\, dx \\
=& \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L} dx
\end{aligned}$$

and

$$\begin{aligned}
k_n =& \frac{1}{2}(a_n + ib_n) \\
=& \frac{1}{2L}\int_{-L}^{L} f(x)\left(\cos\left(\frac{n\pi}{L}x\right) + i\sin\left(\frac{n\pi}{L}x\right)\right)\, dx \\
=& \frac{1}{2L}\int_{-L}^{L} f(x) e^{in\pi x/L} dx
\end{aligned}$$

we see that we can extend the definition of c_n to negative $n \in \mathbb{Z}$ by setting $c_{-n} := k_n$. With this notation we obtain

$$\begin{aligned}
f(x) \quad &= c_0 + \textstyle\sum_{n=1}^{\infty}(c_n e^{in\pi x/L} + k_n e^{-in\pi x/L}) \\
&= c_0 + \textstyle\sum_{n=1}^{\infty}(c_n e^{in\pi x/L} + c_{-n} e^{-in\pi x/L})
\end{aligned}$$

so that the *complex Fourier series of f* is

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \, ,}$$

where

$$\boxed{c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L}\, dx \, .}$$

REMARK 2.13. Notice that just because we have written the Fourier series of a real function f with complex numbers, it does not mean that the function or its Fourier series have all the sudden became complex! In fact, using the definition of c_n (and of k_n), and denoting by $\overline{z} = x - iy$ the complex conjugate of the complex number $z = x + iy$, we have:

$$c_n e^{in\pi x/L} = \frac{1}{2}(a_n - ib_n)e^{in\pi x/L} = \overline{\frac{1}{2}(a_n + ib_n)}\,\overline{e^{-in\pi x/L}} = \overline{k_n e^{-in\pi x/L}} = \overline{c_{-n}e^{-in\pi x/L}}$$

and hence

$$\overline{c_n e^{in\pi x/L}} = c_{-n}e^{-in\pi x/L}\,.$$

Since $z + \overline{z} = 2\Re z = 2x$, we obtain that

$$\begin{aligned}
&c_n e^{in\pi x/L} + c_{-n}e^{-in\pi x/L} \\
&\quad = 2\Re c_n e^{in\pi x/L} = 2\Re\left[\frac{1}{2}(a_n - ib_n)(\sin\left(\frac{n\pi}{L}x\right) + i\sin\left(\frac{n\pi}{L}x\right))\right] \\
&\quad = \Re\big((a_n\cos\left(\frac{n\pi}{L}x\right) + b_n\sin\left(\frac{n\pi}{L}x\right)) + i(a_n\sin\left(\frac{n\pi}{L}x\right) - b_n\cos\left(\frac{n\pi}{L}x\right))\big) \\
&\quad = a_n\cos\left(\frac{n\pi}{L}x\right) + b_n\sin\left(\frac{n\pi}{L}x\right)\,.
\end{aligned} \tag{2.6}$$

Since the Fourier series is absolutely convergent, its terms can be rearranged arbitrarily. This, and the fact that $c_0 = a_0$, imply that

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} &= c_0 + \sum_{n=1}^{\infty}(c_n e^{in\pi x/L} + c_{-n}e^{-in\pi x/L}) \\
&= a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi}{L}x\right) + b_n\sin\left(\frac{n\pi}{L}x\right)\right)\,.
\end{aligned}$$

Example 2.14. We want to find the complex Fourier series of the function $f(x) = e^x$ for $= \pi < x < \pi$, extended by periodicity so as to have $f(x) = f(x + 2\pi)$. We will then derive the real Fourier series from the complex one.

We start by computing the coefficients c_n.

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^x e^{-inx}\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{(1-in)x}\,dx = \frac{1}{2\pi}\,\frac{1}{1-in}e^{(1-in)x}\Big|_{\pi}^{\pi} = (*)
\end{aligned}$$

Since $\cos x = \cos(-x)$, we have

$$e^{\pm in\pi} = \cos(\pm n\pi) + i\sin(\pm n\pi) = (-1)^n\,.$$

Moreover

$$\frac{1}{1-in} = \frac{1+in}{(1-in)(1+in)} = \frac{1+in}{1+n^2}\,,$$

and $\sinh x = \frac{e^x - e^{-x}}{2}$, so that

$$(*) = \frac{1}{2\pi}\frac{1+in}{1+n^2}(e^{in\pi}e^{\pi} - e^{-in\pi}e^{-\pi}) = \frac{1}{2\pi}\frac{1+in}{1+n^2}(-1)^n(e^{\pi} - e^{-\pi}) = \frac{\sinh\pi}{\pi}\frac{1+in}{1+n^2}(-1)^n\,.$$

Hence we can write the complex Fourier series of e^x as

$$e^x = \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{1+in}{1+n^2} e^{inx} .$$

In order to find the real Fourier series, we just need to apply (2.6).

$$\begin{aligned} c_n e^{inx} + c_{-n} e^{-inx} &= 2\frac{\sinh \pi}{\pi} \Re (-1)^n \frac{1+in}{1+n^2} e^{inx} \\ &= \frac{2\sinh \pi}{\pi} (-1)^n \Re \left(\frac{1+in}{1+n^2} (\cos nx + i \sin nx) \right) \\ &= \frac{2\sinh \pi}{\pi} (-1)^n \left(\frac{1}{1+n^2} \cos nx - \frac{n}{1+n^2} \sin nx \right) \end{aligned}$$

Since $c_0 = \frac{2\sinh \pi}{\pi}$, the real Fourier series of e^x with period 2π is

$$e^x = \frac{2\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi} \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{1+n^2} \cos nx - \frac{n}{1+n^2} \sin nx \right) .$$

2.4. Approximation by trigonometric polynomials

We saw already that a trigonometric polynomial is defined as

$$A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx) ,$$

where N is the degree of the polynomial. So, for example the partial sums of a Fourier series are a particular case of a trigonometric polynomial.

Given for simplicity $f(x)$ periodic of period 2π, one wants to find what is the trigonometric polynomial that "best approximates" $f(x)$. We are not interested in pointwise approximation but we want an "overall best approximation".

Let $F(x) = A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx)$ and let us define the *square error* in the approximation of f by F as

$$E = \int_{-\pi}^{\pi} (f - F)^2 \, dx .$$

A careful calculation using the orthogonality relations of the trigonometric system shows the following:

Theorem 2.15. *The trigonometric polynomial of degree N that best approximates a function f on the interval $[-\pi, \pi]$ (i.e. with the smalls square error) is the partial sum of the Fourier series of f. The minimum value E^* of the square error is*

$$E^* = \int_{-\pi}^{\pi} f^2 \, dx - \pi \left[2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \right]$$

and it decreases as N increases.

As an exercise, compute the minimum square error for the sawtooth wave (see Ex. 1 on page 505 of Kreyszig's book).

2.5. Fourier integral

We see now how to extend the concept of Fourier series expansion to functions that are not periodic but defined on the whole real line (and hence cannot be extended by periodicity). If f is periodic of period $2L$, then we can write

$$
\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right) \\
&= \frac{1}{2L}\int_{-L}^{L} f(v)\,dv + \sum_{n=1}^{\infty}\left[\left(\frac{1}{L}\int_{-L}^{L} f(v)\cos\left(\frac{n\pi}{L}v\right)\,dv\right)\cos\left(\frac{n\pi}{L}x\right)\right. \\
&\qquad \left. + \left(\frac{1}{L}\int_{-L}^{L} f(v)\sin\left(\frac{n\pi}{L}v\right)\,dv\right)\sin\left(\frac{n\pi}{L}x\right)\right].
\end{aligned}
$$

We set now $w_n := \frac{n\pi}{L}$ so that $\Delta w = w_{n+1} - w_n = \frac{\pi}{L}$, and we obtain

$$
\begin{aligned}
f(x) &= \frac{1}{2L}\int_{-L}^{L} f(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\left(\int_{-L}^{L} f(v)\cos(w_n v)\,dv\right)\cos(w_n x) + \right. \\
&\qquad \left. \left(\int_{-L}^{L} f(v)\sin(w_n v)\,dv\right)\sin(w_n x)\right] \\
&= \frac{1}{2L}\int_{-L}^{L} f(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\left(\int_{-L}^{L} f(v)\cos(w_n v)\,dv\right)\cos(w_n x)\Delta w + \right. \\
&\qquad \left. \left(\int_{-L}^{L} f(v)\sin(w_n v)\,dv\right)\sin(w_n x)\Delta w\right].
\end{aligned}
$$

We assume that f is *absolutely integrable* on $\mathbb{R}$, that is that

$$\int_{-\infty}^{\infty} |f(x)|\,dx < \infty\,.$$

(We recall that $\int_{-\infty}^{\infty} |f(x)|\,dx = \lim_{\alpha\to-\infty}\int_{\alpha}^{o} |f(x)|\,dx + \lim_{\beta\to\infty}\int_{0}^{\beta} |f(x)|\,dx$.) From this is follows that if $L \to \infty$ the first term goes to 0 and $\Delta w \to 0$, so the sum on the right becomes an integral

$$
\begin{aligned}
f(x) &= \frac{1}{\pi}\int_{0}^{\infty}\left[\left(\int_{-\infty}^{\infty} f(v)\cos wv\,dv\right)\cos wx + \left(\int_{-\infty}^{\infty} f(v)\sin wv\,dv\right)\sin wx\right] dw \\
&= \boxed{\int_{0}^{\infty} [A(w)\cos(wx) + B(w)\sin(wx)]\,dw = f(x)},
\end{aligned}
$$

where

$$\boxed{A(w) := \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(wv)\,dv}$$

and

$$\boxed{B(w) := \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(wv)\,dv}$$

This representation of f is called the *Fourier integral of* f.

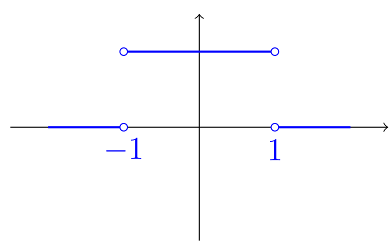
Just like for the Fourier series, one has to be careful to see whether the Fourier integral actually represents the function. This happens if:

(1) f is piecewise continuous in every finite interval;
(2) f has left and right derivatives at the points of discontinuity;
(3) f if absolutely integrable.

Then the Fourier integral represents f at all points where f is continuous and converges toward the average of f at the points of discontinuity. The behaviour is analogous as for the Fourier series.

Remark 2.16. The partial sums of a Fourier series correspond to integrals on finite intervals $\int_0^a$.

Example 2.17. Sometimes one can use Fourier integrals to "compute" integrals in the following sense. Let us consider the function



$$f(x) := \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1\,. \end{cases}$$

Then

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos vw\,dv = \frac{1}{\pi}\int_{-1}^{1}\cos vw\,dv = \frac{\sin wv}{\pi w}\Big|_{-1}^{1} = \frac{2\sin w}{\pi w}\,,$$

and

$$B(w) = 0\,,$$

so that

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos wx\sin w}{w}\,dw\,. \tag{2.7}$$

Since the average of the left and the right limit of f at $x = 1$ is $\frac{1}{2} = \frac{1+0}{2}$, comparing (2.7) with the definition of the function f,

$$\int_0^\infty \frac{\cos wx \sin w}{w}\, dw = \begin{cases} \frac{\pi}{2} & |x| < 1 \\ \frac{\pi}{4} & |x| = 1 \\ 0 & |x| > 1\,. \quad \square \end{cases}$$

This is called *Dirichlet's discontinuous factor*. For $x = 0$ it becomes

$$\int_0^\infty \frac{\sin w}{w}\, dw = \frac{\pi}{2}\,,$$

and

$$\mathrm{Si}(u) = \int_0^u \frac{\sin w}{w}\, dw$$

is called the *sine integral*.

2.5.1. Fourier sine and Fourier cosine integrals. Just like for the Fourier series, we can simplify the formulas of the Fourier integral if the function is even or odd.

If $\boxed{\text{f is even}}$, then

$$\boxed{\begin{aligned} A(w) &= \frac{2}{\pi}\int_0^\infty f(v)\cos wv\, dv \\ B(w) &= 0 \end{aligned}}$$

and

$$\boxed{f(x) = \int_0^\infty A(w)\cos wx\, dw\,,}$$

while if $\boxed{\text{f is odd}}$, then

$$\boxed{\begin{aligned} A(w) &= 0 \\ B(w) &= \frac{2}{\pi}\int_0^\infty f(v)\sin wv\, dv \end{aligned}}$$

and

$$\boxed{f(x) = \int_0^\infty B(w)\sin wx\, dw\,.}$$

2.6. Fourier transform

There is another, and more common, form of the Fourier integral, one that corresponds to the complex form of the Fourier series.

$$\begin{aligned} f(x) &= \int_0^\infty (A(w)\cos wx + B(w)\sin wx)\,dw \\ &= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(v)(\underbrace{\cos wv\cos wx + \sin wv \sin wx}_{\cos(wx-wv)})\,dv\,dw \\ &= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(v)\cos(wx-wv)\,dv\,dw \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(v)\cos(wx-wv)\,dv\,dw\,, \end{aligned}$$

where we used that

$$\cos(wx-wv) = \cos wv\cos wx + \sin wv\sin wx$$

is an even function of w.

Likewise, since $\sin(wx-wv)$ is an odd function of w, we have that

$$\frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(v)\sin(wx-wv)\,dv\,dw = 0\,,$$

so that we can write

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(v)\,\underbrace{(\cos(wx-wv) + i\sin(wx-wv))}_{e^{iw(x-v)}}\,dv\,dw \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(v)e^{iw(x-v)}\,dv\,dw \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(v)e^{-iwv}\,dv\right) e^{iwx}\,dw\,. \end{aligned}$$

Definition 2.18. The *Fourier transform* $\mathcal{F}(f)$ of f is defined as

$$\boxed{\mathcal{F}(f)(w) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(v)e^{-iwv}\,dv\,.}$$

We denote the Fourier transform either as $\mathcal{F}(f)$ or as $\hat{f}(w)$. In either cases we have that

$$\boxed{\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \hat{f}(w)e^{iwx}\,dw \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \mathcal{F}(f)(w)e^{iwx}\,dw\,. \end{aligned}}$$

Just like the Laplace transform, the Fourier transform is an integral transform widely used in the engineering sciences.

REMARK 2.19. Using the Fourier sine and cosine integral, one can define in an analogous way the Fourier sine and cosine transforms.

The above formula gives also a formula for the *inverse Fourier transform* of a function $g(w)$, namely

$$\boxed{\mathcal{F}^{-1}(g)(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(w) e^{iwx}\, dw\,,}$$

where the name "inverse Fourier transform" comes from the fact that if $g = \mathcal{F}(f)$, then

$$\mathcal{F}^{-1}(\mathcal{F}(f)) = f\,.$$

As for the existence of the Fourier transform of a function f, we remark that if f is absolutely integrable and piecewise continuous on finite intervals, then its Fourier transform exists.

EXAMPLE 2.20. We compute the Fourier transform $\hat{f}$ of the function

$$f(x) := \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1\,. \end{cases}$$

Applying the definition, we obtain

$$\begin{aligned} \hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixw}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ixw}\, dx \\ &= \frac{1}{-iw\sqrt{2\pi}} e^{-ixw} \Big|_{-1}^{1} = \frac{-1}{iw\sqrt{2\pi}} (e^{-iw} - e^{iw}) \\ &= \frac{2\sin w}{w\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}} \frac{\sin w}{w}\,. \end{aligned}$$

EXAMPLE 2.21. We compute the Fourier transform of the function

$$f(x) := \begin{cases} e^{-ax} & x > 0 \\ 0 & x < 0 \end{cases},$$

where $a > 0$.

$$\begin{aligned} \hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ax} e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(a+iw)x}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{-(a+iw)} e^{-(a+iw)x} \Big|_{0}^{\infty} = \frac{1}{\sqrt{2\pi}(a+iw)}\,, \end{aligned}$$

where we used the fact that

$$\lim_{x\to\infty} e^{-(a+iw)x} = \lim_{x\to\infty} e^{-ax}(\cos wx + i\sin wx) = 0\,,$$

since $\lim_{x\to\infty} e^{-ax} = 0$ and $\cos wx + i\sin wx$ is bounded in absolute value.

PROPERTIES (Properties of the Fourier transform). (1) *(Linearity)*

$$\mathcal{F}(\alpha f + \beta g) = \alpha\mathcal{F}(f) + \beta\mathcal{F}(g)$$

if all the Fourier transforms exist.

(2) *(Fourier transform of a derivative) If f is continuous on $\mathbb{R}$, $\lim_{|x|\to\infty} f(x) = 0$ and f' is absolutely integrable, then*

$$\mathcal{F}(f'(x)) = iw\mathcal{F}(f(x))\,. \tag{2.8}$$

(3) *If f and g are piecewise continuous, bounded and absolutely integrable, then*

$$\mathcal{F}(f * g) = \sqrt{2\pi}\mathcal{F}(f)\mathcal{F}(g)\,.$$

EXAMPLE 2.22. As an application of the properties of the Fourier transform, we assume we known that

$$\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}}e^{-w^2/4a}\,.$$

and we compute $\mathcal{F}(xe^{-x^2})$.

$$\begin{aligned}\mathcal{F}(xe^{-x^2}) &= \mathcal{F}\left(\frac{d}{dx}\left(-\frac{1}{2}e^{-x^2}\right)\right) = iw\mathcal{F}\left(-\frac{1}{2}e^{-x^2}\right)\\ &= -\frac{iw}{2}\mathcal{F}(e^{-x^2}) = -\frac{iw}{2}\frac{1}{\sqrt{2}}e^{-w^2/4} = \frac{-iw}{2\sqrt{2}}e^{-w^2/4}\,.\end{aligned}$$

CHAPTER 3

Partial Differential Equations

3.1. Introduction and basic definitions

A *partial differential equation* (*PDE* in short) is an equation involving an unknown function and some of its partial derivatives. The unknown function depends on more than one variable, one of which can be the time and the others are space variables.

EXAMPLE 3.1. $u_{xy}u_z + u_{tt} = g(x, y, t)$. □

A PDE is called *linear* if the unknown function u and its derivatives appear with degree one.

EXAMPLE 3.2. The PDE $u_{xy}u_z + u_{tt} = g(x, y, t)$ is not linear, but $u_{xy} + u_z + u_{tt} = g(x, y, t)$ is. □

A linear PDE is called *homogeneous* if each of its terms contains either the function or one of its derivatives.

EXAMPLE 3.3. The PDE $u_{xy} + u_z + u_{tt} = g(x, y, t)$ is not homogeneous, but $u_{xy} + u_z + u_{tt} = 0$ is. □

The *order* of a PDE is the order of the highest derivative in the PDE. We will be mostly concerned with second order PDEs.

EXAMPLE 3.4. (1) One dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}\,.$$

(2) One dimensional heat equation:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}\,.$$

(3) Two dimensional Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,.$$

(4) Two dimensional Poisson equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)\,.$$

(5) Two dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) .$$

(6) Three dimensional Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 .$$

Here c is a positive real constant, x, y, z are spacial variables and the "dimension" refers to the number of spacial variables in the PDE. $\square$

REMARK 3.5. We will see later (for a class of PDEs a bit wider than the linear PDEs of second order) that linear PDEs of the second order can be classified into three groups. In the above Examples 3.4(1)-(4), we could write each equation as

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y) . \tag{3.1}$$

(1) Set $y = ct$ and obtain $u_{yy} - u_{xx} = 0$, that is $A = -1 = -C$, $B = F = 0$.
(2) Set $y = c^2 t$ and obtain $u_y - u_{xx} = 0$, that is $A = 1$, $B = C = 0$, and $F = u_y$.
(3) Here the PDE is already in the form (3.1), with $A = C = 1$ and $B = F = 0$.
(4) Also here the PDE is already in the form (3.1), but now with $A = C = 1$, $B = 0$ and $F = f$.

For the equations in Example 3.4(5) and (6), one could consider a general form similar to (3.1) but with more variables.

Now consider the polynomial

$$Ax^2 + 2Bxy + Cy^2 = L(x, y) , \tag{3.2}$$

where L is a linear function of x and y. We look at the discriminant.

(1) $AC - B^2 < 0$, the curve in (3.2) is a hyperbola;
(2) $AC - B^2 = 0$, the curve in (3.2) is a parabola;
(3) $AC - B^2 > 0$, the curve in (3.2) is an ellipse.

We adopt the same denomination for the corresponding PDE, that are hence *hyperbolic*, *parabolic* or *elliptic*. The reason to classify the second order linear PDEs in this way it that each "group" of PDE has similar features that we will study later.

EXAMPLE 3.6. (1) Wave equations are the prototypes of *hyperbolic* PDEs;
(2) Heat equations are the prototypes of *parabolic* PDEs;
(3) Laplace equations are the prototypes of *elliptic* PDEs. Given that the type of a PDE depends only on the terms of second order, Poisson equations are also elliptic.

This classification is independent of the dimension of the PDE, although our method to derive the classification was not. $\square$

A *solution* of a PDE in a region R is a function differentiable "enough times" in R and satisfying the PDE in R.

WARNING. Sometimes some care might need to be used on the boundary of the region R. This can be done by insuring that R (in fact the boundary ∂R or R) is contained in a slightly larger region where the function has enough derivatives.

Notice that the space of solutions of a PDE can be enormous.

EXAMPLE 3.7. Any function of the form

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

is a solution of the one-dimensional wave equation $u_{tt} = c^2 u_{xx}$. In fact, $u_{xx} = u$ and

$$\begin{aligned} \phi_t &= c\phi' \quad \phi_{tt} = c^2\phi'' \\ \psi_t &= -c\psi' \quad \psi_{tt} = c^2\psi''\,. \end{aligned}$$

so that $u_{tt} = \phi_{tt} + \psi_{tt} = c^2(\phi'' + \psi'') = c^2 u_{xx}$.

This means that any function of $x+ct$ and $x-ct$ is a solution, for example $u(x,t) = (x+ct)^{1/3} + e^{x-ct}$ or $u(x,t) = \cos(x+ct) + (x-ct)^n$, or... □

In order to have uniqueness of solutions, we need to impose *boundary conditions* or *initial conditions*. As the words say, the first are prescribed values of the solution or of its derivatives on the boundary of the region R, while the second are given values of the solution at a given time.

Another tool used to find solutions of a PDE is the following absolutely crucial principle:

SUPERPOSITION PRINCIPLE. *If u_1 and u_2 are solutions of a homogeneous PDE, then $\alpha u_1 + \beta u_2$ is also a solution of the same PDE for all $\alpha, \beta \in \mathbb{R}$.*

It is absolutely essential that the PDE be homogeneous.

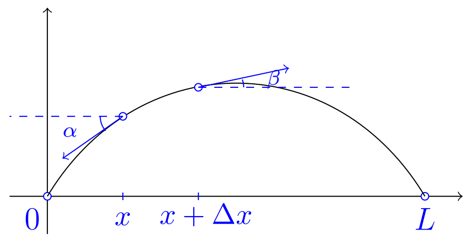
3.2. From a vibrating string to the wave equation

Given a physical system whose behaviour we want to analyse, one of the first step is to device an equation that describes the behaviour. The equation has to be simple enough to be able to solve it (maybe numerically) but also such that the solution will describe faithfully the system.

Take an elastic string stretched to have length L and tighten the endpoints. We set the left endpoint to be at the origin of a system of Cartesian coordinates and denote by $u(x,t)$ the displacement of a point on the string above the point $(x,0)$ and at time t. We assume that

(1) the string movement happens only on one plane, say a vertical one, and every point on the string moves only vertically;
(2) the string is homogenous (that is the mass per unit length ρ is constant), elastic and offers no resistance to bending;
(3) the tension is such that the effect of gravity is negligible.

We consider the forces acting on the string at the points $P(x)$ and $Q(x+\Delta x)$.



The only force acting is the tension and this is tangential to the string. There is no horizontal displacement and hence

$$T_P \cos\alpha = T_Q \cos\beta = \text{constant} =: T > 0\,. \tag{3.3}$$

According to Newton's first law of mechanics, if ρ is the mass of the (undeflected) string per unit length, we have

$$-T_P \sin\alpha + T_Q \sin\beta = \rho \Delta x\, u_{tt}\,. \tag{3.4}$$

Divinding (3.4) by (3.3) we obtain

$$\frac{T_Q \sin\beta}{T_Q \cos\beta} - \frac{T_P \sin\alpha}{T_P \cos\alpha} = \frac{\rho}{T} \Delta x\, u_{tt}\,,$$

that is,

$$\tan\beta - \tan\alpha = \frac{\rho}{T}\Delta x\, u_{tt}\,.$$

But $\tan\beta = u_x\big|_{x+\Delta x}$ and $\tan\alpha = u_x\big|_x$, so that

$$u_x(x+\Delta x, t) - u_x(x,t) = \frac{\rho}{T}\Delta x\, u_{tt}\,.$$

By dividing by Δx we obtain

$$\frac{u_x(x+\Delta x, t) - u_x(x,t)}{\Delta x} = \frac{\rho}{T} u_{tt}\,,$$

which, taking the limit for $\Delta x \to 0$ gives $u_{xx} = \frac{\rho}{T} u_{tt}$. This is the one-dimensional wave equation with $c^2 = \frac{\rho}{T}$.

3.3. Fourier series solution of the one-dimensional wave equation

We want to find the solution of the one-dimensional wave equation $u_{tt} = c^2 u_{xx}$ found in § 3.2, subject to some "reasonable" boundary conditions and initial conditions (where "reasonable" means for example dictated by the observation). We know the boundary condition $u(0,t) = u(L,t) = 0$ for all $t \geq 0$ and it is conceivable that u will be determined by its initial position $u(x,0) = f(x)$ and velocity

$u'(x,0) = g(x)$, for all $0 \leq x \leq L$. Hence we want to solve the system

$$\begin{cases} u_{tt} = c^2 u_{xx} & \\ u(0,t) = u(L,t) = 0 & t \geq 0 \\ u(x,0) = f(x) & 0 \leq x \leq L \\ u_t(x,0) = g(x) & 0 \leq x \leq L\,. \end{cases} \tag{3.5}$$

The first method we want to study consists in three steps:

1. Separation of variables;
2. Determination of "many" intermediary solutions;
3. Use of Fourier series "to put together" the solutions.

3.3.1. Separation of variables. Suppose there exists a solution of the form

$$\boxed{u(x,t) = F(x)G(t)}\,.$$

Then $u_{tt} = F\ddot{G}$ and $u_{xx} = F''G$, so that

$$F\ddot{G} = c^2 F''G\,,$$

from which it follows that

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F}\,.$$

Since the right hand side of the equation does not depend on t, it follows that even the left hand side is independent of t, and hence constant. Thus the right hand side is a constant as well

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k\,,$$

which is equivalent to the system of two equations

$$\begin{cases} F'' = kF \\ \ddot{G} = c^2 kG\,. \end{cases} \tag{3.6}$$

Thus, assuming that there is a solution of the PDE in (3.5) of the form $u(x,t) = F(x)G(t)$, the two functions F and G must satisfy (3.6). The type of solution will depend on the sign of the constant k. We start by solving the ordinary differential equation that has homogeneous boundary conditions. In other words the boundary condition for such a solution $u(x,t) = F(x)G(t)$ get transformed into become

$$u(0,t) = F(0)G(t) = 0 \text{ for every } t \geq 0 \Rightarrow F(0) = 0$$
$$u(L,t) = F(L)G(t) = 0 \text{ for every } t \geq 0 \Rightarrow F(L) = 0\,.$$

3.3.2. Determination of "many" solutions. Hence we start looking for a function $F(x)$ that is a solution of the system

$$F'' = kF\,, \qquad \text{with } F(0) = F(L) = 0\,. \tag{3.7}$$

$k = 0$ Then the ODE becomes $F'' = 0$, which has solution $F(x) = ax + b$. If we impose the conditions $F(0) = F(L) = 0$ we see that $F(x)$ must be identically equal to zero and this is not an interesting solution.

$k > 0$ Then the theory of the solutions of linear second order ODEs with constant coefficients tells us that the general solution is $F(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}$. Imposing the initial conditions we deduce from $F(0) = 0$ that $A + B = 0$ and from $F(L) = 0$ that $Ae^{\sqrt{k}L} + Be^{-\sqrt{k}L} = 0$. Replacing $B = -A$ in the second equation we obtain that $A(e^{\sqrt{k}L} - e^{-\sqrt{k}L}) = 0$. Then either $A = B = 0$ and this is again the zero solution or $e^{2\sqrt{k}L} = 1$, that is impossible since $k \neq 0$.

$k < 0$ In this case we know that

$$F(x) = A\cos(\sqrt{-k}x) + B\sin(\sqrt{-k}x)\,.$$

Again imposing the initial conditions we obtain that $0 = F(0) = A$ and $0 = F(L) = B\sin(\sqrt{-k}L)$. If $B = 0$ then F is again the solution that is identically equal to zero. So it must be $\sin(\sqrt{-k}L) = 0$, that is $\sqrt{-k}L = n\pi$ for any $n \in \mathbb{N}$ or $\sqrt{-k} = \frac{n\pi}{L}$ for any $n \in \mathbb{N}$. Hence for every $n \in \mathbb{N}$ there is a solution

$$F_n(x) = \sin\left(\frac{n\pi}{L}x\right)$$

of (3.7), where $\sqrt{-k} = \frac{n\pi}{L}$.

For these values of k we look for a solution of $\ddot{G} = c^2kG$, that is we look for a solution of $\ddot{G} = -c^2\left(\frac{n\pi}{L}\right)^2 G$. We obtain

$$\begin{aligned} G_n(t) &= B_n\cos\left(\frac{cn\pi}{L}t\right) + B_n^*\sin\left(\frac{cn\pi}{L}t\right) \\ &= B_n\cos(\lambda_n t) + B_n^*\sin(\lambda_n t)\,, \end{aligned}$$

where $\lambda_n = \frac{cn\pi}{L}$.

We have thus found a family of solutions indexed by $n \in \mathbb{N}$, namely

$$\boxed{u_n(x,t) = \big(B_n\cos(\lambda_n t) + B_n^*\sin(\lambda_n t)\big)\sin\left(\frac{n\pi}{L}x\right)\ .} \tag{3.8}$$

The functions u_n are the *eigenfunctions* of the vibrating string with eigenvalues $\lambda_n = \frac{cn\pi}{L}$. The set $\{\lambda_1, \lambda_2, \dots,\}$ is called the *spectrum*.

3.3.3. Use of Fourier series. Each of the eigenfunctions u_n satisfies the PDE in (3.5) and the boundary conditions,

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \quad t \geq 0\,. \end{cases} \tag{3.9}$$

However, most of the times, each one of them will not satisfy the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Since the PDE is homogeneous, we can use the superposition principle. Without worrying too much about convergence issues, we can say that the function

$$\boxed{u(x,t) := \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \big(B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t)\big) \sin\left(\frac{n\pi}{L}x\right)}$$

is also a solution of (3.9) and now we have many coefficients that we can determine so that $u(x,t)$ satisfies also the initial conditions. With this solution the first initial condition about u becomes

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)\,. \tag{3.10}$$

where the series on the right hand side is the sine series of an odd $2L$-periodic function that coincides with f on $[0,L]$. So if we extend f to be odd and $2L$-periodic, we have

$$\boxed{B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right)\, dx},$$

for $n \geq 1$, $n \in \mathbb{N}$. Moreover the initial condition on the derivative u' becomes

$$\begin{aligned} g(x) &= u_t(x,0) \\ &= \sum_{n=1}^{\infty} \big(-\lambda_n B_n \sin(\lambda_n t) + \lambda_n B_n^* \cos(\lambda_n T)\big) \sin\left(\frac{n\pi}{L}x\right)\Big|_{t=0} \\ &= \sum_{n=1}^{\infty} \lambda_n B_n^* \sin\left(\frac{n\pi}{L}x\right)\,. \end{aligned}$$

Just like in the case of (3.10), this is nothing but the Fourier series of the function g, extended to be odd and periodic of period $2L$, so that the formula for the coefficients gives us

$$\lambda_n B_n^* = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right)\, dx\,,$$

or

$$\boxed{B_n^* = \frac{2}{L\lambda_n}\int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right)\, dx}.$$

REMARK 3.8. Let us assume for simplicity that $g = 0$. Recalling that $\lambda_n = \frac{cn\pi}{L}$, the solution $u(x,t)$ becomes

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} B_n \cos(\lambda_n t)\sin\left(\frac{n\pi}{L}x\right) \\
&= \sum_{n=1}^{\infty} B_n \frac{1}{2}\left[\sin\left(\frac{n\pi}{L}x - \lambda_n t\right) + \sin\left(\frac{n\pi}{L}x + \lambda_n t\right)\right] \\
&= \frac{1}{2}\sum_{n=1}^{\infty} B_n \left[\sin\frac{n\pi}{L}(x-ct) + \sin\frac{n\pi}{L}(x+ct)\right] \qquad (3.11)\\
&= \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}(x-ct) + \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}(x+ct) \\
&= \frac{1}{2}[f^*(x-ct) + f^*(x+ct)]\,,
\end{aligned}
$$

where f^* is the odd extension of f to a $2L$-periodic function.

If

(1) f is twice differentiable on $0 < x < L$ and
(2) f has one-sided zero second derivative at the endpoints,

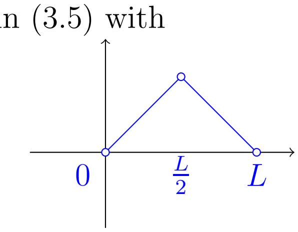
then u is a solution for all $0 \le x \le L$.

If f is only piecewise twice differentiable or the one-sided derivatives are not zero, then u is a solution for all $0 \le x \le L$ except at the points x where f is not twice differentiable.

EXAMPLE 3.9. We want to solve the PDE in (3.5) with

$$
f(x) = \begin{cases} \frac{2k}{L}x & 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \frac{L}{2} < x < L\,. \end{cases}
$$

$$
g(x) = 0
$$



From the last equation (3.11) and from (2.4) in Example 2.12 we obtain immediately that

$$
u(x,t) = \frac{4k}{\pi^2}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^2}\left[\sin\left(\frac{(2j+1)\pi}{L}(x-ct)\right) + \sin\left(\frac{(2j+1)\pi}{L}(x+ct)\right)\right].
$$

(The following picture is taken from the book by Kreyszig.)

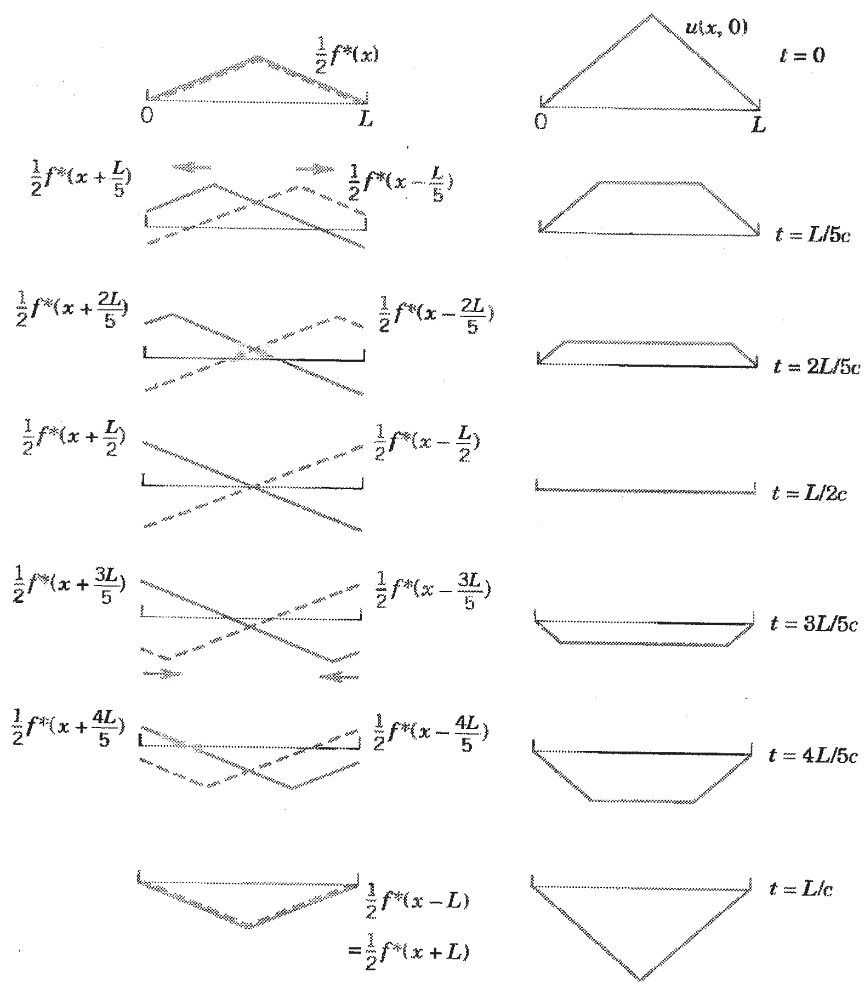


Fig. 288. Solution $u(x, t)$ in Example 1 for various values of t (right part of the figure) obtained as the superposition of a wave traveling to the right (dashed) and a wave traveling to the left (left part of the figure)

3.4. D'Alembert solution of the wave equation and characteristics

Let $u_{tt} = c^2 u_{xx}$ be the wave equation for $-\infty < x < \infty$ and $t > 0$, and let us make the change of variable $v = x + ct$ and $w = x - ct$. Then the solution $u(x,t)$ becomes $u(v,w)$ and

$$\begin{aligned} u_t &= u_v v_t + u_w w_t = u_v c - u_w c\,, \\ u_x &= u_v v_x + u_w w_x = u_v + u_w\,, \end{aligned}$$

so that

$$\begin{aligned} u_{tt} &= u_{vv} v_t c + u_{vw} w_t c - u_{wv} v_t c - u_{ww} w_t c \\ &= u_{vv} c^2 - u_{vw} c^2 - u_{wv} c^2 + u_{ww} c^2 \\ &= u_{vv} c^2 - 2 u_{vw} c^2 + u_{ww} c^2\,, \\ u_{xx} &= u_{vv} + 2u_{vw} + u_{ww}\,. \end{aligned}$$

Plugging in these results into the wave equation $u_{tt} = c^2 u_{xx}$, we obtain

$$c^2(u_{vv} - 2u_{vw} + u_{ww}) = c^2(u_{vv} + 2u_{vw} + u_{ww})\,,$$

that is in these new coordinates the wave equation becomes

$$\boxed{u_{vw} = 0}\,. \tag{3.12}$$

This is now very easy to integrate because from $u_{vw} = 0$ it follows that

$$u_v(v,w) = \int u_{vw}(v,w)\,dw + h(v) = h(v)\,;$$

Hence

$$u(v,w) = \int u_v(v,w)\,dv + \psi(w) = \int h(v)\,dv + \psi(w) = \varphi(v) + \psi(w)\,,$$

for any φ and ψ. Hence

$$\boxed{u(x,t) = \varphi(x+ct) + \psi(x-ct)}$$

is the general solution of the wave equation given in the form of the *D'Alembert solution.*

The point of having reduced the wave equation $u_{tt} = c^2 u_{xx}$ to the form (3.12) was that the new expression preserved the original feature of the differential equation and highlighted some of its features. The expression in (3.12) is called the *normal form* of the equation $u_{tt} = c^2 u_{xx}$. In fact, with an appropriate change of coordinates a 2nd order PDE can be brought into the *normal form*, that is

$$\begin{aligned} u_{vw} &= F^*(v,w,u,u_v,u_w) \text{ if it is } \textit{hyperbolic} \\ u_{vv} &= F^*(v,w,u,u_v,u_w) \text{ if it is } \textit{parabolic} \\ u_{vv} + u_{ww} &= F^*(v,w,u,u_v,u_w) \text{ if it is } \textit{elliptic}\,. \end{aligned}$$

We will look at the case of a hyperbolic PDE to see how to find the appropriate change of coordinates. Take an equation

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0\,,$$

where A, B and C are constants. We look for a solution $u = \varphi(x + \lambda y)$. After taking derivatives one obtains that

$$C\lambda^2 + 2B\lambda + A = 0$$

and, if the equation is hyperbolic, this has two distinct solutions $\lambda_1, \lambda_2 \in \mathbb{R}$. It is easy to see that

$$v = x + \lambda_1 y \qquad w = x + \lambda_2 y$$

is the required change of coordinates that leads to the normal form $u_{vw} = 0$.

Consider now the lines

$$x + \lambda_1 y = c_1 \quad \text{and} \quad x + \lambda_2 y = c_2\,, \tag{3.13}$$

where $c_1, c_2 \in \mathbb{R}$ are constants. It is easy to verify that the slope y' of these two lines satisfies the equation

$$A(y')^2 - 2By' + C = 0\,. \tag{3.14}$$

This is the *characteristic equation* and the lines in (3.13) are the *characteristics* of the equation (3.12).

This method can be applied in general. In other words, as mentioned in § 3.1, the equation

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} = F(u, u_x, u_y, x, y) \tag{3.15}$$

is called *hyperbolic*, *parabolic* or *elliptic* on a region R if $B^2(x,y) - A(x,y)C(x,y)$ is greater than, equal to or less than zero on all points in R. Furthermore, generalizing the property expressed by (3.14), the curves

$$\xi(x,y) = c_1 \quad \text{and} \quad \zeta(x,y) = c_2 \tag{3.16}$$

which are the solutions of the differential equation

$$A(x,y)(y')^2 - 2B(x,y)y' + C(x,y) = 0 \tag{3.17}$$

are said to be the *characteristics* of the partial differential equation in (3.15).

The following theorem tells us what is the change of variable needed to reduce a PDE to its normal form

THEOREM 3.10. *Consider the PDE in* (3.15) *and let*

$$\xi(x,y) = c_1 \quad \textit{and} \quad \zeta(x,y) = c_2$$

be independent solutions of the equation (3.17). *Then:*

(1) *If the given equation is hyperbolic, the change of variables defined by the substitution*

$$v = \xi(x, y) \quad \text{and} \quad w = \zeta(x, y)$$

will reduce it to the normal form $u_{vw} = F^*(u, u_v, u_w, v, w)$.

(2) *If the given equation is parabolic, the change of variables defined by the substitution*

$$v = x \quad \text{and} \quad w = \zeta(x, y)$$

will reduce it to the normal form $u_{vv} = F^*(u, u_v, u_w, v, w)$.

(3) *If the given equation is elliptic, the change of variables defined by the substitution*

$$v = \frac{\xi(x, y) + \zeta(x, y)}{2} \quad \text{and} \quad w = \frac{\xi(x, y) - \zeta(x, y)}{2}$$

will reduce it to the normal form $u_{vv} + u_{ww} = F^*(u, u_v, u_w, v, w)$.

Remark 3.11. If the coefficients are functions of x, y, then the change of coordinates will not be linear.

Example 3.12. We want to transform the differential equation $xu_{xx} - yu_{xy} + u_x = 0$ into normal form. Here $A(x, y) = x$, $2B(x, y) = -y$ and $C(x, y) = 0$, so that $B^(x, y) - A(x, y)C(x, y) = (-y/2)^2 > 0$ and hence the PDE if hyperbolic. The characteristic equation is

$$x(y')^2 + yy' = 0\,,$$

from which it follows that $y'(xy' + y) = 0$ and hence

$$\text{either } y' = 0 \text{ or } xy' + y = 0\,.$$

From $y' = 0$ we obtain $y = c_1$ and from $xy' + y = 0$ we obtain that

$$\begin{aligned} x\frac{dy}{dx} + y = 0 &\Leftrightarrow \frac{dy}{y} + \frac{dx}{x} = 0 \\ &\Leftrightarrow \ln|y| + \ln|x| = c_2' \\ &\Leftrightarrow yx = c_2\,. \end{aligned}$$

Hence if we perform the change of coordinates

$$v = y \qquad w = yx$$

we obtain

$$\begin{aligned} u_x &= u_v v_x + u_w w_x = y u_w \\ u_{xx} &= y^2 u_{ww} \\ u_{xy} &= u_w + y(u_{vw} v_y + u_{ww} w_y) = u_w + y(u_{vw} + x u_{ww})\,. \end{aligned}$$

Plugging this into the original equation gives, after simplifying, $u_{vw} = 0$. Hence $u(v, w) = \varphi(v) + \psi(w)$ or

$$u(x, y) = \varphi(y) + \psi(xy)\,,$$

where φ and ψ are arbitrary. $\square$

Why are the characteristics important? Let us go back to the d'Alembert solution of the wave equation,

$$u(x,t) = \varphi(x+ct) + \psi(x-ct) \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0\,, \tag{3.18}$$

and let us assume that we want to satisfy the initial conditions

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad \text{for all } x \in \mathbb{R} \tag{3.19}$$

The initial value problem consisiting of (3.18) and (3.19) is called *Cauchy problem* for the one dimensional wave equation. A solution of this problem can be interpreted for example as the amplitude of a sound wave propagating in a very long and narrow pipe.

To find the solution of (3.18) and (3.19), observe that for every $y \in \mathbb{R}$

$$\begin{cases} f(y) = & \varphi(y) + \psi(y) \\ g(y) = & c\varphi'(y) - c\psi'(y)\,, \end{cases}$$

from which it follows that

$$\begin{cases} \varphi(y) + \psi(y) & = f(y) \\ \varphi(y) - \psi(y) & = \frac{1}{c}\int_0^y g(s)\,ds + \overbrace{\varphi(0) - \psi(0)}^{=:k_0}\,, \end{cases}$$

$$\Rightarrow \begin{cases} \varphi(y) = \frac{1}{2}f(y) + \frac{1}{2c}\int_0^y g(s)\,ds + \frac{1}{2}k_0 \\ \psi(y) = \frac{1}{2}f(y) - \frac{1}{2c}\int_0^y g(s)\,ds - \frac{1}{2}k_0\,. \end{cases}$$

In particular evaluating the first expression for $y = x + ct$ and the second for $y = x - ct$, we have

$$\begin{aligned} \varphi(x+ct) &= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds + \frac{1}{2}k_0 \\ \psi(x-ct) &= \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)\,ds - \frac{1}{2}k_0 \end{aligned} \tag{3.20}$$

and hence

$$\boxed{u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds\,,}$$

which is the d'Alembert solution of the Cauchy problem (3.18) and (3.19).

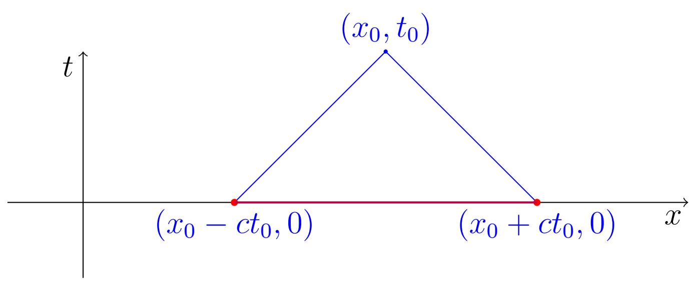
We are interested in investigating what kind of information has an influence on the solution u at the point (x, t). Through any point (x_0, t_0) with $t_0 > 0$, there are exactly two characteristics, namely

$$x - ct = x_0 - ct_0 \quad \text{and} \quad x + ct = x_0 + ct_0 \,.$$

These are straight lines whose intersections with the x-axis are respectively the points $(x_0 - ct_0, 0)$ and $(x_0 + ct_0)$. The triangle with vertices $(x_0 - ct_0, 0)$, $(x_0 + ct_0)$ and (x_0, t_0) is called *characteristic triangle*. Since

$$u(x_0, t_0) = \frac{1}{2}[f(x_0 + ct_0) + f(x_0 - ct_0)] + \frac{1}{2c}\int_{x_0 - ct_0}^{x_0 + ct_0} g(s)\, ds$$

the value of $u(x_0, t_0)$ is determined by the values of f at the vertices of the triangle and of g along the base.



The interval $[x_0 - ct_0, x_0 + ct_0]$ is called the *domain of dependence* of u at (x_0, t_0). Changing f or g outside this domain of dependence will not affect the value $u(x_0, t_0)$.

Now we ask the opposite question, namely what region of the (x, t) in the upper half plane is affected by the initial data on an interval $[a, b]$. The endpoints of the interval define four characteristics

$$x \pm ct = a \quad \text{and} \quad x \pm ct = b \,,$$

whose intersections define six regions indicated in the picture with I, II, III, IV, V and VI. The points that are affected by the initial conditions are exactly the points (x, t) whose domain of dependence $[x - ct, x + ct]$ intersects the interval $[a, b]$ in a non-trivial way. In particular, as shown in the picture, any point in the regions I and III are such that $u(x, t) = 0$.

If a point (x_3, t_3) is in the region IV, then $u(x_3, t_3) = \varphi(x_3 + ct_3) + \psi(x_3 + ct_3)$, where

$$\begin{aligned}
\varphi(x_3 + ct_3) &= \frac{1}{2}f(x_3 + ct_3) + \frac{1}{2c}\int_0^{x_3 + ct_3} g(s)\, ds + \frac{1}{2}k_0 \\
&= \frac{1}{2}f(x_3 + ct_3) + \frac{1}{2c}\int_a^{x_3 + ct_3} g(s)\, ds + \frac{1}{2}k_0 \\
\psi(x_3 - ct_3) &= \frac{1}{2}f(x_3 - ct_3) - \frac{1}{2c}\int_0^{x_3 - ct_3} g(s)\, ds - \frac{1}{2}k_0 = -\frac{1}{2}k_0 \,.
\end{aligned}$$

So, up to the constant k_0, any point in the region IV is influenced only by the characteristics with negative slopes, on in other words only by the left moving wave, and

$$u(x_3,t_3) = \frac{1}{2}f(x_3+ct_3) + \frac{1}{2c}\int_a^{x_3+ct_3} g(s)\,ds\,.$$

Likewise, any point in the region VI is only influenced by the characteristics with positive slope, that is only by the right moving wave. Notice that on the parallel lines $x+ct =$ constant the function $\varphi(x+ct)$ is constant, and on the parallel lines $x-ct =$ constant it is the function $\psi(x-ct)$ to be constant.

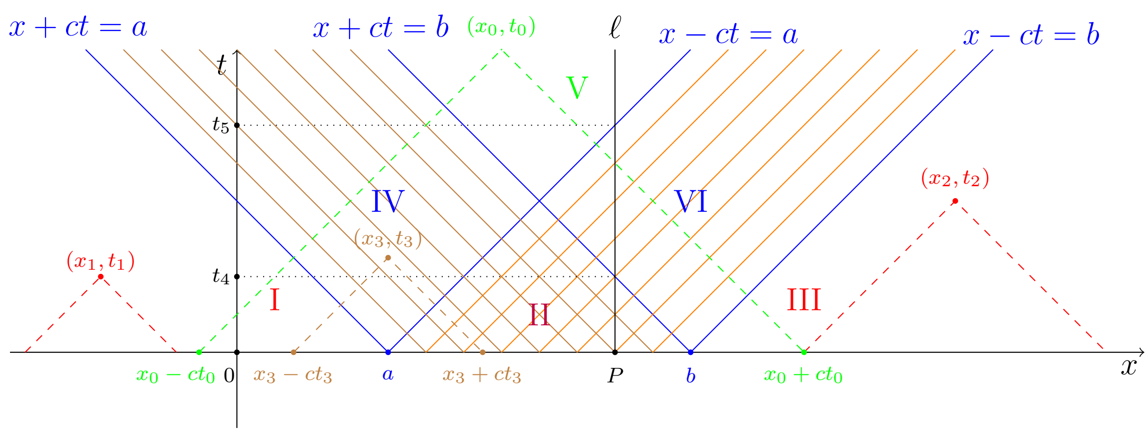
A point in the region VI is on the other hand influenced by both the right moving and the left moving wave. Finally, if (x_0,t_0) is a point in the region V, then the whole interval $[a,b]$ is contained in the domain of dependence. We have

$$f(x_0-ct_0) = f(x_0+ct_0) = 0 \quad \text{and} \quad \int_{x_0-ct_0}^{x_0+ct_0} g(s)\,ds = \int_a^b g(s)\,ds\,,$$

so that

$$u(x_0,t_0) = \frac{1}{2c}\int_a^b g(s)\,ds$$

is hence constant throughout the region.



In general draw a vertical line through the point and see how the point is affected as t grows. We say that the *region of influence* of the interval $[x_1,x_2]$ is the union of the regions II, IV, V and VI.

The behavior of a particular point on the string can be determined qualitatively by moving upward the vertical line that goes through the point. For example, as we move upward along the line ℓ, we see that the point P on the string is affedte by both travelling waves until time t_4, then only by the right travelling wave for $t_4 \leq t \leq t_5$. For values of $t > t_5$ the point is at rest in a position $u(x,t) = \frac{1}{2c}\int_a^b g(s)\,ds$.

REMARK 3.13. It follows from the above considerations that if at some point there is a singularity, it will propagate for all t. This is typical of hyperbolic equations.

3.5. The Heat equation via Fourier series

Assume that we have an insulated bar of length L, positioned along the interval $[0, L]$ of the x-axis, so that the heat flows only in the x-direction. We assume that the temperature at the endpoints of the bar is zero and, given an initial distribution of temperature $f(x)$, we want to find the heat distribution of the bar at time t and position x. In other words, we want now to solve the one-dimensional *heat equation*, that is

$$\begin{cases} u_t = c^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x)\,, \end{cases}$$

where $c^2 = \frac{K}{\sigma\rho}$, with K the thermal conductivity, σ the specific heat and ρ the density of the bar. Here c is the thermal diffusivity that measures the ability of a material to conduct thermal energy relative to its ability to store thermal energy, (this is approximately analogous to whether a material is "cold to the touch").

As in the case of the wave equation, we look for a solution of the form

$$u(x,t) = F(x)G(t)\,.$$

Differentiating this with respect to t once and to x twice, and plugging it into the heat equation, we obtain

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F} = k\,,$$

where k is a constant due to the fact that each of the two fractions cannot depend on the only variable present. This leads to the two differential equations

$$F'' = kF\,, \quad \dot{G} = kc^2 G\,,$$

with initial conditions

$$F(0) = F(L) = 0\,.$$

Solving first

$$F'' = kF\,, \quad F(0) = F(L) = 0\,,$$

as in the case of the wave equation we can easily argue that if $k = 0$ the only possible solution is the one identically zero and that if $k > 0$ there are no solutions. So $k < 0$ and we can set $k = -p^2$, so that the system to solve becomes

$$F'' = -p^2 F\,, \quad F(0) = F(L) = 0\,. \tag{3.21}$$

The general solution is $F(x) = A\cos(px) + B\sin(px)$. Imposing the initial condition $F(0) = 0$ we obtain that $A = 0$; imposing the initial condition $F(L) = 0$ we obtain that $\sin(pL) = 0$, that is that $pL = n\pi$ for $n \in \mathbb{Z}$, $n \geq 1$. Hence $p = \frac{n\pi}{L}$. We have therefore found the infinite family of solutions of (3.21)

$$F_n(x) = \sin\left(\frac{n\pi}{L}x\right), \quad n \in \mathbb{N}.$$

The corresponding differential equation for G is

$$\dot{G} + c^2\left(\frac{n\pi}{L}\right)^2 G = 0$$

whose solution is

$$G_n(t) = B_n e^{-\lambda_n^2 t},$$

where $\lambda_n^2 = \left(\frac{cn\pi}{L}\right)^2$. It follows that the functions

$$u_n(x,t) = B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t} \quad n \in \mathbb{N},$$

with λ_n as above satisfies the conditions

$$\begin{cases} u_t = c^2 u_{xx} \\ u(0,t) = u(L,t) = 0\,. \end{cases}$$

Because of the Superposition Principle, the function

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}} \tag{3.22}$$

is also a solution. By imposing the initial condition $u(x,0) = f(x)$, we obtain

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right).$$

Hence the B_n are the Fourier coefficients of the extension of f to be an odd function periodic of period $2L$, that is

$$\boxed{B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx\,.} \tag{3.23}$$

Hence (3.22) is the solution with the coefficients B_n given by (3.23) We remark that each term in the Fourier series (3.22) goes to zero with time, and the larger the n is, the faster the decay is.

EXAMPLE 3.14. We consider a copper bar of length 80 cm that is laterally insulated. The ends are kept at 0°C and the initial temperature is $f(x)$. We want to find $u(x,t)$ and compute how long it will take for the maximum temperature to drop to 50°C in the two cases in which

(1) $f(x) = 100\sin\left(\frac{\pi x}{80}\right)$, and
(2) $f(x) = 100\sin\left(\frac{3\pi x}{80}\right)$.

We recall that the density ρ of copper is $8.92\,\mathrm{g/cm^3}$, the specific heat σ is $0.092\,\mathrm{cal/(g\times {}^\circ C)}$ and the thermal conductivity K is $0.95\,\mathrm{cal/(cm \times sec \times {}^\circ C)}$, so that

$$c^2 = \frac{K}{\sigma\rho} = \frac{0.95}{0.092 \times 8.92}\,\frac{\mathrm{cal}}{\mathrm{cm \times sec \times {}^\circ C}}\,\frac{\mathrm{g \times {}^\circ C}}{\mathrm{cal}}\,\frac{\mathrm{cm^3}}{\mathrm{g}} = 1.158\frac{\mathrm{cm^2}}{\mathrm{sec}}\,.$$

Solution. We have

(1a) $u(x,0) = 100\sin\left(\frac{\pi x}{80}\right) \Rightarrow B_n = 0$ for $n \geq 2$ and $B_1 = 100$;

(2a) $u(x,0) = 100\sin\left(\frac{3\pi x}{80}\right) \Rightarrow B_n = 0$ for $n = 1, 2$ and $n \geq 4$ and $B_3 = 100$,

hence

(1b) $\lambda_1^2 = c^2\frac{\pi^2}{L^2} = 1.158\frac{9.870}{80^2} \simeq 0.001785\,\mathrm{sec}^{-1}$;

(2b) $\lambda_3^2 = c^2\frac{9\pi^2}{L^2} = 9\lambda_1 \simeq 0.01607\,\mathrm{sec}^{-1}$,

from which it follows that

(1c) $u(x,t) = 100\sin\left(\frac{\pi}{L}x\right)e^{-0.001785t}$;

(2c) $u(x,t) = 100\sin\left(\frac{3\pi}{L}x\right)e^{-0.01607t}$.

Thus

(1d) $u_{\max}(x,t) = 100e^{-0.001785t}$;

(2d) $u_{\max}(x,t) = 100e^{-0.01607t}$,

and hence

(1e) $100e^{-0.001785t} = 50 \Rightarrow t = \ln 0.5/(-0.001785) \simeq 388\,\mathrm{sec}$;

(2e) $100e^{-0.01607t} = 50 \Rightarrow t = \ln 0.5/(-0.01607) \simeq 43\,\mathrm{sec}$.

3.5.1. Steady two-dimensional heat equation and Laplace equation. The heat equation $u_t = c^2\nabla^2 u$ reduces to $\nabla^2 u = 0$ if the solution u does not depend on time. This is the so-called *Laplace equation*. It governs not only the so-called *steady-state* solution of a heat equation (that is the solution after the system reaches its equilibrium and does not change anymore with time), but also many other phenomena in mechanics and electromagnetism, such as for instance the electrostatic potential. This is an example of an *elliptic* equation and it comes with different kinds of boundary conditions. Suppose for example that $\nabla^2 u = 0$ holds on a region R.

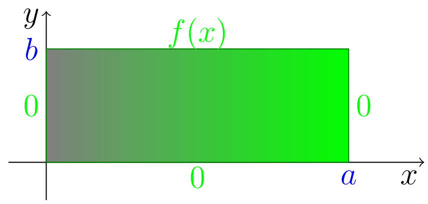
- If u is prescribed on the boundary ∂R of R, then we have the *Dirichlet problem*;
- If the partial derivative u_n of u in the direction n normal to ∂R is prescribed on ∂R, then we have the *Neumann problem*;
- If u is prescribed on one part of ∂R and u_n on another part of ∂R, we have the *Robin problem*.

Dirichlet problem on a rectangle. We assume that the temperature $u(x,y)$ on a rectangle R satisfies $\nabla^2 u = 0$ and if

$$R = \{(x,y) : 0 \leq x \leq a,\, 0 \leq y \leq b\}\,,$$

then

$$u(0,y) = u(a,y) = u(x,0) = 0 \quad u(x,b) = f(x)\,.$$



We look for a solution of the form $u(x,y) = F(x)G(y)$, so that the Laplace equation $\nabla^2 u = u_{xx} + u_{yy} = 0$ becomes

$$F''\,G + F\,G'' = 0\,,$$

and, as usual, there exists a constant k such that

$$\frac{F''}{F} = -\frac{G''}{G} = -k\,.$$

By considering also the boundary conditions we obtain the two problems

$$\begin{cases} F'' = -k\,F \\ F(0) = F(a) = 0 \end{cases} \qquad \text{and} \quad G'' = k\,G\,.$$

and we start by solving the problem with the (homogeneous) boundary conditions, that is the one with unknown F. As in the case of the wave and the heat equation we can see that if $k \leq 0$, the only possible solution is $F(x) \equiv 0$. If on the other hand $k > 0$, then we obtain

$$F(x) = A\cos(\sqrt{k}x) + B\sin(\sqrt{k}x)\,.$$

By setting $F(0) = 0$ we obtain that $A = 0$, and by setting $F(a) = 0$ we obtain that $\sin(\sqrt{k}a) = 0$, that is $\sqrt{k}a = n\pi$ for $n = 1, 2, \ldots$, or $\sqrt{k} = n\pi/a$. Hence we have the infinite family of solutions

$$F_n(x) = \sin\left(\frac{n\pi}{a}x\right)\,, \quad n \in \mathbb{N}\,.$$

The corresponding differential equation for G becomes

$$G'' - \left(\frac{n\pi}{a}\right)^2 G = 0\,,$$

from which we obtain

$$G_n(y) = A_n^* e^{\frac{n\pi}{a}y} + B_n^* e^{-\frac{n\pi}{a}y}\,.$$

By imposing the boundary condition $G_n(0) = 0$ we obtain that $A_n^* = -B_n^*$, that is

$$G_n(y) = A_n^*(e^{\frac{n\pi}{a}y} - e^{-\frac{n\pi}{a}y}) = 2A_n^* \sinh\left(\frac{n\pi}{a}y\right)\,.$$

Thus for every $n \in \mathbb{N}$ we have a solution

$$u_n(x,y) = A_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right) ,$$

and hence by the Superposition Principle

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right)$$

is a solution as well. To satisfy the last condition $u(x,b) = f(x)$, we write

$$f(x) = \sum_{n=1}^{\infty} \left[A_n \sinh\left(\frac{n\pi}{a}b\right)\right] \sin\left(\frac{n\pi}{a}x\right) ,$$

from which we deduce that $A_n \sinh\left(\frac{n\pi}{a}b\right)$ is the Fourier coefficient of the extension of f to be an odd periodic function of period $2a$. In other words

$$A_n \sinh\left(\frac{n\pi}{a}b\right) = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi}{a}x\right)\, dx .$$

Hence the solution of the above Dirichlet problem is

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right) ,}$$

where

$$\boxed{A_n = \frac{2}{a\sinh\left(\frac{n\pi}{a}b\right)}\int_0^a f(x) \sin\left(\frac{n\pi}{a}x\right)\, dx .}$$

3.6. Heat equation on an infinite bar

This method is appropriate when we have an infinite domain, for example a bar of infinite length. In this case the problem to solve is

$$\begin{cases} u_t = c^2 u_{xx} \\ u(x,0) = f(x) . \end{cases}$$

We are going to modify the method that used the Fourier series in order to use the Fourier integral. Then we will see also how one can use the Fourier transform. We look again for a solution of the form

$$u(x,t) = F(x)G(t) ,$$

and we obtain

$$F(x)\dot{G}(t) = c^2 F''(x)G(t) ,$$

from which, as usual,

$$\frac{F''(x)}{F(x)} = \frac{1}{c^2}\frac{\dot{G}(t)}{G(t)} = -k .$$

Hence the differential equations to solve are

$$\begin{cases} F''(x) + kF(x) = 0 \\ \dot{G}(t) + c^2 k G(t) = 0\,. \end{cases}$$

Even without boundary conditions we see that if $k < 0$, then

$$\begin{cases} F(x) = Ae^{\sqrt{-k}x} + Be^{-\sqrt{-k}x} \\ G(t) = e^{-c^2 kt}\,, \end{cases}$$

from which $u(x,t) = e^{-c^2kt}(Ae^{\sqrt{-k}x} + Be^{-\sqrt{-k}x})$ will increase as t increases, which is physically impossible. Thus $k \geq 0$, and we can write $k = p^2$. Then

$$\begin{cases} F_p(x) = A(p)\cos(px) + B(p)\sin(px) \\ G_p(t) = e^{-c^2p^2t}\,. \end{cases}$$

A generalisation of the Superposition Principle leads to the solution

$$u(x,t) = \int_0^\infty [A(p)\cos(px) + B(p)\sin(px)]e^{-c^2p^2t}\,dp\,. \tag{3.24}$$

Since $u(x,0) = f(x)$, then

$$f(x) = \int_0^\infty [A(p)\cos(px) + B(p)\sin(px)]\,dp\,,$$

which is nothing but the Fourier integral of f. Hence we can determine the coefficients $A(p)$ and $B(p)$, namely

$$\begin{aligned} A(p) &= \frac{1}{\pi}\int_{-\infty}^\infty f(v)\cos(pv)\,dv \\ B(p) &= \frac{1}{\pi}\int_{-\infty}^\infty f(v)\sin(pv)\,dv\,. \end{aligned}$$

Plugging these into (3.24) and using the prostapheresis formulas, we obtain

$$\begin{aligned} u(x,t) &= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(v)[\cos(pv)\cos(px) + \sin(pv)\sin(px)]e^{-c^2p^2t}\,dv\,dp \\ &= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(v)\cos(px - pv)e^{-c^2p^2t}\,dv\,dp \qquad (3.25) \\ &= \frac{1}{\pi}\int_{-\infty}^\infty f(v)\left(\int_0^\infty \cos(px - pv)e^{-c^2p^2t}\,dp\right)dv\,. \end{aligned}$$

We assume for the moment that we know that

$$\int_0^\infty \cos(2bs)e^{-s^2}\,ds = \frac{\sqrt{\pi}}{2}e^{-b^2}\,, \tag{3.26}$$

formula that we will discuss later. In order to use (3.26) we set

$$\begin{cases} s^2 = c^2p^2t \\ px - pv = 2bs \end{cases}$$

from which we obtain

$$p = \frac{s}{c\sqrt{t}} \quad \text{and} \quad b = \frac{x-v}{2c\sqrt{t}}\,.$$

Hence applying (3.26) with

$$dp = \frac{ds}{c\sqrt{t}}\,, \quad b = \frac{x-v}{2c\sqrt{t}} \quad \text{and} \quad s^2 = c^2p^2t\,,$$

we obtain

$$\int_0^\infty \cos(px - pv)e^{-c^2p^2t}\,dp = \frac{\sqrt{\pi}}{2c\sqrt{t}} \exp\left[-\left(\frac{x-v}{2c\sqrt{t}}\right)^2\right]\,.$$

Replacing this in (3.25) we obtain

$$\boxed{u(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^\infty f(v) \exp\left[-\left(\frac{x-v}{2c\sqrt{t}}\right)^2\right]\,dv\,.}$$

Before turning to the solution of the same problem using the Fourier transform we need to prove (3.26). This can be obtained using complex analysis methods or as follows: let us define

$$I(b) = \int_0^\infty e^{-s^2}\cos(2bs)\,ds\,.$$

Differentiating with respect to b and using the integration by parts we obtain

$$\begin{aligned} I'(b) &= \int_0^\infty -2se^{-s^2}\sin(2bs)\,ds \\ &= e^{-s^2}\sin(2bs)\Big|_0^\infty - 2b\int_0^\infty e^{-s^2}\cos(2bs)\,ds \\ &= -2bI(b)\,, \end{aligned}$$

so that

$$I(b) = C\,e^{-b^2}\,,$$

where $C = I(0) = \int_0^\infty e^{-s^2}\,ds = \frac{\sqrt{\pi}}{2}$.

Alternative method using the Fourier transform. We start with $u_t = c^2 u_{xx}$, and apply the Fourier transform on both sides $\mathcal{F}(u_t) = c^2\mathcal{F}(u_{xx})$. Note that $u = u(x,t)$ is a function of the two variables x and t and that we are taking the Fourier transform with resect to the variable x. Hence by property (2.8) of the Fourier transform applied twice,

$$\mathcal{F}(u_{xx}) = -w^2\mathcal{F}(u)$$

while

$$\mathcal{F}(u_t(x,t))(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_t(x,t)e^{-iwx}\,dx = \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t)e^{-iwx}\,dx = \frac{\partial \hat{u}}{\partial t}(w,t)\,.$$

Thus we have that

$$\frac{\partial \hat{u}}{\partial t}(w,t) = -c^2w^2\hat{u}(w,t)\,,$$

so that

$$\hat{u}(w,t) = C(w)e^{-c^2w^2t}\,.$$

Since $u(x,0) = f(x)$, then $\hat{u}(w,0) = \hat{f}(w)$, so that $\hat{f}(w) = C(w)$ and

$$\hat{u}(w,t) = \hat{f}(w)e^{-c^2w^2t}\,.$$

By applying the inverse Fourier transform to $\hat{u}$ and $\hat{f}$ we obtain

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)e^{-c^2w^2t}e^{iwx}\,dw \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-ivw}\,dv\right)e^{-c^2w^2t}e^{iwx}\,dw \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\left(\int_{-\infty}^{\infty} e^{-c^2w^2t}e^{i(wx-wv)}\,dw\right)dv \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\left(\int_{-\infty}^{\infty} e^{-c^2w^2t}[\cos(wx-wv)+i\sin(wx-wv)]\,dw\right)dv \qquad (3.27)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\left(\int_{-\infty}^{\infty} e^{-c^2w^2t}\cos(wx-wv)\,dw\right)dv \\
&\qquad + i\frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\left(\int_{-\infty}^{\infty} e^{-c^2w^2t}\sin(wx-wv)\,dw\right)dv \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left(\int_{0}^{\infty} e^{-c^2w^2t}\cos(wx-wv)\,dw\right)dv
\end{aligned}$$

where in the last equality we used that the function $\cos(wx-wv)$ is an even function of w and $\sin(wx-wv)$ is an odd function of w. But this is exactly (3.25), from which the solution $u(x,t)$ was obtained.

Alternative method to the alternative method. From the first equality in (3.27) we obtain

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{-c^2 w^2 t} e^{iwx}\, dw \\
&= \int_{-\infty}^{\infty} \hat{f}(w) \Big(\underbrace{\frac{1}{\sqrt{2\pi}} e^{-c^2 w^2 t}}_{=:\hat{g}^t(w)} \Big) e^{iwx}\, dw \\
&= \sqrt{2\pi} \mathcal{F}^{-1}(\hat{f}\hat{g}^t)(x) \\
&= (f * g^t)(x) \\
&= \int_{-\infty}^{\infty} f(p) g^t(x-p)\, dp\,,
\end{aligned}
\tag{3.28}
$$

where $g^t = \mathcal{F}^{-1}(\hat{g}^t)$. To compute this observe that

$$
\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-w^2/4a}\,,
$$

so that

$$
e^{-ax^2} = \mathcal{F}^{-1}\left(\frac{1}{\sqrt{2a}} e^{-w^2/4a} \right) .
$$

If we set $c^2 t = \frac{1}{4a}$ we obtain $a = \frac{1}{4c^2 t}$ and $\frac{1}{\sqrt{2a}} = \sqrt{2} c \sqrt{t}$, so that

$$
g^t(x) = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}(e^{-w^2 c^2 t}) = \frac{1}{\sqrt{2} c \sqrt{t}} e^{-x^2/4c^2 t} \frac{1}{\sqrt{2\pi}}\,,
$$

and hence from (3.28)

$$
u(x,t) = \int_{-\infty}^{\infty} f(p) g^t(x-p)\, dp = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(p) \exp\left(-\frac{(x-p)^2}{4c^2 t} \right) dp\,.
$$

3.7. Rectangular membrane: the wave equation

We want to study the motion of an elastic membrane that is stretched and fixed along its edge. An example of such situation is a drumhead. We will study two different shapes of membranes, but in both cases we make the following assumptions:

(1) the membrane is homogeneous (that is the mass per unit area is constant) and flexible, and
(2) the tension T per unit length caused by stretching is the same at all points and does not change with time.

Then the deflection $u(x,y,t)$ satisfies the equation $u_{tt} = c^2 \nabla^2 u$, where if we denote by ρ the mass of the undeflected membrane per unit area, we have $c^2 = T/\rho$.

We want to find the solution of the following problem:

$$\tag{3.29} \begin{cases} u_{tt} = c^2\nabla^2 u \text{ on } R = \{(x,y):\ 0 \le x \le a,\ 0 \le y \le b\} \\ u = 0 \text{ on } \partial R \\ u(x,y,0) = f(x,y) \\ u_t(x,y,0) = g(x,y)\,, \end{cases}$$

where $f(x,y)$ is the initial displacement and $g(x,y)$ is the initial velocity.

We look for a solution that satisfies $u(x,y,t) = F(x,y)G(t)$. Then the differential equation $u_{tt} = c^2\nabla^2 u$ becomes

$$F\ddot{G} = c^2(F_{xx} + F_{yy})G\,,$$

from which

$$\frac{\ddot{G}}{c^2 G} = \frac{1}{F}(F_{xx} + F_{yy}) = -\nu^2\,.$$

Moreover

$$F(x,0) = F(0,y) = F(x,b) = F(a,y) = 0\,.$$

Note that we took the constant to be negative because we cannot have that $\frac{1}{F}(F_{xx} + F_{yy}) = \nu^2 > 0$. In fact, suppose that $F_{xx} + F_{yy} = \mu F$, for some $\mu \in \mathbb{R}$. We will show that $\mu < 0$. In fact,

$$\mu \int_0^a \int_0^b F^2 = \int_0^a \int_0^b (F_{xx} + F_{yy})F\,.$$

But

$$\int_0^a F_{xx}F\,dx = F_x\,F\Big|_0^a - \int_0^a F_x^2\,dx = -\int_0^a F_x^2\,dx\,,$$

so that

$$\tag{3.30} \int_0^b \int_0^a F_{xx}F\,dx\,dy = -\int_0^b \int_0^a F_x^2\,dx\,dy\,.$$

Likewise,

$$\tag{3.31} \int_0^a \int_0^b F_{yy}F\,dy\,dx = -\int_0^a \int_0^b F_y^2\,dy\,dx\,.$$

Hence from (3.30) and (3.31) we obtain

$$\mu \int_0^a \int_0^b F^2\,dy\,dx = -\int_0^a \int_0^b (F_x^2 + F_y^2)\,dy\,dx \le 0\,,$$

that is $\mu \le 0$[1].

[1]Notice that this is true for every domain R. That is if $\nabla^2 u = \mu u$ on R and $u = 0$ on ∂R, then $\mu \le 0$.

Hence we need to solve the two differential equations

$$\begin{cases} \ddot{G} + \nu^2 c^2 G = 0 \\ F_{xx} + F_{yy} = -\nu^2 F \,, \end{cases}$$

where the second is called the *Hemholtz equation* for the amplitude function. To solve the equation for F we look for a solution of the form $F(x,y) = H(x)Q(y)$. Then $F_{xx} + F_{yy} = -\nu^2 F$ becomes

$$H_{xx}\,Q + H\,Q_{yy} = -\nu^2 H\,Q\,.$$

This is equivalent to

$$\frac{H_{xx}}{H} + \frac{Q_{yy}}{Q} = -\nu^2\,,$$

or

$$\frac{H_{xx}}{H} = -\frac{1}{Q}(Q_{yy} + \nu^2 Q) = -k^2\,,$$

where we took the constant to be negative because it is easy to see, with the usual reasoning, that a non-negative constant would lead to the solution identically equal to zero. Thus we need to solve the following equations

$$\begin{cases} H_{xx} + k^2 H = 0 \\ Q_{yy} + p^2 Q = 0\,, \end{cases}$$

where $p^2 = \nu^2 - k^2$. Since $F(x,y) = 0$ on ∂R, we infer that

$$H(0) = H(a) = Q(0) = Q(b) = 0\,.$$

Therefore the solutions are

$$\begin{aligned} H_m(x) &= \sin\left(\frac{m\pi x}{a}\right) \text{ for } m \in \mathbb{N} \setminus \{0\} \ \left(\text{and } k = \frac{m\pi}{a}\right) \\ Q_n(x) &= \sin\left(\frac{n\pi y}{b}\right) \text{ for } n \in \mathbb{N} \setminus \{0\} \ \left(\text{and } p = \frac{n\pi}{b}\right)\,. \end{aligned}$$

and

$$F_{mn}(x,y) = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

are the solutions of the Hemholtz equation for the amplitude. Now we have to solve the equation $\ddot{G} + \lambda^2 G = 0$, where $p^2 = \nu^2 - k^2$ and

$$\lambda = c\nu = c\sqrt{p^2 + k^2} = c\sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}} = \pi c\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} =: \lambda_{mn}\,.$$

The corresponding solutions are

$$G_{mn}(t) = B_{mn}\cos(\lambda_{mn} t) + B^*_{mn}\sin(\lambda_{mn} t)\,,$$

so that for every $m, n \in \mathbb{N}$ we have a solution

$$u_{mn}(x,y,t) = \left[B_{mn}\cos(\lambda_{mn} t) + B^*_{mn}\sin(\lambda_{mn} t)\right]\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\,.$$

As usual these are called the *eigenfunctions* corresponding to the *eigenvalue* λ_{mn}. The frequency of u_{mn} is $\lambda_{mn}/2\pi$.

Contrary to what happens in the one-dimensional wave equation, different functions F_{mn} might correspond to the same frequency. In other words there are solutions that have the same frequency but different *nodal lines* (that is curves where the membrane does not move).

EXAMPLE 3.15. We assume $a = b = 1$ in the above discussion, so that $\lambda_{mn} = \pi c\sqrt{m^2+n^2}$. Then $\lambda_{mn} = \lambda_{nm}$, but

$$F_{mn}(x,y) = \sin(m\pi x)\sin(n\pi y) \neq \sin(n\pi x)\sin(m\pi y) = F_{nm}(x,y)\,.$$

For example

$$\begin{aligned} u_{12}(x,y,t) &= \left[B_{12}\cos(\pi c\sqrt{5}t) + B_{12}^*\sin(\pi c\sqrt{5}t)\right]\sin(\pi x)\sin(2\pi y) \\ u_{21}(x,y,t) &= \left[B_{21}\cos(\pi c\sqrt{5}t) + B_{21}^*\sin(\pi c\sqrt{5}t)\right]\sin(2\pi x)\sin(\pi y)\,. \end{aligned}$$

Then

$$u_{12}(x,y,t) = 0 \text{ for every } t \Leftrightarrow \sin(\pi x)\sin(2\pi y) = 0, \text{ for } 0 < x,y < 1 \Leftrightarrow y = \frac{1}{2}\,.$$

On the other hand

$$u_{21}(x,y,t) = 0 \text{ for every } t \Leftrightarrow x = \frac{1}{2}\,. \quad \square$$

We continue with the solution of the system (3.29). By the Superposition Principle, we have a solution

$$\begin{aligned} u(x,y,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(x,y,t) \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[B_{mn}\cos(\lambda_{mn}t) + B_{mn}^*\sin(\lambda_{mn}t)\right]\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\,. \end{aligned}$$

Imposing the initial conditions, we obtain

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\,,$$

that is a double Fourier series of the function f. (This exists and converges if f is twice differentiable with continuity on $\mathbb{R}$.) We can write

$$f(x,y) = \sum_{m=1}^{\infty}\underbrace{\left(\sum_{n=1}^{\infty} B_{mn}\sin\left(\frac{n\pi}{b}y\right)\right)}_{=:K_m(y)}\sin\left(\frac{m\pi}{a}x\right)\,,$$

where

$$K_m(y) = \frac{2}{a}\int_0^a f(x,y)\sin\left(\frac{m\pi x}{a}\right)\,dx$$

are the Fourier coefficients of $f(x,y)$ thought of as a function of x extended to $[-a,a]$ as an odd function. But

$$K_m(y) = \sum_{n=1}^{\infty} B_{mn} \sin\left(\frac{n\pi}{b} y\right)$$

is the Fourier series of $K_m(y)$, so that

$$B_{mn} = \frac{2}{b}\int_0^b K_m(y)\sin\left(\frac{n\pi y}{b}\right)\,dy = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\,dx\,dy\,.$$

Likewise

$$g(x,y) = u_t(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B^*_{mn}\lambda_{mn} t \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\,,$$

so that

$$B^*_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\,dx\,dy\,.$$

for $m,n\in\mathbb{N}$.

3.8. Dirichlet problem on a region with symmetries

We are interested in studying the Laplace equation on a region that has a radial symmetry. The first thing to do is to write our equation in coordinates that are appropriate with respect to this symmetry. We will study the Dirichlet problem on a disk, so that the first step will be to transform the Laplace equation using polar coordinates.

Recall that the polar coordinates are defined as

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad\Leftrightarrow\quad \begin{cases} r = (x^2+y^2)^{1/2} \\ \theta = \arctan\frac{y}{x}\,. \end{cases}$$

Then

$$\begin{aligned} u_x &= u_r r_x + u_\theta \theta_x \\ u_y &= u_r r_y + u_\theta \theta_y\,, \end{aligned}$$

from which

$$\begin{aligned} u_{xx} &= (u_r r_x)_x + (u_\theta\theta_x)_x \\ &= u_{rr} r_x^2 + u_{r\theta}\theta_x r_x + u_r r_{xx} + u_{\theta r} r_x \theta_x + u_{\theta\theta}\theta_x^2 + u_\theta\theta_{xx} \\ &= u_{rr} r_x^2 + 2u_{r\theta} r_x\theta_x + u_{\theta\theta}\theta_x^2 + u_r r_{xx} + u_\theta\theta_{xx}\,. \end{aligned}$$

Since

$$r^2 = x^2+y^2 \Rightarrow 2rr_x = 2x \Rightarrow r_x = \frac{x}{r} \Rightarrow r_{xx} = \frac{r - xr_x}{r^2} = \frac{y^2}{r^3}\,.$$

Moreover

$$y = r\sin\theta \Rightarrow 0 = r_x \sin\theta + r(\cos\theta)\theta_x \Rightarrow \theta_x = -\frac{r_x}{r}\tan\theta = -\frac{r_x}{r}\frac{y}{x} = -\frac{y}{r^2}\,,$$

so that

$$\theta_{xx} = \frac{2yrr_x}{r^4} = \frac{2xy}{r^4}\,.$$

The partial derivatives r_y, r_{yy} and u_{yy} can be obtained from the one above simply by replacing x with y, that is

$$r_y = \frac{y}{r}\,, \quad r_{yy} = \frac{x^2}{r^3}$$

and

$$u_{yy} = u_{rr}r_y^2 + 2u_{r\theta}r_y\theta_y + u_{\theta\theta}\theta_y^2 + u_r r_{yy} + u_\theta\theta_{yy}\,,$$

while

$$\theta_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \frac{x}{r^2} \Rightarrow \theta_{yy} = -\frac{x2rr_y}{r^4} = -\frac{2xy}{r^4}\,.$$

Finally it follows that

$$\begin{aligned} u_{xx}+u_{yy} &= u_{rr}r_x^2 + 2u_{r\theta}r_x\theta_x + u_{\theta\theta}\theta_x^2 + u_r r_{xx} + u_\theta\theta_{xx} \\ &\quad + u_{rr}r_y^2 + 2u_{r\theta}r_y\theta_y + u_{\theta\theta}\theta_y^2 + u_r r_{yy} + u_\theta\theta_{yy} \\ &= u_{rr}(r_x^2+r_y^2) + 2u_{r\theta}(r_x\theta_x + r_y\theta_y) \\ &\quad + u_{\theta\theta}(\theta_x^2+\theta_y^2) + u_r(r_{xx}+r_{yy}) + u_\theta(\theta_{xx}+\theta_{yy}) \\ &= u_{rr}\left(\frac{y^2}{r^2}+\frac{x^2}{r^2}\right) + 2u_{r\theta}\left(\cancel{\frac{x}{r}\left(-\frac{y}{r^2}\right)+\frac{y}{r}\frac{x}{r^2}}\right) \\ &\quad + u_{\theta\theta}\left(\frac{y^2}{r^4}+\frac{x^2}{r^4}\right) + u_r\left(\frac{x^2}{r^3}+\frac{y^2}{r^3}\right) + u_\theta\left(\cancel{\frac{2xy}{r^4}-\frac{2xy}{r^4}}\right) \\ &= u_{rr} + u_{\theta\theta}\frac{1}{r^2} + u_r\frac{1}{r}\,, \end{aligned}$$

that is

$$u_{rr} + u_{\theta\theta}\frac{1}{r^2} + u_r\frac{1}{r} = 0$$

is the Laplace equation in polar coordinates.

Suppose now that we look for the solution of the problem

$$\begin{cases} \nabla^2 u = 0 & \text{on } \{(x,y):\, x^2+y^2 < R^2\} \\ u = f & \text{on } \{(x,y):\, x^2+y^2 = R^2\}\,. \end{cases}$$

Because of the symmetry of the problem and the above calculation we look for a solution of the problem

$$(3.32)\quad \begin{cases} u_{rr} + u_{\theta\theta}\frac{1}{r^2} + u_r\frac{1}{r} = 0 & \text{on } \{(r,\theta) :\, 0 \le r < R,\, 0 \le \theta \le 2\pi\} \\ u(R,\theta) = f(\theta) & \text{on } \{(R,\theta) :\, 0 \le \theta < 2\pi\}\,. \end{cases}$$

We are going to apply the method of separation of variables, so that we look for a solution of the form

$$u(r,\theta) = F(r)G(\theta)\,.$$

Such function will have to satisfy the equation

$$\begin{aligned} & F''\,G + \frac{1}{r}F'\,G + \frac{1}{r^2}F\,G'' = 0 \\ \Rightarrow & r^2F''\,G + rF'\,G + F\,G'' = 0 \\ \Rightarrow & (r^2F'' + rF')\,G = -F\,G'' \\ \Rightarrow & \frac{r^2F'' + rF'}{F} = -\frac{G''}{G} = k \\ \Rightarrow & \begin{cases} r^2F'' + rF' - kF = 0 \\ G'' + kG = 0 \end{cases} \end{aligned}$$

with the conditions

$$(3.33)\qquad G(0) = G(2\pi) \quad \text{and} \quad G'(0) = G'(2\pi)\,.$$

Thus we start with the solution of the differential equation in G and we study as usual the possible signs of k.

$k < 0$ The solution in this case is $G(\theta) = Ae^{\sqrt{-k}\theta} + Be^{-\sqrt{-k}\theta}$. Imposing the conditions (3.33) we obtain

$$\begin{cases} A + B = Ae^{\sqrt{-k}2\pi} + Be^{-\sqrt{-k}2\pi} \\ \sqrt{-k}A - \sqrt{-k}B = \sqrt{-k}Ae^{\sqrt{-k}2\pi} - \sqrt{-k}Be^{-\sqrt{-k}2\pi}\,, \end{cases}$$

that is

$$\begin{cases} A + B = Ae^{\sqrt{-k}2\pi} + Be^{-\sqrt{-k}2\pi} \\ A - B = Ae^{\sqrt{-k}2\pi} - Be^{-\sqrt{-k}2\pi}\,. \end{cases}$$

Adding the two equations one obtains that

$$2A = 2Ae^{\sqrt{-k}2\pi}\,,$$

so that $A = 0$. Replacing this into one of either equations one obtains that also $B = 0$.

$k = 0$ Then $G(\theta) = A\theta + B$. Since $B = G(0) = G(2\pi) = 2\pi A + B$, then $A = 0$. The condition on the derivative of G is of course verified, and hence $G(\theta)$ has to be constant.

$k>0$ In this case the solution is $G(\theta) = A\cos(\sqrt{k}\theta) + B\sin(\sqrt{k}\theta)$. Imposing the conditions (3.33) we obtain

$$\begin{cases} A = A\cos(2\pi\sqrt{k}) + B\sin(2\pi\sqrt{k}) \\ B = -A\sin(2\pi\sqrt{k}) + B\cos(2\pi\sqrt{k}) \, . \end{cases}$$

Multiplying the first equation by B, the second by A and comparing them, we obtain

$$B^2\sin(2\pi\sqrt{k}) = -A^2\sin(2\pi\sqrt{k}) \, .$$

Since $B^2 = -A^2$ is never possible for real A, B, then $\sin(2\pi\sqrt{k}) = 0$, that is $\sqrt{k} \in \mathbb{N}$. Then

$$G_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)$$

is a solution for every $n = 0, 1, 2, \ldots$. Notice that $G_0 = A_0$ is a constant, and hence we recover here the solution found for $k = 0$.

Now we need to find the corresponding solution of the differential equation for F, namely

$$r^2F'' + rF' - n^2F = 0 \, .$$

This is a *Euler equation*[2] of the second order and a solution will be of the form $F(r) = r^\alpha$, where $\alpha \in \mathbb{Q}$ is to be determined. Taking the derivatives and plugging them into the equation of F, we obtain

$$\begin{aligned} &r^2\alpha(\alpha-1)r^{\alpha-2} + r\alpha r^{\alpha-1} - n^2r^\alpha = 0 \\ \Rightarrow &\alpha(\alpha-1) + \alpha - n^2 = 0 \\ \Rightarrow &\alpha^2 - n^2 = 0 \\ \Rightarrow &\alpha = \pm n \, . \end{aligned}$$

Notice that the two solutions r^n and r^{-n} are lineary independent if $n \geq 1$, but coincide if $n = 0$. Without invoking the general theory, note that in this case the Euler equation becomes

$$r^2F'' + rF' = 0 \Rightarrow rF'' + F' = 0 \, ,$$

which has the two solutions $F(r) = \text{constant}$ and $F(r) = \ln r$. But this last solution is not bounded if $r = 0$ and $r = 0$ is a point in our disk, so we disregard this solution. In fact, in general we remark that we want our solution

$$F_n(r) = P_nr^n + Q_nr^{-n}$$

[2]The general form of the Euler equation is

$$y^{(n)} + \frac{b_{n-1}}{r^{n-1}}y^{(n-1)} + \cdots + \frac{b_1}{r}y' + b_0y = 0 \, ,$$

where $y = y(r)$.

to be bounded on the domain, so that we must require that $Q_n = 0$ and hence

$$F_n(r) = P_n r^n \, .$$

Notice that if we were to consider the same problem on a region "outside" the disk, we would have to impose instead that $P_n = 0$.

So, renaming the constants, we have found for every $n = 0, 1, 2, \dots$ the solutions

$$u_n(r,\theta) = r^n (A_n \cos(n\theta) + B_n \sin(n\theta))$$

and hence the solution

$$u(r,\theta) = \sum_{n=0}^{\infty} u_n(r,\theta) = \sum_{n=0}^{\infty} r^n (A_n \cos(n\theta) + B_n \sin(n\theta)) \, . \tag{3.34}$$

Imposing the condition $u(R,\theta) = f(\theta)$, we obtain

$$u(R,\theta) = \sum_{n=0}^{\infty} R^n (A_n \cos(n\theta) + B_n \sin(n\theta)) = f(\theta) \, .$$

Comparing this condition with the Fourier series expansion of f

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta)) \, ,$$

where

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{0i}^{2\pi} f(\varphi) \, d\varphi \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} f(\varphi) \cos(n\varphi) \, d\varphi \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(\varphi) \sin(n\varphi) \, d\varphi \end{aligned}$$

and comparing the coefficients[3] we obtain the following

$$\begin{aligned} A_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(\varphi) \, d\varphi \\ A_n &= \frac{1}{R^n \pi} \int_0^{2\pi} f(\varphi) \cos(n\varphi) \, d\varphi \\ B_n &= \frac{1}{R^n \pi} \int_0^{2\pi} f(\varphi) \sin(n\varphi) \, d\varphi \, . \end{aligned}$$

[3] Note that $\int_0^{2\pi} f(\varphi) \, d\varphi = \int_{-\pi}^{\pi} \tilde{f}(\varphi) \, d\varphi$, where $\tilde{f}$ is the 2π-periodic extension of f.

By replacing these coefficients into the equation (3.34), we obtain

$$
\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)\,d\varphi \\
&\quad + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \int_0^{2\pi} \big[\underbrace{\cos(n\theta)\cos(n\varphi)+\sin(n\theta)\sin(n\varphi)}_{\cos(n(\theta-\varphi))}\big] f(\varphi)\,d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)\,d\varphi + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \int_0^{2\pi} \cos(n(\theta-\varphi)) f(\varphi)\,d\varphi\,.
\end{aligned}
$$

Assuming we can exchange the series and the integral, we obtain

$$
u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\left[1+2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos(n(\theta-\varphi))\right] f(\varphi)\,d\varphi\,. \tag{3.35}
$$

To manipulate the series, observe that if $\alpha\in\mathbb{R}$ and for $|t|<1$,

$$
\sum_{n=1}^{\infty} t^n\cos(n\alpha) = \Re\left(\sum_{n=1}^{\infty} t^n e^{in\alpha}\right) = \Re\left(\frac{te^{i\alpha}}{1-te^{i\alpha}}\right) = \cdots = \frac{t\cos\alpha - t^2}{1-2t\cos\alpha + t^2}\,,
$$

so that

$$
1+2\sum_{n=1}^{\infty} t^n\cos(n\alpha) = \frac{1-t^2}{1-2t\cos\alpha+t^2}\,.
$$

We use this identity with $t=\frac{r}{R}$ and $\alpha=\theta-\varphi$ and substitute the result in (3.35), so that

$$
\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi}\left[1+2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\cos(n(\theta-\varphi))\right] f(\varphi)\,d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{1-\left(\frac{r}{R}\right)^2}{1-2\frac{r}{R}\cos(\theta-\varphi)+\left(\frac{r}{R}\right)^2} f(\varphi)\,d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2rR\cos(\theta-\varphi)+r^2} f(\varphi)\,d\varphi\,.
\end{aligned}
$$

The function

$$
\boxed{K(r,\theta,R,\varphi) := \frac{R^2-r^2}{R^2-2rR\cos(\theta-\varphi)+r^2}}
$$

is called *Poisson integral kernel* and the solution

$$
\boxed{u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} K(r,\theta,R,\varphi) f(\varphi)\,d\varphi}
$$

is given in *Poisson integral form.*

3.9. Mean value property and the maximum principle

Functions that satisfy the Laplace equation $\nabla^2 u = 0$ are called *harmonic* and enjoy very important properties. Suppose for example that u is a function that is harmonic on a region $\mathcal{D}$, that is $\nabla^2 u = 0$ on $\mathcal{D}$: let $(x_0, y_0) \in \mathcal{D}$ be any point and let $a \in \mathbb{R}$ be such that a disk of center (x_0, y_0) and radius a is contained in $\mathcal{D}$. Thus u is tautologically a solution of the Dirichlet problem

$$\begin{cases} \nabla^2 u = 0 & \text{on } \{(r,\theta) : 0 \le r < a\,, 0 \le \theta \le 2\pi\} \\ u = u(a,\theta) & \text{on } \{(a,\theta) : 0 \le \theta \le 2\pi\}\,, \end{cases}$$

and hence can be written in Poisson integral form

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} K(r,\theta,a,\varphi)u(a,\varphi)\,d\varphi\,,$$

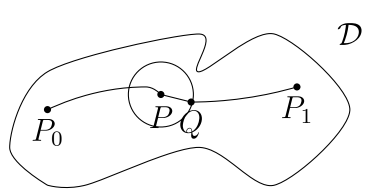
where we are considering here polar coordinates centred at (x_0, y_0). We want to compute the value $u(x_0, y_0)$. To this purpose observe that to the point (x_0, y_0) corresponds $r = 0$ and hence $u(x_0, y_0) = u(0, \theta)$ (the angle θ is undefined, but it will not be a problem). Observe moreover that $K(0, \theta, a, \varphi) = 1$. Thus

$$\begin{aligned} u(x_0,y_0) = u(0,\theta) &= \int_0^{2\pi} K(0,\theta,a,\varphi)u(a,\varphi)\,d\varphi = \frac{1}{2\pi}\int_0^{2\pi} u(a,\varphi)\,d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + a\cos\varphi, y_0 + a\sin\varphi)\,d\varphi\,. \end{aligned}$$

In other words a harmonic function is equal at any point to the average of the values on *any* circle centred at that point. This has in particular the following consequence

Maximum Principle. *If a harmonic function u takes its maximum values on the interior of the region $\mathcal{D}$ where it is harmonic, then it must be constant.*

In fact, let us assume that P_0 is the point in $\mathcal{D}$ where u attains its maximum, $u(P_0) = M$. If u were not to be constant, there would be another point $P_1 \in \mathcal{D}$ where $u(P_1) < M$. Take a path from P_0 to P_1 and let P be the point on the path closest to P_1 where $u(P) = M$. Consider now a circle C with center P all contained in $\mathcal{D}$ and not containing P_1, and let Q be the intersection of the circle C with the path closest to P_1. Then $u|_C \le M$ and in particular $u(Q) < M$, so that $u(P) = M$ cannot be the average of the values of u on the circle C.



3.10. Well-posed and ill-posed problems

Solutions of ordinary differential equations behave well, in the sense that, under certain conditions, one can prove the existence and the uniqueness of the solutions of an initial value problem, as well as the smooth dependence on the initial parameters.

The same statements unfortunately do not hold for partial differential equations, although the same questions arise. Not only we would like to know that a solution of a given problem exist and is unique, but often the boundary conditions and the initial conditions are obtained through measurements. So we would like to know that the solution is not affected too much by the likely imprecision with which boundary conditions and initial conditions were measured.

We say that a problem is *well-posed* if it satifies the following three conditions:

(1) *Existence*: the problem has a solution;
(2) *Uniqueness*: the solution is unique;
(3) *Stability*: the solution depends continuously on the initial conditions and on the boundary conditions.

The problem is *ill-posed* if one of the above properties does not hold. For partial differential equation there is no general theorem like in the case of ordinary differential equations.

Example 3.16. We look for a solution of the differential equation

$$u_x = c_0 u + c_1 ,$$

where $u = u(x, y)$, c_0 is a constant and $c_1 = c_1(x, y)$. We are going to look at different initial conditions, namely

(1) $u(0, y) = f(y)$;
(2) $u(x, 0) = x$ in the case in which $c_1(x, y) = 0$;
(3) $u(x, 0) = e^{c_0 x}$ also in the case $c_1(x, y) = 0$.

(1) Using the formula

$$y(x) = e^{-P(x)} \left(\int_0^x e^{P(\xi)} q(\xi)\, d\xi + y(0) \right)$$

for the solution of a linear ordinary differential equation

$$y' + p(x)y = q(x) ,$$

where

$$P(x) := \int_c^x p(s) ds ,$$

we obtain

$$u(x, y) = e^{c_0 x} \left(\int_0^x e^{-c_0 \xi} c_1(\xi, y)\, d\xi + f(y) \right) ,$$

where we used that $u(0, y) = f(y)$. In this case the problem is well-posed.

(2) The differential equation is now

$$u_x = c_0 u\,,$$

so that

$$u(x,y) = g(y)e^{c_0 x}\,,$$

where g is a function to be determined with the initial conditions. Doing so, we obtain

$$x = u(x,0) = g(0)e^{c_0 x}$$

and hence

$$g(0) = xe^{-c_0 x}\,.$$

But this is not possible since $xe^{-c_0 x}$ is not a constant, and hence the problem has no solution.

(3) Like in (2) the solution is now

$$u(x,y) = g(y)e^{c_0 x}\,,$$

where g is a function to be determined with the initial conditions. Imposing the initial conditions we obtain

$$e^{c_0 x} = u(x,0) = g(0)e^{c_0 x}\,,$$

from which it follows that *any* function $g(y)$ with $g(0) = 1$ gives a different solution

$$u(x,y) = g(y)e^{c_0 x}\,.$$

In this case the problem is not well-posed because the solution is not unique.

Example 3.17. It is easy to verify that the one-dimensional wave equation on an infinite string is a well-posed problem for any initial conditions f, g. Recall in fact the problem

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } t \geq 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x)\,, \end{cases}$$

that has solution

$$u(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy\,.$$

We want to investigate the change in u if f and g are replaced by $f+\delta f$ and $g+\delta g$. With the modified initial conditions the solutions becomes

$$\begin{aligned} u_\delta(x,t) &= \frac{1}{2}(f(x-ct)+\delta f(x-ct)+f(x+ct)+\delta f(x+ct)) \\ &\qquad + \frac{1}{2c}\int_{x-ct}^{x+ct}(g(y)+\delta g(y))\,dy \\ &= u(x,t)+\frac{1}{2}(\delta f(x-ct)+\delta f(x+ct))+\frac{1}{2c}\int_{x-ct}^{x+ct}\delta g(y)\,dy\,, \end{aligned}$$

so that

$$\begin{aligned} |u(x,t)-u_\delta(x,t)| &\le \frac{1}{2}\big|\delta f(x-ct)+\delta f(x+ct)\big| + \frac{1}{2c}\max\big|\delta g(y)\big|\big((x+ct)-(x-ct)\big) \\ &= \frac{1}{2}\delta\big|f(x-ct)+f(x+ct)\big| + t\max\big|\delta g(y)\big|\,. \end{aligned}$$

If $\epsilon > 0$, we can find a $\delta > 0$ such that

$$\frac{1}{2}\delta\big|f(x-ct)+f(x+ct)\big| < \epsilon \text{ and } \max\delta\big|g(y)\big| < \epsilon\,,$$

from which it follows that

$$|u(x,t)-u_\delta(x,t)| < \epsilon + t\epsilon\,.$$

It should be noted that, although this shows continuity on the initial parameter, we also see the the more time goes by, the more the function u_δ differs from u.

Example 3.18. The Neumann problem on a region R is the following

$$\begin{cases} \nabla^2 u = 0 & \text{on } R \\ \frac{\partial u}{\partial n} = g & \text{on } \partial R\,. \end{cases}$$

We see now that a necessary condition for the existence of a solution is that $\int_{\partial R} g = 0$. In fact, using the divergence theorem, we obtain

$$\int_{\partial R} g = \int_{\partial R}\frac{\partial u}{\partial n} = \int_{\partial R}\nabla u\cdot n = \int_R \operatorname{div}(\nabla u)\,dA = \int_R \nabla^2 u = 0\,,$$

which shows that the Neumann problem is not well-posed if $\int_{\partial R} g \ne 0$.