Control Systems I
Recitation 10

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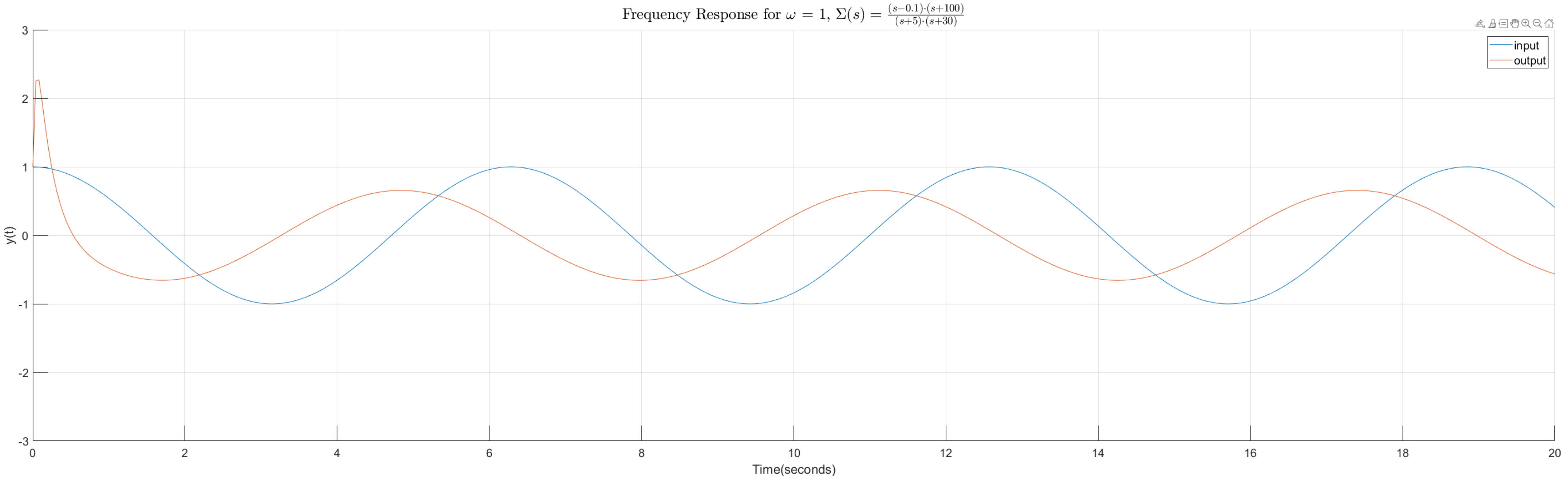
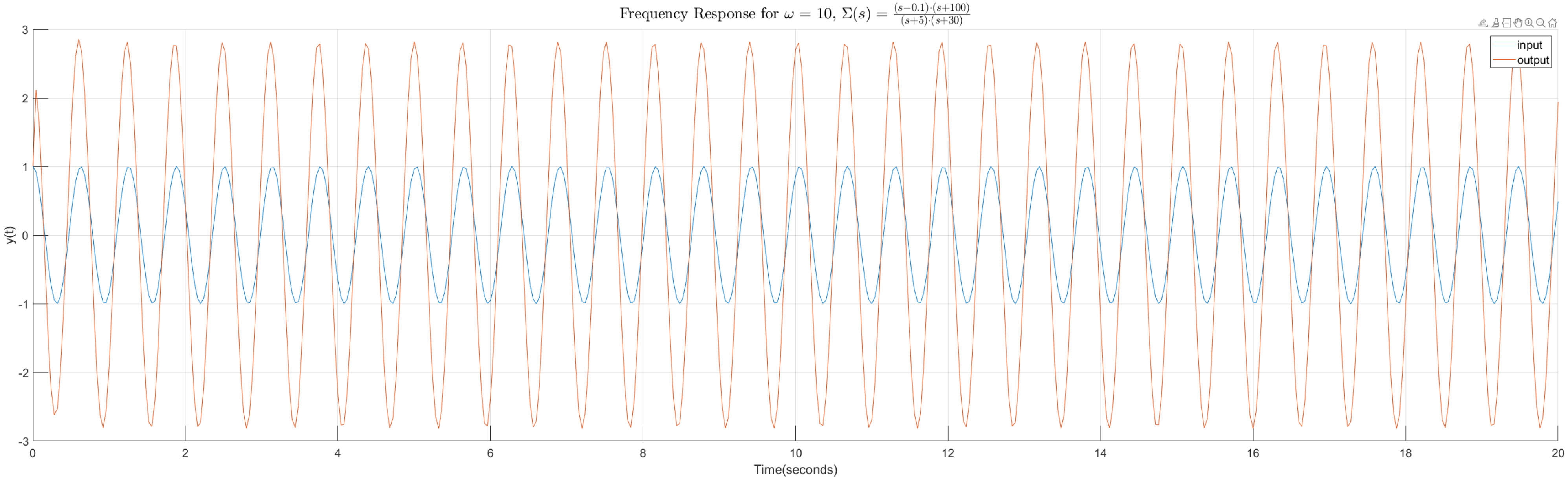
Last Week

Frequency Response

- We want to find the response to a Harmonic Input:
  - $u(t) = \alpha \cdot \cos(\omega \cdot t + \phi)\,, \phi = 0 \text{ in most cases}$
- Resulting Response:
  - $y(t) = |G(j\omega)| \cdot \alpha \cdot \cos\big(\omega \cdot t + \phi + \angle G(j\omega)\big)$
- What do we see?
  - The system oscillates with the same frequency
  - The amplitude is **frequency dependant**
  - The phase shift is **frequency dependant**
- How can we plot $|G(j\omega)|$ and $\angle G(j\omega)$?
  - Bode Plot
  - Polar / Nyquist Plot

Frequency Response





Last Week

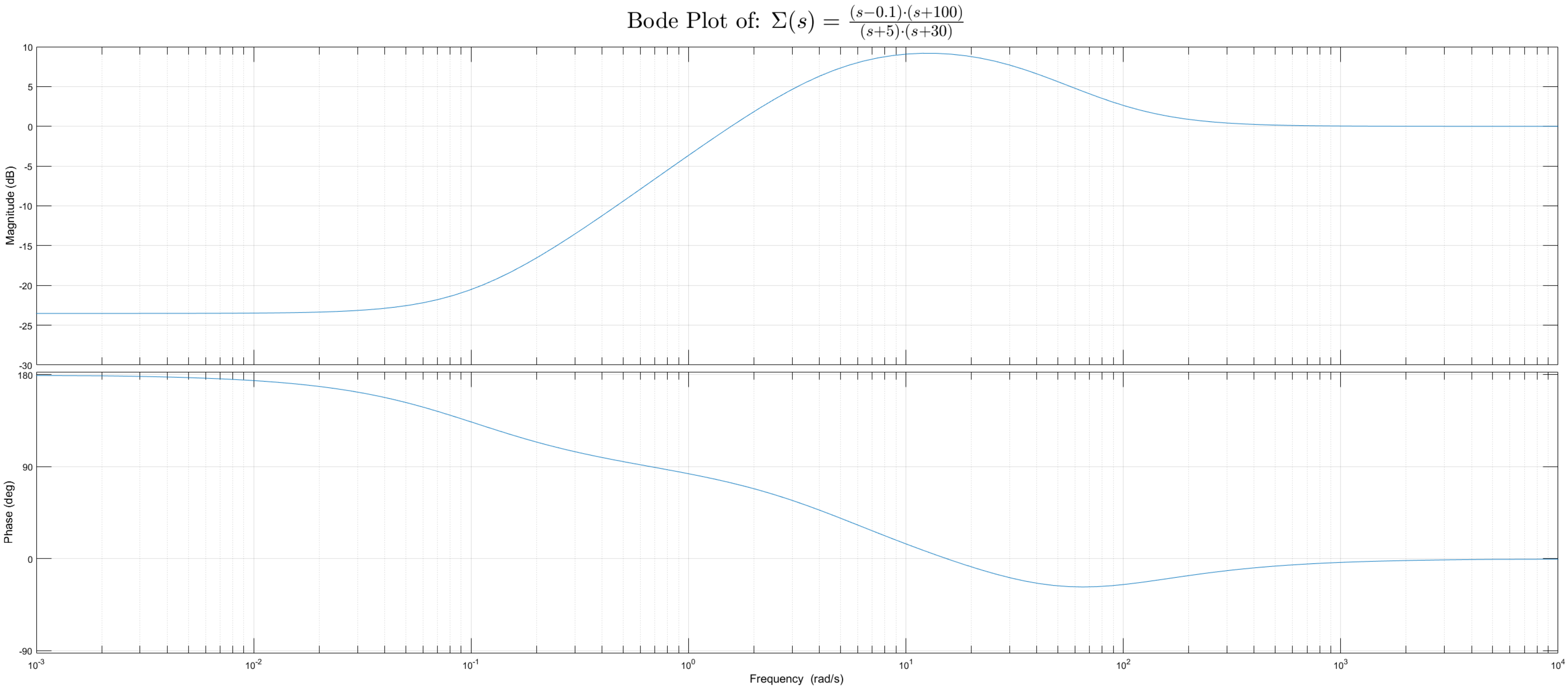
Bode Plot

- Two separate frequency **ex**plicit plots for both $|G(j\omega)|$ and $\angle G(j\omega)$
- Magnitude Plot $|G(j\omega)|$:
  - Logarithmic ω axis and dB(decibel) $|G(j\omega)|$
  - Decibel:
    - $|G(j\omega)|_{dB} = 20 \cdot \log_{10}|G(j\omega)|$
    - $|G(j\omega)| = 10^{\frac{|\Sigma(j\omega)|_{dB}}{20}}$
  - $|G(j\omega)| = \sqrt{\mathrm{Re}\big(G(j\omega)\big)^2 + \mathrm{Im}\big(G(j\omega)\big)^2}$
- Phase Plot $\angle G(j\omega)$:
  - Logarithmic ω axis and linear $\angle G(j\omega)$ (in degrees)
  - $\angle G(j\omega) = \mathrm{arctan2}\left(\frac{\mathrm{Im}(G(j\omega))}{\mathrm{Re}(G(j\omega))}\right)$

Dezimalskala	Dezibelskala
100	40
10	20
5	13.97...
2	6.02...
1	0
$1/\sqrt{2}$	-3.0103
0.1	-20
0.01	-40
0	-Inf

$$\mathrm{atan2}(y,x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0, \\ \frac{\pi}{2} - \arctan\left(\frac{x}{y}\right) & \text{if } y > 0, \\ -\frac{\pi}{2} - \arctan\left(\frac{x}{y}\right) & \text{if } y < 0, \\ \arctan\left(\frac{y}{x}\right) \pm \pi & \text{if } x < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

Bode Plot of a system



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Last Week

Drawing a Bode Plot

- Using Logarithms is very convenient, we can combine different systems
  - Total System: $G(s) = G_1(s) \cdot G_2(s) \cdot \ldots \cdot G_n(s)$
  - Amplitude in decibel: $|\Sigma(s)|_{dB} = |\Sigma_1(s)|_{dB} + |\Sigma_2(s)|_{dB}$
  - Phase: $\angle\Sigma(s) = \angle\Sigma_1(s) + \angle\Sigma_2(s)$
- When drawing combine the effects of poles and zeros of the sub-systems (addition)
  - The effect is at the position of the pole/zero
  - At the pole/zero the phase shift is approx 50% done
  - For multiplicity $k > 1$, the change is multiplied by k

Type	Magnitude Change	Phase Change
Stable Pole	-20 dB/dec	-90°
Unstable Pole	-20 dB/dec	+90°
Minimumphase zero	+20 dB/dec	+90°
Non-minimumphase zero	+20 dB/dec	-90°
Time Delay	0 dB/dec	$-\omega \cdot T$

Bode Plots

Standard Elements – there are a bunch

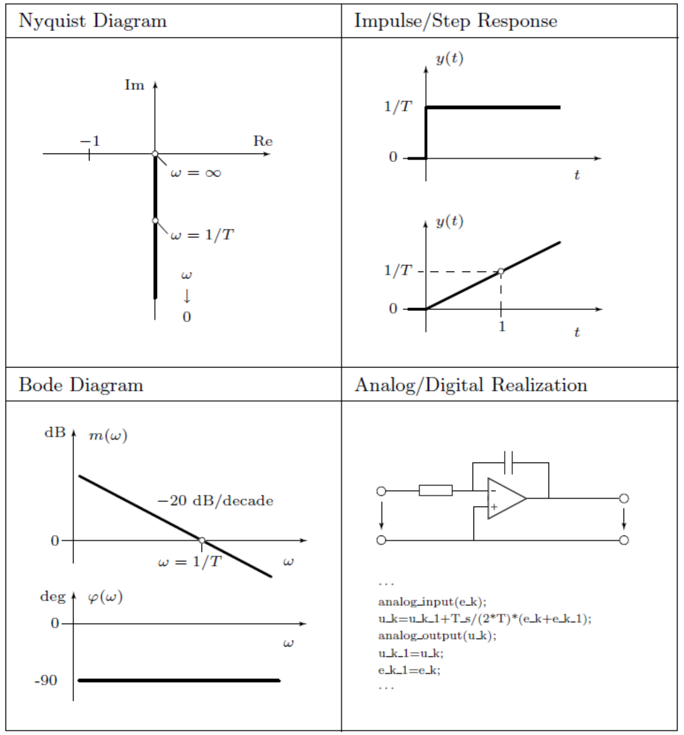
A.1 Integrator Element

Element Acronym: I

Transfer Function: $\Sigma(s) = \frac{1}{T \cdot s}$

Poles/Zeros: $\pi_1 = 0,\ \zeta_1 = \infty$

Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \frac{1}{T} \cdot u(t)$
$y(t) = x(t)$



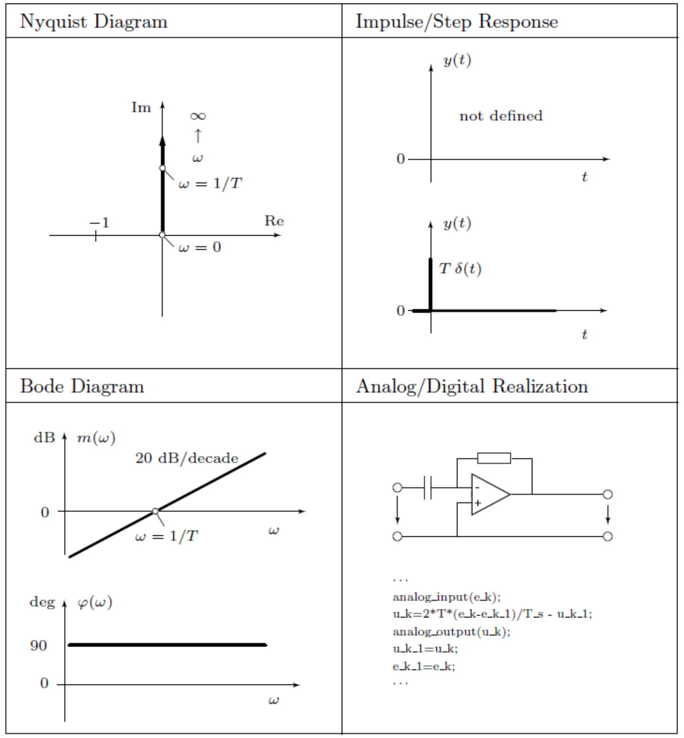
A.2 Differentiator Element

Element Acronym: D

Transfer Function: $\Sigma(s) = T \cdot s$

Poles/Zeros: $\pi_1 = \infty,\ \zeta_1 = 0$

Internal Description: $y(t) = T \cdot \frac{\mathrm{d}}{\mathrm{d}t}u(t)$



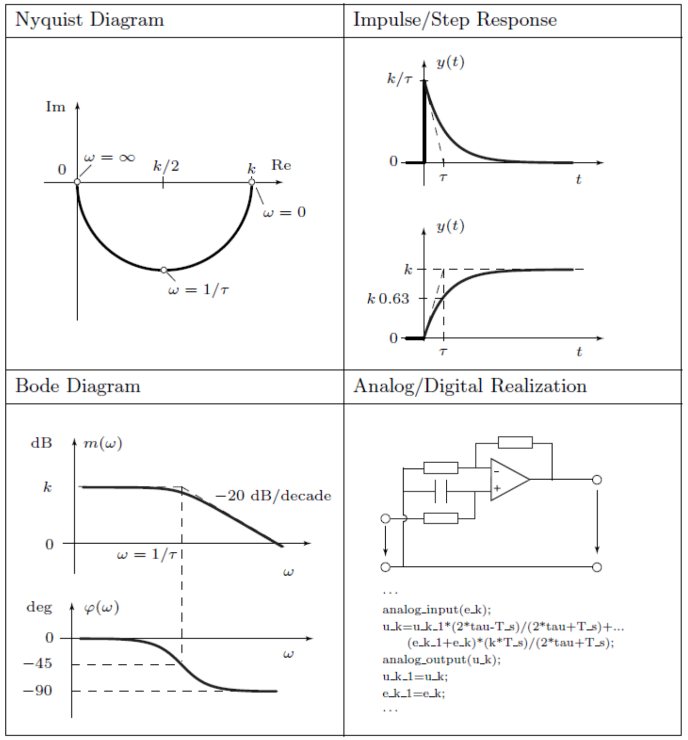
A.3 First-Order Element

Element Acronym: LP-1

Transfer Function: $\Sigma(s) = \frac{k}{\tau \cdot s + 1}$

Poles/Zeros: $\pi_1 = -\frac{1}{\tau},\ \zeta_1 = \infty$

Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -\frac{1}{\tau} \cdot x(t) + \frac{1}{\tau} \cdot u(t)$
$y(t) = k \cdot x(t)$



Bode Plots

Standard Elements – there are a bunch

A.4 Realizable Derivative Element

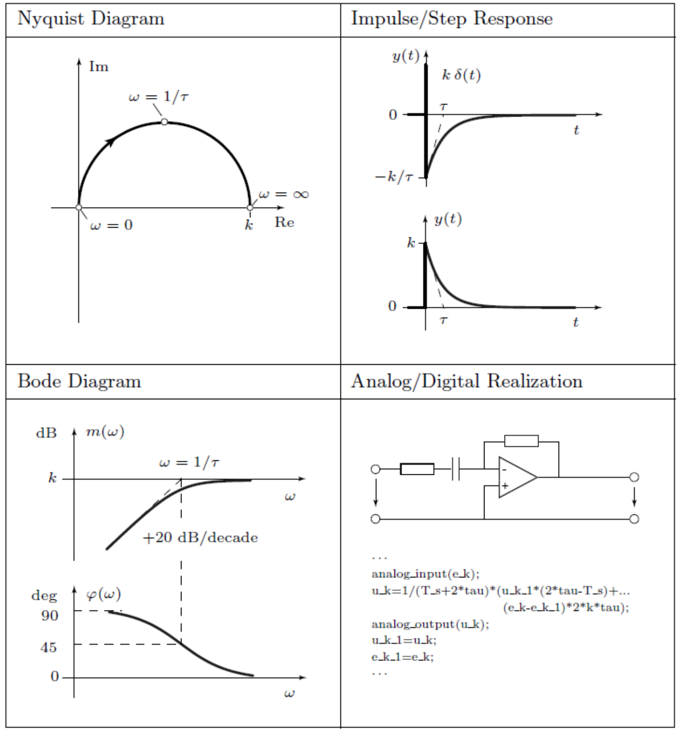
Element Acronym: HP-1

Transfer Function: $\Sigma(s) = k \cdot \frac{\tau \cdot s}{\tau \cdot s + 1} = k \cdot \left(1 - \frac{1}{\tau \cdot s + 1}\right)$

Poles/Zeros: $\pi_1 = -\frac{1}{\tau},\ \zeta_1 = 0$

Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -\frac{1}{\tau} \cdot x(t) + \frac{1}{\tau} \cdot u(t)$

$y(t) = -k \cdot x(t) + k \cdot u(t)$



A.5 Second-Order Element

Element Acronym: LP-2

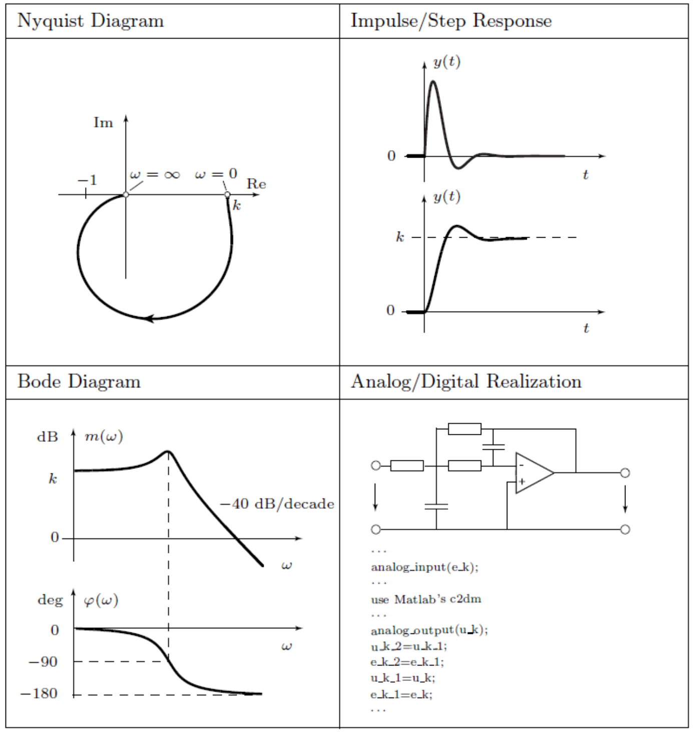
Transfer Function: $\Sigma(s) = k \cdot \frac{\omega_0^2}{s^2 + 2 \cdot \delta \cdot \omega_0 \cdot s + \omega_0^2}$

Poles/Zeros: $\pi_{1,2} = -w_0 \cdot \delta \pm w_0\sqrt{\delta^2 - 1},\ \zeta_{1,2} = \infty$

Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = x_2(t),$

$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = -\omega_0^2 \cdot x_1(t) - 2 \cdot \delta \cdot \omega_0 \cdot x_2(t) + \omega_0^2 \cdot u(t)$

$y(t) = k \cdot x_1(t)$



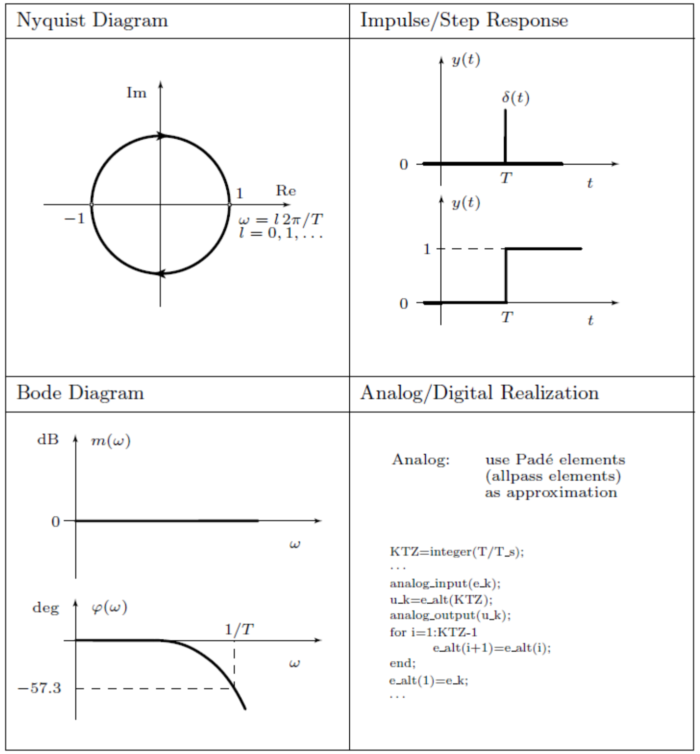
A.10 Delay Element

Element Acronym: -

Transfer Function: $\Sigma(s) = e^{-s \cdot T}$

Poles/Zeros: not a real-rational element

Internal Description: $y(t) = u(t - T)$



Last Week

Bodes Law

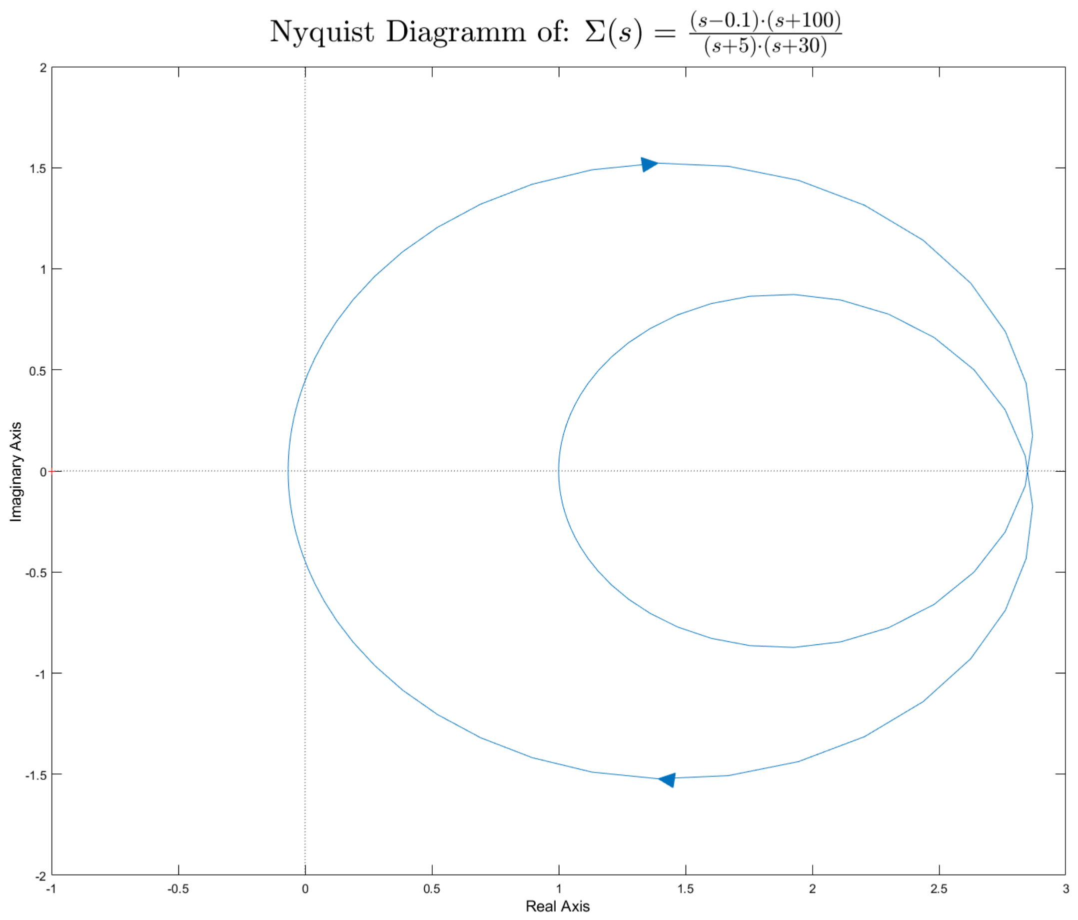
- Phase and Amplitude are not independent
  - $|G(j\omega)|_{dB} = 20\frac{dB}{dec} \cdot \kappa \quad \Rightarrow \quad \angle G(j\omega) \approx \kappa \cdot \frac{\pi}{2}$
- System: $\Sigma(s) = \frac{b_m \cdot s^m + \ldots + b_1 \cdot s + b_0}{s^q \cdot (s^{n-q} + a_{n-k-1} \cdot s^{n-k-1} + \ldots + a_1 \cdot s + a_0)}$
  - Relative degree: $r = n - m$
  - System Type: $q =$ number of integrators

- We further have:
  - For $\omega \rightarrow \infty$: $\frac{\partial |G(j\omega)|_{dB}}{\partial \log_{10}(\omega)} = -r \cdot 20\text{ dB}$, with $r = n - m$ being the relative degree
  - For $\omega \rightarrow 0$: $\angle G(j\omega = 0) = \begin{cases} -q \cdot \frac{\pi}{2}, & \text{for sign}\left(\frac{b_0}{a_0}\right) > 0 \\ -\pi - q \cdot \frac{\pi}{2}, & \text{for sign}\left(\frac{b_0}{a_0}\right) < 0 \end{cases}$

Last Week

Nyquist/Polar Plot

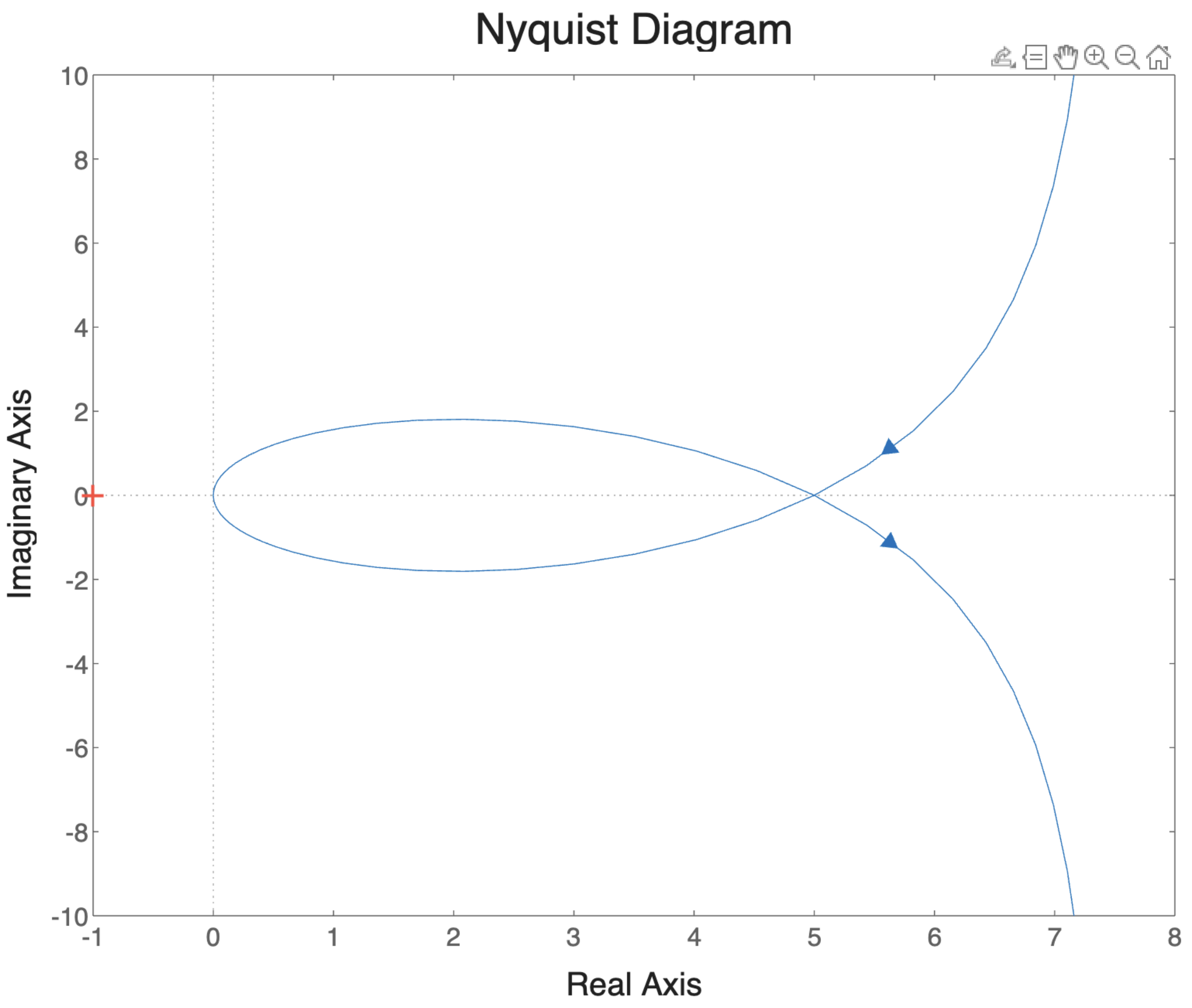
- $|G(s)|$ and $\angle G(s)$ drawn in the complex plane.
  - Polar: $0 \leq \omega < \infty$
  - Nyquist: $-\infty < \omega < \infty$
- The values are now frequency **implicit**
- Drawing usually using Python or Matlab
- Sketching
  - Look at the extremes $\omega \to 0, \omega \to \infty$
    - Use Bodes Law
  - Read values of Bode plot
  - Needs to be qualitatively correct
  - $\omega \in (-\infty, 0]$ is the mirror of $\omega \in [0, \infty)$



Polar / Nyquist Plot

Example – Drawing a Nyquist Plot

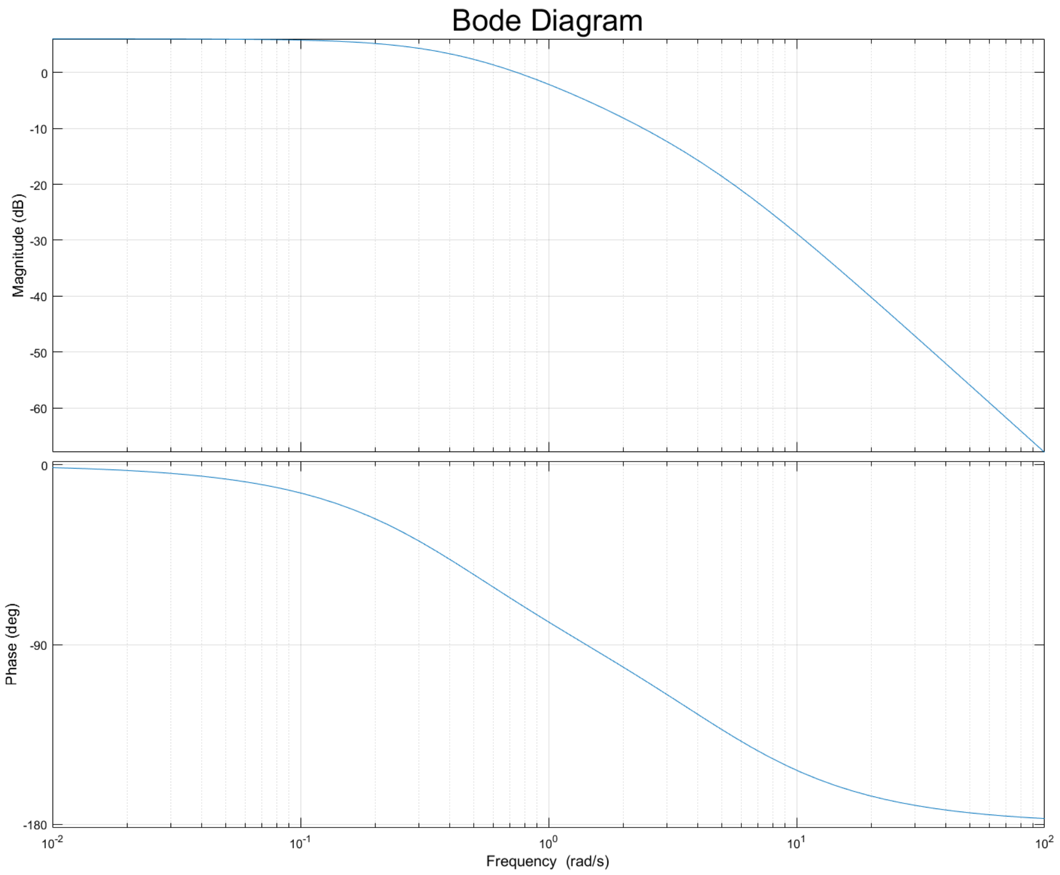
- Draw the Nyquist Plot for:
  - $G(s) = \frac{5(s-0.5)}{s(s+5)}$
- $\omega \to 0^+$:
  - $|G(j\omega)| \to \infty$
  - $\angle G(j\omega) = \begin{cases} -q \cdot \frac{\pi}{2}, & \text{for sign}\left(\frac{b_0}{a_0}\right) > 0 \\ -\pi - q \cdot \frac{\pi}{2}, & \text{for sign}\left(\frac{b_0}{a_0}\right) < 0 \end{cases} = -\frac{3}{2}\pi$
- $\omega \to \infty$:
  - $|G(j\omega)| \to 0$
  - $\angle G(j\omega) \approx \angle \frac{1}{s} = -\frac{\pi}{2}$

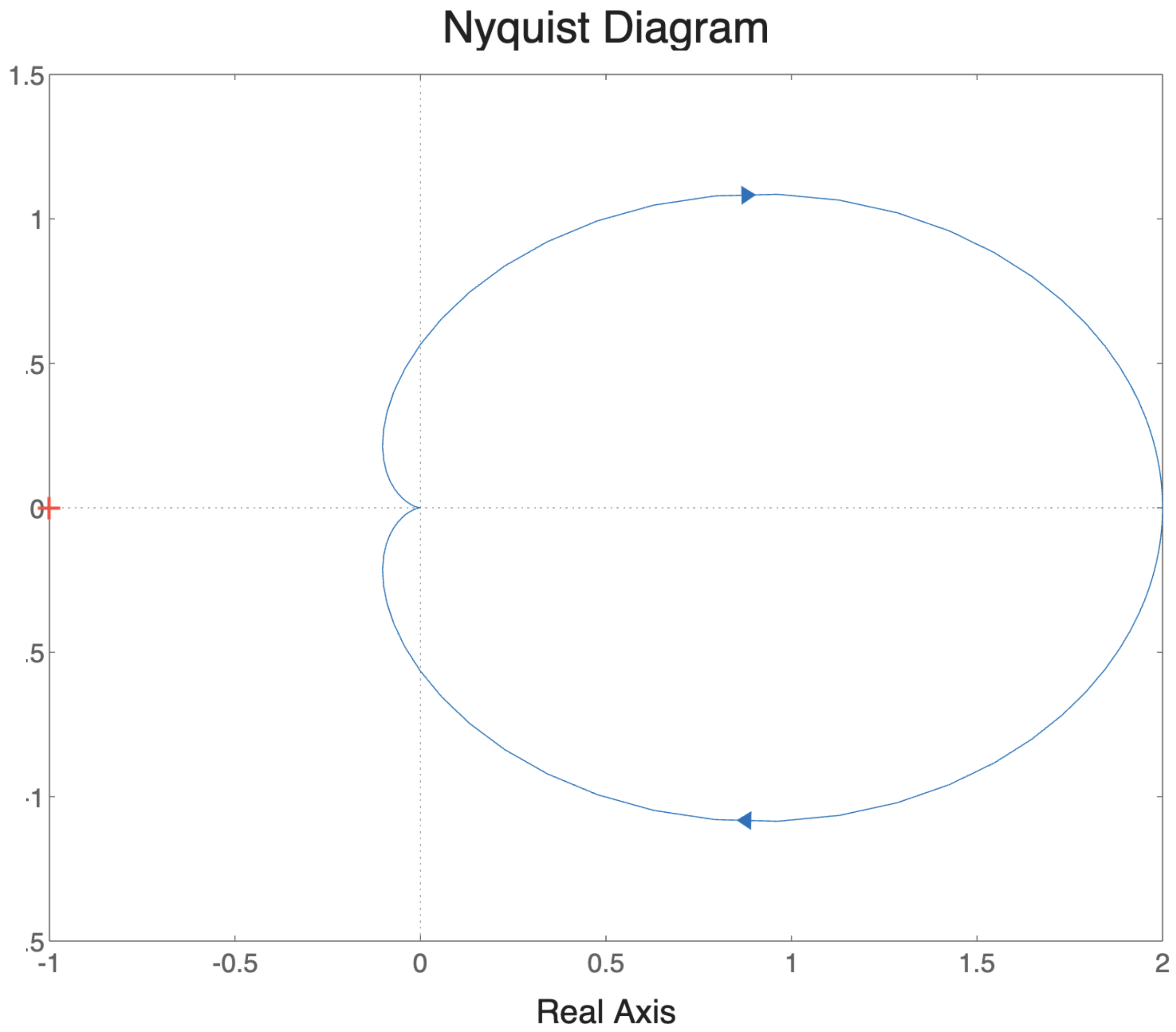


Polar / Nyquist Plot

Example – Drawing a Nyquist Plot

- Draw the Nyquist Plot for the System with the following Bode Plot





Last Week

Bode vs Nyquist/Polar

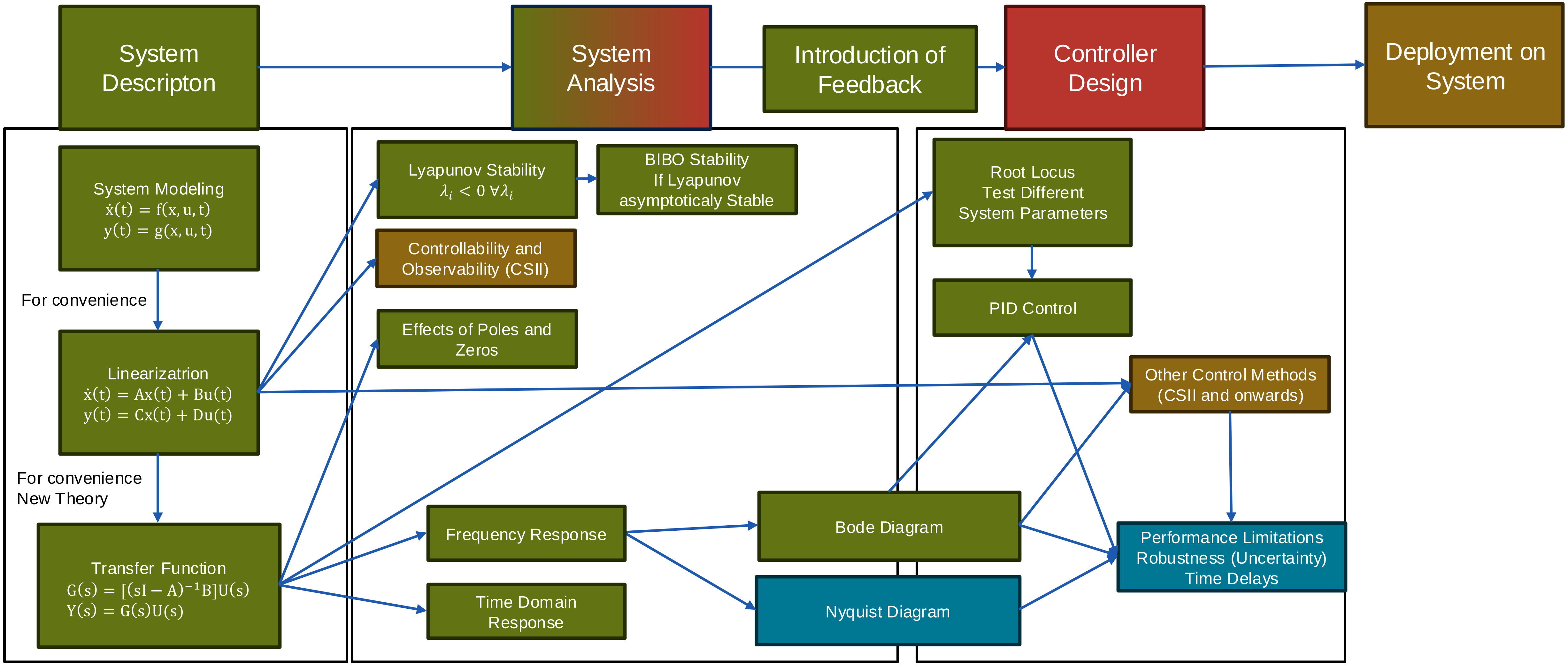
- Graphical representation of $|G(s)|$ and $\angle G(s)$
- Bode Plot:
  - frequency **ex**plicit
  - Logarithmic, decibel and linear axis scale
  - Quantitive analysis
- Nyquist Plot:
  - frequency **im**plicit
  - Linear axis scale
  - Qualitative analysis

Outline

- Nyquist Criterion
  - Cauchy's argument principle
  - Nyquist Condition
  - Nyquist Stability Theorem
  - Counting Encirclements
  - Example
- Stability Margins
  - What?
  - Example

Conceptual Recap

Classical Control Approach



Time Delays

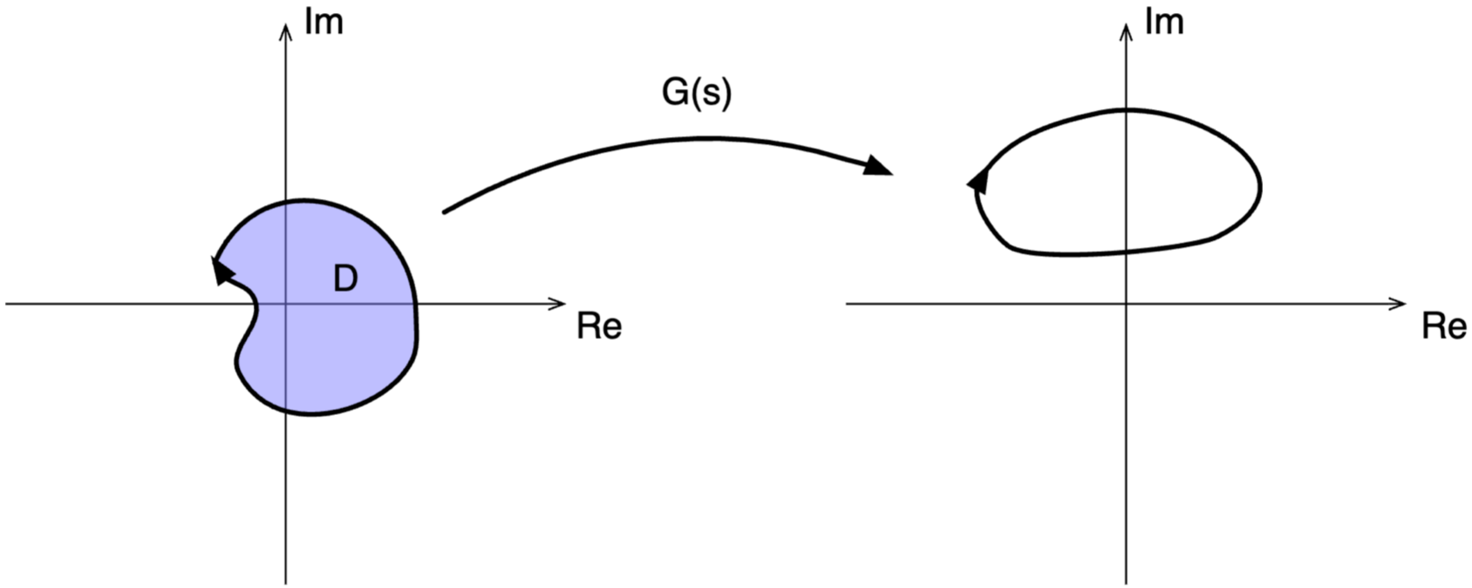
What?

- Whenever an event/transition takes time:
  - Computing a control output using a computer
  - Goods on a conveyor belt with a sensor on the end
  - Long range control (e.g space crafts)
- Definition:
  - A time delay is a **linear** operator that transforms an input signal $t \to u(t)$ into a delayed output signal $y(t) = u(t - T)$, where $T \geq 0$ is the delay.
- Transfer Function:
  - $Y(s) = e^{-sT}U(s) \to G(s) = e^{-sT}$
- Not a polynomial thus root locus is not valid anymore with a system that has time delays!

Nyquist Condition

Cauchy's argument principle

- A Complex Analysis Theorem (don’t read too much into it) (https://www.youtube.com/watch?v=WmfrK4l1TiI&ab_channel=richardpates)
- Given some closed region D in the complex plain we have Γ surrounding that region. If we now apply $G(s)$ to every point on Γ we get $G(\Gamma)$ being another closed curve in the complex plain:
  - Here closed means that the start and end point are the same



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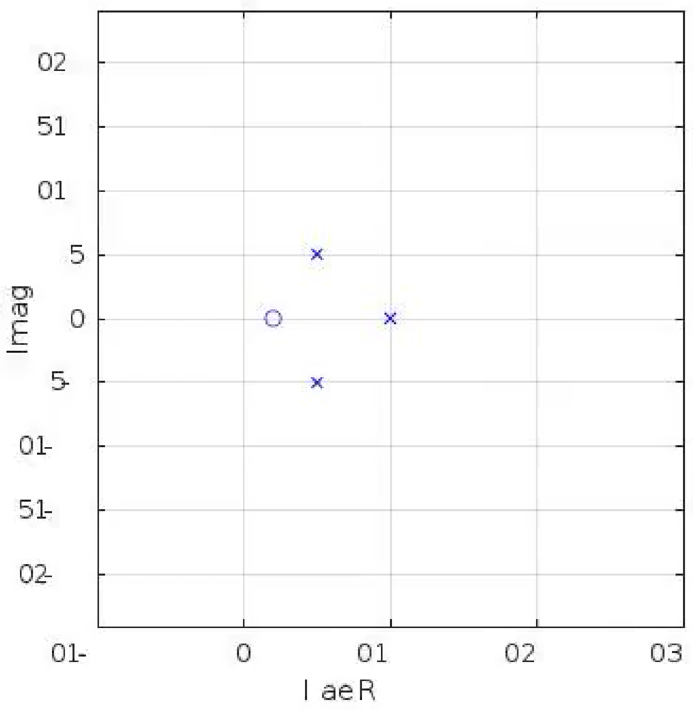
Nyquist Condition

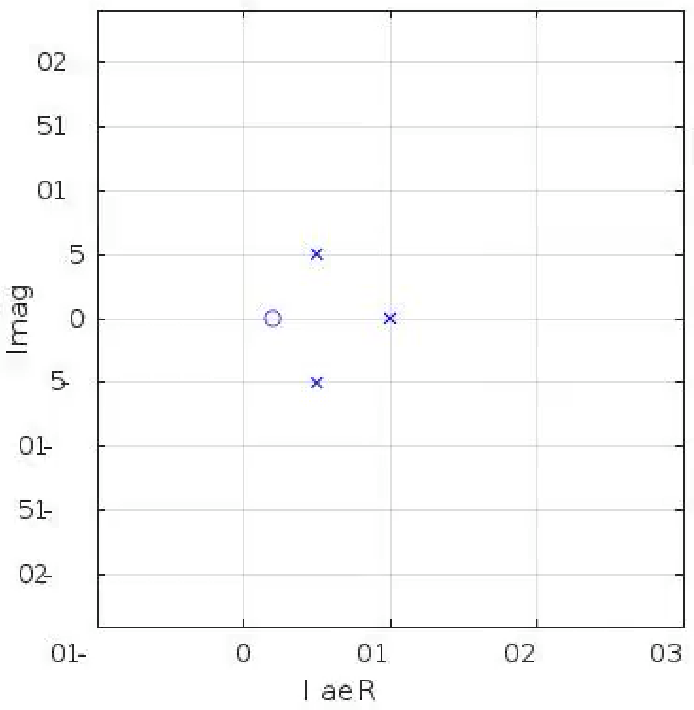
Cauchy's argument principle

- The argument principle:
  - The number N of times that $G(s)$ encircles the origin of the complex plain as s moves along Γ satisfies (*counting positive for encirclements in the same direction as following Γ*)

$$N = Z - P$$

where Z and P are the number of poles and zeros of $G(s)$ contained in D.





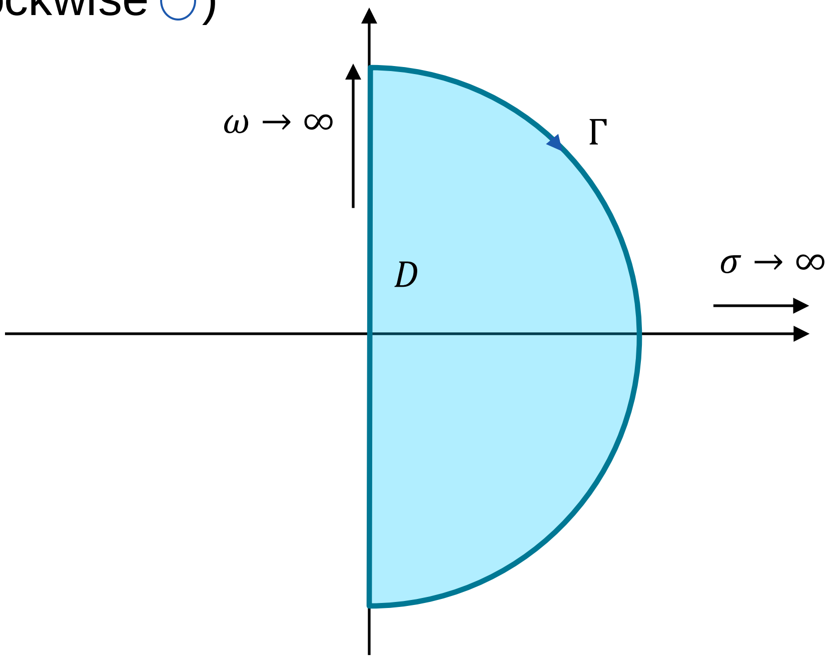
Nyquist Condition

Towards the Nyquist Condition

- How does this help us?
  - From two weeks ago remember: $T(s) = \frac{kL(s)}{1+kL(s)}, S(s) = \frac{1}{1+kL(s)}$
    - Overall system is stable iff $\frac{1}{1+kL(s)}$ has no positive poles = $1 + kL(s)$ has no positive zeros
  - Construct a huge region surrounding the positive half plane and run $1 + kL(s)$ clockwise ↻
    - The curve of $1 + kL(\Gamma)$ now encircles the origin (clockwise ↻)

$$N = Z - P$$

    - Z: number of unstable zeros of $1 + kL(s)$
      - Observe these are the unstable poles of the closed loop system
    - P: number of unstable poles of $1 + kL(s)$
      - Observe that these are the also the unstable poles of $L(s)$



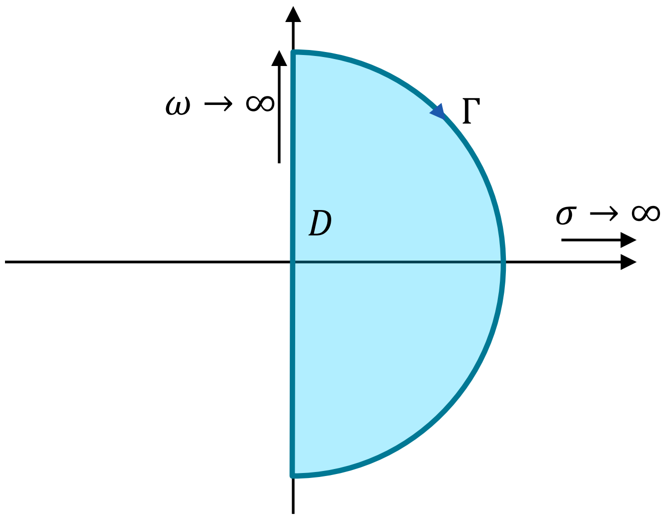
Nyquist Condition

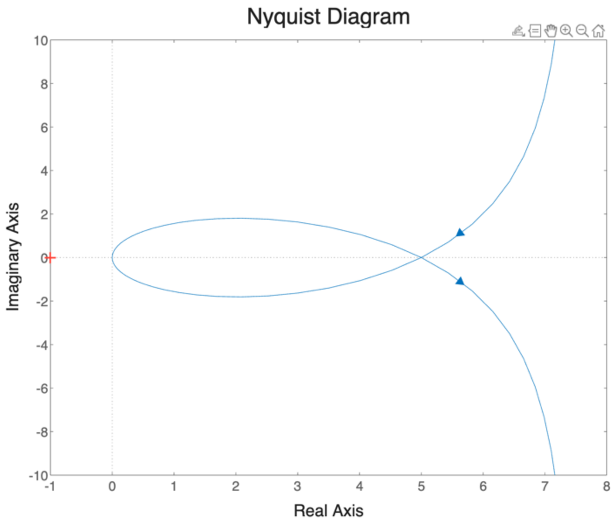
Towards the Nyquist Condition

- We can rewrite this ($1 + kL(s) = 0 \Leftrightarrow \mathrm{L}(s) = -\frac{1}{\mathrm{k}}$)
  - The curve of $L(\Gamma)$ encircles the point $-\frac{1}{k}$ (clockwise ↻)

$$N = Z - P$$

  - Z: number of unstable zeros of $1 + kL(s)$
    - Observe these are the unstable poles of the closed loop system
  - P: number of unstable poles of $1 + kL(s)$
    - Observe that these are the also the unstable poles of $L(s)$
- This huge region D (Nyquist Contour) has $\Gamma: \omega \in (-\infty, \infty)$ and we thus get the **Nyquist plot.**





Nyquist Condition

Nyquist Condition

- We now know that the **Nyquist plot** encircles the point $-\frac{1}{k}$ (clockwise ↻)

$$N = Z - P$$

  - Z: number of unstable zeros of $1 + kL(s)$
    - Observe these are the unstable poles of the closed loop system
  - P: number of unstable poles of $1 + kL(s)$
    - Observe that these are the also the unstable poles of $L(s)$

- Nyquist Criterion:
  - Given an open loop transfer function $kL(s)$ with P poles in the positive half plane (Nyquist contour) and let N be the number of clockwise ↻ encirclements of $-\frac{1}{k}$ by the Nyquist Plot. Then the closed loop system has $Z = N + P$ poles in the positive half plane.

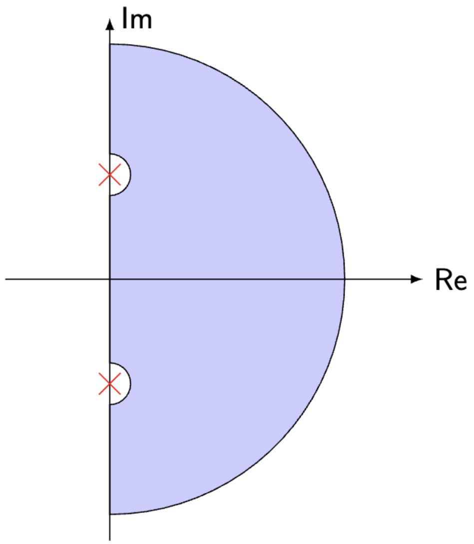
Nyquist Condition

Nyquist Condition

- Nyquist Criterion:
  - Given an open loop transfer function $kL(s)$ with P poles in the positive half plane (Nyquist contour) and let N be the number of clockwise ↻ encirclements of $-\frac{1}{k}$ by the Nyquist Plot. Then the closed loop system has $Z = N + P$ poles in the positive half plane.
- For stability we now want $Z = 0 \;\rightarrow \mathrm{N} = -\mathrm{P}$,
  - **Nyquist Stability Theorem:**
    - A closed-loop system is stable if for $kL(s)$ the following holds:

$$\mathrm{n}_c = n_p$$

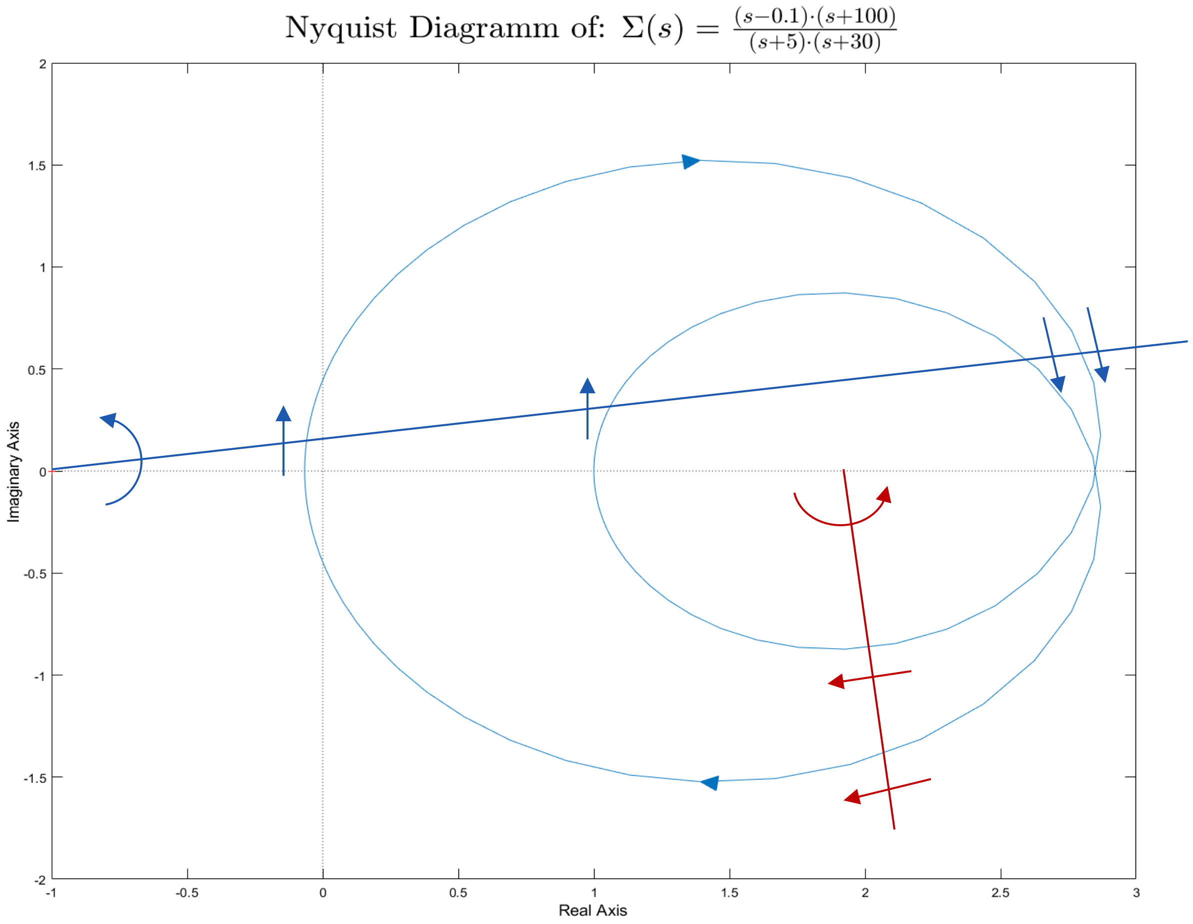
    - n_c: number of *counter-clockwise* ↺ encirclements of $-\frac{1}{k}$ by the Nyquist Plo
    - n_p: number of poles with positive real part of $L(s)$
  - Valid only if no nonminimum phase - unstable pole cancellation was done!
- Things to keep in mind:
  - Avoid zeros on the imaginary axis by excluding them
  - k is usually 1 and backed into $L(s) \rightarrow$ everything is with respect to -1



Nyquist Condition

How to count encirclements

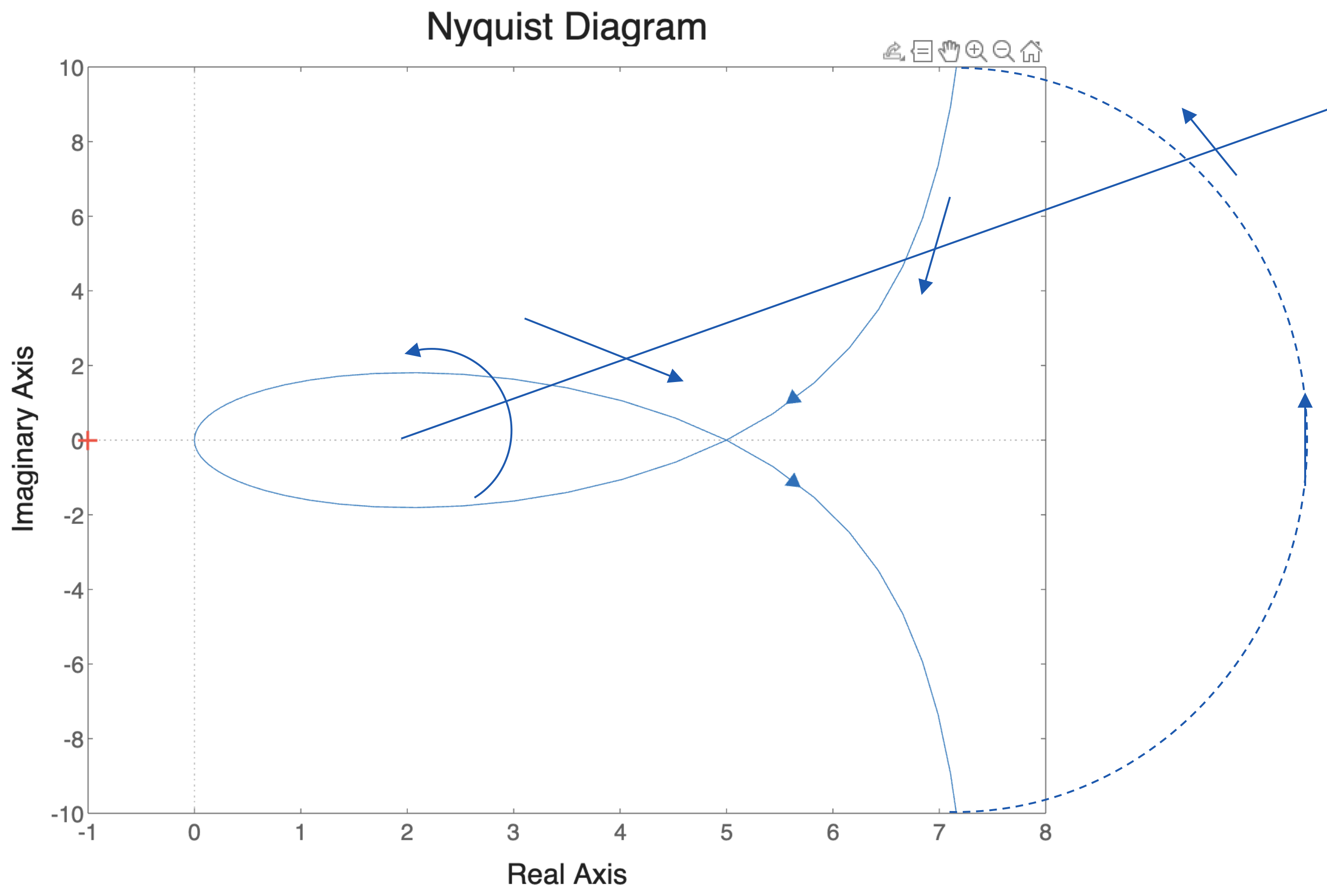
- Draw a line outwards from the point $-\frac{1}{k}$
- Draw the crossings of the Nyquist plot with this line (keep the direction in mind)
- Add the number of crossings (counter-clockwise positive, clockwise negative)
- Example:
  - Encirclements around -1:
    - 0 since 2 CCW and 2 CW
  - Encirclements around 2:
    - -2 since 0 CCW and 2 CW



Nyquist Condition

How to count encirclements - Infinity

- Take $\lim_{\varepsilon \to 0} \angle L(\varepsilon e^{j\theta}) = f(\theta)$
- Now look at what happens for $\theta: -\frac{\pi}{2} \to \frac{\pi}{2}$
  - $\lim_{\varepsilon \to 0} \angle L(\varepsilon e^{j\theta}): f\left(-\frac{\pi}{2}\right) \to f\left(\frac{\pi}{2}\right)$
- Close the loop accordingly
- Example: $G(s) = \frac{5(s-0.5)}{s(s+5)}$
  - $\lim_{\varepsilon \to 0} \angle L(\varepsilon e^{j\theta}) = \lim_{\varepsilon \to 0} \angle \frac{5(\varepsilon e^{j\theta}-0.5)}{\varepsilon e^{j\theta}(\varepsilon e^{j\theta}+5)}$
  - $= \lim_{\varepsilon \to 0} \angle \frac{5(-0.5)}{\varepsilon e^{j\theta}(5)} = \lim_{\varepsilon \to 0} \angle - \frac{0.5}{\varepsilon} e^{-j\theta} = \theta$
  - For $\theta: -\frac{\pi}{2} \to \frac{\pi}{2}$:
    - $\lim_{\varepsilon \to 0} \angle L(\varepsilon e^{j\theta}): -\frac{\pi}{2} \to \frac{\pi}{2}$
- Encirclements around 2:
    - -1 since 1 CCW and 2 CW



Nyquist Condition

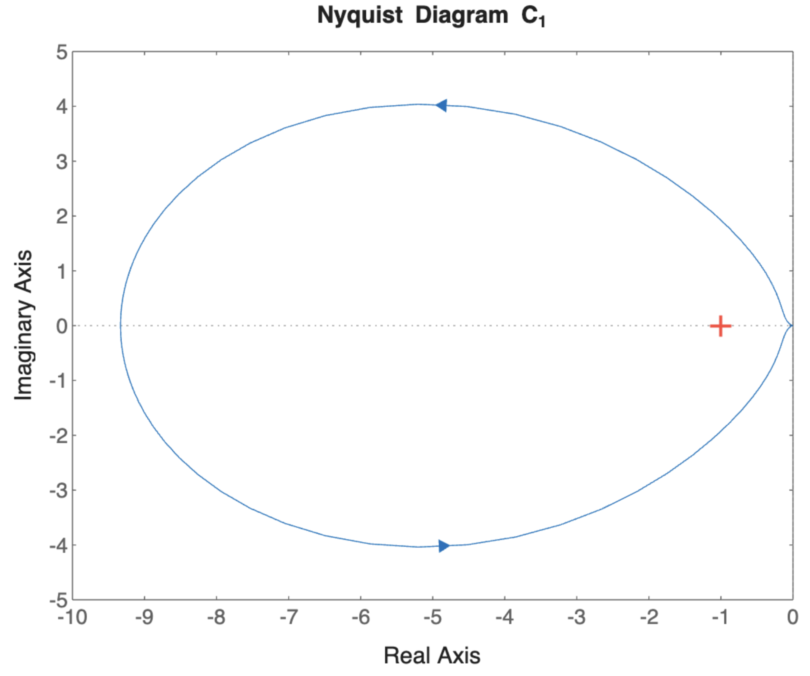
Example

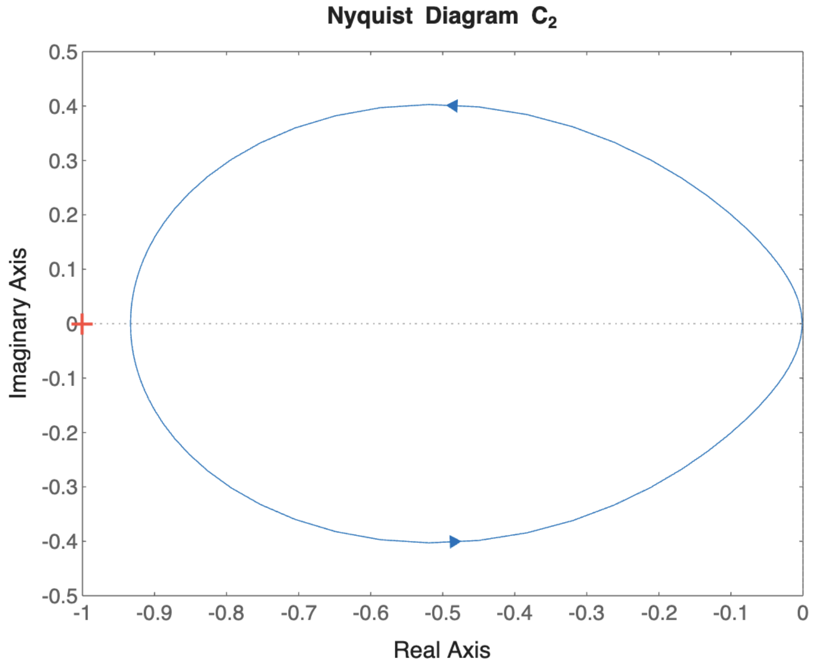
- Consider the inverted pendulum (upright position):
- $\dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{3g}{2L} & -\frac{3c_f}{mL^2} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix} u,\ y = [1 \quad 0]x$
- Using $g = 10, m = 1, L = \frac{3}{2}, c_f = \frac{9}{4}$ we get
  - $G(s) = \frac{1.33}{s^2+3s-10}$
- Consider a PD controller
  - $C(s) = k_p + \frac{k_d s}{T_f s+1}$
- We have two controllers with poles of the open loop $L(s) = G(s)C(s)$:
  - C_1: $k_p = 70, k_d = 10, T_f = 0.001$
    - $p_1 = -100,\ p_2 = -5,\ p_3 = 2$
  - C_2: $k_p = 7, k_d = 1, T_f = 0.001$
    - $p_1 = -1000,\ p_2 = -5,\ p_3 = 2$

Nyquist Condition

Example

- We have two controllers with poles of the open loop $L(s) = G(s)C(s)$:
  - C_1: $k_p = 70, k_d = 10, T_f = 0.001$
    - $p_1 = -100,\ p_2 = -5,\ p_3 = 2$
  - C_2: $k_p = 7, k_d = 1, T_f = 0.001$
    - $p_1 = -1000,\ p_2 = -5,\ p_3 = 2$
- Given the Nyquist Plots which controller stabilizes the system?
  - C_1:
    - $n_p = 1$, we have 1 unstable pole
    - $n_c = 1$, we have 1 CCW encirclement of -1
    - Closed loop system is **stable**
  - C_2:
    - $n_p = 1$, we have 1 unstable pole
    - $n_c = 0$, we have 0 CCW encirclement of -1
    - Closed loop system is **unstable**



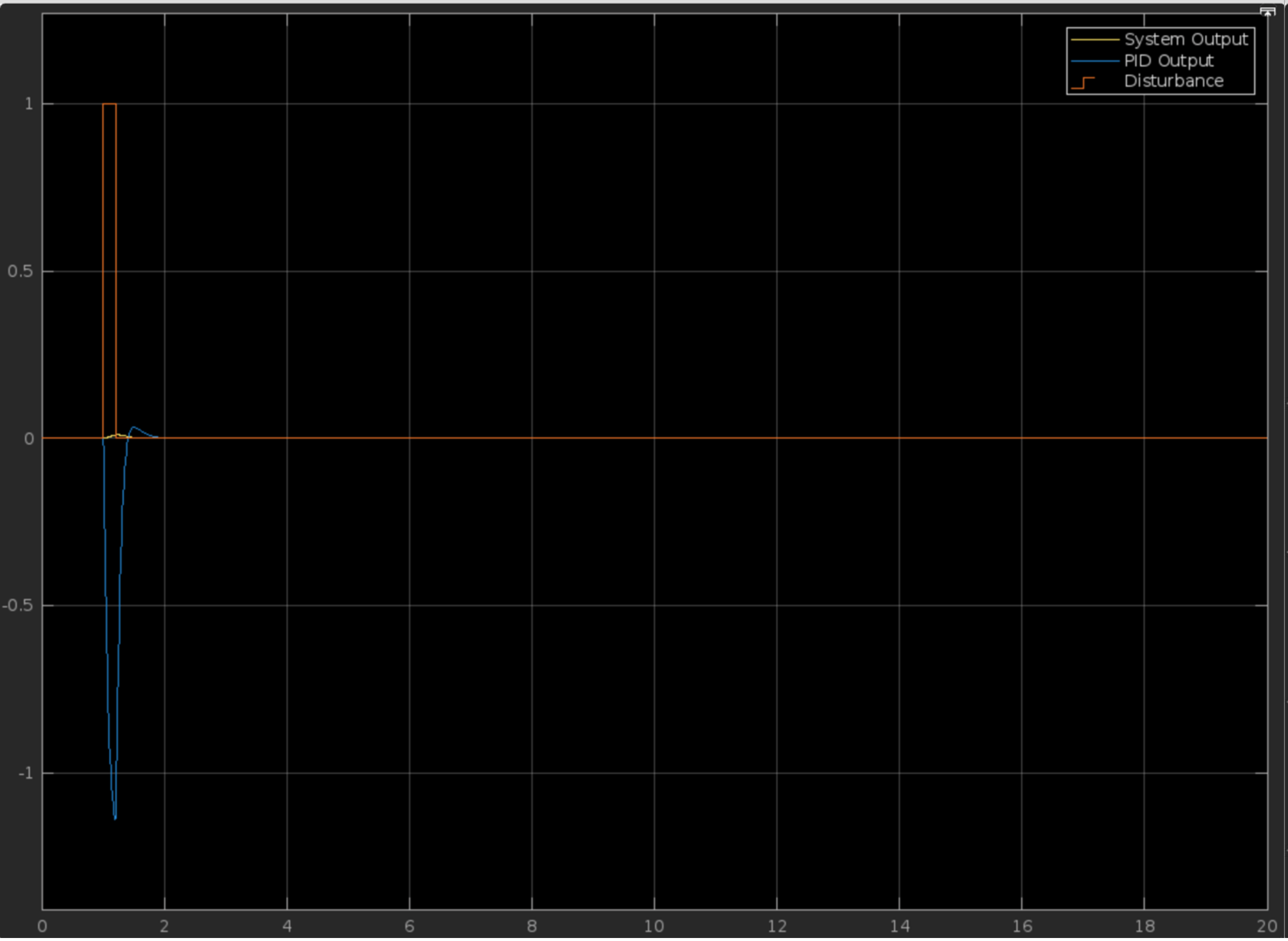


Nyquist Condition

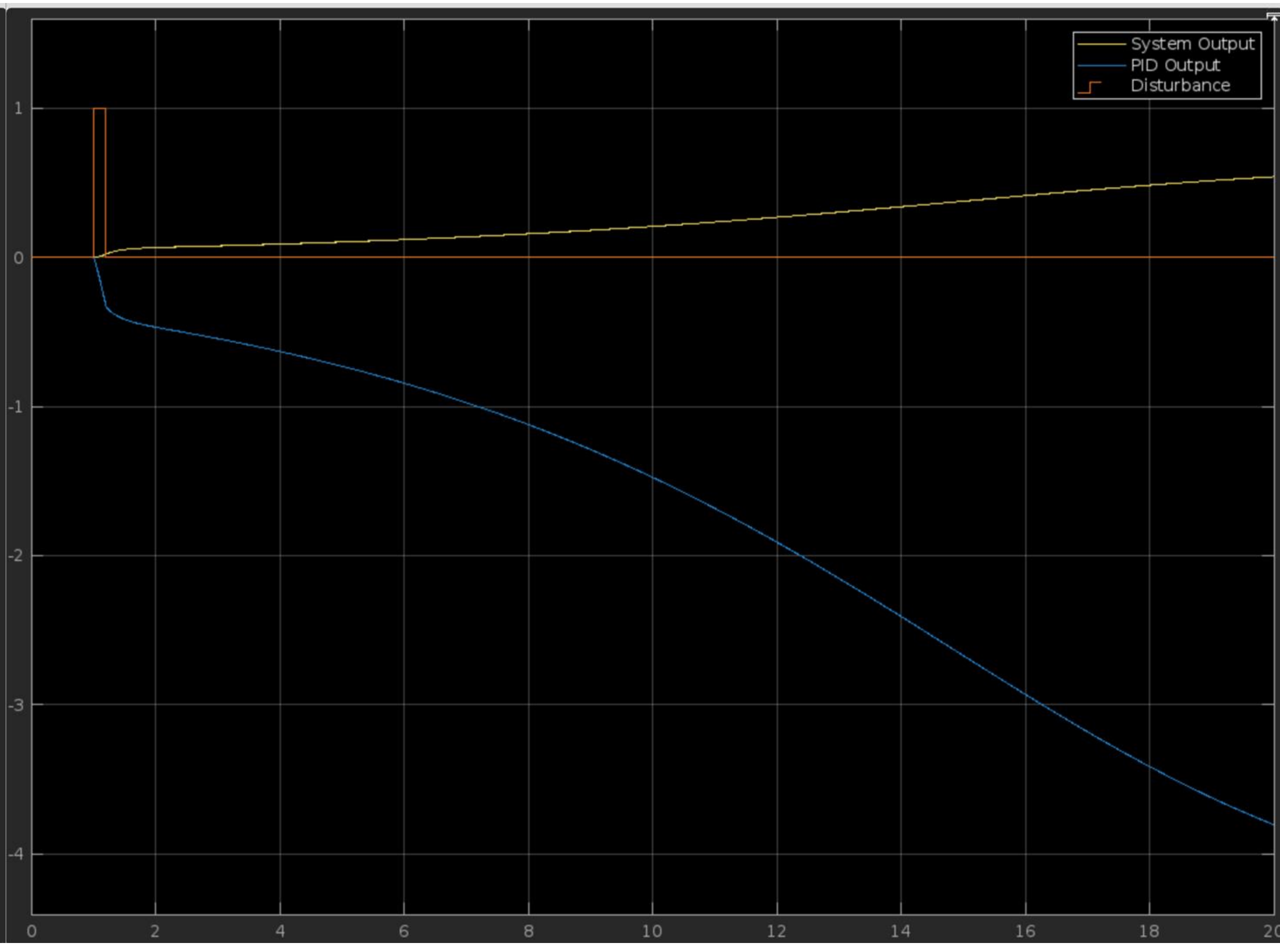
Example

- Impulse Disturbance

C_1:



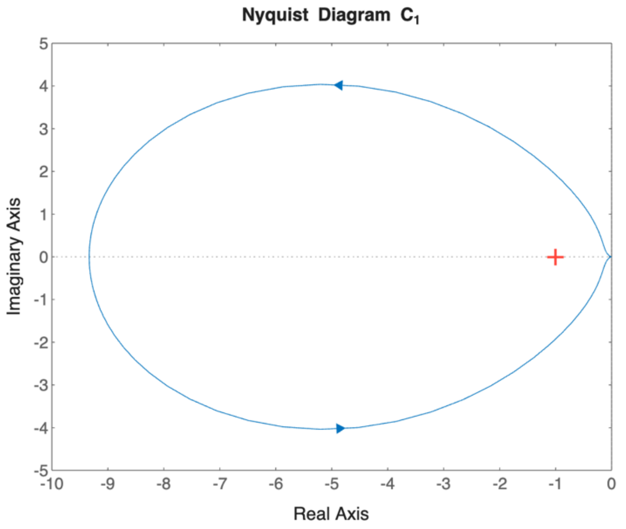
C_2:

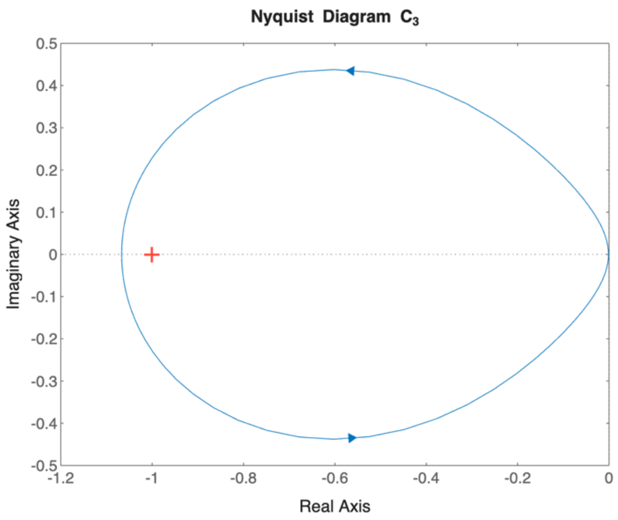


Stability Margins

What?

- Number of surroundings of the point -1 $(k = 1)$ is very important for stability
  - What happens if that changes? -> system becomes unstable (see example before)
- Is there a difference between a large encirclement and small encirclement?



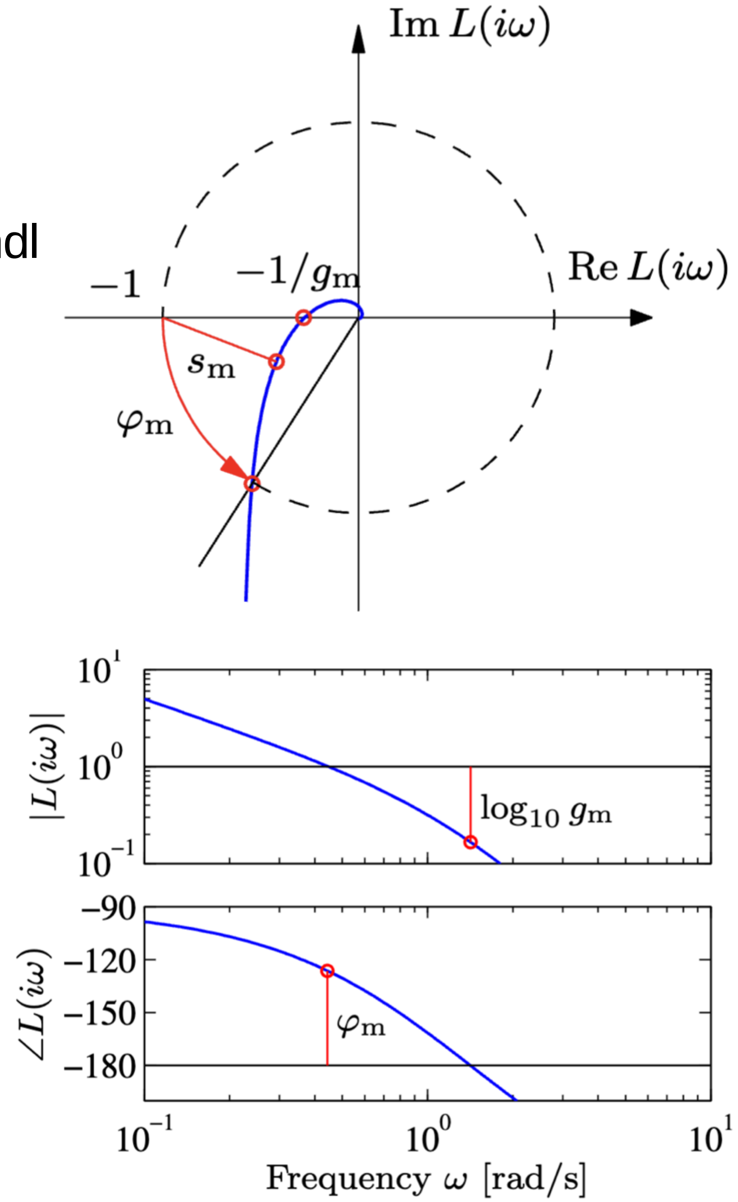


  - Yes, there is!!
  - The further away from the point -1 the Nyquist plot is, the more robust is a system
  - Need a metric to define how far away the system is from the point -1 → Stability Margins

Stability Margins

What?

- The stability margins tell us how far away we are from a -1 crossing:
  - Tells us how much modelling errors the closed loop system can handl before going unstable!
- We have:
  - g_m: gain margin
    - How much more can we "blow up" the system
    - At $\angle L(j\omega_g) = -180 \rightarrow g_m = \frac{1}{|L(j\omega_g)|}$
  - φ_m: phase margin
    - How much phase shift/lag can the system handle
    - At $|L(j\omega_c)| = 1 \rightarrow \varphi_m = \angle L(j\omega_g) + 180°$
    - ω_c = cross-over frequency

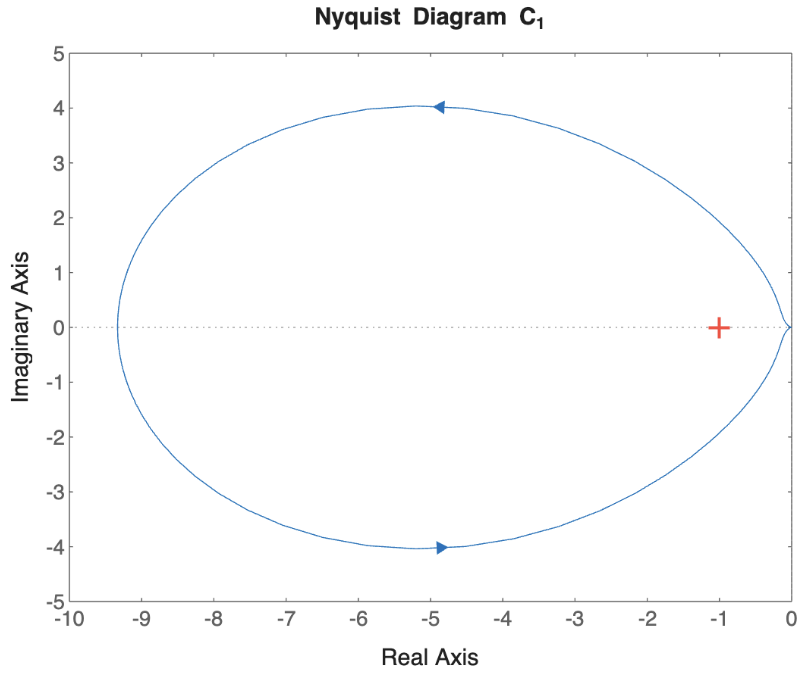


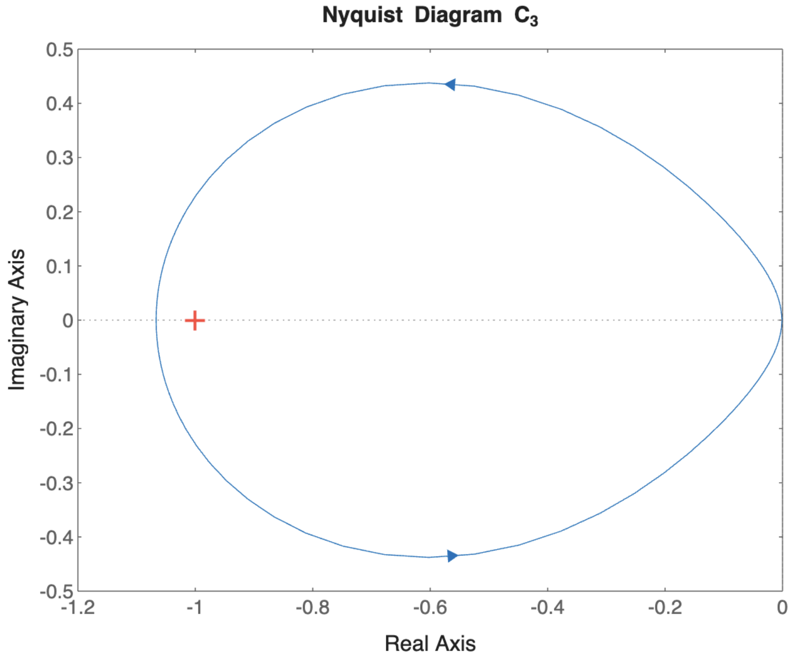
- We often need to find a trade-off between performance and robustness:
  - We can either have a robust system or a system that performs well

Stability Margins

Example

- Consider again the inverted pendulum with two controllers:
  - C_1: $k_p = 70, k_d = 10, T_f = 0.001$
    - $p_1 = -100,\ p_2 = -5,\ p_3 = 2$
  - C_3: $k_p = 8, k_d = 1, T_f = 0.001$
    - $p_1 = -1000,\ p_2 = -5,\ p_3 = 2$
- We see C_1 is way further away from -1 than C_3
  - C_1 is much more robust than C_3
- What happens when we made a mistake during modelling, and we designed our controller according to the model?
  - Suppose $m_{real} = m_{model} * 1.1$
  - What happens with the system response?



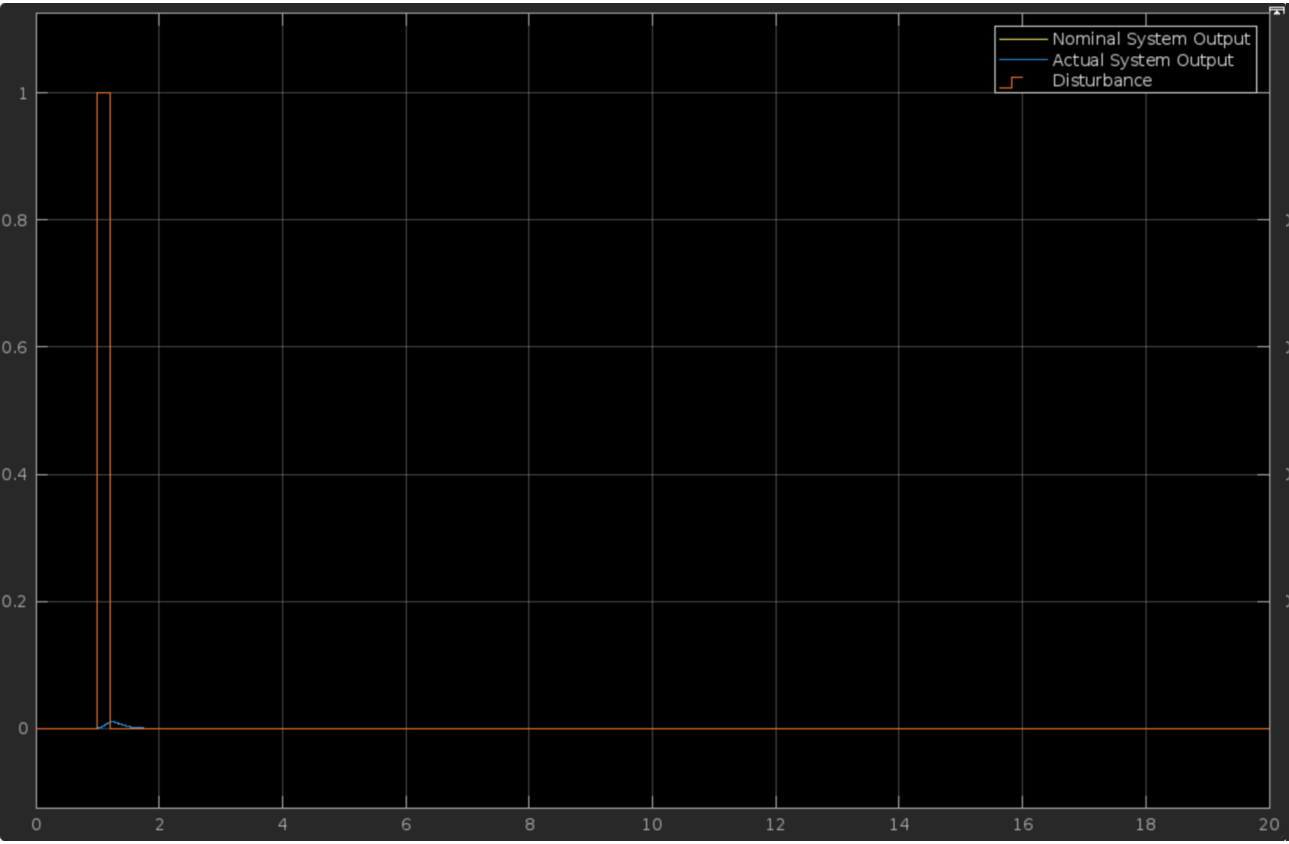


Stability Margins

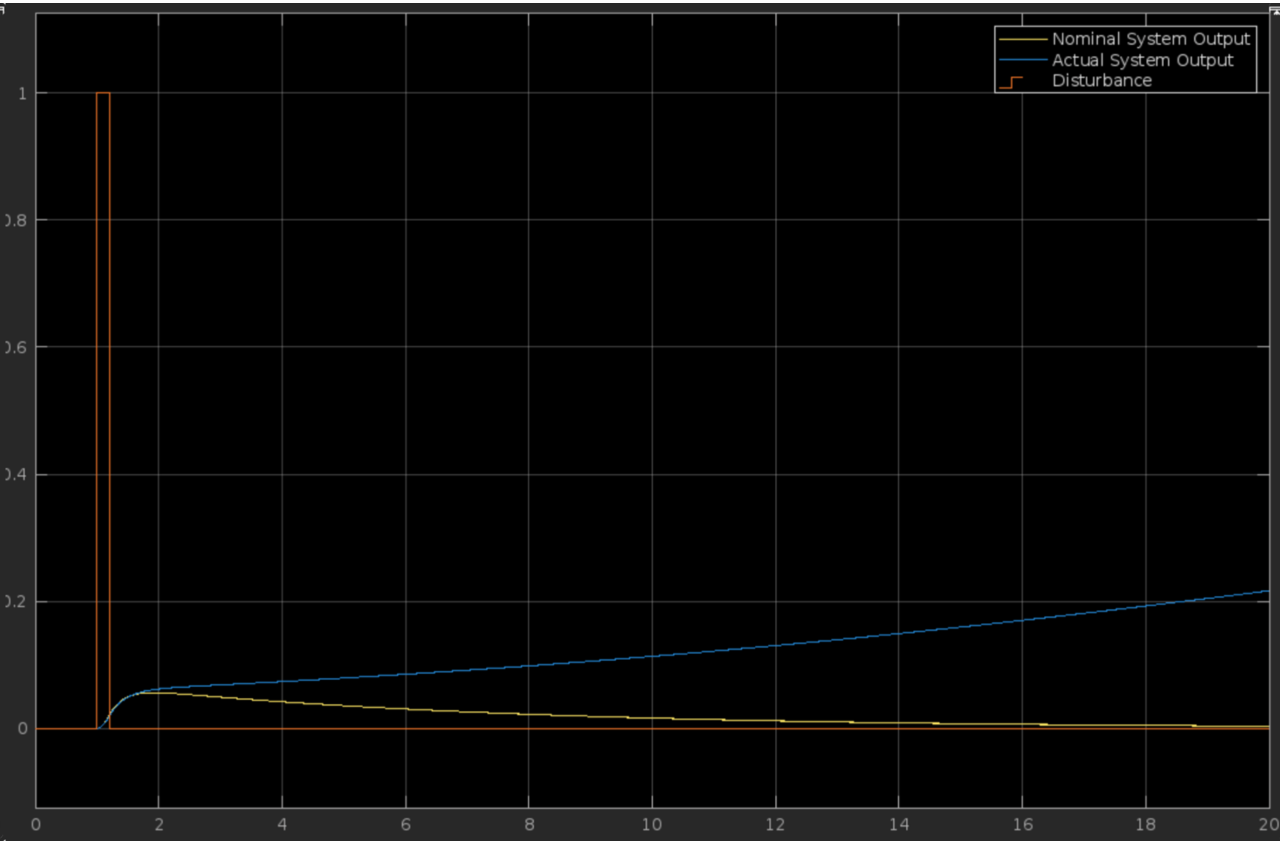
Example

- We see C_1 can handle this modelling error while C_3 can't.
  - Both nominal systems are stable but only C_1 can control the actual system!!

C_1:



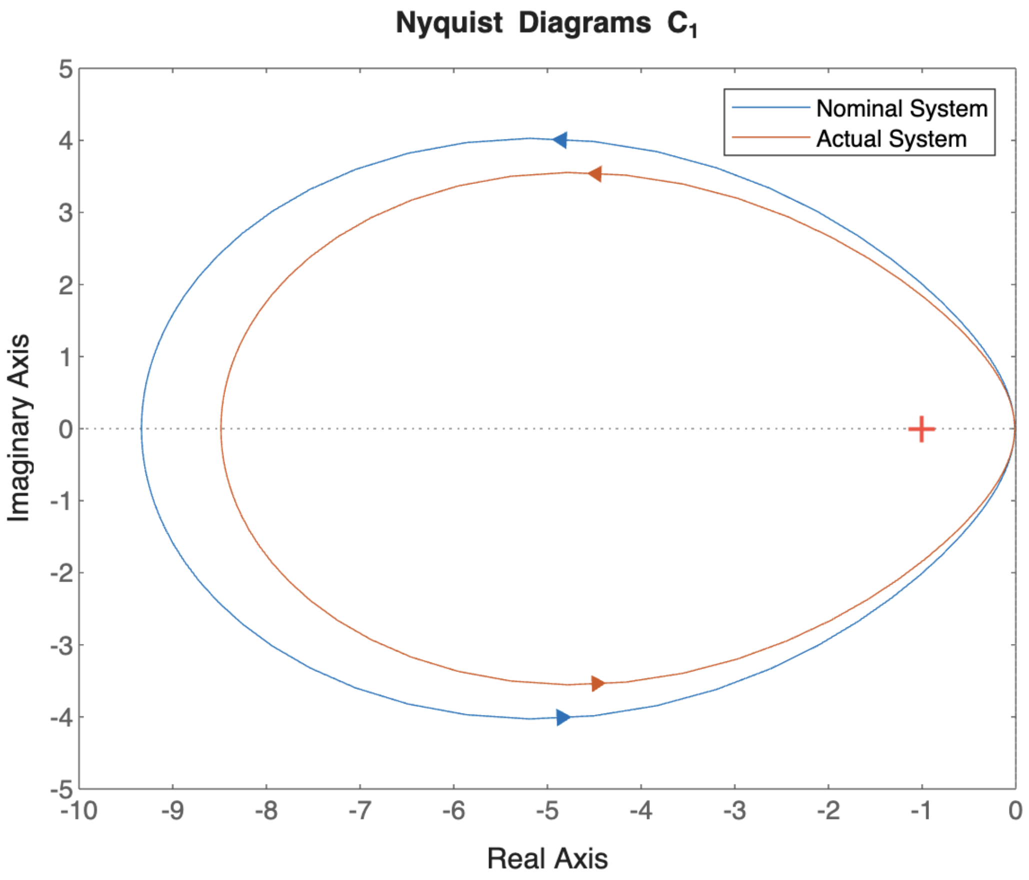
C_3:

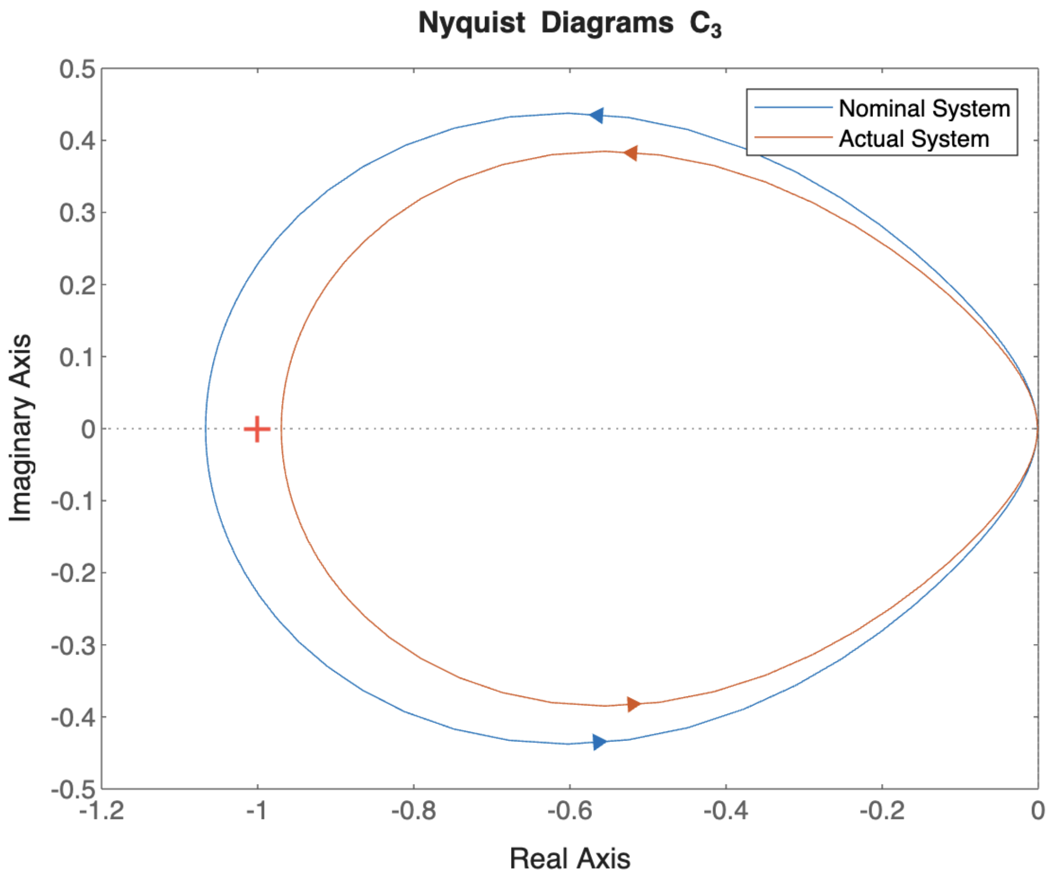


Stability Margins

Example

- We see C_1 can handle this modelling error while C_3 can't.
  - This can also be seen in the Nyquist Plots.





Exercise 09

What to do?

- 1:
  - Do
- 2:
  - Do
- 3:
  - Do
- 4:
  - Do one