

Control Systems I

Recitation 10

<https://n.ethz.ch/~jgeurts/>
jgeurts@ethz.ch

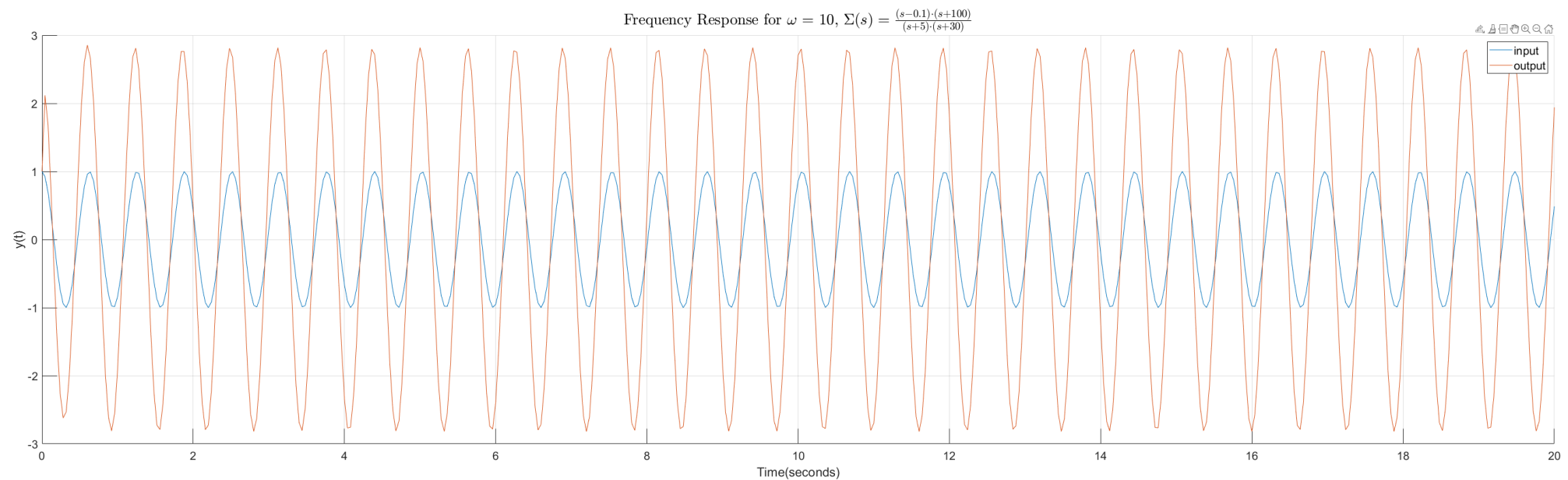
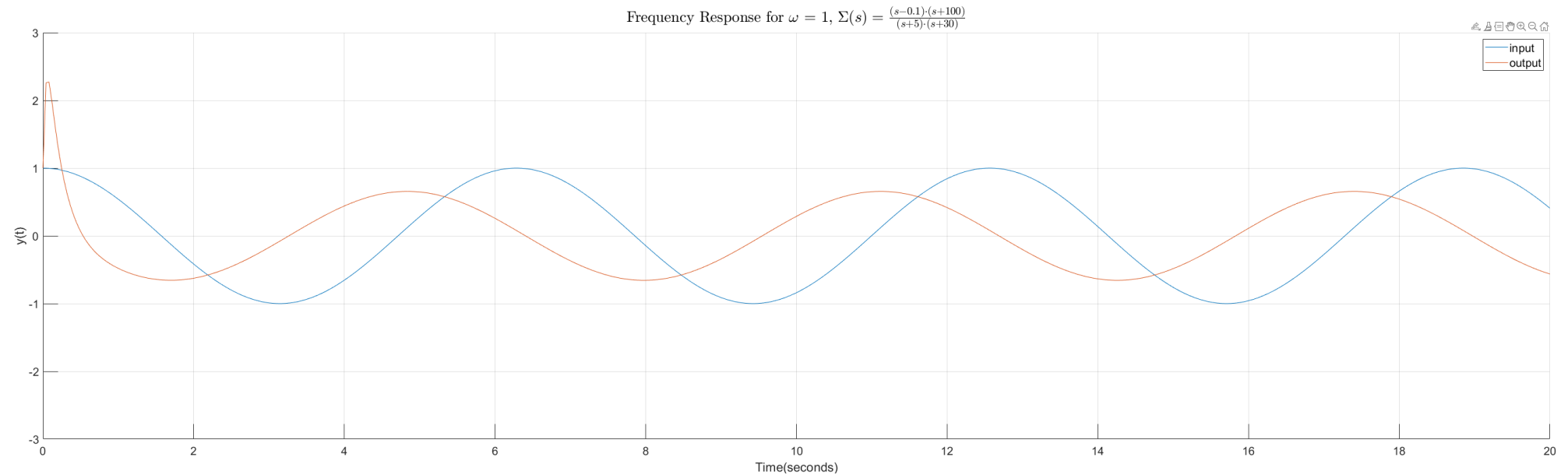


Last Week

Frequency Response

- We want to find the response to a Harmonic Input:
 - $u(t) = \alpha \cdot \cos(\omega \cdot t + \phi)$, $\phi = 0$ in most cases
- Resulting Response:
 - $y(t) = |G(j\omega)| \cdot \alpha \cdot \cos(\omega \cdot t + \phi + \angle G(j\omega))$
- What do we see?
 - The system oscillates with the same frequency
 - The amplitude is **frequency dependant**
 - The phase shift is **frequency dependant**
- How can we plot $|G(j\omega)|$ and $\angle G(j\omega)$?
 - Bode Plot
 - Polar / Nyquist Plot

Frequency Response



Last Week

Bode Plot

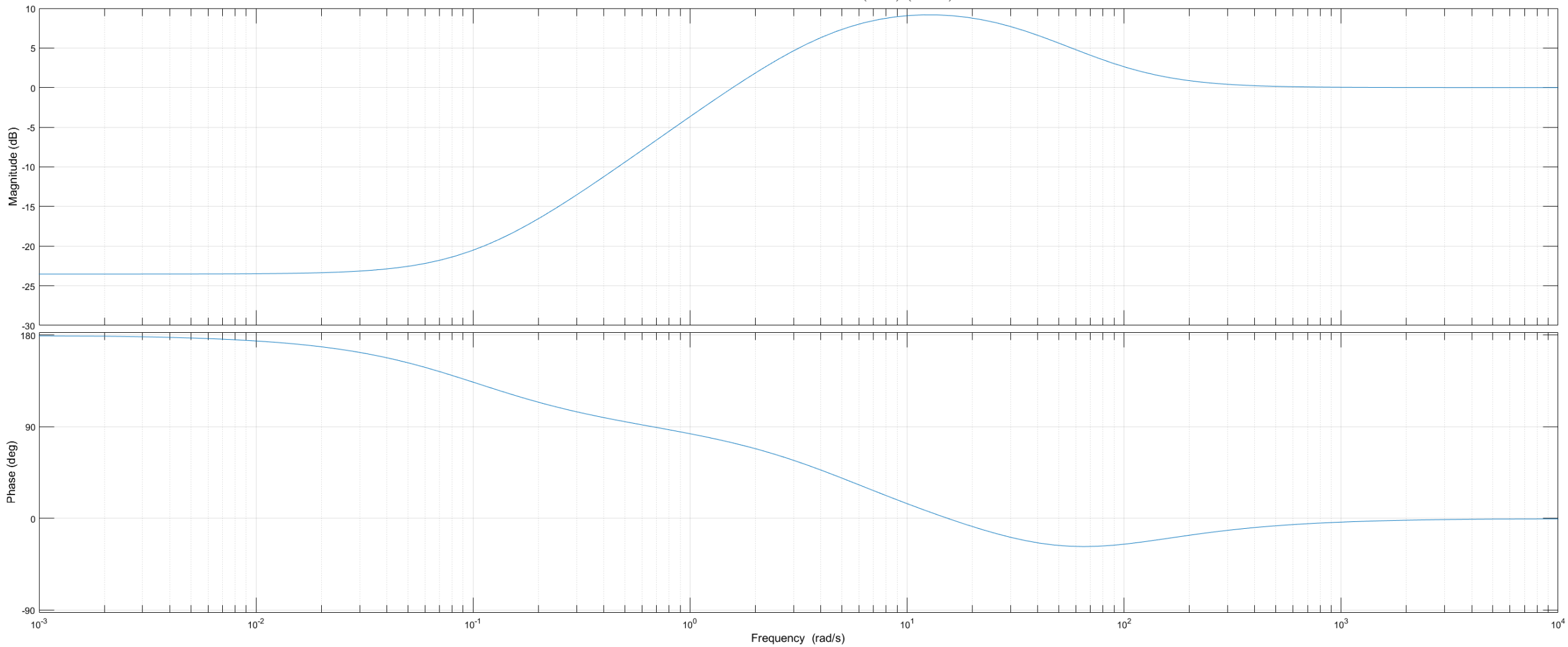
- Two separate frequency **explicit** plots for both $|G(j\omega)|$ and $\angle G(j\omega)$
- Magnitude Plot $|G(j\omega)|$:
 - Logarithmic ω axis and dB(decibel) $|G(j\omega)|$
 - Decibel:
 - $|G(j\omega)|_{\text{dB}} = 20 \cdot \log_{10}|G(j\omega)|$
 - $|G(j\omega)| = 10^{\frac{|G(j\omega)|_{\text{dB}}}{20}}$
 - $|G(j\omega)| = \sqrt{\text{Re}(G(j\omega))^2 + \text{Im}(G(j\omega))^2}$
- Phase Plot $\angle G(j\omega)$:
 - Logarithmic ω axis and linear $\angle G(j\omega)$ (in degrees)
 - $\angle G(j\omega) = \arctan2\left(\frac{\text{Im}(G(j\omega))}{\text{Re}(G(j\omega))}\right)$

Dezimalskala	Dezibelskala
100	40
10	20
5	13.97...
2	6.02...
1	0
$1/\sqrt{2}$	-3.0103
0.1	-20
0.01	-40
0	-Inf

$$\text{atan2}(y, x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0, \\ \frac{\pi}{2} - \arctan\left(\frac{x}{y}\right) & \text{if } y > 0, \\ -\frac{\pi}{2} - \arctan\left(\frac{x}{y}\right) & \text{if } y < 0, \\ \arctan\left(\frac{y}{x}\right) \pm \pi & \text{if } x < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

Bode Plot of a system

Bode Plot of: $\Sigma(s) = \frac{(s-0.1) \cdot (s+100)}{(s+5) \cdot (s+30)}$



Last Week

Drawing a Bode Plot

- Using Logarithms is very convenient, we can combine different systems
 - Total System: $G(s) = G_1(s) \cdot G_2(s) \cdot \dots \cdot G_n(s)$
 - Amplitude in decibel: $|\Sigma(s)|_{\text{dB}} = |\Sigma_1(s)|_{\text{dB}} + |\Sigma_2(s)|_{\text{dB}}$
 - Phase: $\angle \Sigma(s) = \angle \Sigma_1(s) + \angle \Sigma_2(s)$
- When drawing combine the effects of poles and zeros of the sub-systems (addition)
 - The effect is at the position of the pole/zero
 - At the pole/zero the phase shift is approx 50% done
 - For multiplicity $k > 1$, the change is multiplied by k

Type	Magnitude Change	Phase Change
Stable Pole	-20 dB/dec	-90°
Unstable Pole	-20 dB/dec	+90°
Minimumphase zero	+20 dB/dec	+90°
Non-minimumphase zero	+20 dB/dec	-90°
Time Delay	0 dB/dec	$-\omega \cdot T$

Bode Plots

Standard Elements – there are a bunch

A.1 Integrator Element

Element Acronym: **I**

Transfer Function: $\Sigma(s) = \frac{1}{T \cdot s}$

Poles/Zeros: $\pi_1 = 0, \zeta_1 = \infty$

Internal Description: $\frac{d}{dt}x(t) = \frac{1}{T} \cdot u(t)$
 $y(t) = x(t)$

A.2 Differentiator Element

Element Acronym: **D**

Transfer Function: $\Sigma(s) = T \cdot s$

Poles/Zeros: $\pi_1 = \infty, \zeta_1 = 0$

Internal Description: $y(t) = T \cdot \frac{d}{dt}u(t)$

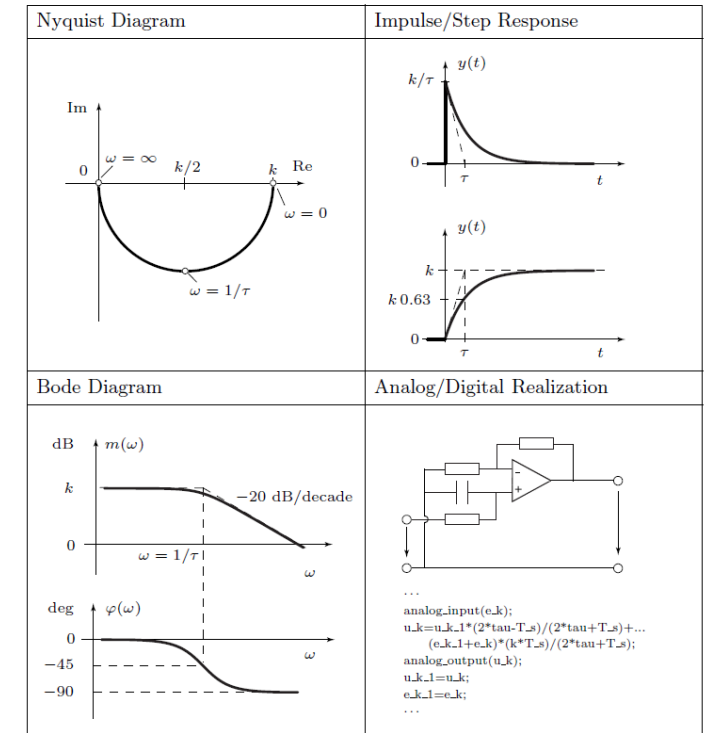
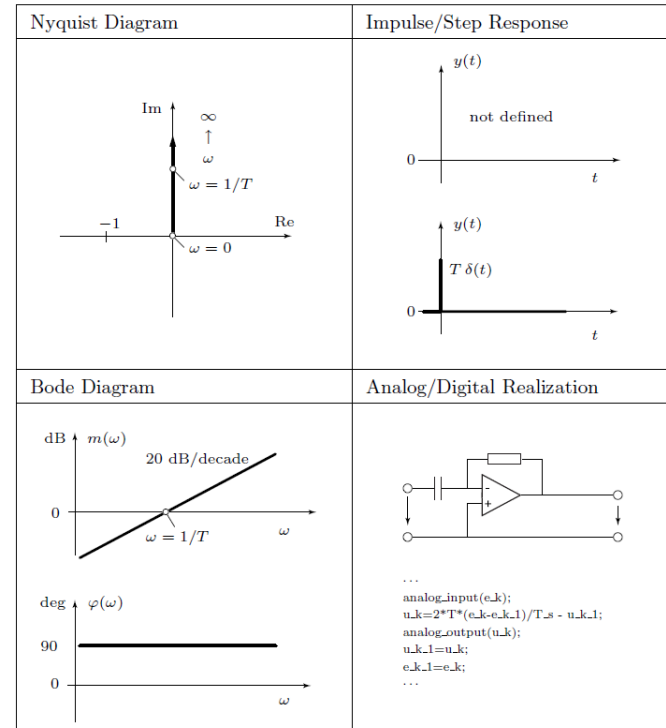
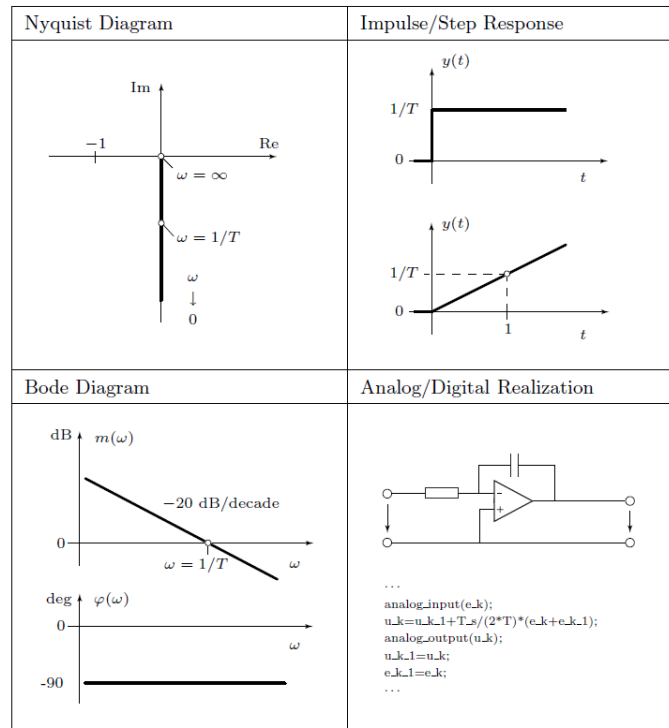
A.3 First-Order Element

Element Acronym: **LP-1**

Transfer Function: $\Sigma(s) = \frac{k}{\tau \cdot s + 1}$

Poles/Zeros: $\pi_1 = -\frac{1}{\tau}, \zeta_1 = \infty$

Internal Description: $\frac{d}{dt}x(t) = -\frac{1}{\tau} \cdot x(t) + \frac{1}{\tau} \cdot u(t)$
 $y(t) = k \cdot x(t)$



Bode Plots

Standard Elements – there are a bunch

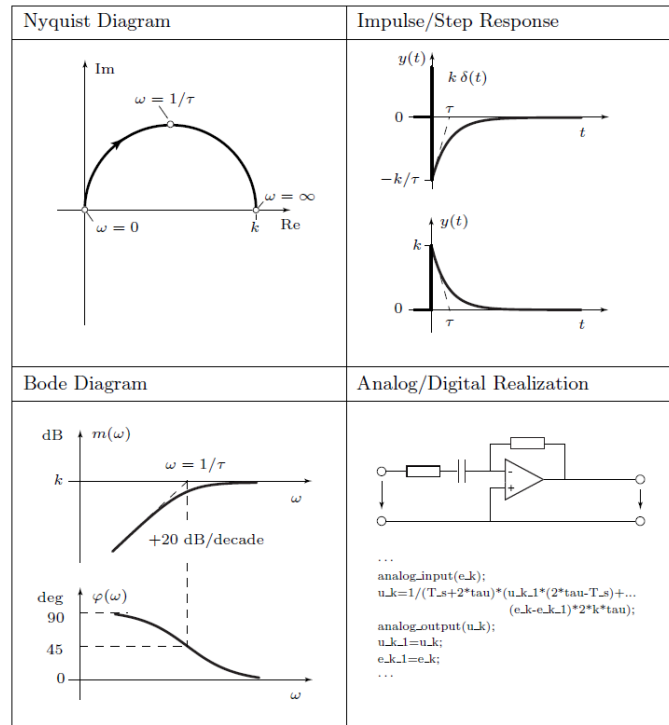
A.4 Realizable Derivative Element

Element Acronym: HP-1

Transfer Function: $\Sigma(s) = k \cdot \frac{\tau \cdot s}{\tau \cdot s + 1} = k \cdot \left(1 - \frac{1}{\tau \cdot s + 1}\right)$

Poles/Zeros: $\pi_1 = -\frac{1}{\tau}, \zeta_1 = 0$

Internal Description: $\frac{d}{dt}x(t) = -\frac{1}{\tau} \cdot x(t) + \frac{1}{\tau} \cdot u(t)$
 $y(t) = -k \cdot x(t) + k \cdot u(t)$



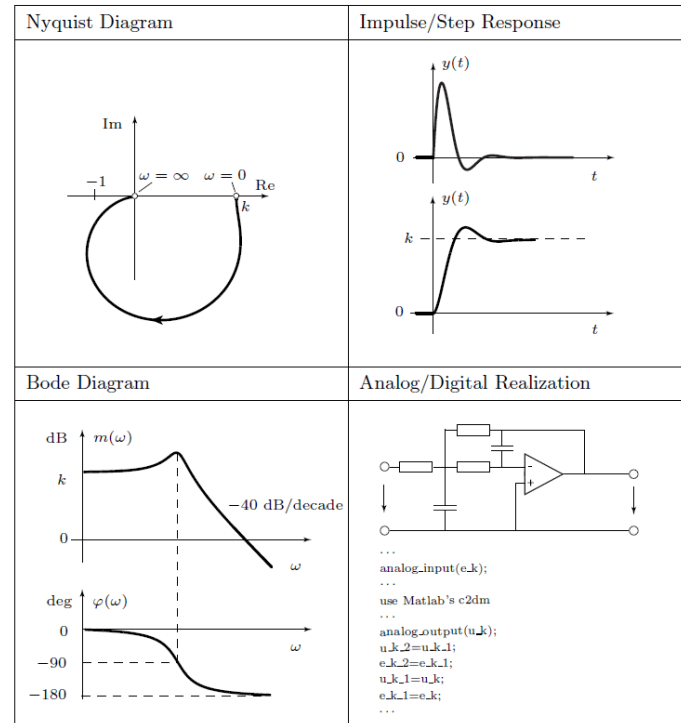
A.5 Second-Order Element

Element Acronym: LP-2

Transfer Function: $\Sigma(s) = k \cdot \frac{\omega_0^2}{s^2 + 2 \cdot \delta \cdot \omega_0 \cdot s + \omega_0^2}$

Poles/Zeros: $\pi_{1,2} = -w_0 \cdot \delta \pm w_0 \sqrt{\delta^2 - 1}, \zeta_{1,2} = \infty$

Internal Description: $\frac{d}{dt}x_1(t) = x_2(t),$
 $\frac{d}{dt}x_2(t) = -\omega_0^2 \cdot x_1(t) - 2 \cdot \delta \cdot \omega_0 \cdot x_2(t) + \omega_0^2 \cdot u(t)$
 $y(t) = k \cdot x_1(t)$



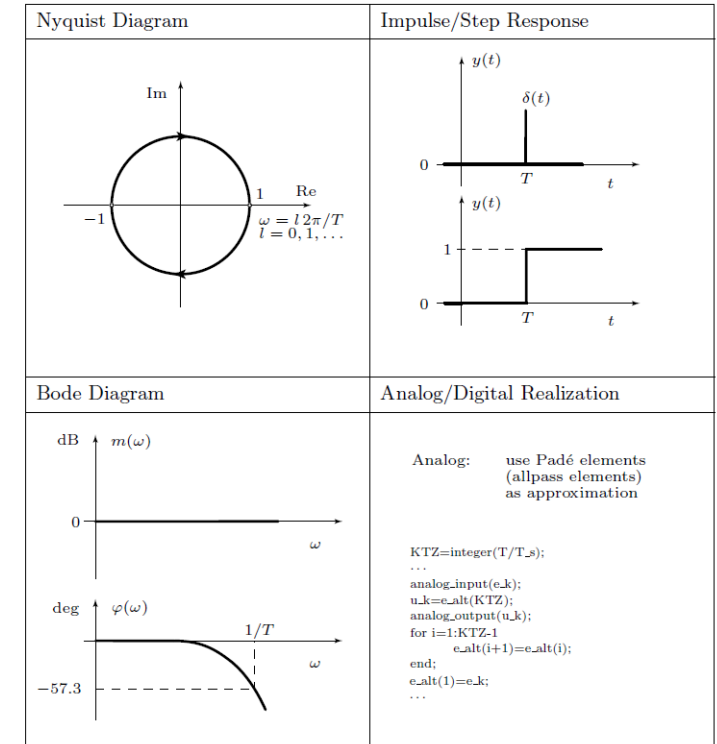
A.10 Delay Element

Element Acronym: -

Transfer Function: $\Sigma(s) = e^{-s \cdot T}$

Poles/Zeros: not a real-rational element

Internal Description: $y(t) = u(t - T)$



Last Week

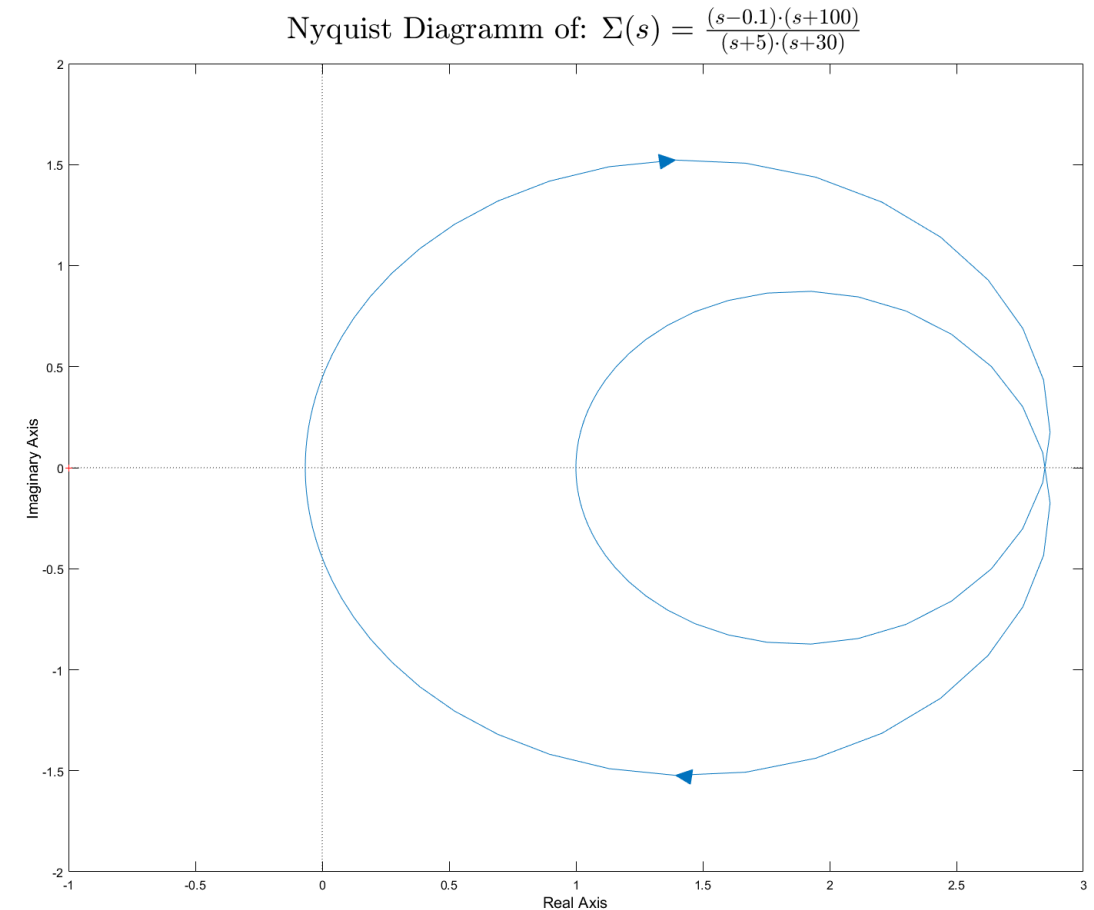
Bodes Law

- Phase and Amplitude are not independent
 - $|G(j\omega)|_{\text{dB}} = 20 \frac{\text{dB}}{\text{dec}} \cdot \kappa \Rightarrow \angle G(j\omega) \approx \kappa \cdot \frac{\pi}{2}$
- System: $\Sigma(s) = \frac{b_m \cdot s^m + \dots + b_1 \cdot s + b_0}{s^q \cdot (s^{n-q} + a_{n-q-1} \cdot s^{n-q-1} + \dots + a_1 \cdot s + a_0)}$
 - Relative degree: $r = n - m$
 - System Type: $q = \text{number of integrators}$
- We further have:
 - For $\omega \rightarrow \infty$: $\frac{\partial |G(j\omega)|_{\text{dB}}}{\partial \log_{10}(\omega)} = -r \cdot 20 \text{ dB}$, with $r = n - m$ being the relative degree
 - For $\omega \rightarrow 0$: $\angle G(j\omega = 0) = \begin{cases} -q \cdot \frac{\pi}{2}, & \text{for } \text{sign}\left(\frac{b_0}{a_0}\right) > 0 \\ -\pi - q \cdot \frac{\pi}{2}, & \text{for } \text{sign}\left(\frac{b_0}{a_0}\right) < 0 \end{cases}$

Last Week

Nyquist/Polar Plot

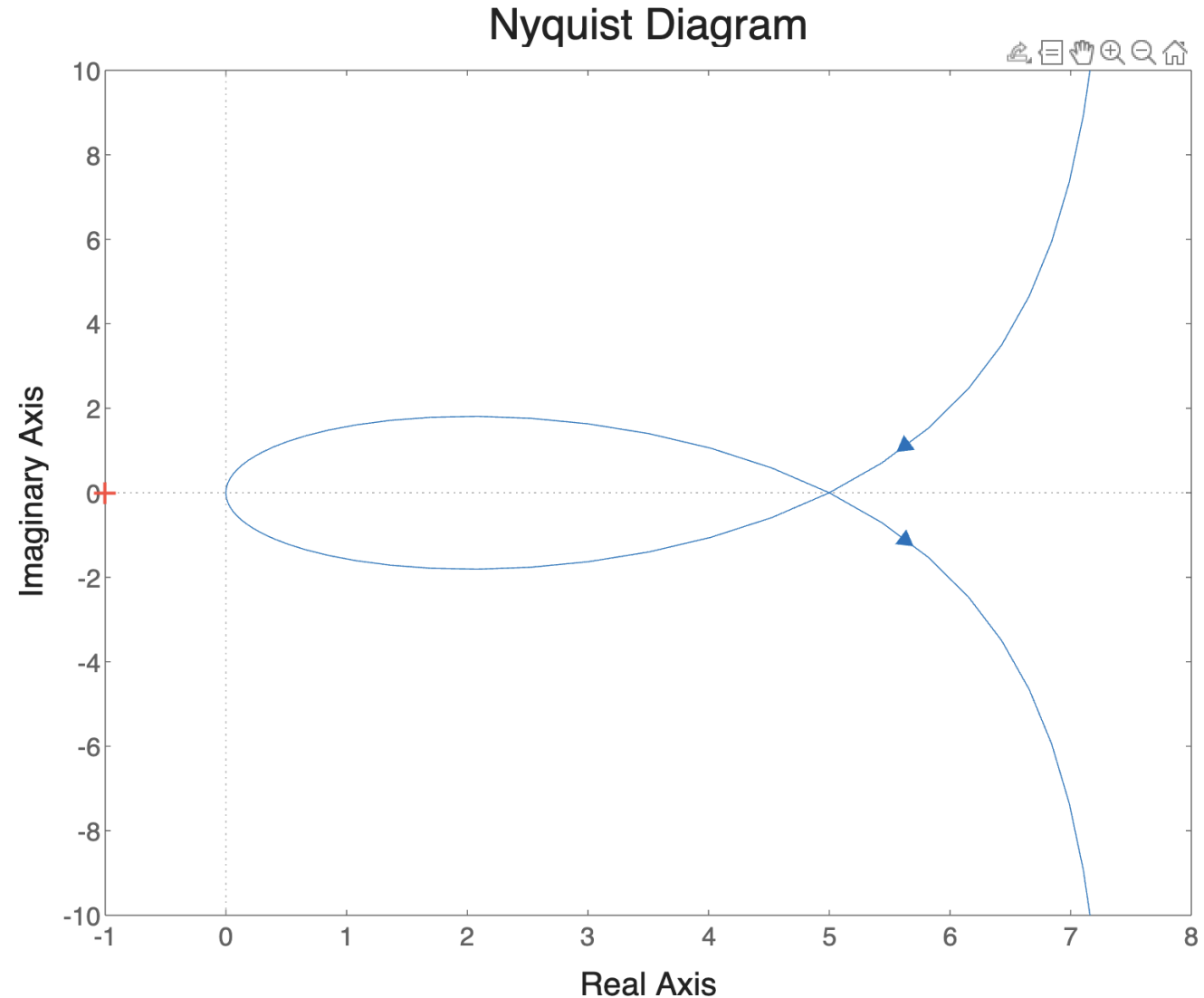
- $|G(s)|$ and $\angle G(s)$ drawn in the complex plane.
 - Polar: $0 \leq \omega < \infty$
 - Nyquist: $-\infty < \omega < \infty$
- The values are now frequency **implicit**
- Drawing usually using Python or Matlab
- Sketching
 - Look at the extremes $\omega \rightarrow 0, \omega \rightarrow \infty$
 - Use Bodes Law
 - Read values of Bode plot
 - Needs to be qualitatively correct
 - $\omega \in (-\infty, 0]$ is the mirror of $\omega \in [0, \infty)$



Polar / Nyquist Plot

Example – Drawing a Nyquist Plot

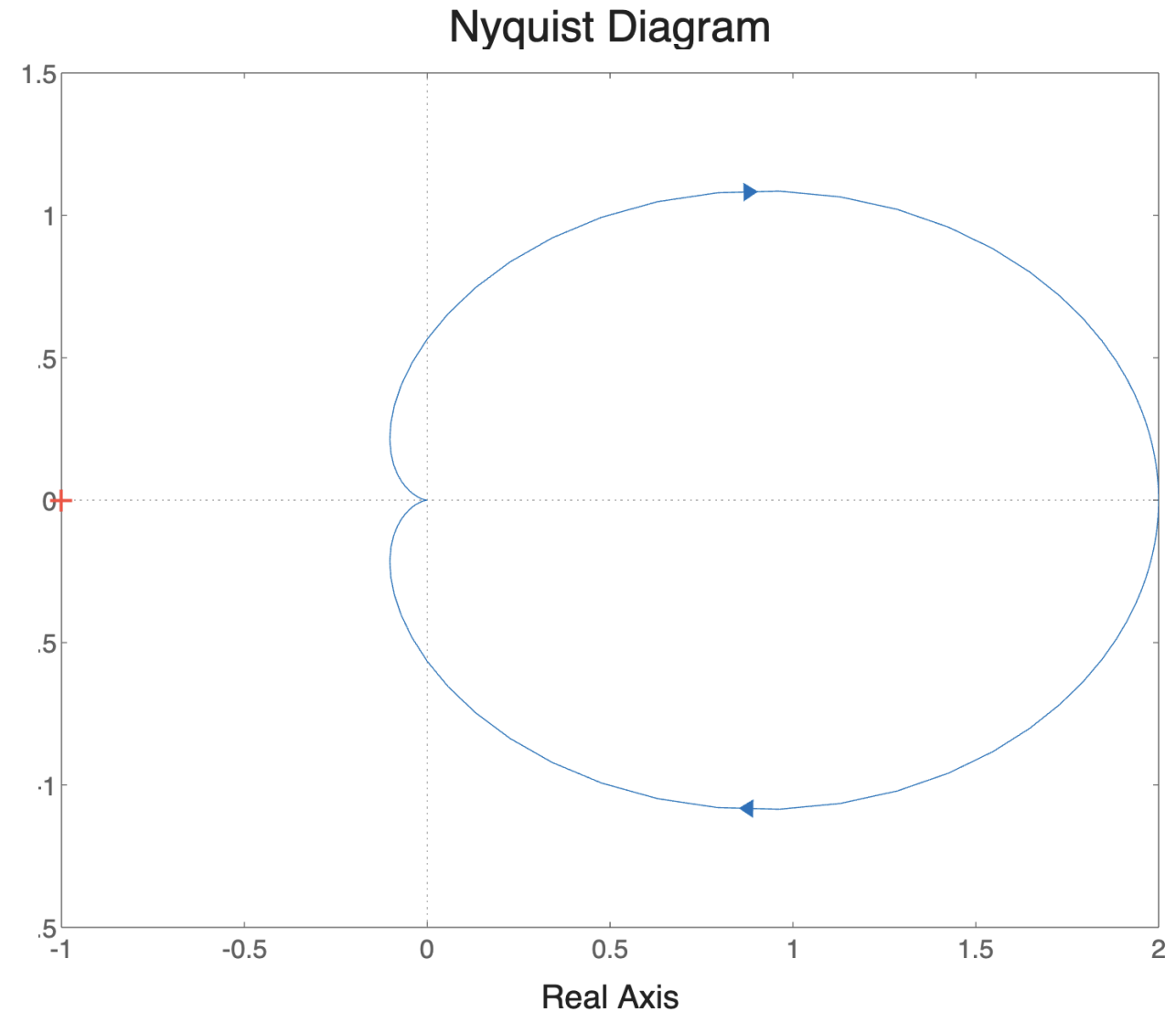
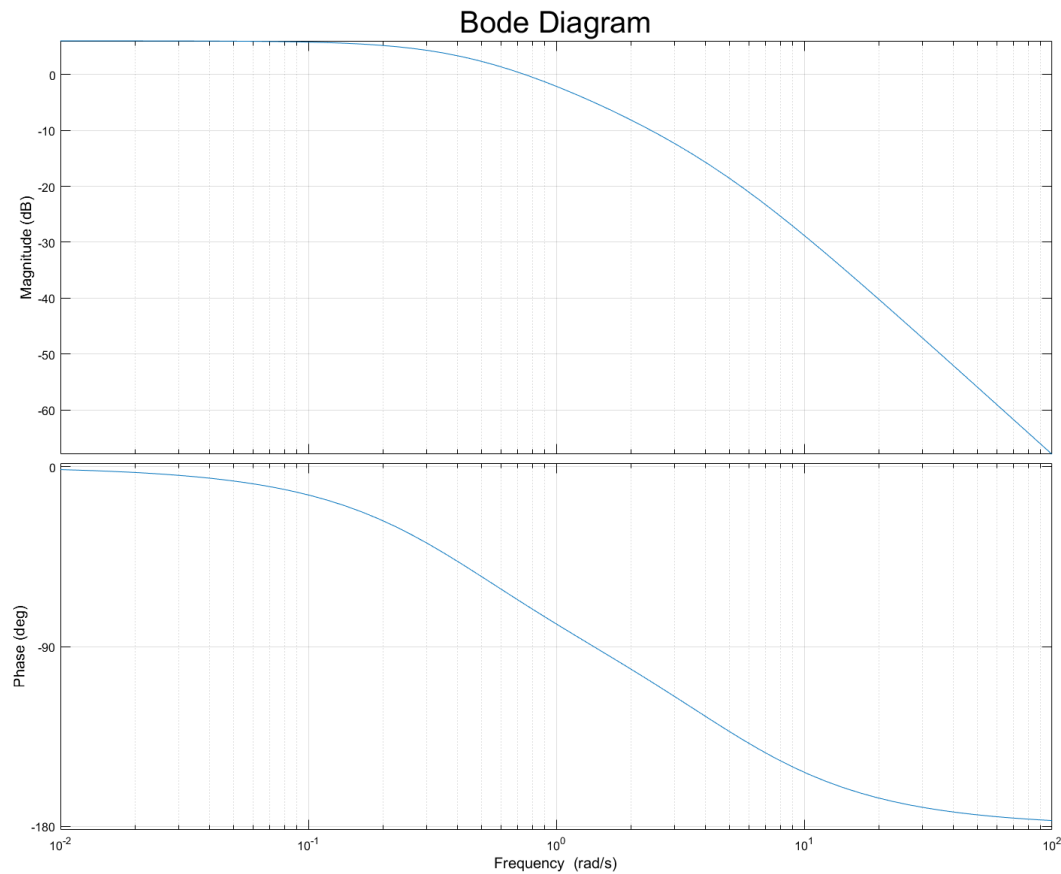
- Draw the Nyquist Plot for:
 - $G(s) = \frac{5(s-0.5)}{s(s+5)}$
- $\omega \rightarrow 0^+$:
 - $|G(j\omega)| \rightarrow \infty$
 - $\angle G(j\omega) = \begin{cases} -q \cdot \frac{\pi}{2}, & \text{for } \text{sign}\left(\frac{b_0}{a_0}\right) > 0 \\ -\pi - q \cdot \frac{\pi}{2}, & \text{for } \text{sign}\left(\frac{b_0}{a_0}\right) < 0 \end{cases} = -\frac{3}{2}\pi$
- $\omega \rightarrow \infty$:
 - $|G(j\omega)| \rightarrow 0$
 - $\angle G(j\omega) \approx \angle \frac{1}{s} = -\frac{\pi}{2}$



Polar / Nyquist Plot

Example – Drawing a Nyquist Plot

- Draw the Nyquist Plot for the System with the following Bode Plot



Last Week

Bode vs Nyquist/Polar

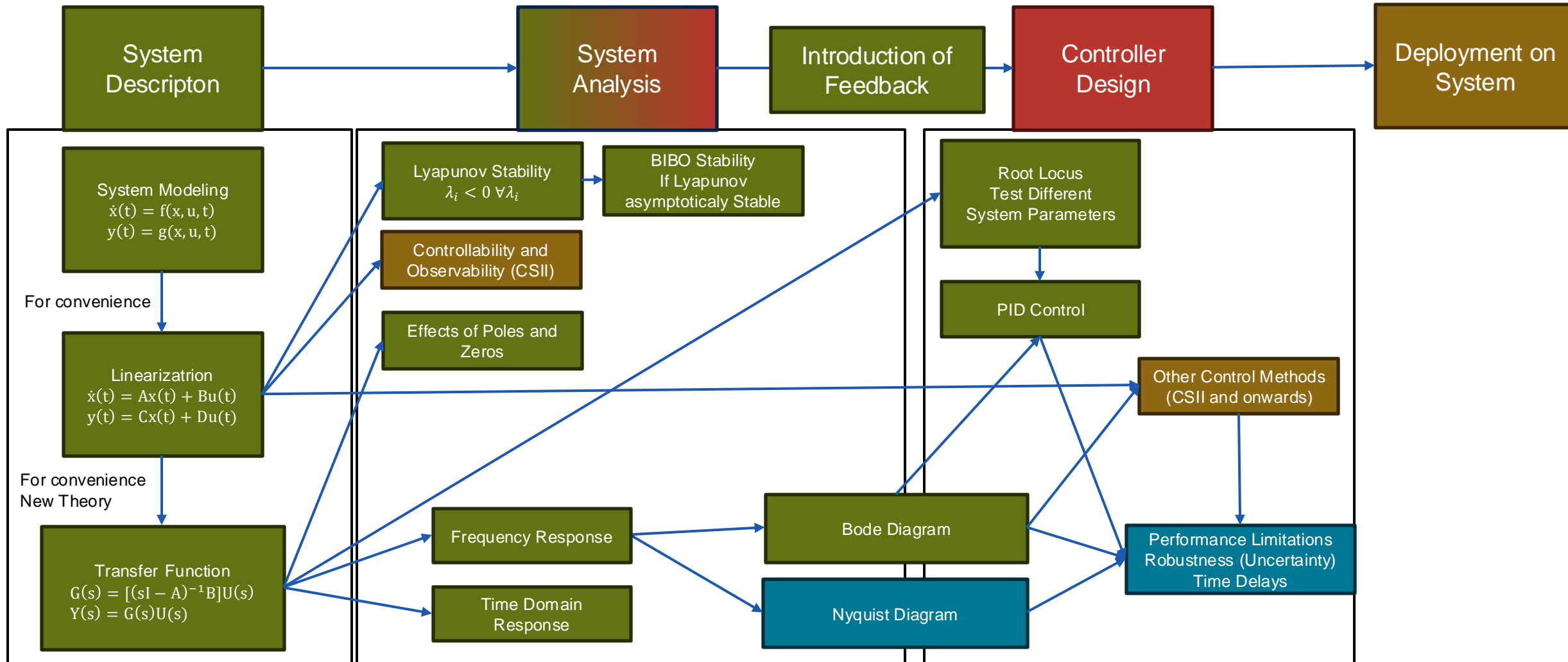
- Graphical representation of $|G(s)|$ and $\angle G(s)$
- Bode Plot:
 - frequency **explicit**
 - Logarithmic, decibel and linear axis scale
 - Quantitative analysis
- Nyquist Plot:
 - frequency **implicit**
 - Linear axis scale
 - Qualitative analysis

Outline

- Nyquist Criterion
 - Cauchy's argument principle
 - Nyquist Condition
 - Nyquist Stability Theorem
 - Counting Encirclements
 - Example
- Stability Margins
 - What?
 - Example

Conceptual Recap

Classical Control Approach



Time Delays

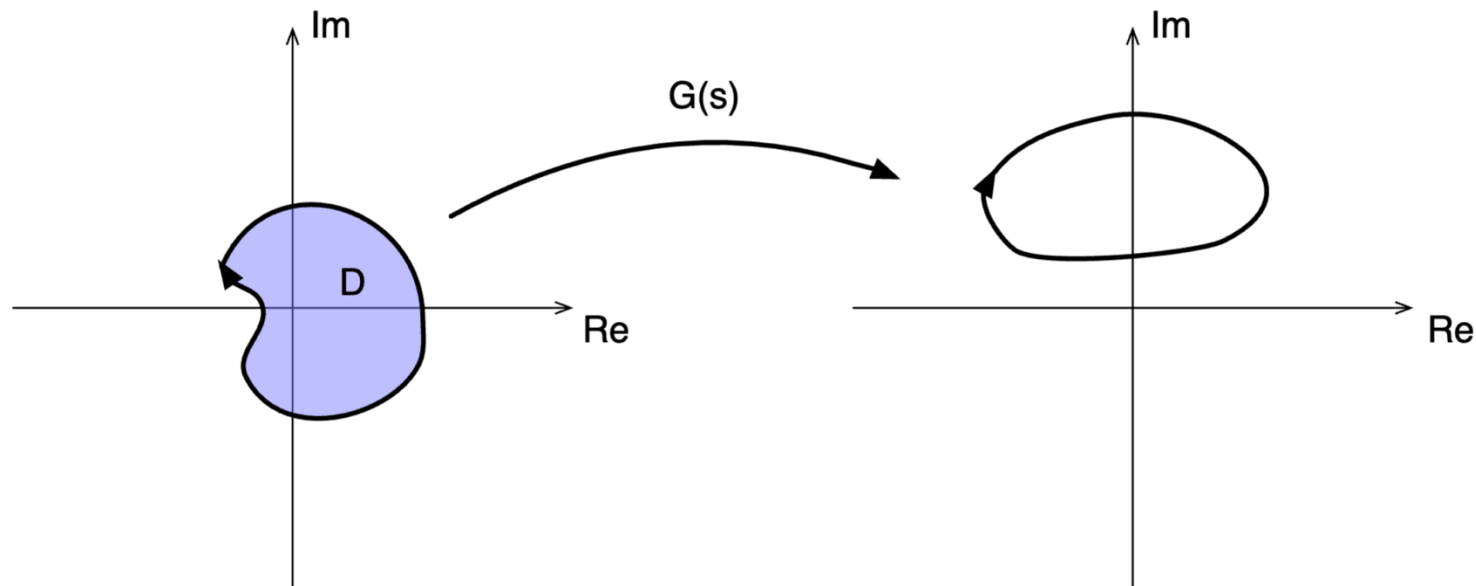
What?

- Whenever an event/transition takes time:
 - Computing a control output using a computer
 - Goods on a conveyor belt with a sensor on the end
 - Long range control (e.g space crafts)
- Definition:
 - A time delay is a **linear** operator that transforms an input signal $t \rightarrow u(t)$ into a delayed output signal $y(t) = u(t - T)$, where $T \geq 0$ is the delay.
- Transfer Function:
 - $Y(s) = e^{-sT}U(s) \rightarrow G(s) = e^{-sT}$
- Not a polynomial thus root locus is not valid anymore with a system that has time delays!

Nyquist Condition

Cauchy's argument principle

- A Complex Analysis Theorem (don't read too much into it) (https://www.youtube.com/watch?v=WmfrK4i1Til&ab_channel=richardpates)
- Given some closed region D in the complex plain we have Γ surrounding that region. If we now apply $G(s)$ to every point on Γ we get $G(\Gamma)$ being another closed curve in the complex plain:
 - Here closed means that the start and end point are the same



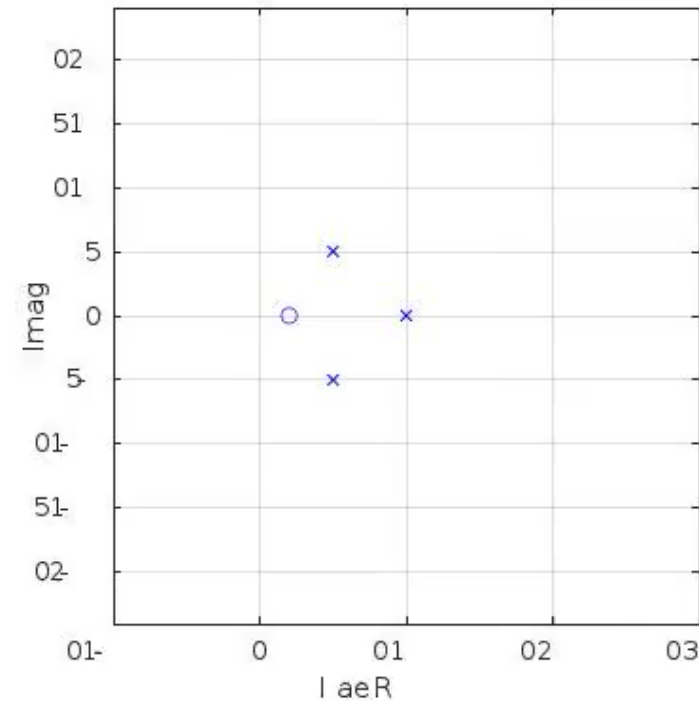
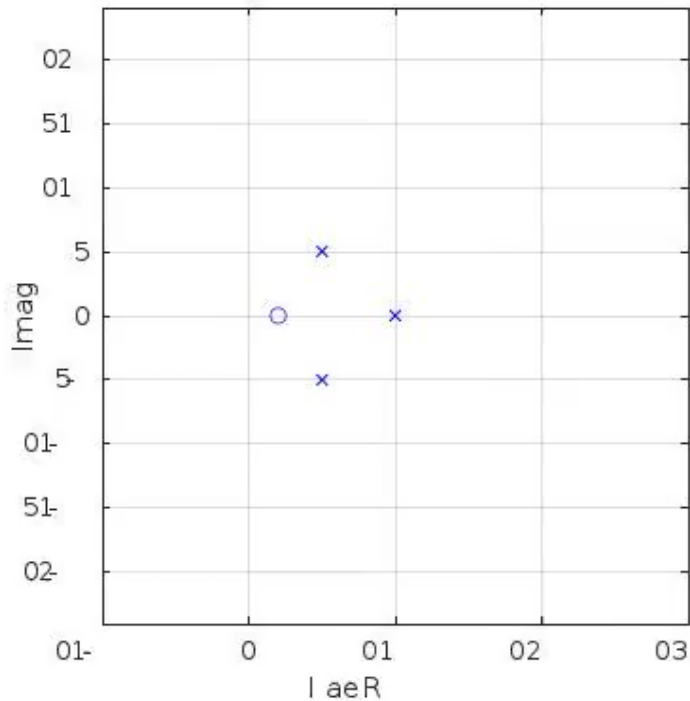
Nyquist Condition

Cauchy's argument principle

- The argument principle:
 - The number N of times that $G(s)$ encircles the origin of the complex plane as s moves along Γ satisfies (*counting positive for encirclements in the same direction as following Γ*)

$$N = Z - P$$

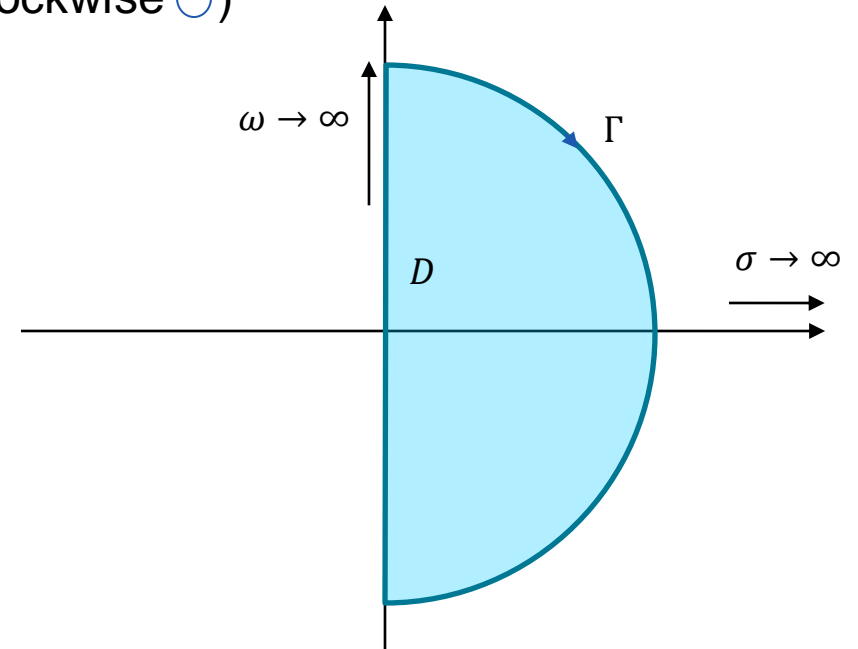
where Z and P are the number of poles and zeros of $G(s)$ contained in D .



Nyquist Condition

Towards the Nyquist Condition

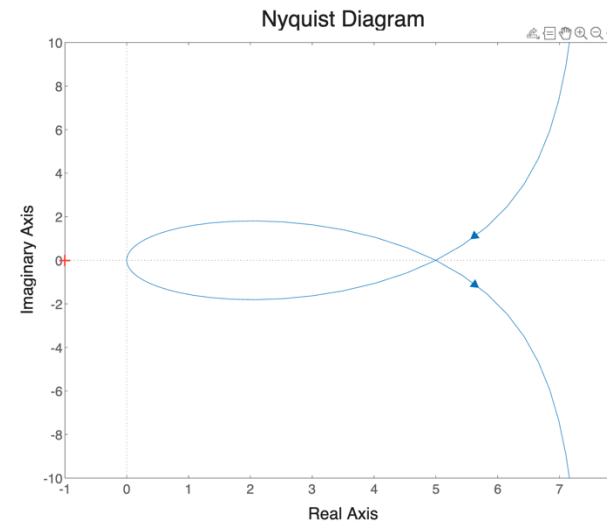
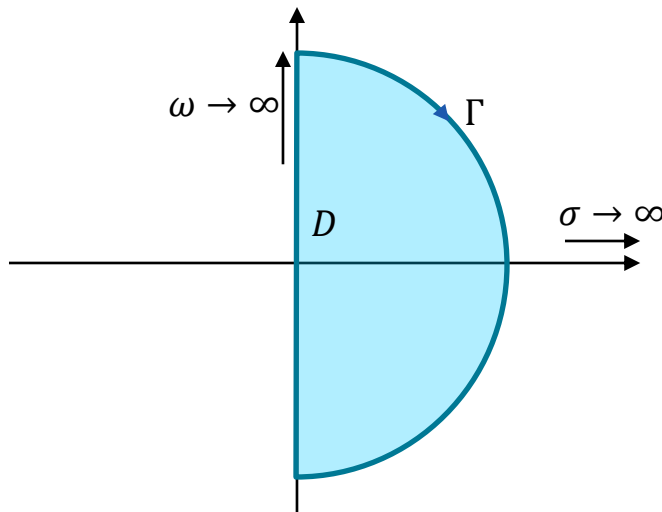
- How does this help us?
 - From two weeks ago remember: $T(s) = \frac{kL(s)}{1+kL(s)}$, $S(s) = \frac{1}{1+kL(s)}$
 - Overall system is stable iff $\frac{1}{1+kL(s)}$ has no positive poles = $1 + kL(s)$ has no positive zeros
 - Construct a huge region surrounding the positive half plane and run $1 + kL(s)$ clockwise ↻
 - The curve of $1 + kL(\Gamma)$ now encircles the origin (clockwise ↻)
$$N = Z - P$$
 - Z : number of unstable zeros of $1 + kL(s)$
 - Observe these are the unstable poles of the closed loop system
 - P : number of unstable poles of $1 + kL(s)$
 - Observe that these are also the unstable poles of $L(s)$



Nyquist Condition

Towards the Nyquist Condition

- We can rewrite this $(1 + kL(s) = 0 \Leftrightarrow L(s) = -\frac{1}{k})$
 - The curve of $L(\Gamma)$ encircles the point $-\frac{1}{k}$ (clockwise ↻)
$$N = Z - P$$
 - Z : number of unstable zeros of $1 + kL(s)$
 - Observe these are the unstable poles of the closed loop system
 - P : number of unstable poles of $1 + kL(s)$
 - Observe that these are the also the unstable poles of $L(s)$
- This huge region D (Nyquist Contour) has $\Gamma: \omega \in (-\infty, \infty)$ and we thus get the **Nyquist plot**.



Nyquist Condition

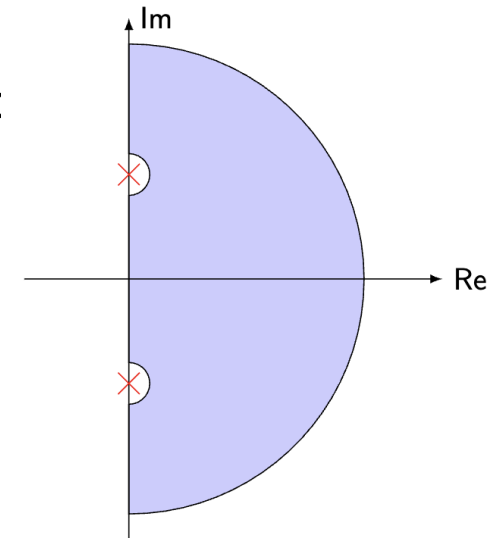
Nyquist Condition

- We now know that the **Nyquist plot** encircles the point $-\frac{1}{k}$ (clockwise ↻)
$$N = Z - P$$
 - Z : number of unstable zeros of $1 + kL(s)$
 - Observe these are the unstable poles of the closed loop system
 - P : number of unstable poles of $1 + kL(s)$
 - Observe that these are the also the unstable poles of $L(s)$
- Nyquist Criterion:
 - Given an open loop transfer function $kL(s)$ with P poles in the positive half plane (Nyquist contour) and let N be the number of clockwise ↻ encirclements of $-\frac{1}{k}$ by the Nyquist Plot. Then the closed loop system has $Z = N + P$ poles in the positive half plane.

Nyquist Condition

Nyquist Condition

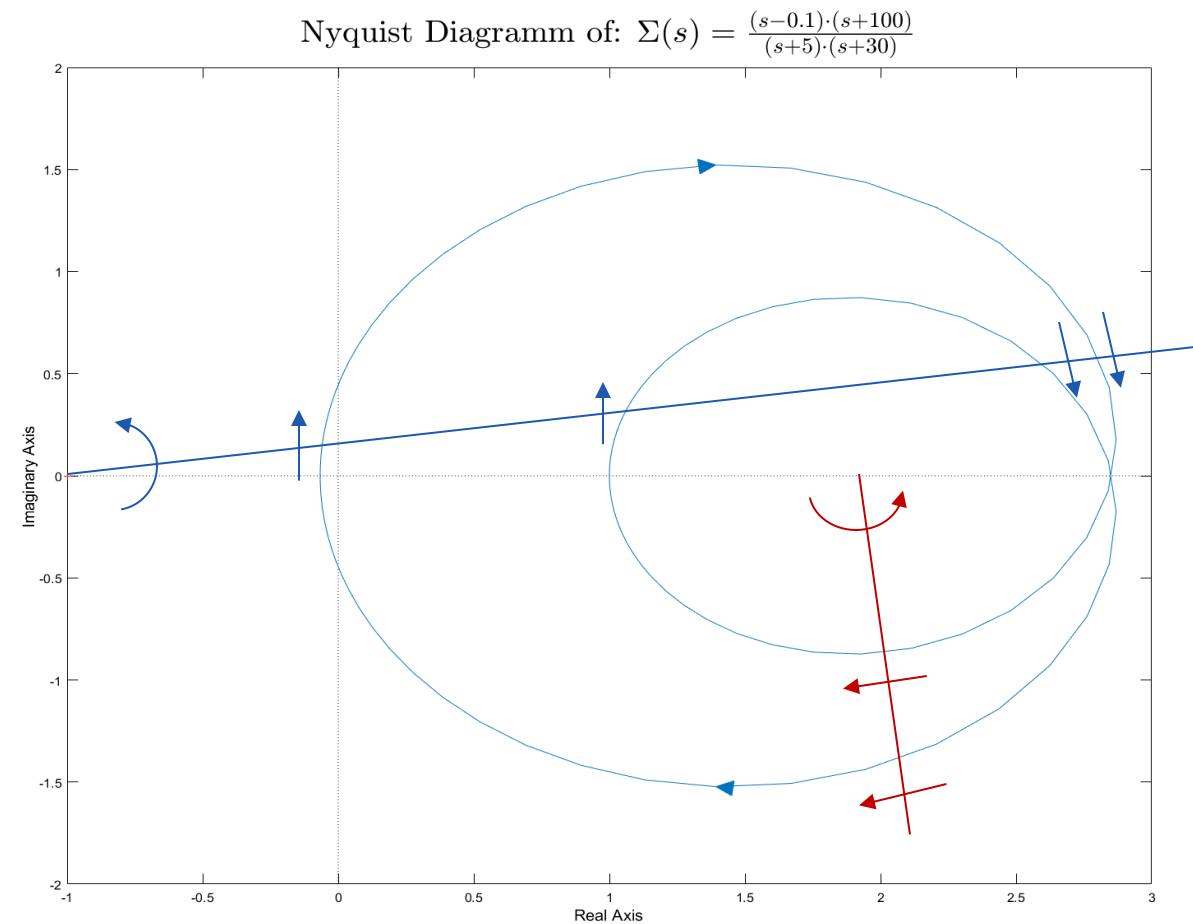
- Nyquist Criterion:
 - Given an open loop transfer function $kL(s)$ with P poles in the positive half plane (Nyquist contour) and let N be the number of clockwise ↻ encirclements of $-\frac{1}{k}$ by the Nyquist Plot. Then the closed loop system has $Z = N + P$ poles in the positive half plane.
- For stability we now want $Z = 0 \rightarrow N = -P$,
 - **Nyquist Stability Theorem:**
 - A closed-loop system is stable if for $kL(s)$ the following holds:
$$n_c = n_p$$
 - n_c : number of *counter-clockwise* ↻ encirclements of $-\frac{1}{k}$ by the Nyquist Plc
 - n_p : number of poles with positive real part of $L(s)$
 - Valid only if no nonminimum phase - unstable pole cancellation was done!
- Things to keep in mind:
 - Avoid zeros on the imaginary axis by excluding them
 - k is usually 1 and backed into $L(s) \rightarrow$ everything is with respect to -1



Nyquist Condition

How to count encirclements

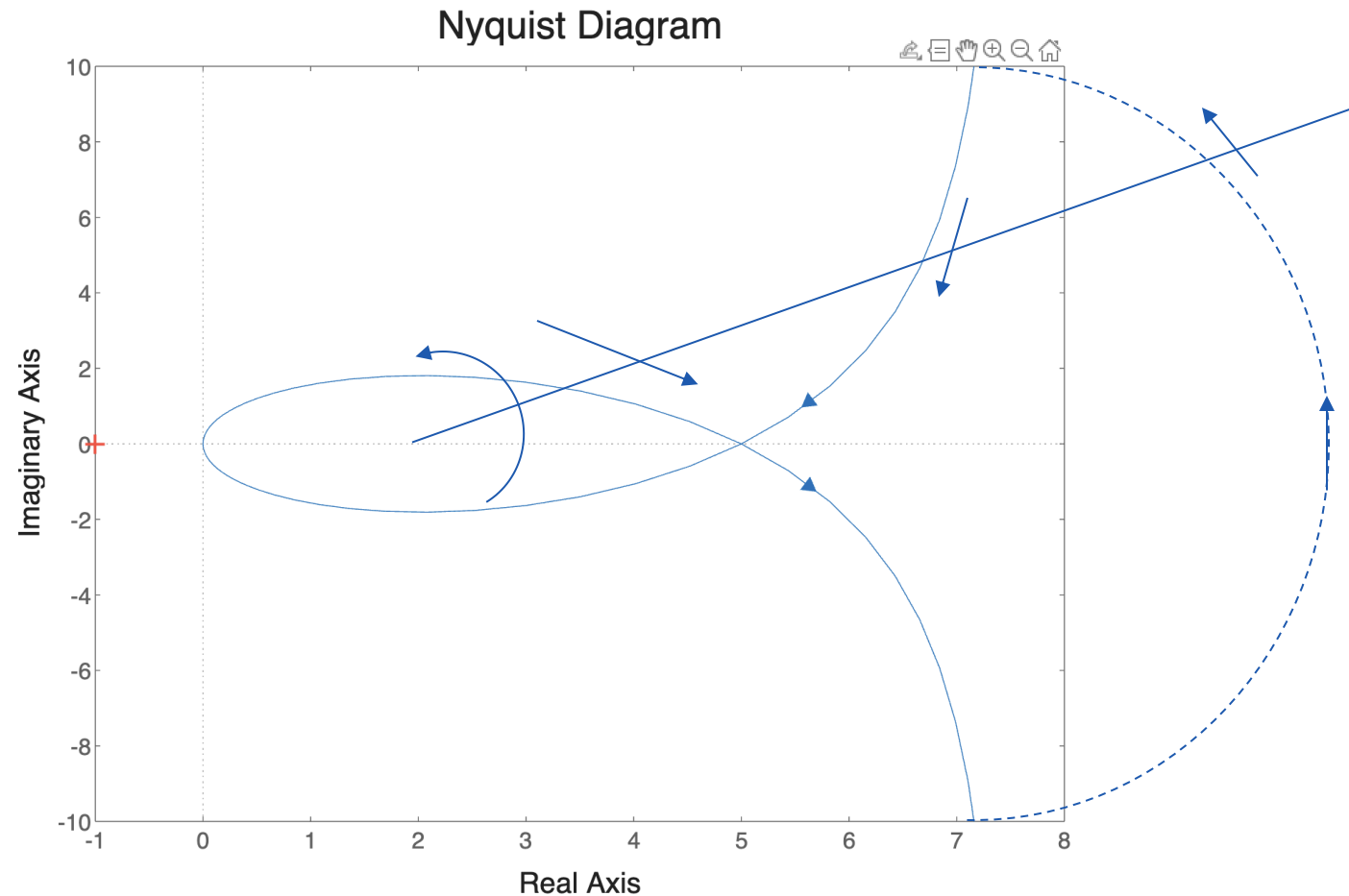
- Draw a line outwards from the point $-\frac{1}{k}$
- Draw the crossings of the Nyquist plot with this line (keep the direction in mind)
- Add the number of crossings (counter-clockwise positive, clockwise negative)
- Example:
 - Encirclements around -1 :
 - 0 since 2 CCW and 2 CW
 - Encirclements around 2:
 - -2 since 0 CCW and 2 CW



Nyquist Condition

How to count encirclements - Infinity

- Take $\lim_{\varepsilon \rightarrow 0} \angle L(\varepsilon e^{j\theta}) = f(\theta)$
- Now look at what happens for $\theta: -\frac{\pi}{2} \rightarrow \frac{\pi}{2}$
 - $\lim_{\varepsilon \rightarrow 0} \angle L(\varepsilon e^{j\theta}): f\left(-\frac{\pi}{2}\right) \rightarrow f\left(\frac{\pi}{2}\right)$
- Close the loop accordingly
- Example: $G(s) = \frac{5(s-0.5)}{s(s+5)}$
 - $\lim_{\varepsilon \rightarrow 0} \angle L(\varepsilon e^{j\theta}) = \lim_{\varepsilon \rightarrow 0} \angle \frac{5(\varepsilon e^{j\theta} - 0.5)}{\varepsilon e^{j\theta}(\varepsilon e^{j\theta} + 5)}$
 - $= \lim_{\varepsilon \rightarrow 0} \angle \frac{5(-0.5)}{\varepsilon e^{j\theta}(5)} = \lim_{\varepsilon \rightarrow 0} \angle -\frac{0.5}{\varepsilon} e^{-j\theta} = \theta$
 - For $\theta: -\frac{\pi}{2} \rightarrow \frac{\pi}{2}$:
 - $\lim_{\varepsilon \rightarrow 0} \angle L(\varepsilon e^{j\theta}): -\frac{\pi}{2} \rightarrow \frac{\pi}{2}$
- Encirclements around 2:
 - -1 since 1 CCW and 2 CW



Nyquist Condition

Example

- Consider the inverted pendulum (upright position):
- $\dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{3g}{2L} & -\frac{3c_f}{mL^2} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$
- Using $g = 10, m = 1, L = \frac{3}{2}, c_f = \frac{9}{4}$ we get
 - $G(s) = \frac{1.33}{s^2 + 3s - 10}$
- Consider a PD controller
 - $C(s) = k_p + \frac{k_d s}{T_f s + 1}$
- We have two controllers with poles of the open loop $L(s) = G(s)C(s)$:
 - $C_1: k_p = 70, k_d = 10, T_f = 0.001$
 - $p_1 = -100, p_2 = -5, p_3 = 2$
 - $C_2: k_p = 7, k_d = 1, T_f = 0.001$
 - $p_1 = -1000, p_2 = -5, p_3 = 2$

Nyquist Condition

Example

- We have two controllers with poles of the open loop $L(s) = G(s)C(s)$:

- C_1 : $k_p = 70, k_d = 10, T_f = 0.001$

- $p_1 = -100, p_2 = -5, p_3 = 2$

- C_2 : $k_p = 7, k_d = 1, T_f = 0.001$

- $p_1 = -1000, p_2 = -5, p_3 = 2$

- Given the Nyquist Plots which controller stabilizes the system?

- C_1 :

-

-

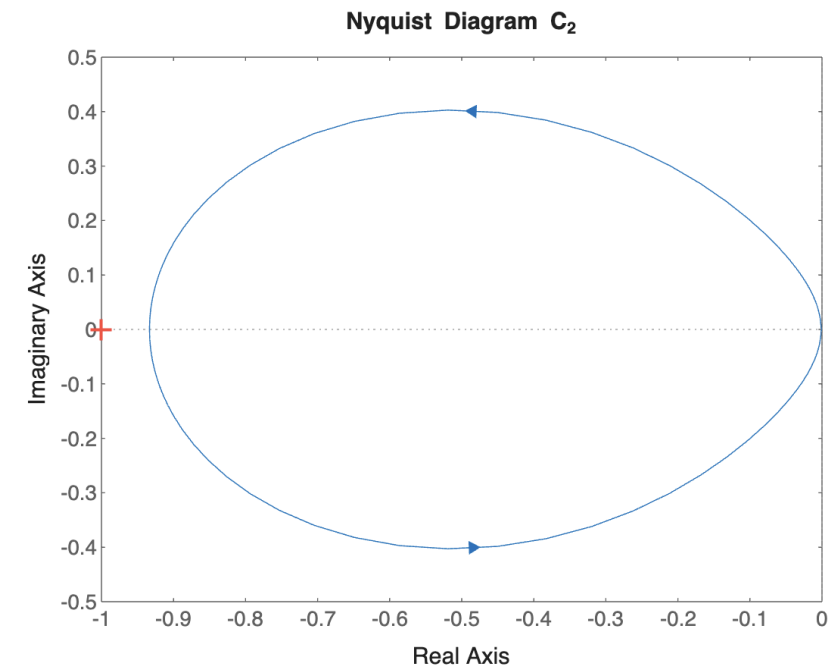
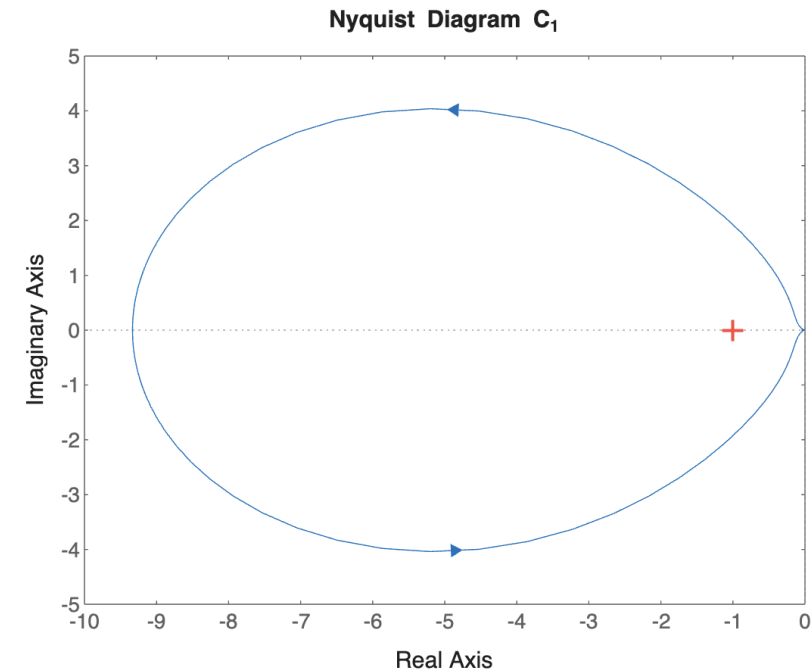
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- C_2 :

-

-

-

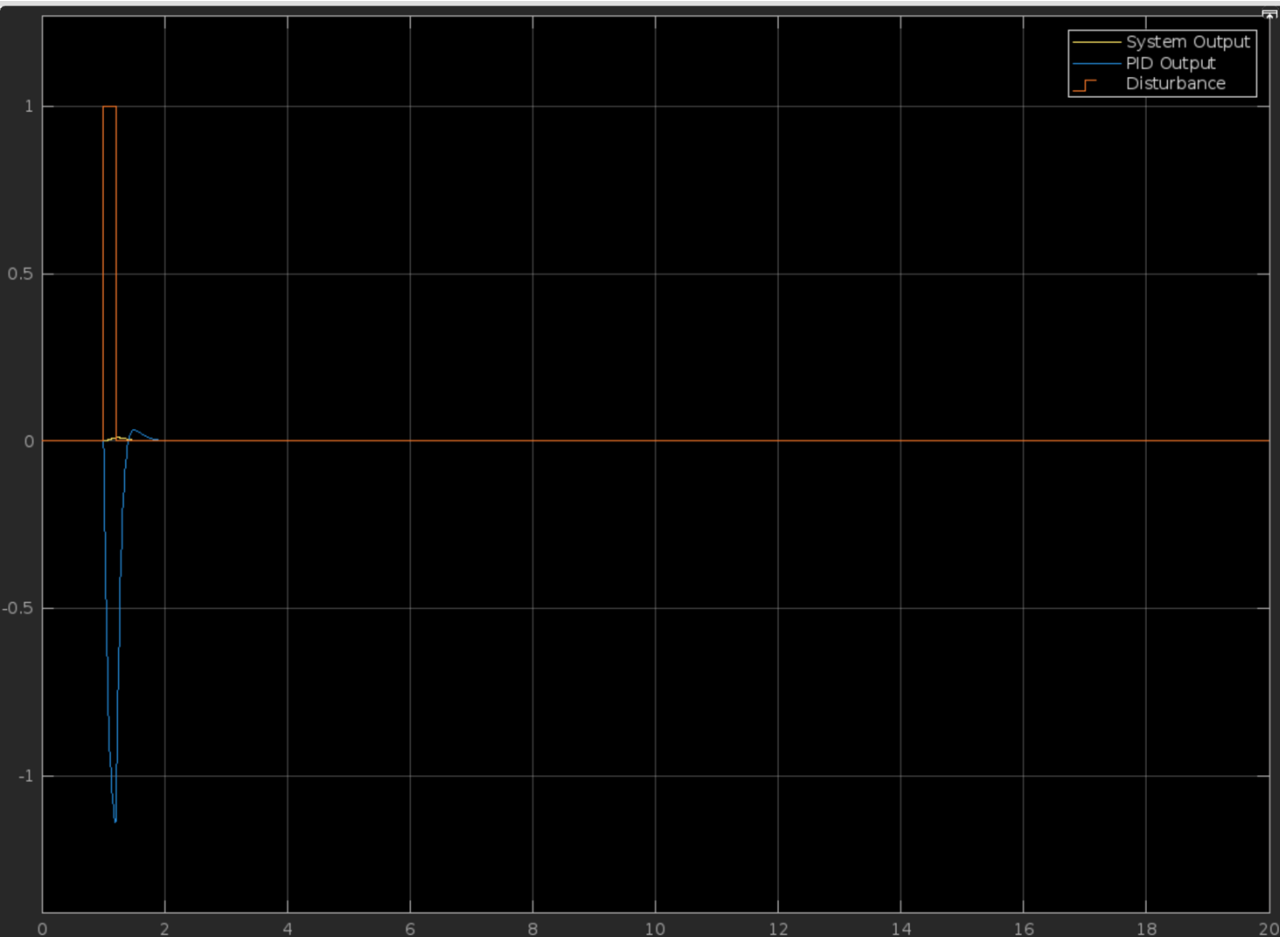


Nyquist Condition

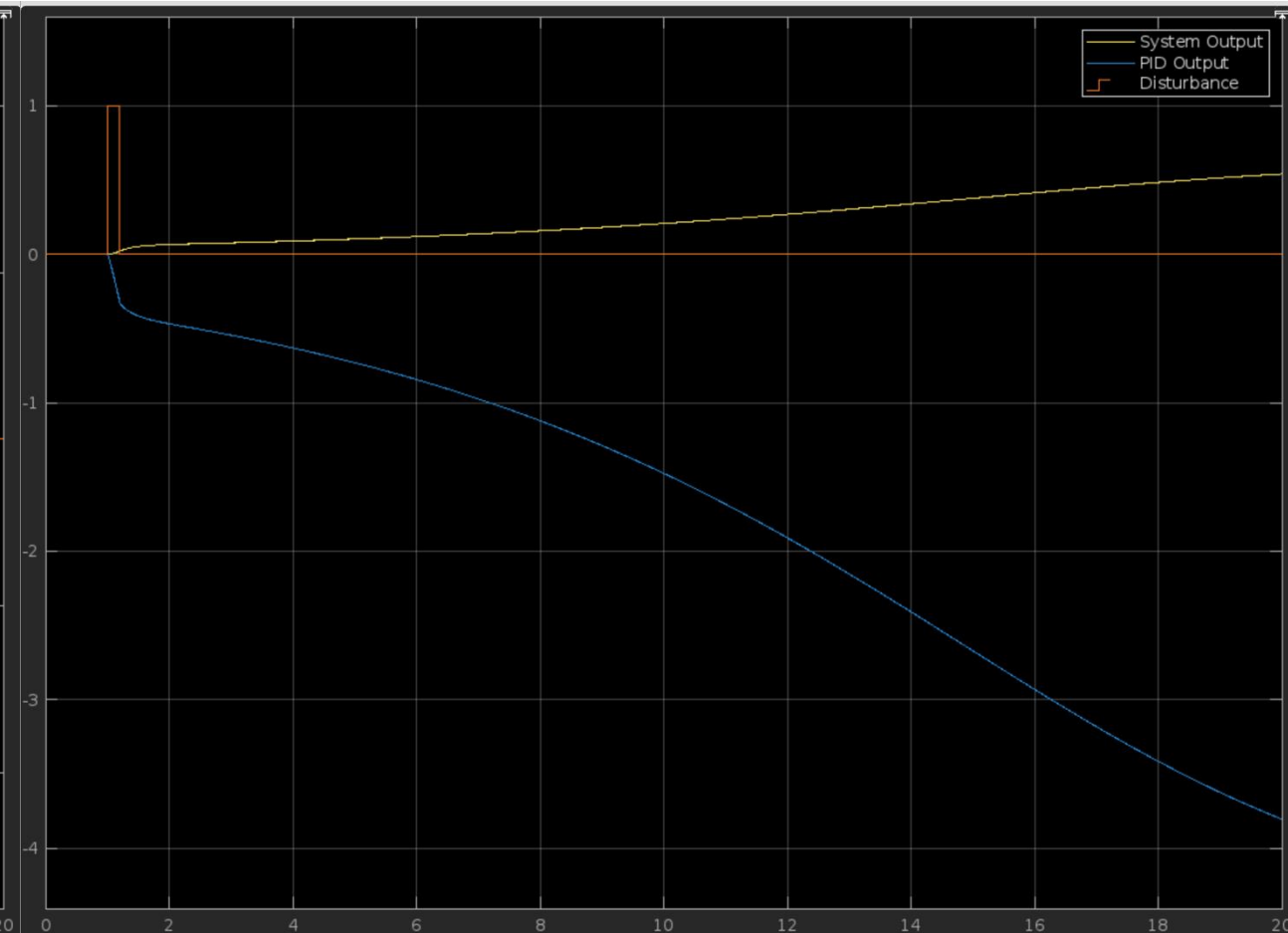
Example

- Impulse Disturbance

$C :$



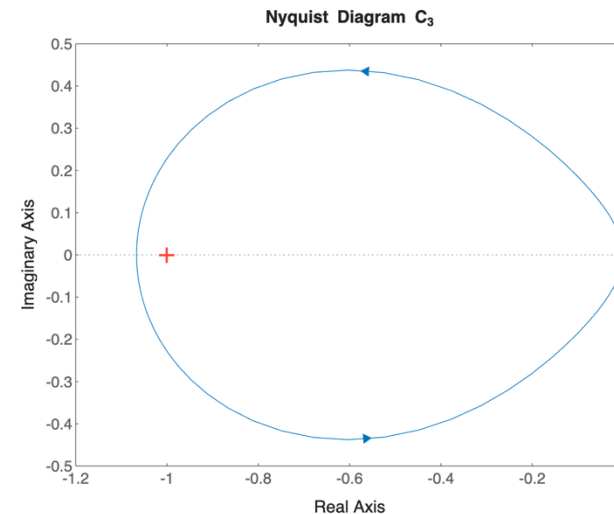
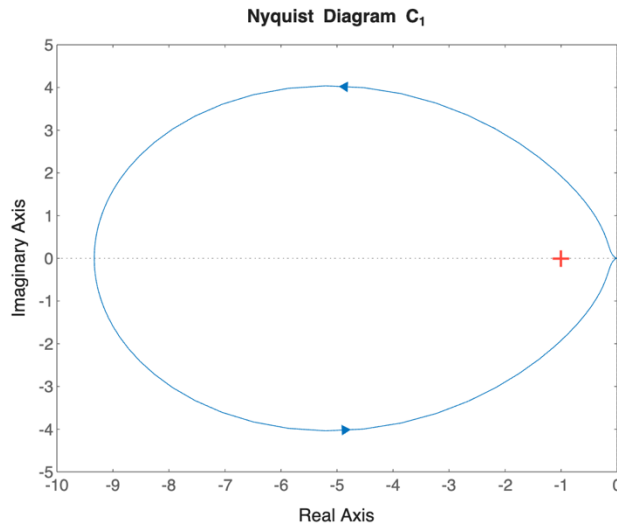
$C :$



Stability Margins

What?

- Number of surroundings of the point -1 ($k = 1$) is very important for stability
 - What happens if that changes? \rightarrow system becomes unstable (see example before)
- Is there a difference between a large encirclement and small encirclement?

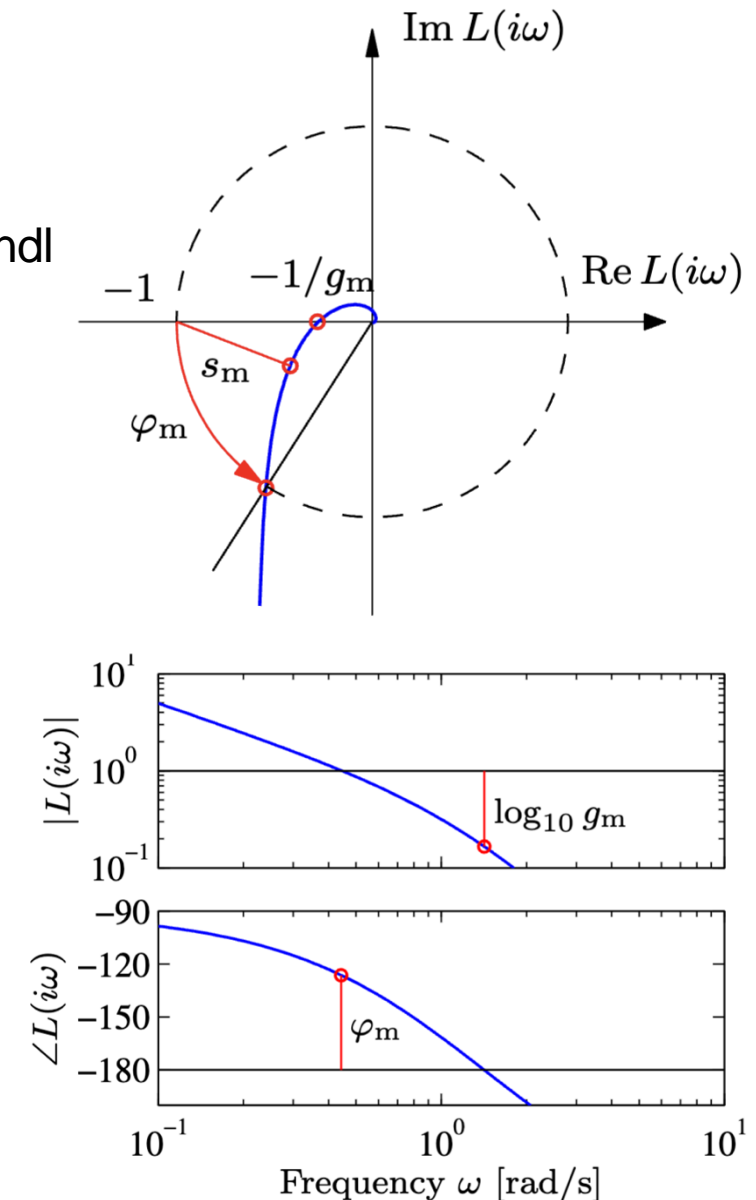


- Yes, there is!!
- The further away from the point -1 the Nyquist plot is, the more robust is a system
- Need a metric to define how far away the system is from the point $-1 \rightarrow$ Stability Margins

Stability Margins

What?

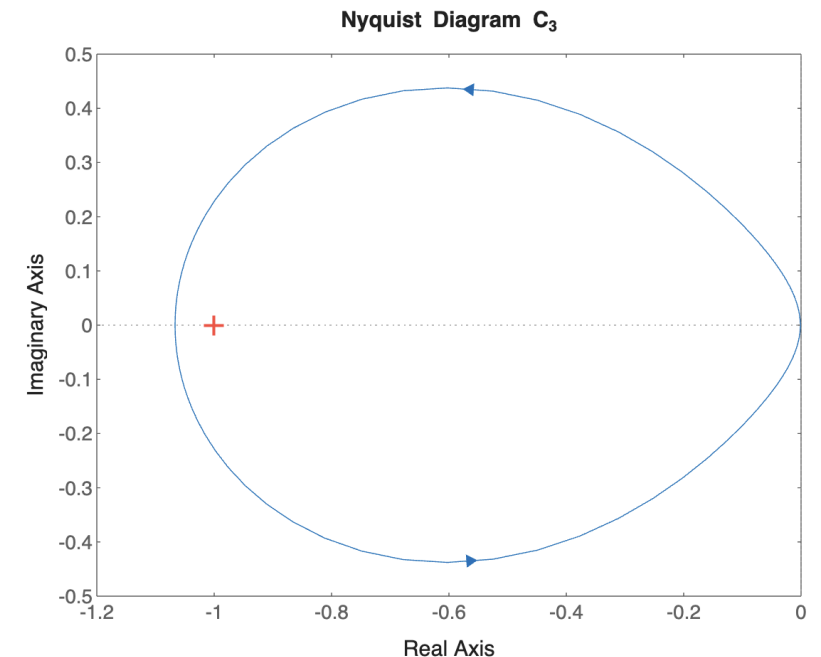
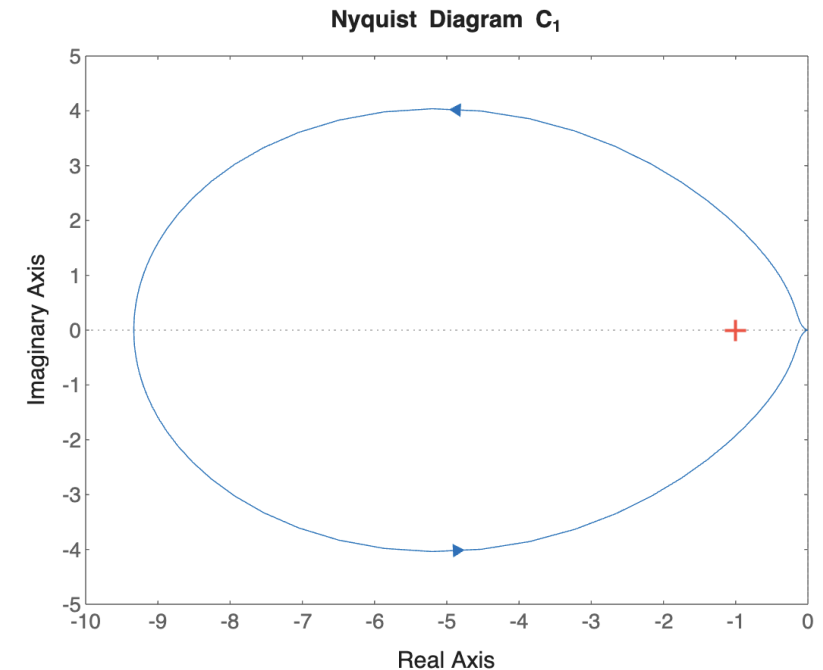
- The stability margins tell us how far away we are from a -1 crossing:
 - Tells us how much modelling errors the closed loop system can handle before going unstable!
- We have:
 - g_m : gain margin
 - How much more can we “blow up” the system
 - At $\angle L(j\omega_g) = -180 \rightarrow g_m = \frac{1}{|L(j\omega_g)|}$
 - φ_m : phase margin
 - How much phase shift/lag can the system handle
 - At $|L(j\omega_c)| = 1 \rightarrow \varphi_m = \angle L(j\omega_g) + 180^\circ$
 - ω_c = cross-over frequency
- We often need to find a trade-off between performance and robustness:
 - We can either have a robust system or a system that performs well



Stability Margins

Example

- Consider again the inverted pendulum with two controllers:
 - C_1 : $k_p = 70, k_d = 10, T_f = 0.001$
 - $p_1 = -100, p_2 = -5, p_3 = 2$
 - C_3 : $k_p = 8, k_d = 1, T_f = 0.001$
 - $p_1 = -1000, p_2 = -5, p_3 = 2$
- We see C_1 is way further away from -1 than C_3
 - C_1 is much more robust than C_3
- What happens when we made a mistake during modelling, and we designed our controller according to the model?
 - Suppose $m_{real} = m_{model} * 1.1$
 - What happens with the system response?

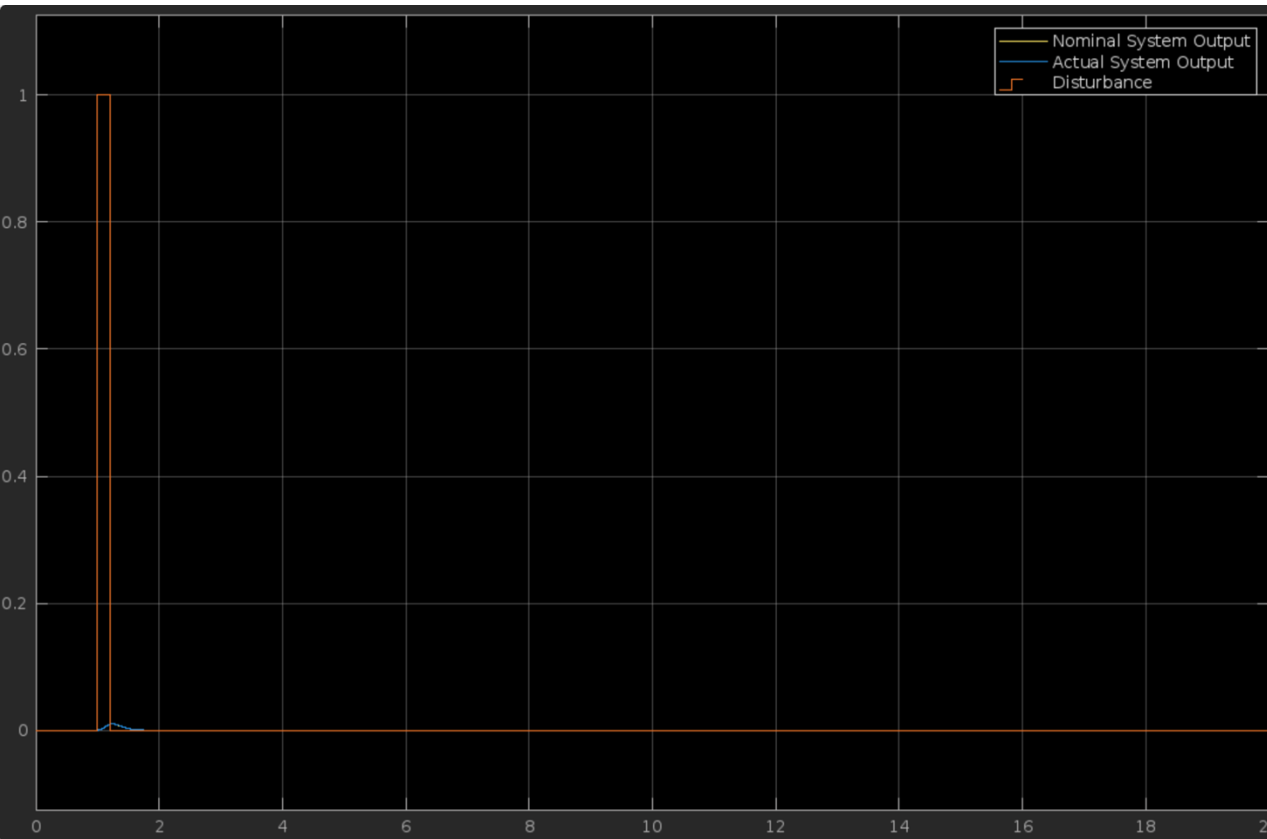


Stability Margins

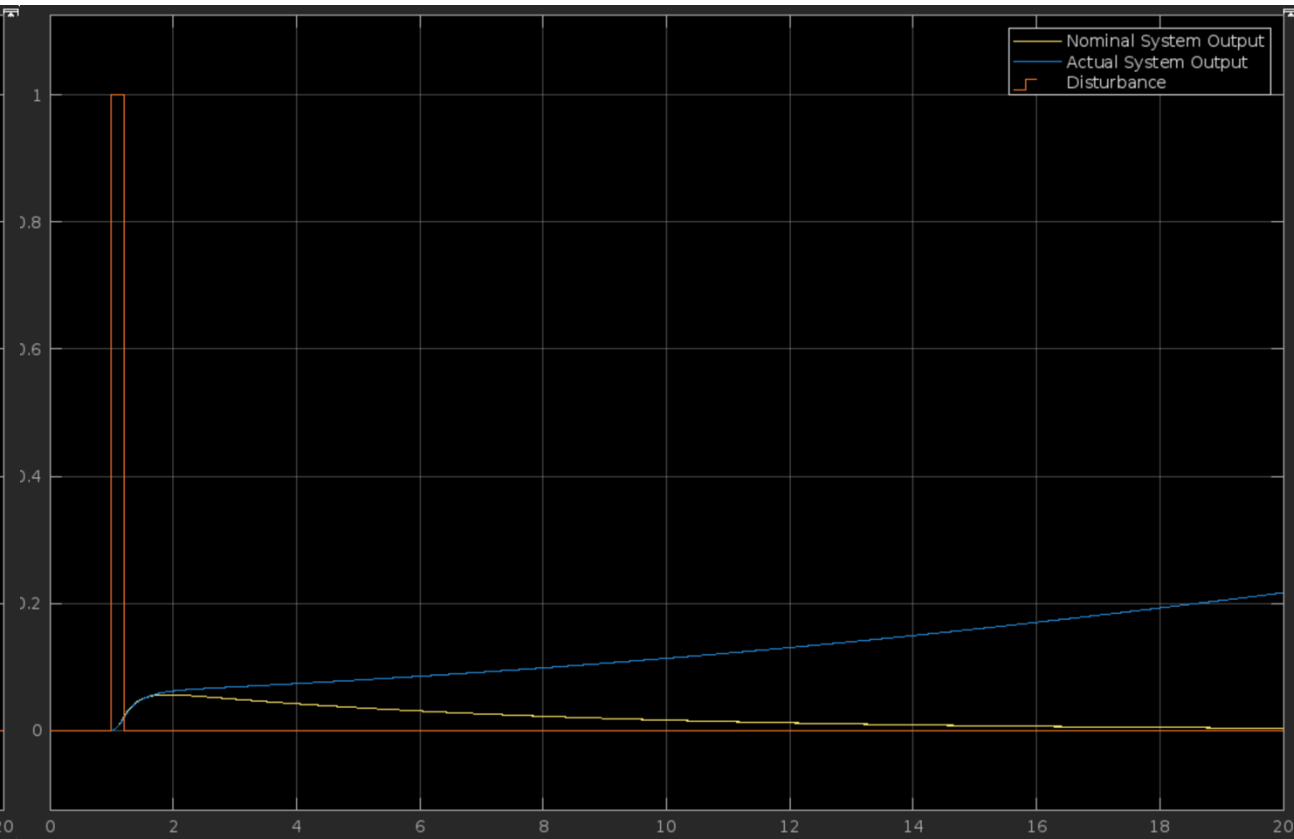
Example

- We see C_1 can handle this modelling error while C_3 can't.
 - Both nominal systems are stable but only C_1 can control the actual system!!

C_1 :



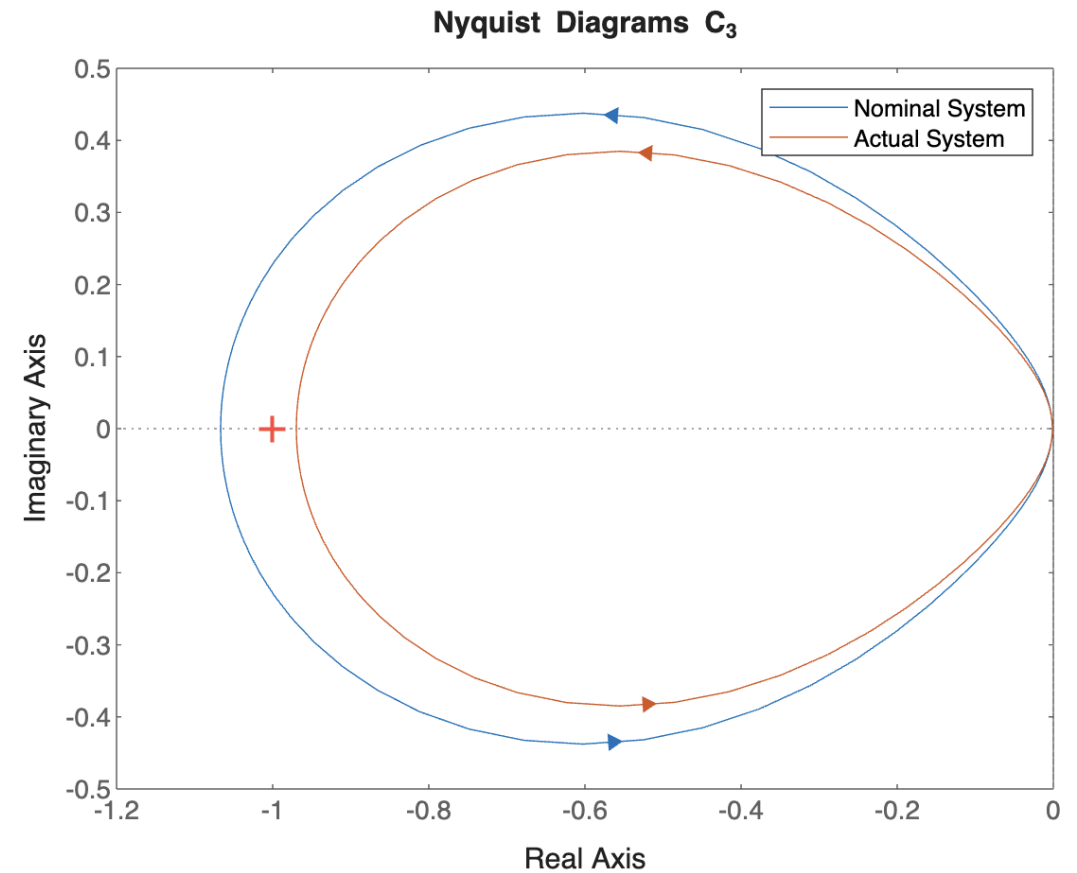
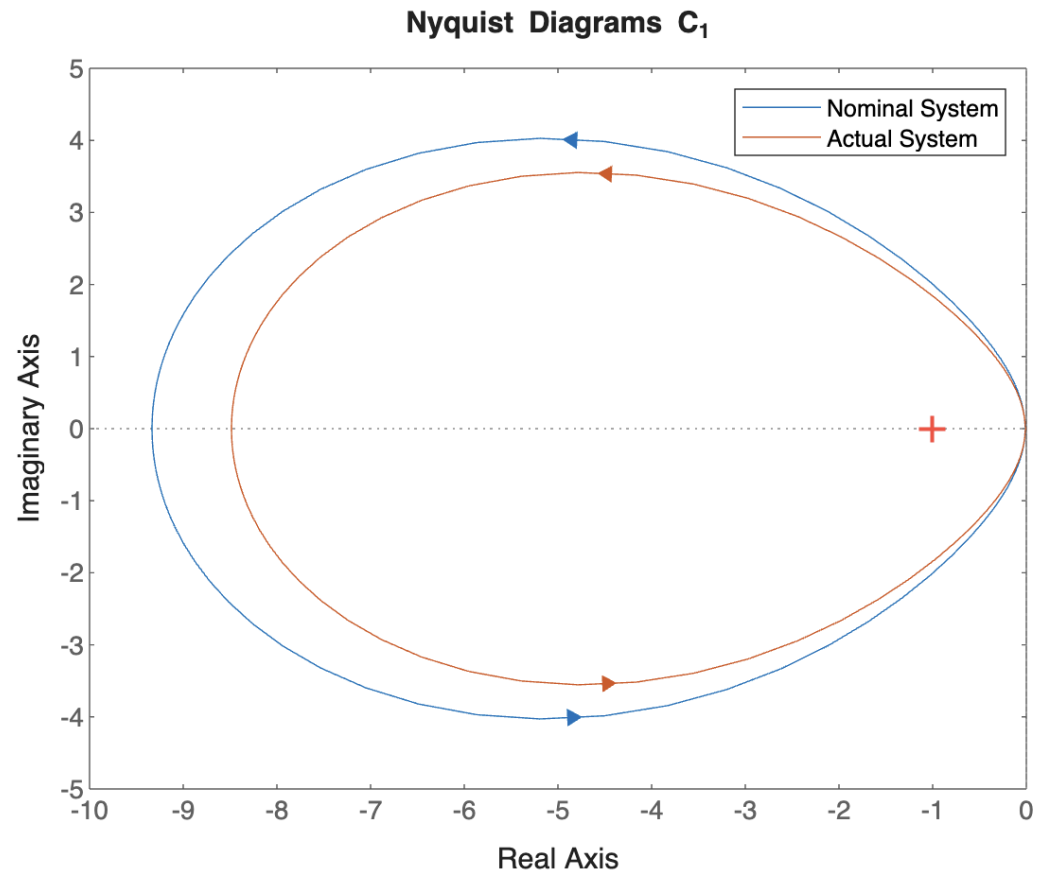
C_3 :



Stability Margins

Example

- We see C_1 can handle this modelling error while C_3 can't.
 - This can also be seen in the Nyquist Plots.



Exercise 09

What to do?

- 1:
 - Do
- 2:
 - Do
- 3:
 - Do
- 4:
 - Do one