Eidgenössische Technische Hochschule Zürich
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Analysis III
Notes

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Summary based on lecture notes from the
Department of Mechanical Engineering
ETH Zürich

Preface

This document is not meant to replace a lecture or the official lecture notes. Its purpose is more to provide engineering students supplementary material for the course. Since I am not a mathematician, I cannot garantee that everything is formulated mathematically correct. Most of the content shall be seen as summary and application guide where I tried to simplify and explain the most important concepts when a bit more depth is required in my opinion. If you spot any mistakes, please contact me in the tutorial session or via email. I'd generally really appreciate feedback (huwylerf@ethz.ch).

Thanks for reading and hopefully it helps one or the other student.

Contents

1 Laplace Transform

The Laplace transform transforms a function of time from the time domain to the frequecy domain. Hence, the variable changes from t to s, where s is a complex frequency.

$$\mathscr{L}(f(t))(s) = \int_0^\infty e^{-st} f(t) \cdot dt \tag{1.1}$$

Example:

$$f(t) = 1 \qquad \mathscr{L}(1) = \int_0^\infty e^{-st} 1 \cdot dt = \left[-\frac{1}{s} e^{-st} \right]_{t=0}^{t=\infty} = \frac{1}{s} \tag{1.2}$$

Some important transformations are given in the following table:

Table 1: Common Laplace Transforms

Time Domain	**Frequency Domain**
$\delta(t)$	1
$\delta(t-a)$	e^{-as}
$u(t)$	$\frac{1}{s}$
$u(t-a)$	$\frac{1}{s}e^{-as}$
$u(t) \cdot t$	$\frac{1}{s^2}$
$u(t) \cdot t^n$	$\frac{n!}{s^{n+1}}$
$u(t) \cdot e^{at}$	$\frac{1}{s-a}$
$u(t) \cdot t^n \cdot e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$u(t) \cdot \sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$u(t) \cdot \cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$u(t) \cdot \cosh(at)$	$\frac{s}{s^2-a^2}$
$u(t) \cdot \sinh(at)$	$\frac{a}{s^2-a^2}$

Be careful, some transformations require a differentiation between different cases.

Example:

$$f(t) = e^{at} \qquad \mathscr{L}(f(t)) = \int_0^\infty e^{-st} e^{at} \cdot dt = \int_0^\infty e^{(a-s)t} \cdot dt = \left[\frac{1}{a-s} e^{(a-s)t} \right]_0^\infty \tag{1.3}$$

$$\begin{cases} a \geq s: & \infty \\ a \leq s: & -\frac{1}{a-s} \end{cases} \tag{1.4}$$

1.1 Heaviside Function

The heaviside function is very common in complex analysis and is defined as follows:

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases} \tag{1.5}$$

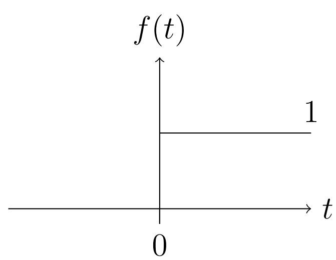


Figure 1: Heaviside function

1.2 Dirac Measure

The diract measure or impulse is very common in control theory and is a theoretical construct for which ϵ becomes zero. One can look at the function in the following way: It is infinity for $t = 0$ and zero everywhere else.

For $\lim_{\epsilon \to 0}$

$$\delta(t) = \begin{cases} \frac{1}{\epsilon} & 0 \leq t \leq \epsilon \\ 0 & else \end{cases} \tag{1.6}$$

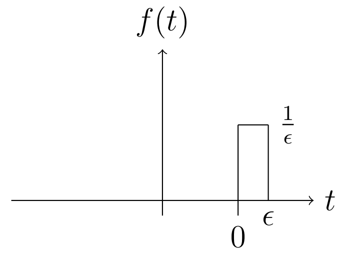


Figure 2: Diract measure

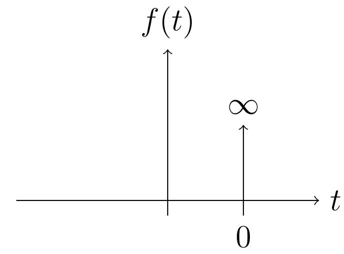


Figure 3: Diract measure in practice

There are the following properties of the Dirac measure:

- $$\delta(0) = \frac{d}{dt} u(0) \tag{1.7}$$

- $$\delta(\epsilon) \geq 0 \tag{1.8}$$

- $$\int_{-\infty}^{\infty} \delta(t) \cdot dt = 1 \tag{1.9}$$

- $$\lim_{\epsilon \to 0} \delta(t) = 0 \qquad \forall \;\; t \neq 0 \tag{1.10}$$

- For an arbitrary function $f(t)$, the following holds:

$$\lim_{\epsilon \to 0} \int_{0}^{\infty} f(t) \cdot \delta(t-a) \cdot dt = f(a) \tag{1.11}$$

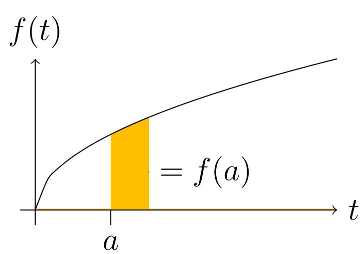


Figure 4: Property of the Dirac Measure

1.3 Properties

1.3.1 Derivative of Transform

The derivatie of a Laplace transform can be derived as follows:

$$\frac{d}{ds}\mathscr{L}(f(t))(s) = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t) \cdot dt\right] = -\int_0^\infty te^{-st} f(t) \cdot dt = -\mathscr{L}(t \cdot f(t))(s) \tag{1.12}$$

$$\boxed{\frac{d}{ds}\mathscr{L}(f(t))(s) = -\mathscr{L}(t \cdot f(t))(s)} \tag{1.13}$$

Example:

Find the inverse transform of $\mathscr{L}(f(t))$

$$\mathscr{L}(f(t))(s) = \frac{s}{(s^2-16)^2} \tag{1.14}$$

In this case, the <u>transform looks like a derivative of the transformation of another function</u> $g(t)$.

$$\mathscr{L}(f(t)) = \mathscr{L}'(g(t)) = \frac{d}{ds}\mathscr{L}(g(t)) \tag{1.15}$$

Integrating the transformation gives $\mathscr{L}(g(t))$.

$$\mathscr{L}(g(t)) = \int \mathscr{L}'(g(t)) \cdot ds = \int \frac{s}{(s^2-16)^2} \cdot ds = -\frac{1}{2}\left(\frac{1}{s^2-16}\right) \tag{1.16}$$

Rearranging the transformation makes an easy inverse transform possible.

$$g(t) = \mathscr{L}^{-1}\left(-\frac{1}{8}\left(\frac{4}{s^2-16}\right)\right) = -\frac{1}{8}\sinh(4t) \tag{1.17}$$

From equation (1.13), we know that the following holds:

$$\mathscr{L}(f(t)) = \mathscr{L}'(g(t)) = -\mathscr{L}(t \cdot g(t)) \tag{1.18}$$

Hence, we have

$$f(t) = -tg(t) = \frac{1}{8}t\sinh(4t) \tag{1.19}$$

1.3.2 Linearity

From basic properties of the linearity of integrals follows that the Laplace transform is linear as well.

$$\boxed{\mathscr{L}(\alpha f(t)) + \beta g(t))(s) = \alpha \mathscr{L}(f(t))(s) + \beta \mathscr{L}(g(t))(s)} \tag{1.20}$$

Example:

$$\mathscr{L}(3t^2 + 5t - 1) = 3\mathscr{L}(t^2) + 5\mathscr{L}(t) - \mathscr{L}(1) \tag{1.21}$$

$$= 3\frac{2}{s^3} + 5\frac{1}{s^2} - \frac{1}{s} \tag{1.22}$$

1.3.3 Shifting Theorem

$$\boxed{\mathscr{L}(e^{at} f(t))(s) = \mathscr{L}(f(t))(s-a)} \tag{1.23}$$

Example:

$$\mathscr{L}(5e^{-\frac{\sqrt{2}}{4}t}) = 5\frac{1}{s + \frac{\sqrt{2}}{4}} = \frac{20}{4s + \sqrt{2}} \tag{1.24}$$

1.3.4 2nd Shifting Theorem

$$\mathscr{L}\Big(u(t-a)f(t-a)\Big)(s) = e^{-as}\mathscr{L}(f(t))(s) \tag{1.25}$$

$$\mathscr{L}\Big(f(t) \cdot u(t-a)\Big)(s) = e^{-as} \cdot \mathscr{L}(f(t+a))(s) \tag{1.26}$$

Similarily, the following holds for the inverse.

$$\mathscr{L}^{-1}\left(e^{-as}\mathscr{L}(f(t))\right)(t) = u(t-a)f(t-a) = \begin{cases} f(t-a) & if \quad t > a \\ 0 & if \quad t \leq a \end{cases} \tag{1.27}$$

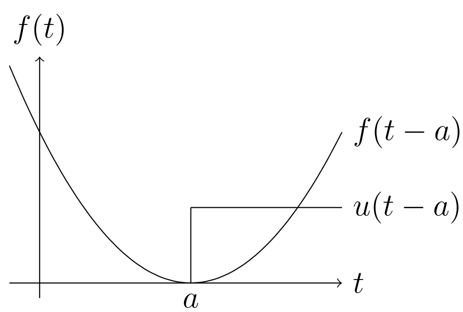


Figure 5: Shifted function and heaviside function

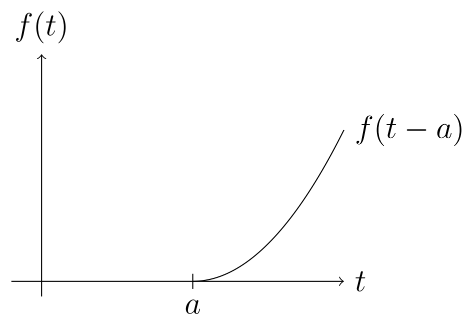


Figure 6: Multiplied: $u(t-a)f(t-a)$

Example:

$$\mathscr{L}^{-1}(2e^{-s}\frac{1}{s+1}) \tag{1.28}$$

One can account for the e^{-s} with the shifting theorem and hence ignore it for now. The back transform of the main function is then:

$$\frac{1}{s+1} \rightarrow e^{-t} \tag{1.29}$$

Together with the factor two and the shifting theorem one gets:

$$f(t) = 2u(t-1)e^{-(t-1)} \tag{1.30}$$

1.3.5 Combined Shifting Theorem

Sometimes, it is convenient to use both shifting theorems for a transformation. The general combined form is written below:

$$\boxed{\mathscr{L}\left(e^{a(t-b)}f(t-b)u(t-b)\right)(s) = e^{-bs}\mathscr{L}(f(t))(s-a) \tag{1.31}}$$

1.3.6 Transform of Derivatives

For the n^{th} derivative of a function $f^{(n)} = \frac{d^n f}{dx^n}$, the following holds:

$$\boxed{\mathscr{L}(f^{(n)}(t))(s) = s^n \mathscr{L}(f(t))(s) - \sum_{j=0}^{n-1} s^{n-1-j} f^{(j)}(0)} \tag{1.32}$$

$$\mathscr{L}(f'(t))(s) = s\mathscr{L}(f(t))(s) - f(0) \tag{1.33}$$

$$\mathscr{L}(f''(t))(s) = s^2\mathscr{L}(f(t))(s) - sf(0) - f'(0) \tag{1.34}$$

Example:
Derivation of $\mathscr{L}(\sin(\omega t))$

$$f(t) = \sin(\omega t) \qquad f'(t) = \omega\cos(\omega t) \qquad f''(t) = -\omega^2\sin(\omega t) = -\omega^2 f(t) \tag{1.35}$$

$$\mathscr{L}(f''(t)) = s^2\mathscr{L}(f(t)) - s\underbrace{f(0)}_{0} - \underbrace{f'(0)}_{\omega} \tag{1.36}$$

$$f(t) = -\frac{1}{\omega^2}f''(t) \quad \Rightarrow \quad \mathscr{L}(f(t)) = -\frac{1}{\omega^2}\mathscr{L}(f''(t)) \tag{1.37}$$

$$\mathscr{L}(f(t)) = -\frac{1}{\omega^2}\underbrace{(s^2\mathscr{L}(f(t)) - \omega)}_{\mathscr{L}(f''(t))} \tag{1.38}$$

$$\Rightarrow \quad \mathscr{L}(\sin(\omega t))(s) = \frac{\omega}{\omega^2 + s^2} \tag{1.39}$$

1.3.7 Laplace Transform of Integrals

$$\boxed{\mathscr{L}\left(\int_0^t f(x)\cdot dx\right)(s) = \frac{1}{s}\mathscr{L}(f(t))(s)} \tag{1.40}$$

1.3.8 Convolution

Apart from basic operations like addition and multiplication, there are other operations which are a bit more complex. One of them is the so-called convolution which is defined by an integral. The convolution of two functions is usually written with a star $*$.

$$\boxed{f(t) * g(t) = \int_0^t f(r)g(t-r) \cdot dr \qquad t \geq 0} \tag{1.41}$$

The convolution has the following properties:

- $$f(t) * g(t) = g(t) * f(t) \tag{1.42}$$
- $$f(t) * (g(t) + h(t)) = f(t) * g(t) + f(t) * h(t) \tag{1.43}$$
- $$f(t) * (g(t) * h(t)) = (f(t) * g(t)) * h(t) \tag{1.44}$$
- $$(\alpha f(t)) * g(t) = \alpha(f(t) * g(t)) \tag{1.45}$$
- $$f(t) * 0 = 0 \tag{1.46}$$
- $$f(t) * 1 = \int_0^t f(r) \cdot dr \tag{1.47}$$
- $$\mathscr{L}(f(t) * g(t)) = \mathscr{L}(f(t)) \cdot \mathscr{L}(g(t)) \tag{1.48}$$

$$f(t) * g(t) = \mathscr{L}^{-1}\Big(\mathscr{L}(f(t)) \cdot \mathscr{L}(g(t))\Big) \tag{1.49}$$

Note: Generally $f(t) * f(t) \ngeq 0$

1.4 Solving Ordinary Differential Equations

One can transform an ordinary differential equation to the frequency domain, where the equation becomes algebraic. It is hence simple to simplify the equation. Once the function can be written in an explicit form, one can transform the function back to the time domain. With the derivatives of section 1.3.6, one can easily transform an equation. For simplicity, we call $\mathscr{L}(y(t)) = Y(s)$.

Example:

$$y''(t) - y(t) = t \qquad y(0) = 1 \quad y'(0) = 1 \tag{1.50}$$

This equation is now transformed to the frequency domain:

$$s^2 Y(s) - s - 1 - Y(s) = \frac{1}{s^2} \tag{1.51}$$

This is an algebraic equation and can easily be simplified.

$$Y(s) = \frac{\frac{1}{s^2} + s + 1}{s^2 - 1} = \frac{1}{s^2-1} + \frac{1}{s^2(s+1)(s-1)} + \frac{s}{(s+1)(s-1)} \tag{1.52}$$

$$= \frac{1/2}{s-1} - \frac{1/2}{s+1} - \frac{1}{s^2} - \frac{1/2}{s+1} + \frac{1/2}{s-1} + \frac{1/2}{s+1} + \frac{1/2}{s-1} \tag{1.53}$$

$$= \frac{3/2}{s-1} - \frac{1/2}{s+1} - \frac{1}{s^2} \tag{1.54}$$

Back transform of this function of s yields the function in the time domain:

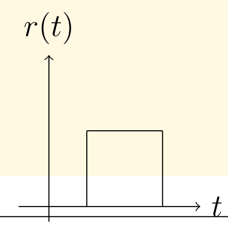
$$y(t) = u(t) \cdot \left(\frac{3}{2}e^t - \frac{1}{2}e^{-t} - t\right) \tag{1.55}$$

Example:
Solve:

$$y''(t) + 3y'(t) + 2y(t) = r(t) \tag{1.56}$$

$$y(0) = y'(0) = 0 \tag{1.57}$$

$$r(t) = u(t-1) - u(t-2) = \begin{cases} 1 & 1 \le t \le 2 \\ 0 & else \end{cases} \tag{1.58}$$



Transforming the equation yields:

$$s^2Y(s) + 3sY(s) + 2Y(s) = \mathscr{L}(r(t)) = e^{-s}\frac{1}{s} - e^{-2s}\frac{1}{s} \tag{1.59}$$

$$(s^2 + 3s + 2)Y(s) = \frac{1}{s}(e^{-s} - e^{-2s}) \tag{1.60}$$

$$Y(s) = \frac{1}{s(s+1)(s+3)}(e^{-s} - e^{-2s}) \tag{1.61}$$

For now, one only looks at the fraction and ignores the exponential functions because one can account for them later with a shifting theorem. Partial fraction decomposition yields:

$$\frac{1}{s(s+1)(s+3)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)} \tag{1.62}$$

The inverse Laplace transform of this is:

$$\mathscr{L}^{-1}\left(\frac{1}{s(s+1)(s+3)}\right) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \tag{1.63}$$

Using the 2nd shifting theorem from section 1.3.4 to account for the exponential functions yields:

$$\begin{aligned} y(t) =& u(t-1)\frac{1}{2} - u(t-1)e^{-(t-1)} + u(t-1)\frac{1}{2}e^{-2(t-1)} - u(t-2)\frac{1}{2} + u(t-2)e^{-(t-2)} \\ &- u(t-2)\frac{1}{2}e^{-2(t-2)} \end{aligned} \tag{1.64}$$

This is already the solution. Usually, one is required to write the function in an even nicer way with cases so that one could easily plot it. In order to do that, it is helpful to look at the heaviside functions first. In this case, we have two heaviside functions around 1 and 2. This means that there are three intervals: from zero to 1, from 1 to 2 and from 2 to ∞. Writing down the cases is then quite easy. One has to pay attention though, otherwise mistakes are easily made.

$$y(t) = \begin{cases} 0 & t < 1 \\ \frac{1}{2} - e^{-t+1} + \frac{1}{2}e^{-2t+2} & 1 \leq t \leq 2 \\ -e^{-t+1} + \frac{1}{2}e^{-2t+2} + e^{-t+2} - \frac{1}{2}e^{-2t+4} & t > 2 \end{cases} \tag{1.65}$$

Example:
Solve:

$$y(t) - \underbrace{\int_0^t y(r)\sin(t-r)\cdot dr}_{y(t)*\sin(t)} = t \tag{1.66}$$

Transformation of the equation yields:

$$Y(s) - Y(s)\underbrace{\mathscr{L}(\sin(t))}_{\frac{1}{s^2+1}} = \frac{1}{s^2} \tag{1.67}$$

Simplifying gives:

$$Y(s) = \frac{1}{s^2}\cdot\frac{1}{1-\frac{1}{s^2+1}} = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4} \tag{1.68}$$

The solution is then:

$$y(t) = t + \frac{1}{6}t^3 \tag{1.69}$$

2 Fourier Series

The whole idea of Fourier Series is very simple. One tries to express a function as a sum of different sine and cosine functions. One of the advantages of that is that the function then can be easily integrated because the integral of sine an cosine functions is known. A function is called **P-Periodic** if $f(x) = f(x+p)$ holds.

Examples:

- sin(x),cos(x) 2π-periodic
- tan(5x) $\frac{\pi}{5}$-periodic
- x^2, x^3, e^x not periodic
- $\cos(n\frac{\pi}{L}x)$, $\sin(n\frac{\pi}{L}x)$ 2L-periodic for $n \in \mathbb{Z}$

Important Integrals:

$$\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\cdot dx = \begin{cases} 0 & \text{if } n \neq m \\ L & \text{if } n = m \neq 0 \\ 2L & \text{if } n = m = 0 \end{cases} \tag{2.1}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\cdot dx = \begin{cases} 0 & \text{if } n \neq m \\ L & \text{if } n = m \neq 0 \\ 0 & \text{if } n = m = 0 \end{cases} \tag{2.2}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\cdot dx = 0 \tag{2.3}$$

Also good to know:

$$\int_{-\pi}^{\pi} \cos(Kx)\cdot dx = 0 \qquad \forall K \in \mathbb{N}\setminus\{0\} \tag{2.4}$$

$$\int_{-\pi}^{\pi} \cos(Kx)\cdot dx = 2\pi \qquad K = 0 \tag{2.5}$$

2.1 Definition

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cdot\cos\left(\frac{n\pi}{L}x\right) + b_n\cdot\sin\left(\frac{n\pi}{L}x\right)\right] \tag{2.6}$$

$$\begin{aligned} a_0 &= \frac{1}{2L}\cdot\int_{-L}^{L} f(x)\cdot dx \\ a_n &= \frac{1}{L}\cdot\int_{-L}^{L} f(x)\cdot\cos\left(\frac{n\pi}{L}x\right)\cdot dx \\ b_n &= \frac{1}{L}\cdot\int_{-L}^{L} f(x)\cdot\sin\left(\frac{n\pi}{L}x\right)\cdot dx \end{aligned} \tag{2.7}$$

A function can be brought into the form of equation (2.6). Therefore, one needs to calculate the Fourier coefficients first.

Derivation Coefficients

Integrate the function from $-L$ to L and use the definition of the Fourier series.

$$\int_{-L}^{L} f(x) \cdot dx = \underbrace{\int_{-L}^{L} a_0 \cdot dx}_{2La_0} + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right) \cdot dx}_{0} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right) \cdot dx}_{0} \quad (2.8)$$

$$\to a_0 = \frac{1}{2L} \cdot \int_{-L}^{L} f(x)dx \quad (2.9)$$

Multiply the function with $\cos(\frac{\pi}{L}x)$ and integrate from -L to L. (Note that we multiply with the function with $\cos(\frac{n\pi}{L}x)$. Because we aim to find a_1 we choose $n = 1$)

$$\begin{aligned}\int_{-L}^{L} f(x) \cos\left(\frac{\pi}{L}x\right) \cdot dx =& a_0 \underbrace{\int_{-L}^{L} \cos\left(\frac{\pi}{L}x\right) \cdot dx}_{0} + a_1 \underbrace{\int_{-L}^{L} \cos\left(\frac{\pi}{L}\right) \cos\left(\frac{\pi}{L}x\right) \cdot dx}_{L} \\ &+ \sum_{n=2}^{\infty} \underbrace{\int_{-L}^{L} a_n \cos\left(\frac{\pi}{L}x\right) \cos\left(\frac{n\pi}{L}\right) \cdot dx}_{0} \\ &+ \sum_{n=1}^{\infty} \underbrace{\int_{-L}^{L} b_n \cos\left(\frac{\pi}{L}x\right) \sin\left(\frac{n\pi}{L}\right) \cdot dx}_{0}\end{aligned} \quad (2.10)$$

$$\to a_1 = \frac{1}{L} \cdot \int_{-L}^{L} f(x) \cdot \cos\left(\frac{\pi}{L}x\right) \cdot dx \quad (2.11)$$

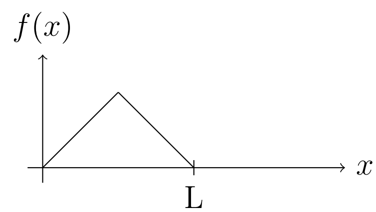
2.2 Even and Uneven Functions

- f is called **even** if $f(x) = f(-x)\ \forall x$
  Examples: $x^2, x^4, cos(x)$
- f is called **uneven** if $f(x) = -f(-x)\ \forall x$
  Examples: $x, x^3, sin(x)$

For the coefficients of a Fourier series of a function f, the following holds:

- If f is even, $b_n = 0\ \forall n$
- If f is uneven, $a_n = 0\ \forall n$

Often, it is asked that one extends a function and either does an even or odd extension. First, an arbitrary function is given in the positive domain. This function then goes from zero to L.



As a first step, one mirrors the function so that one gets an even or odd function.

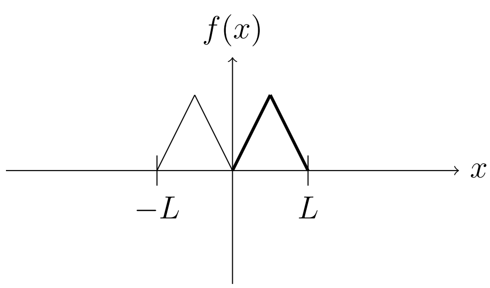


Figure 7: Even Extension

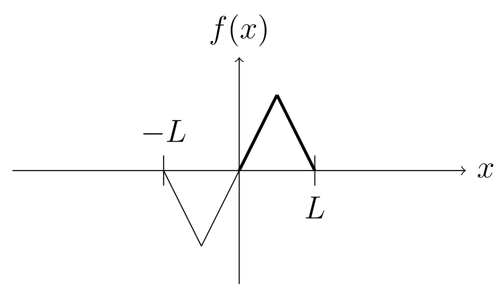
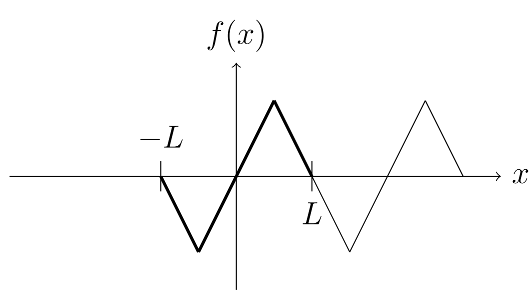


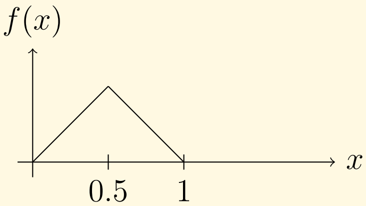
Figure 8: Uneven Extension

These functions are $2L-$periodic. If one wants to know how the function goes on after L, one can just "copy" the function from $-L$ to L and "paste" it after L. Note that depending on the function, discontinuities can occur at L.



Example: Calculate the Fourier series of the even extension of $f(x)$.

$$f(x) = \begin{cases} 2x & 0 \leq x \leq 0.5 \\ 2(1-x) & 0.5 \leq x \leq 1 \end{cases} \tag{2.12}$$



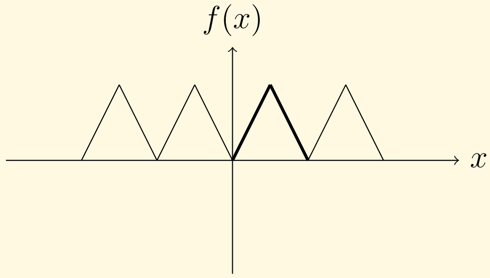


Figure 9: Even Extension

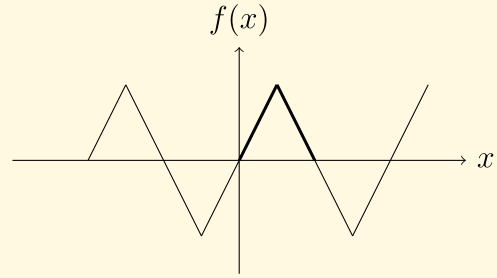


Figure 10: Uneven Extension

Because we look at the even extension of the function, we know that $b_n = 0 \; \forall n$.

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x) \cdot dx = \frac{1}{2} \tag{2.13}$$

Because the cosine function as well as our function $f(x)$ are even, we can integrate from zero to L and multiply the integral with two.

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x) \cdot dx = 2\int_{0}^{1} f(x)\cos(n\pi x) \cdot dx \tag{2.14}$$

Because our function is defined on different intervals, we can split the integral and calculate its value.

$$a_n = 2\int_{0}^{1/2} 2x\cos(n\pi x) \cdot dx + 2\int_{1/2}^{1} 2(1-x)\cos(n\pi x) \cdot dx \tag{2.15}$$

Integrating and plugging in the boundaries yields:

$$a_n = \frac{8}{n^2\pi^2}\cos(\frac{n\pi}{2}) - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos(n\pi) \tag{2.16}$$

Then it is usually helpful to make a table and check the values for the first values of n. In this case, we check for $n = 1, 2, 3, 4$. We get the same values for $n = 5, 6, 7, 8$ because the function is 2π-periodic. It is often helpful to quickly sketch a sine or cosine function to see why this is the case. We get zero for $n = 1, 3, 4$ and $-\frac{16}{n^2\pi^2}$ for $n = 2$. Since the function is 2π-periodic, the values repeat for every four values of n. Hence, the integral is non-zero for $n = 2, 6, 10 \ldots$. The last step is now to find an explicit formulation of this sequence. $2(2j + 1)$ with $j = 0, 1, 2, 3 \ldots$

$$= \begin{cases} -\frac{16}{n^2\pi^2} & \text{for } n = 2(2j+1) \text{ with } j = 0,1,2,3\ldots \\ 0 & \text{else} \end{cases} \tag{2.17}$$

Now the coefficients are known and one can write down the Fourier Series.

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} \cos(2(2j+1)\pi x) \tag{2.18}$$

Note: If $f(x)$ is an even function, the following holds:

$$\int_{-L}^{L} f(x) \cdot dx = 2 \int_{0}^{L} f(x) \cdot dx \tag{2.19}$$

For the multiplication of different functions, the following properties hold:

$$\begin{aligned} f(-x)g(-x) &= f(x)g(x) \qquad \text{even} \cdot \text{even=even} \\ f(-x)g(-x) &= -f(x)g(x) \qquad \text{even} \cdot \text{odd=odd} \\ f(-x)g(-x) &= (-f(x))(-g(x)) = f(x)g(x) \qquad \text{odd} \cdot \text{odd=even} \end{aligned} \tag{2.20}$$

2.3 Complex Fourier Series

Just as a quick reminder, some basic formulas for complex numbers:

$$\mathring{\imath}^2 = -1 \qquad \frac{1}{\mathring{\imath}} = -\mathring{\imath} \tag{2.21}$$

$$e^{\mathring{\imath}\varphi} = \cos(\varphi) + \mathring{\imath}\sin(\varphi) \qquad e^{-\mathring{\imath}\varphi} = \cos(\varphi) - \mathring{\imath}\sin(\varphi) \tag{2.22}$$

Rearranging and combining these formulas gives complex formulations of sine and cosine functions.

$$\cos(\varphi) = \frac{1}{2}(e^{\mathring{\imath}\varphi} + e^{-\mathring{\imath}\varphi}) \tag{2.23}$$

$$\sin(\varphi) = \frac{1}{2\mathring{\imath}}(e^{\mathring{\imath}\varphi} - e^{-\mathring{\imath}\varphi}) \tag{2.24}$$

Similar as for the real Fourier Series, the definition of the complex one is given by:

$$f(x) = \sum_{m=-\infty}^{\infty} c_m e^{\mathring{\imath} m \frac{\pi}{L} x} \tag{2.25}$$

$$c_m = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-\mathring{\imath} m \frac{\pi}{L} x} \cdot dx \tag{2.26}$$

Example:

$$f(x) = e^x \qquad -\pi \leq x \leq \pi \tag{2.27}$$

First, the coefficients for the series are calculated.

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x e^{-\mathring{\imath} m x} \cdot dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x(1-\mathring{\imath} m)} \cdot dx = \frac{1}{2\pi(1-\mathring{\imath} m)} e^{x(1-\mathring{\imath} m)} \Bigg|_{x=-\pi}^{x=\pi} \tag{2.28}$$

$$= \frac{1}{2\pi} \left(\frac{1+\mathring{\imath} m}{1+m^2} \right) \left(e^{\pi} e^{-\mathring{\imath} \pi m} - e^{-\pi} e^{\mathring{\imath} \pi m} \right) \tag{2.29}$$

Since m is an integer and we know the values for sine and cosine functions, it can be helpful to rewrite the complex exponential function and express it with sine and cosine functions.

$$c_m = \frac{1}{2\pi} \left(\frac{1+\mathring{\imath} m}{1+m^2} \right) \left(e^{\pi} \Big(\cos(m\pi) - \mathring{\imath} \sin(m\pi) \Big) - e^{-\pi} \Big(\cos(m\pi) + \mathring{\imath} \sin(m\pi) \Big) \right) \tag{2.30}$$

From looking at the graph of a sine and cosine function, we know that $\sin(m\pi) = 0$ and $\cos(m\pi) = (-1)^m$ for integers m.

$$c_m = \frac{1}{2\pi} \left(\frac{1+\mathring{\imath} m}{1+m^2} \right) \left(e^{\pi}(-1)^m - e^{-\pi}(-1)^m \right) \tag{2.31}$$

$$f(x) = \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} \left(\frac{1+\mathring{\imath} m}{1+m^2} \right) \left(e^{\pi}(-1)^m - e^{-\pi}(-1)^m \right) e^{\mathring{\imath} m x} \tag{2.32}$$

There is a connection between the real Fourier coefficients a_n, b_n and the complex ones c_n.

- **Complex → Real**

$$a_0 = c_0 \qquad a_n = c_n + c_{-n} \qquad b_n = \mathring{\imath} \cdot (c_n - c_{-n}) \tag{2.33}$$

- **Real → Complex**

$$c_0 = a_0 \qquad c_n = \frac{1}{2} \left(a_n - \mathring{\imath} \cdot b_n \right), \quad n > 0 \qquad c_{-n} = \frac{1}{2} \left(a_n + \mathring{\imath} \cdot b_n \right) \tag{2.34}$$

3 Fourier Integral

As you may have noticed in the previous section, one can only write a function as a Fourier series if it is 2L-periodic (L has to finite). If this is not the case, one can either define a finite interval on which the function is looked at and then make it periodic outside this interval or use a Fourier integral instead. The Fourier integral is equivalent to the Fourier Series only that the function is not periodic and defined from $-\infty$ to ∞. The Fourier integral is defined as follows:

$$f(x) = \int_0^\infty \Big(A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\Big) \cdot d\omega \tag{3.1}$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\omega x) \cdot dx \tag{3.2}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\omega x) \cdot dx \tag{3.3}$$

Note that for even functions $B(\omega) = 0$ and that for odd functions $A(\omega) = 0$.

3.1 Discontinuities

If a function is only piecewise continuous, the locations of the discontinuities have to be treated separately. The Fourier integral can only be calculated if the function is absolutely integrable ($\int_{-\infty}^{\infty} |f(x)|dx < \infty$) and if a left and a right derivative exists at the discontinuity.

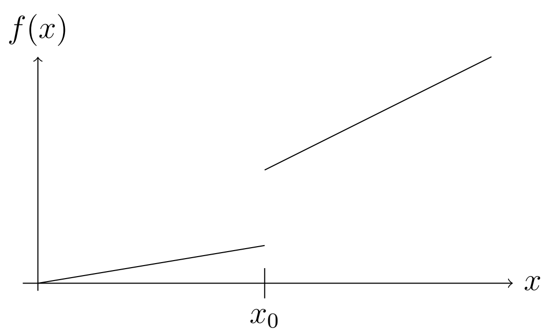


Figure 11: Piecewise continuous function

The function is then given by the Fourier integral $\mathscr{I}\{f(x)\}$ where f(x) is continuous. At the discontinuities x_0, the Fourier integral is given by:

$$\mathscr{I}\{f(x)\}(x_0) = \frac{1}{2}\left(\lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x)\right) \tag{3.4}$$

4 Fourier Transform

A Fourier transform decomposes a function of time, which is usually a signal or data set, into its individual frequencies. One can imagine that the time signal is composed of different sinuidal functions with different frequencies, like in a Fourier series. The transform extracts then the individual frequencies and gives an array of complex values for the data set. The transformation can be calculated with the following integral:

$$\boxed{\hat{f}(y) = \mathscr{F}\{f\}(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iyt} \cdot dt \qquad (4.1)}$$

The Fourier transform hence yields a function $\hat{f}(y)$ and since $\mathscr{F} : \mathbb{R} \mapsto \mathbb{C}$, a complex value corresponds to every frequency y. The absolute value of this complex number represents the amount of that frequency present in the time signal. The argument of the number describes the phase offset from a basic cosine function with which the signal is present. If a certain fequency spectrum is given, one can also convert it back into a time signal. This is done with the inverse Fourier transform which is given by:

$$\boxed{f(t) = \mathscr{F}^{-1}\{\hat{f}\}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(y) e^{iyt} \cdot dy \qquad (4.2)}$$

The Fourier transform of a time signal with one single frequency yields a single peak in the frequency plot. This peak is at the frequency of the signal. To illustrate that, the Fourier transform of a cosine function is shown below.

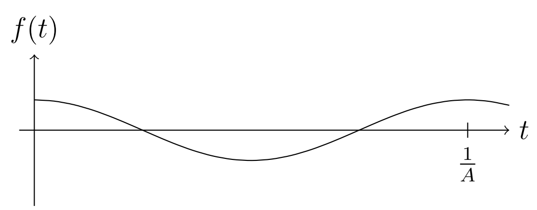


Figure 12: $f(t) = \cos(2\pi A t)$

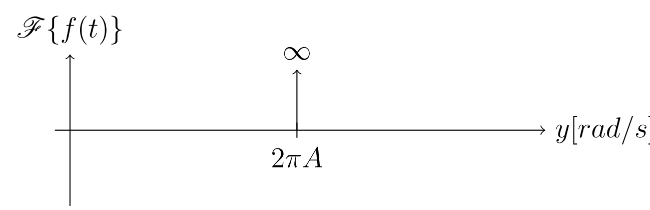


Figure 13: $\mathscr{F}\{f(t)\} = \frac{1}{\sqrt{2\pi}}[\delta(y-2\pi A)+\delta(y+2\pi A)]$

Since the cosine function is defined on $\mathbb{R}$ and $\mathbb{R}$ is taken as domain of the function, the signal appears infinitely often. For this reason, the peak goes to infinity which is described by the dirac measure. Since this dirac impulse is purely real, the argument is zero. This means that the the time signal has no phase offset compared to a basic cosine function. For a simple cosine function with a phase shift of B the Fourier transform has the following form:

$$f(t) = \cos(2\pi At + B) \qquad \mathscr{F}\{f(t)\} = \sqrt{\frac{\pi}{2}}\left[e^{-\mathring{\imath}B}\delta(y - 2\pi A) + e^{\mathring{\imath}B}\delta(2\pi A + y)\right] \tag{4.3}$$

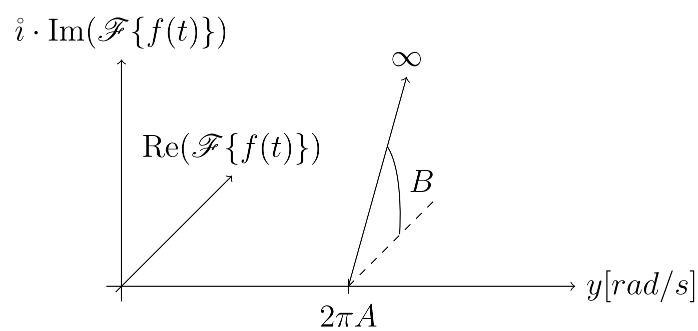


Figure 14: Fourier transform in the complex plane, dependent on the frequency y

The Fourier transform of a sine function $f(t) = \sin(2\pi At)$ is given by $\mathscr{F}\{f(t)\} = \frac{\mathring{\imath}}{\sqrt{2\pi}}(\delta(y - 2\pi A) - \delta(y + 2\pi A))$ which is purely imaginary. The argument is hence 90° which is exactly the offset between a sine and a cosine function. If multiple frequencies are present in a signal, the Fourier frequency spectrum shows multiple peaks at the signals' frequencies.

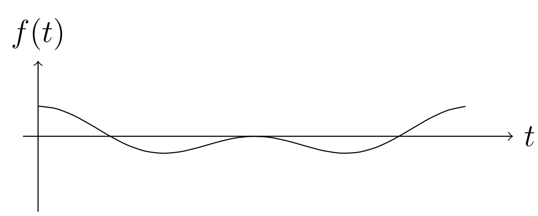


Figure 15: $f(t) = \cos(2\pi At) + \cos(2\pi Bt)$

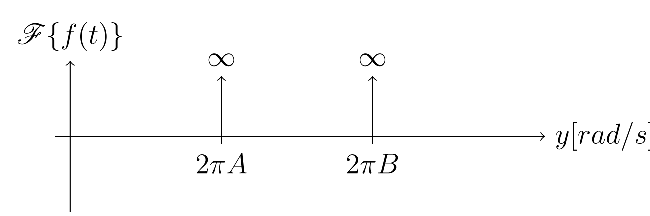


Figure 16: $\mathscr{F}\{f(t)\} = \frac{1}{\sqrt{2\pi}}[\delta(y-2\pi A)+\delta(y+2\pi A) + \delta(y - 2\pi B) + \delta(y + 2\pi B)]$

The Fourier transform can be calculated for any arbitrary function like for example an exponential function (this works because the function can be written as a Fourier series).

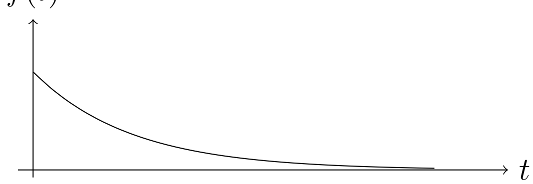


Figure 17: $f(t) = e^{-a|t|}$

Figure 18: $\mathscr{F}\{f(t)\} = \frac{2a}{a^2+y^2}$

Example:

Calculate the Fourier transform of $f(x)$.

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{else} \end{cases} \tag{4.4}$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \cdot dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega x} \cdot dx \tag{4.5}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{(-i\omega)} e^{-i\omega x} \Big|_{x=-1}^{x=1} = \frac{i}{\omega\sqrt{2\pi}} (e^{-i\omega} - e^{i\omega}) \tag{4.6}$$

$$= \frac{i}{\omega\sqrt{2\pi}} (\cos(\omega) - i\sin(\omega) - \cos(\omega) - i\sin(\omega)) = \frac{2}{\omega\sqrt{2\pi}} \sin(\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin(\omega)}{\omega} \tag{4.7}$$

4.1 Properties

- **Linearity**
  $f \mapsto \mathscr{F}\{f(t)\}$ is linear
  $\mathscr{F}\{\alpha f(t) + \beta g(t)\} = \alpha \cdot \mathscr{F}\{f(t)\} + \beta \cdot \mathscr{F}\{g(t)\}$
- **Derivative**
  $\mathscr{F}\{f'(t)\}(\omega) = i\omega \mathscr{F}\{f(t)\}(\omega)$
  $\mathscr{F}\left\{f''(x)\right\}(\omega) = -\omega^2 \cdot \mathscr{F}\{f(x)\}(\omega)$
- **Convolution**
  $\mathscr{F}\{f(t) * g(t)\}(\omega) = \sqrt{2\pi} \mathscr{F}\{f(t)\} \mathscr{F}\{g(t)\}$
  $\mathscr{F}^{-1}\{\hat{f}(\omega)\hat{g}(\omega)\} = \frac{1}{\sqrt{2\pi}} f(t) * g(t)$

- **x-Shift**
  $\mathscr{F}\{f(x-a)\}(\omega) = e^{-ia\omega} \cdot \mathscr{F}\{f(x)\}$
- **y-Shift**
  $\mathscr{F}\left\{e^{iax} \cdot f(x)\right\}(\omega) = \mathscr{F}\{f(x)\}(\omega - a) = \hat{f}(\omega - a)$
- **Modulation**
  $\mathscr{F}^{-1}\{\hat{f}(\omega - a)\}(x) = e^{iax} \cdot f(x)$
- **ω-Derivative**
  $\mathscr{F}(x \cdot f(x)(\omega) = \mathring{i}\frac{d}{d\omega}\mathscr{F}(f(x))(\omega)$
  $\mathscr{F}(x^2 \cdot f(x))(\omega) = -\frac{d^2}{d\omega^2}\mathscr{F}(f(x))(\omega)$

Example:
Given is the Fourier transform of the function $f(x)$:

$$\mathscr{F}(f(x))(\omega) = \frac{1}{\sqrt{2\pi}} \frac{3}{(5 + \mathring{i}\omega)} \tag{4.8}$$

Find now the value of the following integral:

$$\int_{-\infty}^{\infty} f(x) \cdot dx = ? \tag{4.9}$$

First, we know the value of the transform and write that down.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-\mathring{i}\omega x} \cdot dx = \frac{1}{\sqrt{2\pi}} \frac{3}{(5 + \mathring{i}\omega)} \tag{4.10}$$

This must hold for every ω. To eliminate the term with the exponential function, we choose $\omega = 0$.

$$\rightarrow \int_{-\infty}^{\infty} f(x) \cdot dx = \frac{3}{5} \tag{4.11}$$

Example:
Given is the same Fourier transform of the function $f(x)$:

$$\mathscr{F}(f(x))(\omega) = \frac{1}{\sqrt{2\pi}} \frac{3}{(5 + \mathring{i}\omega)} \tag{4.12}$$

Find now the following integral:

$$\int_{-\infty}^{\infty} x f(x) \cdot dx = ? \tag{4.13}$$

By using the ω-derivative property of the Fourier transform, we know that:

$$\mathscr{F}(xf(x))(\omega) = \mathring{\imath}\frac{d}{d\omega}\mathscr{F}(f(x))(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xf(x)e^{-\mathring{\imath}\omega x}\cdot dx = \mathring{\imath}\frac{d}{d\omega}\left(\frac{1}{\sqrt{2\pi}}\frac{3}{(5+\mathring{\imath}\omega)}\right) \quad (4.14)$$

Hence, we have:

$$\int_{-\infty}^{\infty} xf(x)e^{-\mathring{\imath}\omega x}\cdot dx = \frac{3}{(5+\mathring{\imath}\omega)^2} \quad (4.15)$$

Again, we eliminate the exponential function by chosing $\omega = 0$. This gives:

$$\int_{-\infty}^{\infty} xf(x)\cdot dx = \frac{3}{25} \quad (4.16)$$

Example: Given is the same Fourier transform of the function $f(x)$:

$$\mathscr{F}(f(x))(\omega) = \frac{1}{\sqrt{2\pi}}\frac{3}{(5+\mathring{\imath}\omega)} \quad (4.17)$$

Find now the following integral:

$$\int_{-\infty}^{\infty} x^2 f(x)\cdot dx = ? \quad (4.18)$$

Like before, we write use the ω-derivative property of Fourier transform:

$$\mathscr{F}(x^2 f(x))(\omega) = -\frac{d^2}{d\omega^2}\mathscr{F}(f(x))(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2 f(x)e^{-\mathring{\imath}\omega x}\cdot dx = -\frac{d^2}{d\omega^2}\left(\frac{1}{\sqrt{2\pi}}\frac{3}{(5+\mathring{\imath}\omega)}\right) \quad (4.19)$$

Hence we get:

$$\int_{-\infty}^{\infty} x^2 f(x)e^{-\mathring{\imath}\omega x}\cdot dx = \frac{6}{(5+\mathring{\imath}\omega)^3} \quad (4.20)$$

Finally, we chose $\omega = 0$ agian. Therefore, the integral is given by:

$$\int_{-\infty}^{\infty} x^2 f(x)\cdot dx = \frac{6}{125} \quad (4.21)$$

4.2 Logistic Function

The logistic or sigmoidal function is very commonly used to filter the frequency specturum of a dataset. The function is given by:

$$f(x) = \frac{L}{1+e^{-k(x-x_0)}} \quad (4.22)$$

k descibes the steepness of the curve, x_0 descibes the midpoint of the sigmoidal and L descibes the height of the function.

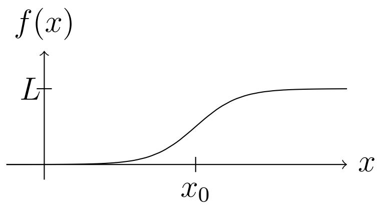


Figure 19: Logistic / Sigmoidal Function

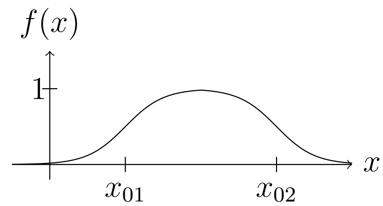


Figure 20: To sigmoidal curves combined

$$f(x) = \frac{1}{1 + e^{-k(x - x_{01})}} \cdot \left(1 - \frac{1}{1 + e^{-k(x - x_{02})}}\right) \tag{4.23}$$

Equation (4.23) shows a typical bandpass filter. The function is multiplied with the Fourier transform and only frequencies between x_{01} and x_{02} are kept.

4.3 Convolution in the Frequency Domain

One of the most useful properties of the Fourier transform is that the convolution of two functions becomes a simple multiplication in the frequency domain. A common application of this property is filtering a certain signal. One can use a logistic function as descibed in section 4.2 to create a Low-Pass or High-Pass filter. If one combines two sigmoidal functions, one can also design a filter which only lets some frequencies through. In figure 21, we see a function that has a small high frequency noise. To filter this noise, one can take the Fourier transform of the signal, multiply it with a logistic function and transform it back into the time domain. This yields the signal without the noise.

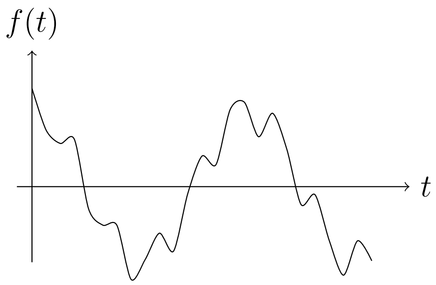


Figure 21: Unfiltered Signal

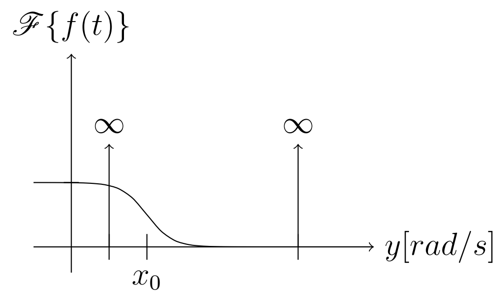


Figure 22: Sigmoidal function multiplied with the Fourier transform in the frequency domain

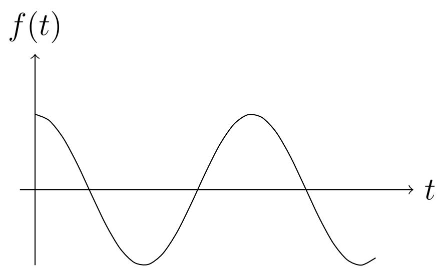


Figure 23: Filtered Signal

5 Differential Equations

5.1 Repetition Ordinary Differential Equations

As many principles of solving ordinary differential equations (ODEs) also apply to partial differential equations, which are covered in this course, it might be helpful to quickly recall some basics about solving ODEs. This is only a very short repetiton of some basic methods and it is hence advised to look at the notes from Analysis II if a repetition with more depth is required. It is essential to keep in mind that the result of a differential equation is always a function, unlike for commonly used algebraic equations where the solution is a set of numbers. The functions name is chosen arbitrarily. Some common choices are: $f, x, u \dots$ Those functions are usually functions of either time t or spacial coordinates $x, y, z \dots$. For an ODE, the equation is always of the following form:

$$\boxed{F(X,t) = G(t) \qquad (5.1)}$$

Here, x is the unknown function and t is the variable of the function x. The equation is called **homogeneous** if G(t)=0.

Examples:

$$\frac{dX}{dt} \cdot t^2 = \sin(t) \tag{5.2}$$

$$\frac{d^2X}{dt^2} + 3\frac{dX}{dt} = 0 \tag{5.3}$$

$$\frac{dX}{dt} + \sin(X) = t \tag{5.4}$$

Note that only example equation (5.3) is homogeneous. The other two examples have terms which are only dependent on the variable (t) but not on the function (X). Solving the equations above gives functions $X(t) = \dots$. Often, the dependency of the function on the variable is implied and therefore not written in equations (for example $\sin(X)$ instead of $\sin(X(t))$) The general concept of solving ODEs is always the same.

- Find a solution for the homogeneous version of the equation
- Find a particular solution for the inhomogeneous equation
  - Variation of Constants
  - Ansatz
  - ...
- Use boundary or initial conditions to determine constants

5.1.1 First Order Separable Differential Equation

Separable differential equations have the following general form:

$$\frac{dY}{dx} = \frac{g(x)}{h(Y)} \tag{5.5}$$

One can separate Y and x in this case and integrate with respect to x i.e. Y afterwards.

$$\int h(Y) \cdot dY = \int g(x) \cdot dx \tag{5.6}$$

Some rearanging brings the equation from above to the form $Y(x) = \dots$.

Example:

$$\frac{dY}{dx} = -\frac{x}{Y} \tag{5.7}$$

$$\int Y \cdot dY = -\int x \cdot dx \tag{5.8}$$

$$\frac{1}{2}Y^2 = -\frac{1}{2}x^2 + C_1 \tag{5.9}$$

$$Y(x) = \pm\sqrt{2C_1 - x^2} \tag{5.10}$$

Note that if one divides while rearanging the equation, it is necessary to look at the case of the term being zero separately.

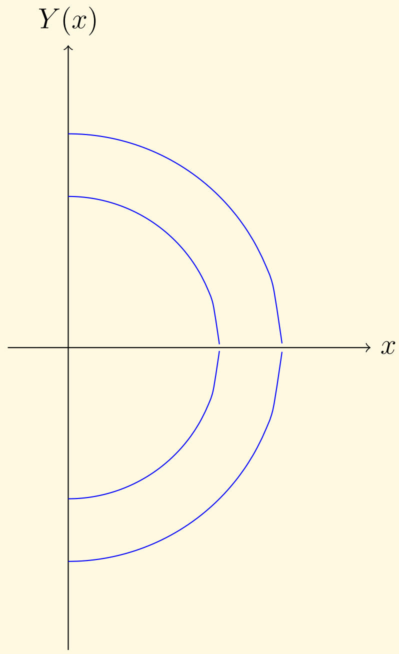


Figure 24: $Y(x)$ for $C_1 = 1, 2$

Example:

$$\frac{dY}{dx} = aY + b \qquad a, b \in \mathbb{R} \quad a \neq 0 \tag{5.11}$$

$$\frac{1}{aY+b} \cdot dY = 1 \cdot dx \quad \rightarrow \quad \int \frac{1}{aY+b} \cdot dY = \int 1 \cdot dx \tag{5.12}$$

$$\frac{1}{a} \ln(|aY+b|) = x + C_1 \quad \rightarrow \quad |aY+b| = e^{a(x+C_1)} = \underbrace{e^{aC_1}}_{C} e^{ax} \tag{5.13}$$

$$Y(x) = \frac{1}{a}(Ce^{ax} - b) \tag{5.14}$$

The case where one divides by zero during the procedure is being looked at separately. This would be the case if:

$$aY + b = 0 \quad \rightarrow \quad Y(x) = -\frac{b}{a} \tag{5.15}$$

In this case, this "special" solution is already included in the general one (for $C = 0$). Remember that C is determined by using the boundary conditions.

5.1.2 Substitution

It is often the case that differential equations are easier to solve, if one uses a substitution. Finding a useful substitution is not always easy but sometimes makes an equation separable.

Example:

$$\frac{dY}{dx} = (2x + 3Y)^2 \tag{5.16}$$

Define $u(x)$:

$$u(x) = 2x + 3Y \quad \rightarrow \quad Y = \frac{1}{3}(u(x) - 2x) \tag{5.17}$$

Derivative of Y in terms of $u(x)$

$$\frac{dY}{dx} = \frac{1}{3}(u'(x) - 2) \quad \rightarrow \quad \frac{1}{3}(u'(x) - 2) = u^2 \tag{5.18}$$

Combine with equation (5.16) and solve for $u(x)$

$$u'(x) = 3u^2 + 2 = \frac{du}{dx} \quad \rightarrow \quad \int \frac{1}{3u^2 + 2} \cdot du = \int 1 \cdot dx \tag{5.19}$$

$$\frac{1}{\sqrt{6}} \arctan\left(\sqrt{\frac{3}{2}} u\right) = x + C \quad \rightarrow \quad u = \sqrt{\frac{3}{2}} \tan(\sqrt{6}(x + C)) \tag{5.20}$$

Solve for $Y(x)$

$$u = 2x + 3Y = \sqrt{\frac{3}{2}} \tan(\sqrt{6}(x + C)) \tag{5.21}$$

$$Y(x) = \frac{1}{3}\left[\frac{\sqrt{3}}{2} \tan(\sqrt{6}(x + C)) - 2x\right] \tag{5.22}$$

5.1.3 Particular Solution

As mentioned before, there are different possibilities of getting the particular solution to an equation. A very common one is looking for a suitable Ansatz. It is usually a good idea to try functions which are similar to the inhomogeneous term (polinomials, trigonometric functions, exponential functions . . .).

Example:

$$\frac{dY}{dx} = \frac{1}{x}Y + 4x^2 \tag{5.23}$$

First, the homogeneous version is solved:

$$\frac{dY}{dx} = \frac{1}{x}Y \quad \rightarrow \quad Y_h(x) = C_1 x \tag{5.24}$$

Then an Ansatz with a constant A is tried:

$$Y_p = Ax^3 \tag{5.25}$$

Inserted into the equation, one gets:

$$3Ax^2 = \frac{1}{x}Ax^3 + 4x^2 \quad \rightarrow \quad a = 2 \quad \rightarrow \quad Y_p = 2x^2 \tag{5.26}$$

According to the superposition principle, the solution is then the sum of the homogeneous and the particular solution.

$$Y(x) = \underbrace{C_1 x}_{Y_h} + \underbrace{2x^2}_{Y_p} \tag{5.27}$$

Variation of Constants

Alternatively, one could also use the **variation of constants** principle. There, one assumes that the integration constant is not an actual constant but also a function of the variable.

An equation is given by:

$$y'(x) = p(x)y + q(x) \tag{5.28}$$

The homogeneous solution $y_h(x)$ is found for the homogeneous equation $y'(x) = p(x)y$. It is now assumed that the total solution is given by $y(x) = y_h(x) + y_p(x) = C(x) \cdot y_h(x)$, where $C(x)$ is an unknown function.

$$y(x) = C(x) \cdot y_h(x) \quad \rightarrow \quad y'(x) = C'(x) \cdot y_h(x) + C(x) \cdot y_h'(x) \tag{5.29}$$

$$y_h(x) = Ce^{\int p(x) \cdot dx} \qquad y_h'(x) = Cp(x)e^{\int p(x) \cdot dx} \tag{5.30}$$

Inserting $y(x)$ and $y'(x)$ into the equation gives:

$$C'(x) \cdot y_h(x) + C(x) \cdot y_h'(x) = p(x)C(x) \cdot y_h(x)) + q(x) \tag{5.31}$$

$$C'(x) \cdot e^{\int p(x) \cdot dx} + \cancel{C(x) \cdot p(x) e^{\int p(x) \cdot dx}} = \cancel{p(x)C(x) \cdot e^{\int p(x) \cdot dx}} + q(x) \tag{5.32}$$

$$C(x) = \int q(x) e^{-\int p(x) \cdot dx} \cdot dx \tag{5.33}$$

$$y(x) = C(x) e^{\int p(x) \cdot dx} \tag{5.34}$$

Example:

$$\frac{dY}{dx} = \frac{1}{x} Y + 4x^2 \tag{5.35}$$

The homogeneous solution from above is taken and it is assumed that C_1 is now a function of x.

$$Y(x) = C(x) \cdot x \tag{5.36}$$

The derivative of Y is then:

$$Y'(x) = C'(x) \cdot x + C(x) \tag{5.37}$$

This can be plugged into the equation:

$$C'(x) \cdot x + C(x) = \frac{1}{x} \underbrace{C(x) \cdot x}_{Y} + 4x^2 \quad \rightarrow \quad C'(x) = 4x \tag{5.38}$$

Integrating $C'(x)$ and plugging it back in the Ansatz gives the solution.

$$C(x) = \frac{4}{3} x^3 + C_1 \quad \rightarrow \quad Y(x) = x \cdot (2x^2 + C_1) = \underbrace{2x^2}_{Y_p} + \underbrace{C_1 x}_{Y_h} \tag{5.39}$$

Variation of Constants (Second Order)

For an equation of the form $f(x, y, y', y'') = q(x)$, the solutions of the homogeneous equation $f(x, y, y', y'') = 0$ are given by y_1, y_2. Calculating the integrals of equation (5.42) and (5.43) gives the functions $C_1(x)$ and $C_2(x)$. The solution is then given as:

$$y(x) = C_1(x) \cdot y_1(x) + C_2(x) \cdot y_2(x) \tag{5.40}$$

$$W = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \tag{5.41}$$

$$C_1(x) = -\int \frac{q(x)y_2(x)}{W(x)} \cdot dx \tag{5.42}$$

$$C_2(x) = \int \frac{q(x)y_1(x)}{W(x)} \cdot dx \tag{5.43}$$

Once the solution is found, it is usually a good idea to quickly take the derivatives and insert them into the equation to check if the solution actually solves it.

5.1.4 Exact Differential Equations

For a function $g : (x, y) \mapsto g(x, y)$, the contour lines are given by: $g(x, y) = const.$ Which differential equation describes this contour curve? The derivative of $g(x, y) = const.$ is

$$\boxed{g_x(x, y) + g_y(x, y)y' = 0} \tag{5.44}$$

This can be rearanged to:

$$y' = -\frac{g_x(x, y)}{g_y(x, y)} = f(x, y) \tag{5.45}$$

Example:
Derive the equation:

$$g(x, y) = x^2 - y^2 = c \tag{5.46}$$

$$2x - 2y \cdot y' = 0 \tag{5.47}$$

If a differential equation comes from a function $g(x, y)$, it has the following form:

$$\varphi(x, y) + \psi(x, y)y' = 0 \tag{5.48}$$

Comparing this to equation (5.44) yields $g_x = \varphi$ and $g_y = \psi$. This is equivalent to:

$$\nabla g(x, y) = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} g_x \\ g_y \end{pmatrix} \tag{5.49}$$

Recall Schwarz's theorem:

$$f(x, y)_{xy} = f(x, y)_{yx} \tag{5.50}$$

Hence, the following must hold:

$$\varphi_y = \psi_x \qquad \Leftrightarrow \qquad \text{rot} \begin{pmatrix} \varphi \\ \psi \\ 0 \end{pmatrix} = \vec{0} \tag{5.51}$$

Theorem: If there exists a function $g(x, y)$ for an equation of the form $\varphi(x, y) + \psi(x, y)y' = 0$ and the domain is simply connected, the equation is called exact. The solutions are contour lines of g.

Example:

$$\underbrace{(2x^2 - y^2 + y)}_{\varphi} + \underbrace{(-2xy + x - 4y)}_{\psi} y' = 0 \tag{5.52}$$

First it is important to check if $g_{xy} = g_{yx}$ i.e. $\varphi_y = \psi_x$.

$$\varphi_y = -2y + 1 = \psi_x \tag{5.53}$$

As the domain is chosen to be $\mathbb{R}^2$, which is simply connected, the equation is exact. Now, one simply needs to integrate φ and ψ.

$$g_x = \varphi = 2x^2 - y^2 + y \qquad g = \frac{2}{3}x^3 - y^2x + xy + A(y) \tag{5.54}$$

$$g_y = \psi = -2xy + x - 4y \qquad g = -xy^2 + xy - 2y^2 + B(x) \tag{5.55}$$

Combining the two equations yields:

$$g(x, y) = \frac{2}{3}x^3 - xy^2 + xy - 2y^2 + const. \tag{5.56}$$

The solutions are now contour lines of $g(x, y)$. The constant is found with the boundary conditions. For example it is given that y(1)=2:

$$-\frac{28}{3} = \frac{2}{3}x^3 - xy^2 + xy - 2y^2 \tag{5.57}$$

Solving this for y gives the result:

$$y(x) = \frac{x^2 \pm \sqrt{x^2 + 4(x+2)(\frac{2}{3}x^3 + \frac{28}{3})}}{2(x+2)} \tag{5.58}$$

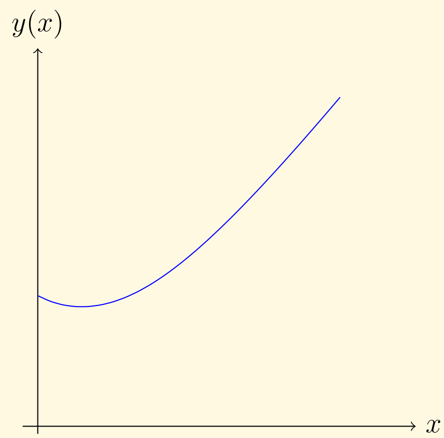


Figure 25: Function plot of the solution $y(x)$

5.1.5 Linear Differential Equations of Higher Orders

A differential equation of nth order for a function $x \mapsto y(x)$ is of the following form:

$$f(x, y, y', y'', \ldots, y^{(n)}) = 0 \tag{5.59}$$

Such an equation has exactly one solution $y(x)$ with n parameters that satisfies n boundary or initial conditions.

Linear Dependence

Linear dependence i.e. independence is an important property of functions. As known from the Linear Algebra lectures, a group of vectors is linear dependent if one can form the zero vector as a linear combination without having all coefficients equal to zero. The same also holds for functions. If a linear combination of functions gives zero for all x with coefficients $\neq 0$, the functions are linear dependent.

Theorem:
Let the set of functions $f_i(x)$ be differentiable on $[a, b]$. If the Wronskian is nonzero for some $x \in [a, b]$ then $f_i(x)$ are linearly independent on $[a, b]$. If $f_i(x)$ are linearly dependent, the Wronskian is zero for all $x \in [a, b]$. The Wronskian is defined as:

$$W\left(f_1, \ldots, f_n\right)(x) = \det \begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix} \tag{5.60}$$

Example:

$$C_1 \sin(x) + C_2 \cos(x) = 0 \tag{5.61}$$

$$W = \det \begin{pmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{pmatrix} = -\sin^2(x) - \cos^2(x) = -(\sin^2(x) + \cos^2(x)) = -1 \tag{5.62}$$

The Wronski determinante is always -1. Hence, sine and cosine functions are linearly independent.

Example:

$$C_1 x + C_2(x^2 + 3x) = 0 \tag{5.63}$$

$$W = \det \begin{pmatrix} x & x^2 + 3x \\ 1 & 2x + 3 \end{pmatrix} = 2x^2 + 3x - x^2 - 3x = x^2 \tag{5.64}$$

This is only zero for $x = 0$. The functions are hence linearly independent.

Example:

$$C_1x + C_2(x^2 + 3x) + C_3x^2 = 0 \tag{5.65}$$

$$W = \det \begin{pmatrix} x & x^2+3x & x^2 \\ 1 & 2x+3 & 2x \\ 0 & 2 & 2 \end{pmatrix} = x(4x + 6 - 4x) - (2x^2 + 6x - 2x^2) = 0 \tag{5.66}$$

Since this is zero for all x, the functions are linearly dependent.

Theorem:
For a homogeneous linear differential equation, the following holds:

- Every linear combination of solutions solves the differential equation.
- If n linear independent solutions are found, the array of functions with n parameters forms the general solution.

Theorem: The general solution of an inhomogeneous equation has the form of:

$$y(x) = y_h(x) + y_p(x) \tag{5.67}$$

5.1.5.1 Homogeneous Linear Differential Equation with Constant Coefficients

The general form of such equations is:

$$\boxed{y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y_1' + a_0y = 0} \tag{5.68}$$

There are n linear independent solutions to this equation. The solution is found with an Ansatz $y(x) = C_ie^{\lambda_i x}$. Plugged into the equation, this yields:

$$\underbrace{e^{\lambda x}}_{\neq 0}\underbrace{(\lambda^n + a_{n-1}\lambda^{n-1} + \ldots a_1\lambda + a_0)}_{0} = 0 \tag{5.69}$$

Hence, one has to solve the polynomial of λ to find the values of λ_i. This equation is also called the **characteristic equation**. The general procedure to find solutions is:

1. Replace $y^{(n)}$ with λ^n to get the characteristic equation.
2. Solve the characteristic equation to find the values of λ_i
3. Write down the solution:
   - If all zeros of the characteristic equation are different and real, the solution is:

$$y_h(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \cdots + C_n e^{\lambda_n x} \tag{5.70}$$

   - If two zeros are the same, i.e. $\lambda_1 = \lambda_2$, the solution is:

$$y_h(x) = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x} + C_3 e^{\lambda_3} + \dots \tag{5.71}$$

   - If m zeros are the same, the solution is:

$$y_h(x) = \left(C_1 + C_2 x + C_3 x^2 + \cdots + C_m x^{m-1}\right) e^{\lambda_m x} + \cdots \tag{5.72}$$

   - If $\lambda_i \in \mathbb{C}$, then $\bar{\lambda}_i$ is also a zero. The solution is then for $\lambda_{1,2} = \alpha \pm \beta \mathring{\imath}$:

$$y_h(x) = e^{\alpha x}\Bigg[C_1 \sin(\beta x) + C_2 \cos(\beta x)\Bigg] + \cdots \tag{5.73}$$

   - If multiple complex zeros are the same ($\lambda_1 = \lambda_2 = \cdots = \lambda_m = \alpha \pm \beta \mathring{\imath}$), the solution is:

$$\begin{aligned} y_h(x) =& e^{\alpha x}\Bigg(C_1 \sin(\beta x) + C_2 \cos(\beta x) + x\Bigg[C_3 \sin(\beta x) + C_4 \cos(\beta x)\Bigg] + \cdots \\ &+ x^{m-1}\Bigg[C_{2m+1} \sin(\beta x) + C_{2m+2} \cos(\beta x)\Bigg]\Bigg) + \cdots \end{aligned} \tag{5.74}$$

5.1.5.2 Homogeneous Eulerian Differential Equation

Eulerian differential equations have the following form:

$$x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} \dots a_3 x^3 y''' + a_2 x^2 y'' + a_1 x y' + a_0 y = 0 \tag{5.75}$$

To find a solution, the Ansatz $y = x^\lambda$ is taken. Plugged into the equation, this yields:

$$x^{\lambda-n}(\lambda^{[n]} + a_{n-1}\lambda^{[n-1]} \dots a_1 \lambda^{[1]} + a_0) = 0 \tag{5.76}$$

$\lambda^{[i]}$ denotes the following:

$$\lambda^{[n]} = \lambda(\lambda-1)(\lambda-2)\dots(\lambda-(n-1)) = \prod_{i=0}^{n-1}(\lambda - i) \tag{5.77}$$

1. Write down the index polynomial and find its zeros.
2. Write down the solution
   - If all λ_i are different:

$$y(x) = C_1 x^{\lambda_1} + C_2 x^{\lambda_2} + C_3 x^{\lambda_3} + \dots \tag{5.78}$$

   - If m zeros are the same i.e. $\lambda_1 = \lambda_2 = \dots = \lambda_m$:

$$\begin{aligned} y(x) =& C_1 x^{\lambda_1} + C_2 \Big[\log(x)\Big] x^{\lambda_1} + C_3 \Big[\log(x)\Big]^2 x^{\lambda_1} + \dots \\ &+ C_m \Big[\log(x)\Big]^{m-1} x^{\lambda_1} + C_{m+1} x^{\lambda_{m+1}} + \dots \end{aligned} \tag{5.79}$$

- If m complex zeros of the form $\alpha \pm \mathring{\imath}\beta$ are the same:

$$\begin{aligned} y(x) =& C_1 x^\alpha \cos(\beta \log(x)) + C_2 x^\alpha \sin(\beta \log(x)) \\ &+ C_3 \left[\log(x) \right] x^\alpha \cos(\beta \log(x)) \\ &+ C_4 \left[\log(x) \right] x^\alpha \sin(\beta \log(x)) + \cdots + \\ &+ C_{2m-1} \left[\log(x) \right]^{m-1} x^\alpha \cos(\beta \log(x)) \\ &+ C_{2m} \left[\log(x) \right]^{m-1} x^\alpha \sin(\beta \log(x)) \end{aligned} \tag{5.80}$$

5.1.6 Systems of Differential Equations

A system first order differential equations has the general form of:

$$\begin{cases} y_1' = f_1(x, y_1, \dots, y_n) \\ y_2' = f_2(x, y_1, \dots, y_n) \\ \vdots \end{cases} \tag{5.81}$$

The solution is a set of functions that satisfy all equations. There are boundary/initial conditions for all equations. If the system is only dependend on $y_1, y_2, y_3 \dots$ but not on terms of x, the system is called **autonomous**.

5.1.6.1 Higher Order Equation ⇒ First Order System

Sometimes, it is convenient to rewrite a higher order differential equation into a first order system of equations. Therefore, one chooses $y_1 = y, y_2 = y', y_3 = y'' \dots$ For a general equation of the form:

$$y^{(n)} = a_0 y + a_1 y' + a_2 y'' + \dots \tag{5.82}$$

One can rewrite the equation into the following system:

$$\begin{pmatrix} y \\ y' \\ y'' \\ \vdots \\ y^{(n-1)} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} \qquad \begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \\ y''' \\ \vdots \\ y^{(n)} \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ y_4 \\ \vdots \\ a_0y_1 + a_1y_2 + a_2y_3 \dots \end{pmatrix} \tag{5.83}$$

$$\underbrace{\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ \vdots \\ y_n' \end{pmatrix}}_{\vec{y}'} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix}}_{\vec{y}} \tag{5.84}$$

5.1.6.2 First Order System $\Rightarrow$ Higher Order Equation

5.1.6.3 Linear Autonomous System of Differential Equations

A very common form of systems of differential equtions is a linear autonomous system with constant coefficients. It has the form of

$$\dot{\vec{y}} = A\vec{y} + \vec{b} \tag{5.85}$$

The solutions is given by

$$\boxed{\vec{x}(t) = \sum_i C_i \vec{v}_i e^{\lambda_i t}} \tag{5.86}$$

where C_i are the constants given by the initial conditions, $\vec{v}_i$ are the Eigenvectors of A and λ_i are the Eigenvalues of A.

Example:

$$\begin{cases} \dot{y}_1 = 1y_1 + 3y_2 \\ \dot{y}_2 = 2y_1 + 2y_2 \\ y_1(0) = 0 \\ y_2(0) = 5 \end{cases} \qquad A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \tag{5.87}$$

$\det(A-\lambda\mathbb{I}) = 0$ yields the Eigenvalues $\lambda_1 = 4, \lambda_2 = -1$. $(A-\lambda\mathbb{I})\vec{x} = 0$ gives the eigenvectors $\vec{v}_1 = (1\ 1)^T, \vec{v}_2 = (3\ \ -2)^T$

$$\vec{y}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + C_2 \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{-t} \tag{5.88}$$

Using the initial conditions gives $C_1 = 3, C_2 = -1$.

$$\rightarrow \vec{y}(t) = \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^{4t} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} e^{-t} \tag{5.89}$$

5.2 Partial Differential Equations

Unlike ordinary differential equations, partial differential equations are equations of functions that are dependent on more than one variable. Hence, all derivatives are partial derivatives. Usually, a notation with the partial deltas is used. Because this notation requires a lot of writing, a short hand notation is commonly used. There, a subscript denotes the variable of the partial derivative.

Example:

$$\frac{\partial f}{\partial x_i} = f_{x_i} \qquad \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \qquad \frac{\partial^2 f}{\partial x^2} = f_{xx} \tag{5.90}$$

Superposition Principle

If u_1 and u_2 are solutions of the homogeneous PDE, $\alpha u_1 + \beta u_2$ are also solutions.

Additional Conditions

To get a unique solution, one needs boundary and initial conditions.

5.2.1 Classification of 2D 2nd Order PDEs

For many partial differential equations, no analytical solution is known. In order to solve a PDE, it can be helpful to look up, if any solution for an equation of a similar form is known. To make it easier to find a solution to an equation, the equations are classified. A two dimensional partial differential equation of 2nd order has the following general form:

$$Au_{\xi\xi}(\xi,\eta) + 2Bu_{\xi\eta}(\xi,\eta) + Cu_{\eta\eta}(\xi,\eta) + Fu_{\eta}(\xi,\eta) = 0 \tag{5.91}$$

- $AC - B^2 < 0$: **Hyperbolic** equation (Example: Wave equation)
- $AC - B^2 = 0$: **Parabolic** equation (Example: Heat equation)
- $AC - B^2 > 0$: **Eliptic** equation (Example: Laplace equation)

5.2.2 Pseudo PDEs

These partial differential equations are not actually called "Pseudo PDEs" but I just call them like that because they are solved like ODEs. One can easily spot such equations because they are about functions of multiple variables but only involve derivatives with respect to one variable. In this case, the variable which does not occur in any derivative can be treated as a normal coefficient. The following examples shall illustrate how to solve such equations.

Example:

$$u(x,y)_{yy} = 4x \cdot u(x,y)_y \tag{5.92}$$

This equation has no derivatives with respect to x. Hence, we can treat x as a coefficient. First we substitute $u_y = v$.

$$v_y = 4xv \qquad \frac{dv}{dy} = 4xv \tag{5.93}$$

$$\frac{1}{v} \cdot dv = 4x \cdot dy \qquad \ln(v) = 4xy + C(x) \tag{5.94}$$

$$\rightarrow v(x,y) = C_1(x)e^{4xy} = u_y = \frac{du}{dy} \tag{5.95}$$

$$u(x,y) = \int C_1(x)e^{4xy} \cdot dy = \frac{C_1(x)}{4x}e^{4xy} + C_2(x) \tag{5.96}$$

Example:

$$u(x,y)_{yy} + 2x \cdot u(x,y)_y + u(x,y) = 0 \tag{5.97}$$

$$\lambda^2 + 2x\lambda + 1 = 0 \tag{5.98}$$

$$\lambda_{1,2} = \frac{-2x \pm \sqrt{4x^2 - 4}}{2} = -x \pm \sqrt{x^2 - 1} \tag{5.99}$$

$$u(x, y) = Ae^{\lambda_1 y} + Be^{\lambda_2 y} = Ae^{(-x+\sqrt{x^2-1})y} + Be^{(-x-\sqrt{x^2-1})y} \tag{5.100}$$

5.2.3 General Solution of the Wave Equation

The wave equation is given by:

$$u(x, t)_{tt} = c^2 u(x, t)_{xx} \tag{5.101}$$

The general solution is given by:

$$\boxed{u(x, t) = \phi(x + ct) + \psi(x - ct)} \tag{5.102}$$

If one wants to calculate derivatives of those functions, one must not forget to apply the chain rule.

$$\phi_t = \phi' \cdot c \qquad \phi_{tt} = \phi'' \cdot c^2 \qquad \psi_t = -c \cdot \psi' \qquad \psi_{tt} = c^2 \cdot \psi'' \qquad u_{xx} = \phi'' + \psi'' \tag{5.103}$$

$$\rightarrow u_{tt} = \phi_{tt} + \psi_{tt} = c^2(\phi'' + \psi'') = c^2 u_{xx} \tag{5.104}$$

Examples:

$$u(x, t) = (x + ct)^2 + \sqrt{x - ct} \tag{5.105}$$

$$u(x, t) = sin(x + ct) + e^{x-ct} \tag{5.106}$$

5.2.4 1D-Wave Equation with Fourier Series

Given is the wave equation and the corresponding initial and boundary conditions.

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0, t) = u(L, t) = 0 \\ u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{cases} \tag{5.107}$$

Solving these kind of problems is not specifically difficult. One has to be careful to find all solutions though, otherwise one can easily forget some. Because it is a rather long process, it can get quite confusing midway through the calculation if one loses one's train of thought. For this reason I advise you to solve such equations always step by step and write everything down nicely. This will not only secure you partial credit but will also help to keep an overview over the solution. It is usually not advisable to skip steps or do them in your head because writing them down does not take very long compared to looking for a mistake once something doesn't work out anymore. The process how I solve a problem like that is shown with the example from above.

Step 1
First an ansatz is used. In this case, it is the **separation of variables** which is probably the most commonly used ansatz for most PDEs.

Ansatz: Separation of Variables

$$u(x,t) = F(x)G(t) \tag{5.108}$$

The required derivatives are then calculated with the Ansatz and are then plugged into the equation.

$$u_{tt} = F\ddot{G} \qquad u_{xx} = F''G \tag{5.109}$$

$$u_{tt} = c^2 u_{xx} \qquad \rightarrow \qquad F\ddot{G} = c^2 F''G \tag{5.110}$$

Rearanging gives

$$\frac{1}{c^2}\frac{\ddot{G}(t)}{G(t)} = \frac{F''(x)}{F(x)} = const. = k \tag{5.111}$$

Because equation (5.111) must hold $\forall x, t$ it can only be constant.

$$\begin{cases} F''(x) = kF(x) \\ \ddot{G}(t) = c^2 kG(t) \end{cases} \tag{5.112}$$

Step 2
Now the separate ordinary differential equations are solved. It is important to make a case differentiation for the values of k. Once a solution is obtained, the boundary conditions must be fulfilled.

$F''(x) = kF(x)$

- **k=0**

$$F''(x) = 0 \rightarrow F(x) = ax + b \tag{5.113}$$

Using the boundary conditions gives:

$$u(0,t) = F(0)G(t) = 0 \ \forall t \geq 0 \ \rightarrow F(0) = 0 \tag{5.114}$$

$$u(L,t) = F(L)G(t) = 0 \ \forall t \geq 0 \ \rightarrow F(L) = 0 \tag{5.115}$$

$$F(0) = F(L) = 0 \rightarrow F(x) = 0 \tag{5.116}$$

$$\boxed{F(x) = 0} \tag{5.117}$$

- **k>0**

$$F''(x) - kF(x) = 0 \ \rightarrow \ \lambda^2 - k = 0 \ \rightarrow \ \lambda = \pm\sqrt{k} \tag{5.118}$$

$$\rightarrow \ F(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x} \tag{5.119}$$

Using the boundary conditions gives:

$$u(0,t) = F(0)G(t) = 0 \ \forall t \geq 0 \ \rightarrow F(0) = 0 \tag{5.120}$$

$$u(L,t) = F(L)G(t) = 0 \ \forall t \geq 0 \ \rightarrow F(L) = 0 \tag{5.121}$$

$$F(0) = 0: \ A + B = 0 \ \rightarrow \ B = -A \tag{5.122}$$

$$F(L) = 0: \ \underbrace{A}_{=0}\underbrace{(e^{\sqrt{k}L} - e^{-\sqrt{k}L})}_{=0} = 0 \tag{5.123}$$

In order to satisfy the previous equation, either A=0 ($\rightarrow$ A=B=0) or the sum of exponential functions is zero. In that case $e^{2\sqrt{k}L} = 0$ which is only possible if k=0. Because we look at the case where k>0, this is not possible. Hence the only solution for this case is A=B=0.

$$\boxed{F(x) = 0} \tag{5.124}$$

- **k<0**

$$F''(x) - kF(x) = 0 \ \rightarrow \ \lambda^2 - k = 0 \ \rightarrow \ \lambda = \pm\sqrt{k} \tag{5.125}$$

$$\rightarrow \; F(x) = A\cos(\sqrt{-k}x) + B\sin(\sqrt{-k}x) \tag{5.126}$$

Using the boundary conditions gives:

$$u(0,t) = F(0)G(t) = 0 \; \forall t \geq 0 \; \rightarrow F(0) = 0 \tag{5.127}$$

$$u(L,t) = F(L)G(t) = 0 \; \forall t \geq 0 \; \rightarrow F(L) = 0 \tag{5.128}$$

$$F(0) = A = 0 \qquad F(L) = B\sin(\sqrt{-k}L) = 0 \tag{5.129}$$

The second boundary condition is satisfied if B=0 or if $\sin(\sqrt{-k}L) = 0$. The later one is the case if $\sqrt{-k}L = n\pi$. Rewritten: $\sqrt{-k} = \frac{n\pi}{L}$

$$\boxed{F(x) = 0} \qquad \boxed{F(x) = B\sin\left(\frac{n\pi}{L}x\right)} \tag{5.130}$$

$\ddot{G}(t) = c^2 k G(t)$

- **k=0**
  Because $F(x) = 0$, G(t) is not relevant as $u(x,t) = F(x)G(t)$
- **k>0**
  Because $F(x) = 0$, G(t) is not relevant as $u(x,t) = F(x)G(t)$
- **k<0**

$$\ddot{G}(t) - kc^2 G(t) = 0 \;\rightarrow\; \lambda^2 - kc^2 = 0 \;\rightarrow\; \lambda = \pm c\sqrt{k} = \pm \mathring{\imath} c\sqrt{-k} \tag{5.131}$$

$$G(t) = C\cos(c\sqrt{-k}t) + D\sin(c\sqrt{-k}t) \tag{5.132}$$

Before, we defined that $\sqrt{-k} = \frac{n\pi}{L}$. This can be plugged into our function G(t) which then gives:

$$\boxed{G(t) = C\cos\left(c\frac{n\pi}{L}t\right) + D\sin\left(c\frac{n\pi}{L}t\right)} \tag{5.133}$$

Step 3:
Combine the individual solutions.

$$u(x,t) = F(x)G(t) = B\sin\left(\frac{n\pi}{L}x\right)\left(C\cos\left(c\frac{n\pi}{L}t\right) + D\sin\left(c\frac{n\pi}{L}t\right)\right) \tag{5.134}$$

The function above solves the equation for every n. Since linear combinations of solutions are solutions as well, we have to sum over all n to get the most general form of the solution.

Note that all other cases ($k = 0, k > 0$) had the solution $u(x,t) = 0$ which is included in the solution above already.

$$u(x,t) = F(x)G(t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)\left(C_n \cos\left(c\frac{n\pi}{L}t\right) + D_n \sin\left(c\frac{n\pi}{L}t\right)\right) \tag{5.135}$$

Step 4:

Now, the initial conditions are used to find a unique solution.

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \tag{5.136}$$

Plugging the first initial condition into the combined solution gives:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)\left(C_n \underbrace{\cos\left(c\frac{n\pi}{L}t\right)}_{=1} + D_n \underbrace{\sin\left(c\frac{n\pi}{L}t\right)}_{=0}\right) \tag{5.137}$$

To make everything simpler, we can summarise the constants $B_nC_n = C_{1n}, B_nD_n = C_{2n}$. This gives:

$$f(x) = \sum_{n=1}^{\infty} C_{1n} \sin\left(\frac{n\pi}{L}x\right) \tag{5.138}$$

This looks like a Fourier series of the function $f(x)$ where $a_n = 0$. Hence the function is uneven. From equation (2.7), we know how to get the coefficients C_{1n}.

$$C_{1n} = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L}x\right) \cdot dx \tag{5.139}$$

Because the function is uneven, we can integrate from zero to L and multiply the integral with 2 instead of integrating from -L to L. The same procedure is now done with the second initial condition.

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(-\frac{cn\pi}{L}C_{1n}\sin\left(\frac{cn\pi}{L}t\right) + \frac{cn\pi}{L}C_{2n}\cos\left(\frac{cn\pi}{L}t\right)\right) \tag{5.140}$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \underbrace{C_{2n}\frac{cn\pi}{L}}_{B_n} \sin\left(\frac{n\pi}{L}x\right) \tag{5.141}$$

We can summarise the term in front of the sine function as B_n because it's just a term dependent on n. The equation then looks again like a Fourier Series of an uneven function. The coefficients C_{2n} can be found by solving the following integral.

$$B_n = \frac{1}{L}\int_{-L}^{L} g(x) \sin\left(\frac{n\pi}{L}x\right) \cdot dx = C_{2n}\frac{cn\pi}{L} \tag{5.142}$$

$$C_{2n} = \frac{1}{cn\pi} \int_{-L}^{L} g(x) \sin\left(\frac{n\pi}{L} x\right) \cdot dx \tag{5.143}$$

Once C_{1n}, C_{2n} are known, the solution can be written down.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L} x\right) \left(C_{1n} \cos\left(c \frac{n\pi}{L} t\right) + C_{2n} \sin\left(c \frac{n\pi}{L} t\right)\right)} \tag{5.144}$$

Note:

If the function $f(x)$ of the initial condition is already a sine function, there is no need to calculate the Fourier coefficients with the integrals. The coefficients can be taken directely from the function.

Example:

The initial condition is given by:

$$\begin{cases} u(x,0) = f(x) = k \sin(\pi x) - \frac{1}{2} \sin(2\pi x) \\ u_t(x,0) = g(x) = 0 \end{cases} \tag{5.145}$$

$$f(x) = \sum_{n=1}^{\infty} C_{1n} \sin\left(\frac{n\pi}{L} x\right) = C_{11} \sin(\pi x) + C_{12} \sin(2\pi x) + C_{13} \sin(3\pi x) + \dots \tag{5.146}$$

Comparing the coefficients gives:

$$\rightarrow \qquad C_{11} = k, C_{12} = -\frac{1}{2}, C_{1n} = 0 \text{ for } n = 3, 4, 5, \dots \tag{5.147}$$

For the second initial condition, see that the integral (5.143) is zero if $g(x) = 0$. The solution can hence just be written down.

$$u(x,t) = k \cos(\pi t) \sin(\pi x) - \frac{1}{2} \cos(2\pi t) \sin(2\pi x) \tag{5.148}$$

5.2.5 Heat Equation on a Finite Bar

The equation is given with the following boundary and initial conditions:

$$\begin{cases} u_t = c^2 u_{xx} \\ u(x,0) = f(x) \\ u(0,t) = u(L,t) = 0 \end{cases} \tag{5.149}$$

Step 1:
First an Ansatz is chosen. In this case, the Ansatz is again the separation of variables.

$$u(x,t) = F(x)G(t) \tag{5.150}$$

$$u_t = F(x)\dot{G}(t) \qquad u_{xx} = F''(x)G(t) \tag{5.151}$$

Inserting this into the equation gives:

$$F(x)\dot{G}(t) = c^2 F''(x)G(t) \tag{5.152}$$

$$\rightarrow \frac{\dot{G}(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = const. = -k^2 \tag{5.153}$$

Because the equation above must hold for every t and x, it can only be constant. We choose this constant to be $-k^2$. This choice is arbitrary but is turns out to be convenient later on.

Step 2:
First the F-equation is solved. $F''(x) + k^2 F(x) = 0$

- **k=0**

$$F''(x) = 0 \qquad\qquad F(x) = ax + b \tag{5.154}$$

  Using the boundary conditions gives:

$$u(0,t) = F(0)G(t) = 0 \ \forall t \geq 0 \ \rightarrow F(0) = 0 \tag{5.155}$$

$$u(L,t) = F(L)G(t) = 0 \ \forall t \geq 0 \ \rightarrow F(L) = 0 \tag{5.156}$$

$$F(0) = a = 0 \qquad F(L) = \underbrace{a}_{0} L + b = 0 \qquad a = b = 0 \tag{5.157}$$

$$\boxed{F(x) = 0} \tag{5.158}$$

- **k≠0**

$$F''(x) + k^2 F(x) = 0 \tag{5.159}$$

$$\lambda^2 + k^2 = 0 \qquad \lambda = \pm \mathring{\imath} k \tag{5.160}$$

$$F(x) = A\cos(kx) + B\sin(kx) \tag{5.161}$$

Using the boundary conditions gives:

$$u(0,t) = F(0)G(t) = 0\ \forall t \geq 0\ \rightarrow F(0) = 0 \tag{5.162}$$

$$u(L,t) = F(L)G(t) = 0\ \forall t \geq 0\ \rightarrow F(L) = 0 \tag{5.163}$$

$$F(0) = A\underbrace{\cos(0)}_{=1} + B\underbrace{\sin(0)}_{=0} = 0 \qquad \rightarrow A = 0 \tag{5.164}$$

$$F(L) = B\sin(kL) = 0 \qquad \rightarrow B = 0 \text{ or } \sin(kL) = 0 \tag{5.165}$$

If $\sin(kL) = 0$, the values of kL are equal to $kL = n\pi$. This gives $k = \frac{n\pi}{L}$.

$$\boxed{F(x) = B\sin\left(\frac{n\pi}{L}x\right)} \tag{5.166}$$

Note that here, we do not differenciate between $k < 0$ and $k > 0$ because the constant is k^2 either way.

Step 3: Now we solve the G-Equation. $\dot{G}(t) + k^2c^2G(t) = 0$

- **k=0**
  In this case, F(x) is zero. Hence the function G(t) is not relevant.

- **k≠0**

$$\dot{G}(t) + k^2c^2G(t) = 0 \qquad \lambda + k^2c^2 = 0 \qquad \rightarrow \lambda = -k^2c^2 \tag{5.167}$$

$$G(t) = Ae^{-k^2c^2t} \tag{5.168}$$

Using the definition of k from the F-equation gives:

$$\boxed{G(t) = Ae^{-\frac{n^2\pi^2}{L^2}c^2t}} \tag{5.169}$$

Step 4:
In the next step, both solutions are combined.

$$u(x,t) = F(x)G(t) = Ae^{-\frac{n^2\pi^2}{L^2}c^2 t}\sin\left(\frac{n\pi}{L}x\right) \tag{5.170}$$

Don't forget that we are looking for the complete general solution. That is why we have to include all possible solutions as a linear combination. Therefore, we sum up over all n.

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2}{L^2}c^2 t}\sin\left(\frac{n\pi}{L}x\right) \tag{5.171}$$

Step 5: Now, the initial conditions are used to determine a unique solution.

$$u(x,0) = f(x) \tag{5.172}$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) = f(x) \tag{5.173}$$

This again looks like a Fourier series of an uneven function. The coefficients A_n can be found by solving the following integral:

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\cdot dx \tag{5.174}$$

The solution is:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2}{L^2}c^2 t}\sin\left(\frac{n\pi}{L}x\right)} \tag{5.175}$$

5.2.6 Laplace Equation on a Rectangle

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = u(a,y) = u(x,0) = 0 \\ u(x,b) = f(x) \end{cases} \tag{5.176}$$

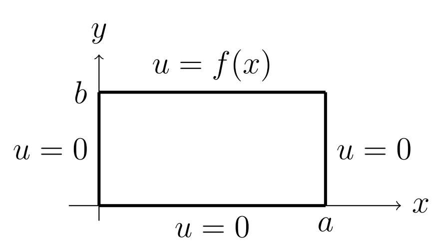


Figure 26: Boundary conditions

Step 1:

Again, the separation of variables Ansatz is used to solve the equation.

$$u(x,y) = F(x)G(y) \tag{5.177}$$

$$u_{xx} = F''(x)G(y) \qquad u_{yy} = F(x)G''(y) \tag{5.178}$$

Inserted into the equation, this gives:

$$F''(x)G(y) = -F(x)G''(y) \qquad \frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = -k^2 \tag{5.179}$$

Step 2:

F-Equation

- **k=0**

$$F''(x) = 0 \qquad F(x) = Ax + B \tag{5.180}$$

Using the boundary conditions gives:

$$u(0,y) = F(0)G(y) = 0 \ \forall y \geq 0 \ \rightarrow F(0) = 0 \tag{5.181}$$

$$u(a,y) = F(a)G(y) = 0 \ \forall y \geq 0 \ \rightarrow F(a) = 0 \tag{5.182}$$

$$F(0) = B = 0 \qquad F(a) = Aa = 0 \qquad A = B = 0 \tag{5.183}$$

$$\boxed{F(x) = 0} \tag{5.184}$$

- **k≠0**

$$F''(x) + k^2F(x) = 0 \qquad \lambda^2 + k^2 = 0 \qquad \lambda = \pm \mathring{\imath}k \tag{5.185}$$

$$F(x) = A\cos(kx) + B\sin(kx) \tag{5.186}$$

Using the boundary conditions gives:

$$u(0,y) = F(0)G(y) = 0 \ \forall y \geq 0 \ \rightarrow F(0) = 0 \tag{5.187}$$

$$u(a,y) = F(a)G(y) = 0 \ \forall y \geq 0 \ \rightarrow F(a) = 0 \tag{5.188}$$

$$\begin{aligned} &F(0) = A\underbrace{\cos(kx)}_{=1} + B\underbrace{\sin(kx)}_{=0} = 0 && \rightarrow A = 0 \\ &F(a) = B\sin(ka) = 0 && \rightarrow B = 0 \text{ or } ka = n\pi \end{aligned} \tag{5.189}$$

$$F(x) = B\sin\left(\frac{n\pi}{a}x\right) \tag{5.190}$$

Step 3:
G-Equation

- **k=0**
  Since $F(x) = 0$ in this case, we don't need to look for the solution of $G(y)$.

- **k≠0**

$$G''(y) - k^2G(y) = 0 \tag{5.191}$$

$$\lambda^2 - k^2 = 0 \qquad \lambda = \pm k \tag{5.192}$$

$$G(y) = Ae^{ky} + B^{-ky} \tag{5.193}$$

Sometimes it is convenient to rewrite exponential functions into hyperbolic functions. The procedure of solving the equation is the same and the result will also stay the same.

$$\begin{aligned} Ae^{ky} + Be^{-ky} &= \left(\frac{C_1}{2} + \frac{C_2}{2}\right)e^{ky} + \left(-\frac{C_1}{2} + \frac{C_2}{2}\right)e^{-ky} \\ &= C_1\frac{e^{ky} - e^{-ky}}{2} + C_2\frac{e^{ky} + e^{-ky}}{2} = C_1\sinh(ky) + C_2\cosh(ky) \end{aligned} \tag{5.194}$$

Using the 3rd boundary condition gives:

$$u(x, 0) = F(x)G(0) = 0 \ \forall x \geq 0 \ \rightarrow G(0) = 0 \tag{5.195}$$

$$G(0) = C_1\underbrace{\sinh(0)}_{0} + C_2\underbrace{\cosh(0)}_{=1} = 0 \qquad C_2 = 0 \tag{5.196}$$

$$\boxed{G(y) = C_1\sinh\left(\frac{n\pi}{a}y\right)} \tag{5.197}$$

Step 4:

Now the separate solutions are combined. Again, keep in mind that we want all solutions and hence sum over all n. The constants C_1 and B are summarised as C_n.

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi}{a}y\right) \sin\left(\frac{n\pi}{a}x\right) \tag{5.198}$$

Step 5:

Using the last boundary condition will give us a unique solution.

$$u(x,b) = f(x) \tag{5.199}$$

$$u(x,b) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi}{a}b\right) \sin\left(\frac{n\pi}{a}x\right) = f(x) \tag{5.200}$$

This looks like a Fourier series again. The coefficients C_n can hence be found by solving the following integral:

$$C_n = \frac{1}{\sinh\left(\frac{n\pi}{a}b\right)} \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi}{a}x\right) \cdot dx \tag{5.201}$$

The solution is then:

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi}{a}y\right) \sin\left(\frac{n\pi}{a}x\right)} \tag{5.202}$$

Remark:

How to get from equation (5.200) to equation (5.201)? For an uneven function, the definition of the Fourier series and its coefficients was:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \qquad b_n = \frac{1}{L} \cdot \int_{-L}^{L} f(x) \cdot \sin\left(\frac{n\pi}{L}x\right) \cdot dx \tag{5.203}$$

$C_n \sinh\left(\frac{n\pi}{a}b\right)$ is a number dependent on n. We can summarise it as b_n. We then have:

$$u(x,b) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{a}x\right) = f(x) \tag{5.204}$$

According to the definition in equation (5.203) we get:

$$b_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi}{a}x\right) \cdot dx = C_n \sinh\left(\frac{n\pi}{a}b\right) \tag{5.205}$$

Rearranging the equation above gives equation (5.201).

5.2.7 D'Alembert Solution of the Wave Equation

The D'Alembert solution is used if no boundary conditions are given. It is based on the idea of a substitution with the varibales $v = x + ct$ and $w = x - ct$. Using those variables in the wave equations gives

$$u_{vw} = 0 \tag{5.206}$$

Integrating gives:

$$u(v, w) = \varphi(v) + \psi(w) = u(x, t) = \varphi(x + ct) + \psi(x - ct) \tag{5.207}$$

Now the following initial conditions are given:

$$\begin{cases} u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{cases} \tag{5.208}$$

The solution is then:

$$\boxed{u(x, t) = \frac{1}{2}\Big(f(x + ct) + f(x - ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s) \cdot ds} \tag{5.209}$$

Example:

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x, 0) = \sin(x) \\ u_t(x, 0) = e^x \end{cases} \tag{5.210}$$

$$u(x, t) = \frac{1}{2}\Big(\sin(x + ct) + \sin(x - ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} e^s \cdot ds \tag{5.211}$$

$$u(x, t) = \frac{1}{2}\Big(\sin(x + ct) + \sin(x - ct)\Big) + \frac{1}{2c}\Big(e^{x+ct} - e^{x-ct}\Big) \tag{5.212}$$

Remark: The D'Alembert solution is based on the method of characteristics which will be discussed in the next section.

5.2.8 Method of Characteristics

The method of characteristics can be used for equations that can be brought into the following form:

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y) \tag{5.213}$$

To solve such equations, one has to simply follow the steps described below:

Step 1:
Bring the equation into the following form:

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y) \tag{5.214}$$

Step 2:
Determine the type of the equation:

$$\begin{aligned} AC - B^2 &= 0 && \text{Parabolic} \\ AC - B^2 &< 0 && \text{Hyperbolic} \\ AC - B^2 &> 0 && \text{Elliptic} \end{aligned} \tag{5.215}$$

Step 3:
Write down the characteristic equation:

$$A(y')^2 - 2B(y') + C = 0 \tag{5.216}$$

$$y' = \frac{B \pm \sqrt{B^2 - AC}}{A} = \lambda_{1,2} \tag{5.217}$$

Set

$$y = \lambda_1 x + C_1 \qquad y = \lambda_2 x + C_2 \tag{5.218}$$

Step 4:
Determine C as a function of x,y.

$$\begin{aligned} C_1 &= y - \lambda_1 x && \rightarrow \Phi(x, y) = C_1 = y - \lambda_1 x \\ C_2 &= y - \lambda_2 x && \rightarrow \Psi(x, y) = C_2 = y - \lambda_2 x \end{aligned} \tag{5.219}$$

Step 5:
Choose new variables depending on the type of the equation:

$$\begin{aligned} &\textbf{Hyperbolic:} && v = \Phi && w = \Psi \\ &\textbf{Parabolic:} && v = x && w = \Phi = \Psi \\ &\textbf{Elliptic:} && v = \frac{1}{2}(\Phi + \Psi) && w = \frac{1}{2}(\Phi - \Psi) \end{aligned} \tag{5.220}$$

Step 6:

Calculate the derivatives:

$$
\begin{aligned}
u_x &= v_x u_v + w_x u_w \\
u_y &= v_y u_v + w_y u_w \\
u_{xx} &= v_x^2 u_{vv} + v_{xx} u_v + w_x^2 u_{ww} + w_{xx} u_w + 2 v_x w_x u_{vw} \\
u_{yy} &= v_y^2 u_{vv} + v_{yy} u_v + w_y^2 u_{ww} + w_{yy} u_w + 2 v_y w_y u_{vw} \\
u_{xy} &= v_x v_y u_{vv} + v_{xy} u_v + w_x w_y u_{ww} + w_{xy} u_w + u_{vw}(v_y w_x + v_x w_y)
\end{aligned}
\tag{5.221}
$$

Step 7:
Substitute the newly found derivatives into the equation. This will give something of the following form:

$$u_{vw} = f(v, w) \tag{5.222}$$

This can be simply integrated to find u(v,w).

$$u(v, w) = \int\int f(v, w) \cdot dvdw + \varphi(v) + \psi(w) \tag{5.223}$$

Step 8:
In the last step, v and w are substituted back with x and y.

Example:
Solve

$$u_{xx} + u_{xy} = x \tag{5.224}$$

We can directly determine $A = 1, B = 1/2, C = 0$. Hence, the equation is hyperbolic. We then determine the lambda values.

$$\lambda_{1,2} = \frac{1/2 \pm \sqrt{1/4}}{1} = 1, 0 \tag{5.225}$$

$$y = x + C_1 \qquad y = C_2 \tag{5.226}$$

C is then expressed as a function of x and y.

$$C_1 = y - x = \Phi \qquad C_2 = y = \Psi \tag{5.227}$$

The new variables v,w are chosen.

$$v = \Phi = y - x \qquad w = \Psi = y \tag{5.228}$$

The derivatives are calculated with respect to the new coordinates.

$$u_{xx} = u_{vv} \qquad u_{xy} = -u_{vv} - u_{vw} \tag{5.229}$$

The old variables x and y are expressed by using the new variables.

$$x = y - v = w - v \qquad y = w \tag{5.230}$$

$$u_{xx} + u_{xy} = u_{vv} - u_{vv} - u_{vw} = -u_{vw} = x = w - v \tag{5.231}$$

$$\rightarrow u_{vw} = v - w \tag{5.232}$$

We then integrate twice

$$u_v = -\frac{1}{2}w^2 + vw + \chi(v) \tag{5.233}$$

$$u = -\frac{1}{2}w^2 v + \frac{1}{2}v^2 w + X(v) + \xi(w) \tag{5.234}$$

In a last step, we substitute the variables back to the old ones.

$$\begin{aligned} u(x,y) &= -\frac{1}{2}y^2(y-x) + \frac{1}{2}(y-x)^2 y + X(v) + \xi(w) \\ &\boxed{u(x,y) = \frac{1}{2}(x^2 y - y^2 x) + X(y-x) + \xi(y)} \end{aligned} \tag{5.235}$$

To check if this solution solves the equation one can simply take the derivatives and check

$$\begin{aligned} u_x &= \frac{1}{2}(2xy - y^2) - X'(y-x) \\ u_{xx} &= y + X''(y-x) \\ u_{xy} &= -y + x - X''(y-x) \end{aligned} \tag{5.236}$$

$$u_{xx} + u_{xy} = y + X''(y-x) + -y + x - X''(y-x) = x \tag{5.237}$$

Hence, the equation is solved.

5.2.9 Wave Equation with different Boundary Conditions

Given is the wave equation, this time with different boundary conditions. This may be confusing at first, but the way to solve it is exactly the same.

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u_x(0,t) = u_x(L,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \tag{5.238}$$

Step 1 of the solution procedure can be done like in section 5.2.4. In step 2, where the boundary conditions are used the case differentiation is done again. The boundary conditions imply in this case the following:

$$\begin{aligned} u_x(0,t) &= F'(0)G(t) = 0 \qquad \forall t \qquad \rightarrow F'(0) = 0 \\ u_x(L,t) &= F'(L)G(t) = 0 \qquad \forall t \qquad \rightarrow F'(L) = 0 \end{aligned} \tag{5.239}$$

Step 2:
F-Equation

- **k=0**

$$F(x) = ax + b \qquad F'(x) = a \qquad F'(0) = F'(L) = 0 \rightarrow a = 0 \tag{5.240}$$

The function is following equal to a constant.

$$\boxed{F(x) = A} \tag{5.241}$$

- **k>0**

$$F(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x} \qquad F'(x) = A\sqrt{k}e^{\sqrt{k}x} - B\sqrt{k}e^{-\sqrt{k}x} \tag{5.242}$$

$$\begin{aligned} F'(0) = 0 \qquad & 0 = A\sqrt{k} - B\sqrt{k} \rightarrow A = B \\ F'(L) = 0 \qquad & 0 = A\sqrt{k}e^{\sqrt{k}L} - A\sqrt{k}e^{-\sqrt{k}L} \rightarrow A = 0 \end{aligned} \tag{5.243}$$

$$\boxed{F(x) = 0} \tag{5.244}$$

- **k<0**

$$F(x) = A\cos(\sqrt{-k}x) + B\sin(\sqrt{-k}x) \qquad F'(x) = -\sqrt{-k}A\sin(\sqrt{-k}x) + \sqrt{-k}B\cos(\sqrt{-k}x) \tag{5.245}$$

$$\begin{aligned} F'(0) &= 0 \qquad \sqrt{-k}B = 0 \to B = 0 \\ F'(L) &= 0 \qquad -\sqrt{-k}A\sin(\sqrt{-k}L) = 0 \end{aligned} \tag{5.246}$$

Hence, either $A = 0$ or $\sqrt{-k}L = n\pi$

$$\boxed{F(x) = 0} \qquad \boxed{F(x) = A\cos\left(\frac{n\pi}{L}x\right)} \tag{5.247}$$

Now the G-Equation is solved:

- **k=0**

$$\ddot{G} = 0 \qquad \to \boxed{G(t) = At + B} \tag{5.248}$$

- **k>0**
  Because $F(x) = 0$, $G(t)$ is not relevant as $u(x,t) = F(x)G(t)$

- **k<0**

$$\boxed{G(t) = C\cos(c\sqrt{-k}t) + D\sin(c\sqrt{-k}t)} \tag{5.249}$$

Step 3:
Combining the solutions to get the general solution. Remeber to sum over all n to get all solutions. Unlike in section 5.2.4, here the cases for $k = 0$ has to be included in the solution as well as it is non-zero. Hence we multiply the F and G solution and add it to the solution for $k < 0$.

$$\boxed{u(x,t) = C_1 t + C_2 + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}x\right)\left(A_n\cos\left(\frac{cn\pi}{L}t\right) + B_n\sin\left(\frac{cn\pi}{L}t\right)\right)} \tag{5.250}$$

The following steps with the initial conditions can be done like in section 5.2.4.

5.2.10 2D Wave Equation

$$\begin{cases} u_{tt} = c^2(u_{xx} + u_{yy}) \\ u(\partial R) = 0 \\ u(x, y, 0) = f(x, y) \\ u_t(x, y, 0) = g(x, y) \end{cases} \tag{5.251}$$

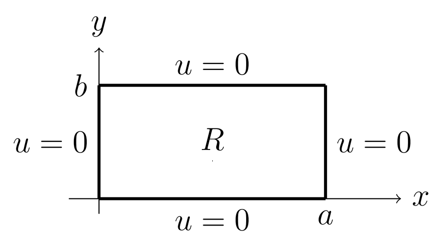


Figure 27: Boundary conditions

Step 1:
A separation of variables Ansatz is made:

$$u(x, y, t) = F(x, y)G(t) \tag{5.252}$$

Inserted into the equation, this gives:

$$\ddot{G}(t)F(x, y) = c^2(F_{xx}(x, y) + F_{yy}(x, y))G(t) \tag{5.253}$$

$$\rightarrow \frac{\ddot{G}(t)}{c^2G(t)} = \frac{F_{xx}(x, y) + F_{yy}(x, y)}{F(x, y)} = -k^2 \tag{5.254}$$

Step 2:
Solving both equations separately:
F-Equation

$$F_{xx} + F_{yy} = -k^2F \tag{5.255}$$

Sub-Ansatz:

$$F(x, y) = H(x)Q(y) \tag{5.256}$$

$$\rightarrow H''(x)Q(y) + H(x)Q''(y) = -k^2H(x)Q(y) \tag{5.257}$$

$$\frac{H''(x)}{H(x)} + \frac{Q''(y)}{Q(y)} = -k^2 \tag{5.258}$$

In order that equation (5.258) holds $\forall x, y$, both fractions have to be constant as well.
Q-Equation

$$\frac{Q''(y)}{Q(y)} = -p^2 \tag{5.259}$$

- **k=0**

$$Q''(y) = 0 \qquad Q(y) = Ay + B \tag{5.260}$$

  Using the boundary condition gives:

$$\begin{aligned} u(x,0,t) &= H(x)Q(0)G(t) = 0 \qquad \forall x,t \qquad \rightarrow Q(0) = 0 \\ u(x,b,t) &= H(x)Q(b)G(t) = 0 \qquad \forall x,t \qquad \rightarrow Q(b) = 0 \end{aligned} \tag{5.261}$$

$$\rightarrow \boxed{Q(y) = 0} \tag{5.262}$$

- **k≠0**

$$Q(y) = A\cos(py) + B\sin(py) \tag{5.263}$$

$$\begin{aligned} Q(0) &= 0 \qquad \rightarrow A = 0 \\ Q(b) &= 0 \qquad \rightarrow p = \frac{n\pi}{b} \end{aligned} \tag{5.264}$$

$$\boxed{Q(y) = A\sin\left(\frac{n\pi}{b}y\right)} \tag{5.265}$$

H-Equation

$$\frac{H''(x)}{H(x)} = -k^2 + p^2 = -h^2 \tag{5.266}$$

- **k=0**
  Is not relevant as Q(y)=0 and hence u(x,y,t)=0.

- **k≠0**
  We can combine p and k in ways so that $h = 0$ or that $h \neq 0$. For this reason we differenciate between different cases for h.

  - **h=0 (p=k)**

$$H(x) = Ax + B \qquad H(0) = H(a) = 0 \rightarrow A = B = 0 \tag{5.267}$$

$$\boxed{H(x) = 0} \tag{5.268}$$

– **h≠ 0**

$$H(x) = A\sin(hx) + B\cos(hx) \tag{5.269}$$

$$H(0) = H(a) = 0 \qquad \rightarrow A = 0, h = \frac{m\pi}{a} \tag{5.270}$$

$$\boxed{H(x) = B\sin\left(\frac{m\pi}{a}x\right)} \tag{5.271}$$

G-Equation

$$\ddot{G}(t) = -c^2k^2G(t) \tag{5.272}$$

- **k=0**
  Is not relevant as $Q(y) = 0$ for that case.
- **k≠0**

$$G(t) = A\cos(ckt) + B\sin(ckt) \tag{5.273}$$

Step 3:
Combining the Q and H solution to the F solution gives:

$$F(x,y) = H(x)Q(y) = C_1\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \tag{5.274}$$

Now, the F-solution is combined with the G-solution. Again, we sum over all m, n to get the complete solution.

$$\boxed{u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{m\pi}{a}x\right)\Big(A_{nm}\cos(ck_{nm}t) + B_{nm}\sin(ck_{nm}t)\Big)} \tag{5.275}$$

Step 4:
Now the initial conditions are used:

$$u(x,y,0) = f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{nm}\sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{m\pi}{a}x\right) \tag{5.276}$$

Rearranging this equation gives:

$$f(x,y) = \sum_{m=1}^{\infty}\sin\left(\frac{m\pi}{a}x\right)\underbrace{\sum_{n=1}^{\infty}A_{nm}\sin\left(\frac{n\pi}{b}y\right)}_{K_n(y)} \tag{5.277}$$

$$f(x,y)=\sum_{m=1}^{\infty}K_n(y)\sin\left(\frac{m\pi}{a}x\right)\rightarrow K_n(y)=\frac{2}{a}\int_0^a f(x,y)\sin\left(\frac{m\pi}{a}x\right)\cdot dx \tag{5.278}$$

This gives $K_n(y)$. Now we want to get back to A_{nm}.

$$K_n(y)=\sum_{n=1}^{\infty}A_{nm}\sin\left(\frac{n\pi}{b}y\right)\rightarrow A_{nm}=\frac{2}{b}\int_0^b K_n(y)\sin\left(\frac{n\pi}{b}y\right)\cdot dy \tag{5.279}$$

Hence the coefficients are found by calculating the following two dimensional integral:

$$\boxed{A_{nm}=\frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cdot dxdy} \tag{5.280}$$

The coefficients B_{nm} are found by using the second boundary condition. Remember that $k_{nm}^2=h^2+p^2=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2$ and hence k is also dependent on n and m.

$$u_t(x,y,0)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}ck_{nm}B_{nm}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)=g(x,y) \tag{5.281}$$

Analogously like before, one can rearrange the equation and eventually gets:

$$\boxed{B_{nm}=\frac{4}{abck_{n,m}}\int_0^b\int_0^a g(x,y)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cdot dxdy} \tag{5.282}$$

Once the coefficients are calculated, one can write the solution down (see equation 5.275).

5.2.11 Heat Equation on an Infinite Bar

Given is the heat equation with the following initial condition:

$$\begin{cases}u_t=c^2u_{xx}\\ u(x,0)=f(x)\end{cases} \tag{5.283}$$

This time no boundary conditions are given. The solution is done with a Fourier transformation.

$$\mathscr{F}(u_{xx})=-\omega^2\mathscr{F}(u) \tag{5.284}$$

$$\mathscr{F}(u_t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u_te^{-i\omega x}\cdot dx=\frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}ue^{-i\omega x}\cdot dx=\frac{\partial\mathscr{F}(u)}{\partial t} \tag{5.285}$$

Transforming the equation gives:

$$\mathscr{F}(u_t) = \mathscr{F}(c^2 u_{xx}) \tag{5.286}$$

$$\frac{\partial \hat{u}}{\partial t} = -\omega^2 c^2 \hat{u} \tag{5.287}$$

$$\rightarrow \hat{u}_t = -\omega^2 c^2 \hat{u} \tag{5.288}$$

This equation can easily be solved.

$$\rightarrow \hat{u}(\omega, t) = C_1(\omega) e^{-c^2\omega^2 t} \tag{5.289}$$

Keep in mind that the integration constant can be a function of ω since we integrate with respect to time. In a next step, the initial condition is transformed.

$$\mathscr{F}(u(x,0)) = \mathscr{F}(f(x)) \qquad \hat{u}(x,0) = \hat{f}(\omega) = C_1(\omega)e^0 = C_1(\omega) \tag{5.290}$$

$$\hat{u}(\omega, t) = \hat{f}(\omega) e^{-c^2\omega^2 t} \tag{5.291}$$

A simple inverse Fourier transform yields the function in the regular spatial domain.

$$\boxed{u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-c^2\omega^2 t} e^{i\omega x} \cdot d\omega} \tag{5.292}$$

5.2.12 Dirichlet Problem

For a space with d spatial dimensions, a Laplace differential operator is defined as follows:

Laplace Operator

$$\nabla^2 u(x) = \Delta u(x) = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} \tag{5.293}$$

Example:

$$d = 3 \qquad \Delta u(x) = u_{xx} + u_{yy} + u_{zz} \tag{5.294}$$

Dirichlet Problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \Omega \\ u(x) = f(x) & \forall x \in \partial\Omega \end{cases} \tag{5.295}$$

In this case, x is a general set of d coordinates ($x_1, x_2, x_3, \ldots$ i.e. $x, y, z, \ldots$) and Ω is the d-dimensional region on which the equation is solved. This region could be a circle in 2D, a sphere in 3D etc. The boundary of this region is depicted with $\partial\Omega$. Such problems have only analytical solutions if Ω has symmetries (for example a rectangle or sphere). An exception is a circle for a 2 dimensional space for which no analytical solution exists.

We now look at a problem in polar coordinates:

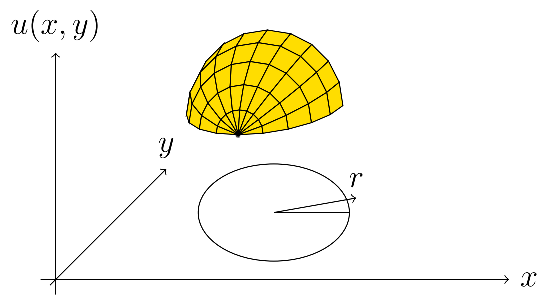
$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad dxdy \to rdrd\theta \tag{5.296}$$

The Laplace operator is now expressed in terms of r, θ. Remember that the partial derivative of a function with transformed coordinates is given by:

$$\frac{\partial \Phi(Y)}{\partial x_j} = \sum_i \frac{\partial \Phi(Y)}{\partial y_i} \frac{\partial y_i}{x_j} \tag{5.297}$$

Here, X is the old coordinate system and Y is the new coordinate system.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \tag{5.298}$$



$$\begin{aligned} u_x &= (u_r r_x) + (u_\theta \theta_x) \\ u_{xx} &= (u_r r_x)_x + (u_\theta \theta_x)_x \\ &= u_{rr} r_x^2 + u_{r\theta}\theta_x r_x + u_r r_{xx} + u_{\theta r} r_x \theta_x + u_{\theta\theta}\theta_x^2 + u_\theta \theta_{xx} \\ &= u_{rr} r_x^2 + 2u_{r\theta} r_x \theta_x + u_{\theta\theta}\theta_x^2 + u_r r_{xx} + u_\theta \theta_{xx} \end{aligned} \tag{5.299}$$

After a quite elaborate calculation, one gets:

$$\boxed{\Delta u = u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r} \tag{5.300}$$

The equation to solve is then:

$$\begin{cases} u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r = 0 & \forall r \in (0, R) \\ u(R, \theta) = f(\theta) & \theta \in (0, 2\pi) \end{cases} \tag{5.301}$$

Step 1:
Separation of variables

$$u(r, \theta) = F(r)G(\theta) \tag{5.302}$$

$$\Delta u = F''(r)G(\theta) + \frac{1}{r^2} F(r)G''(\theta) + \frac{1}{r} F'(r)G(\theta) = 0 \tag{5.303}$$

This equation is then multiplied with $\frac{r^2}{F(r)G(\theta)}$

$$r^2 \frac{F''(r)}{F(r)} + r\frac{F'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)} = k \tag{5.304}$$

Step 2:
G-Equation

$$G''(\theta) + kG(\theta) = 0 \tag{5.305}$$

Our boundary condition is that $G(0) = G(2\pi)$ and that $G'(0) = G'(2\pi)$ because the function must be 2π-periodic.

- k=0

$$G''(\theta) = 0 \qquad G(\theta) = A\theta + B \tag{5.306}$$

$$\begin{aligned} &G(0) = B = G(2\pi) = 2\pi A + B &&\to A = 0 \\ &G'(0) = A = G'(2\pi) = A \end{aligned} \tag{5.307}$$

$$\boxed{G(\theta) = B} \tag{5.308}$$

- k<0

$$G(\theta) = Ae^{\sqrt{k}\theta} + Be^{-\sqrt{k}\theta} \tag{5.309}$$

$$G(0) = A + B = G(2\pi) = Ae^{\sqrt{k}2\pi} + Be^{-\sqrt{k}2\pi} \tag{5.310}$$

$$\boxed{G(\theta) = 0} \tag{5.311}$$

- k>0

$$G(\theta) = A\cos(\sqrt{k}\theta) + B\sin(\sqrt{k}\theta) \tag{5.312}$$

$$G(0) = A = G(2\pi) = A\cos(2\sqrt{k}\pi) + B\underbrace{\sin(2\sqrt{k}\pi)}_{=0} \tag{5.313}$$

To make the sine term zero, the term $\sqrt{k} = n$ has to be a natural number.

$$\boxed{G(\theta) = A\cos(n\theta) + B\sin(n\theta)} \tag{5.314}$$

F-Equation

$$r^2F''(r) + rF'(r) - kF(r) = 0 \tag{5.315}$$

- k=0

$$r^2F''(r) + rF'(r) = 0 \qquad F(r) = A\ln(r) + B \tag{5.316}$$

$F(r)$ must be bounded so that $F(0)$ is finite. Hence, A can only be zero. The solution in this case is then only a constant $u(\theta, r) = A$. Because there will be a constant in the solution for $k > 0$ as well, we dont need to mind about this case.

- k<0
  Not important as $G(\theta) = 0$.

- k>0

$$r^2F''(r) + rF'(r) - n^2F(r) = 0 \tag{5.317}$$

This is an eulerian differential equation with $\lambda = \pm n$. The solution is hence:

$$F(r) = Ar^n + Br^{-n} \tag{5.318}$$

The function must be finite when r goes to zero. Hence, B must be zero.

$$\boxed{F(r) = Ar^n} \tag{5.319}$$

Step 3:
Connecting the solutions gives:

$$\boxed{u(r,\theta) = \sum_{n=0}^{\infty} r^n (A_n \cos(n\theta) + B_n \sin(n\theta))} \tag{5.320}$$

Step 4:
Boundary conditions:

$$u(R,\theta) = \sum_{n=0}^{\infty} R^n (A_n \cos(n\theta) + B_n \sin(n\theta)) = f(\theta) \tag{5.321}$$

Rewriting this as a Fourier series yields:

$$\begin{aligned} A_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \cdot d\theta \\ A_n &= \frac{1}{R^n \pi} \int_0^{2\pi} f(\theta) \cos(n\theta) \cdot d\theta \\ B_n &= \frac{1}{R^n \pi} \int_0^{2\pi} f(\theta) \sin(n\theta) \cdot d\theta \end{aligned} \tag{5.322}$$

Remark
Why is the integral from 0 to 2π ? Originally, the integral is from $-\pi$ to $+\pi$. Commonly, the integral is multiplied by 2 and is evaluated from 0 to π. In the case described above, the function $f(\theta)$ has to be 2π-perdiodic (If you go 360° around, you get the same function value). Since sine, cosine and the function f are 2π-periodic one can integrate from 0 to 2π instead.

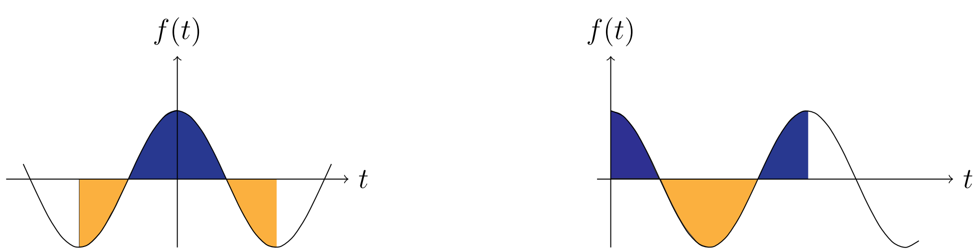


Figure 28: Integral from 0 to 2π is the same as from $-\pi$ to π