Control Systems I
Recitation 11

https://n.ethz.ch/~jgeurts/
jgeurts@ethz.ch

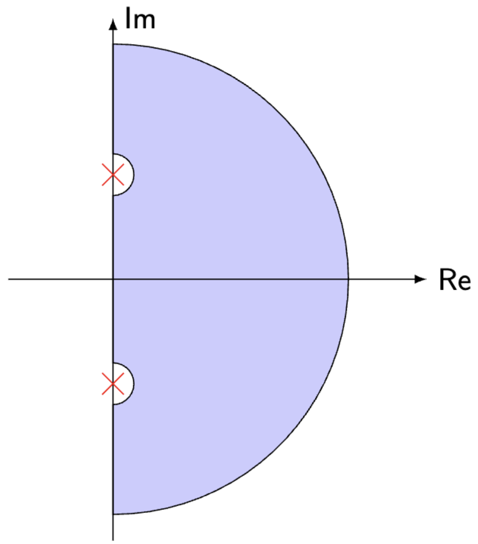
Last Week

Nyquist Condition

- Nyquist Criterion:
  - Given an open loop transfer function $kL(s)$ with P poles in the positive half plane (Nyquist contour) and let N be the number of clockwise encirclements of $-\frac{1}{k}$ by the Nyquist Plot. Then the closed loop system has $Z = N + P$ poles in the positive half plane.
- **Nyquist Stability Theorem:** (Z=0)
  - A closed-loop system is stable if for $kL(s)$ the following holds:

$$\mathrm{n}_c = n_p$$

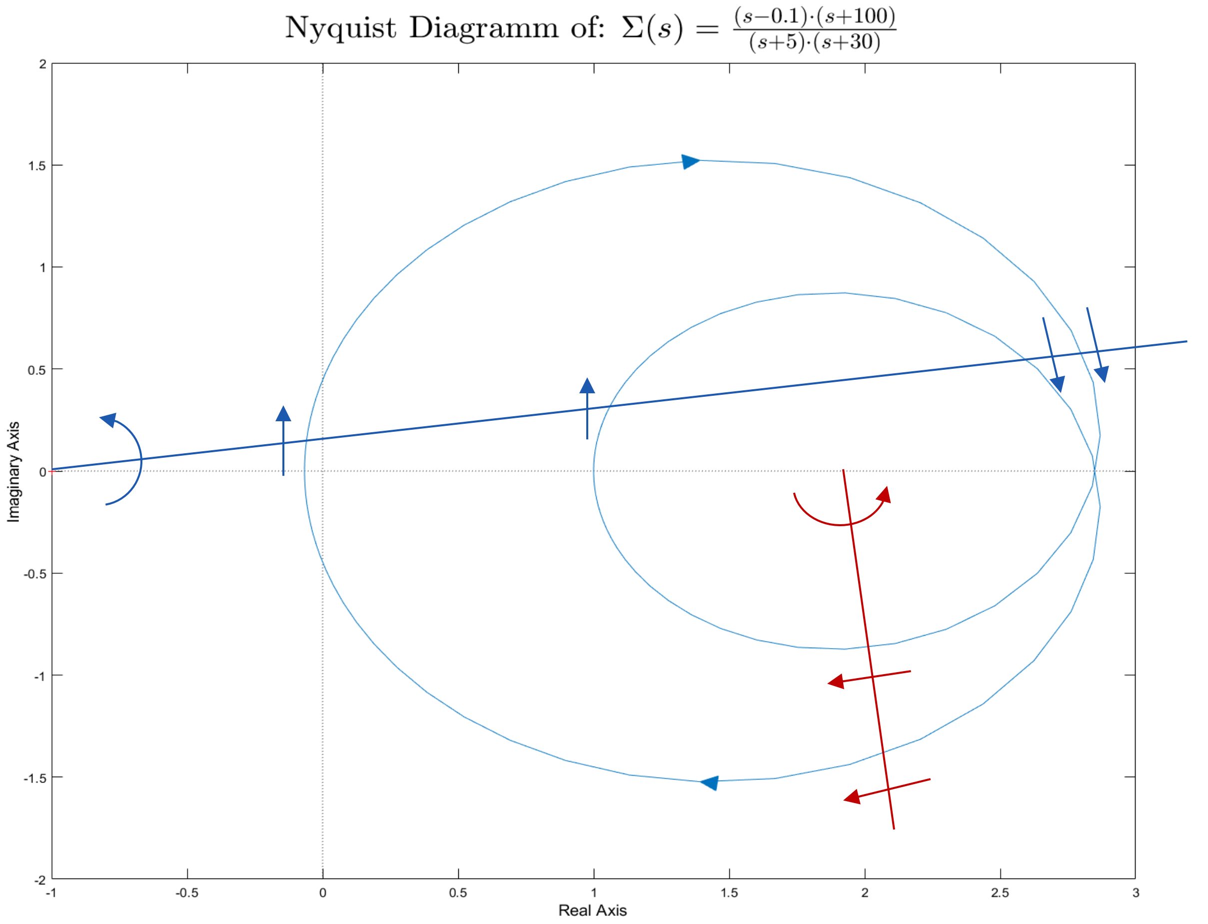
  - n_c: number of *counter-clockwise* encirclements of $-\frac{1}{k}$ by the Nyquist Plot
  - n_p: number of poles with positive real part of $L(s)$
- Valid only if no nonminimum phase - unstable pole cancellation was done!
- Things to keep in mind:
  - Avoid zeros on the imaginary axis by excluding them
  - k is usually 1 and backed into $L(s)$ → everything is with respect to -1



Last Week

How to count encirclements

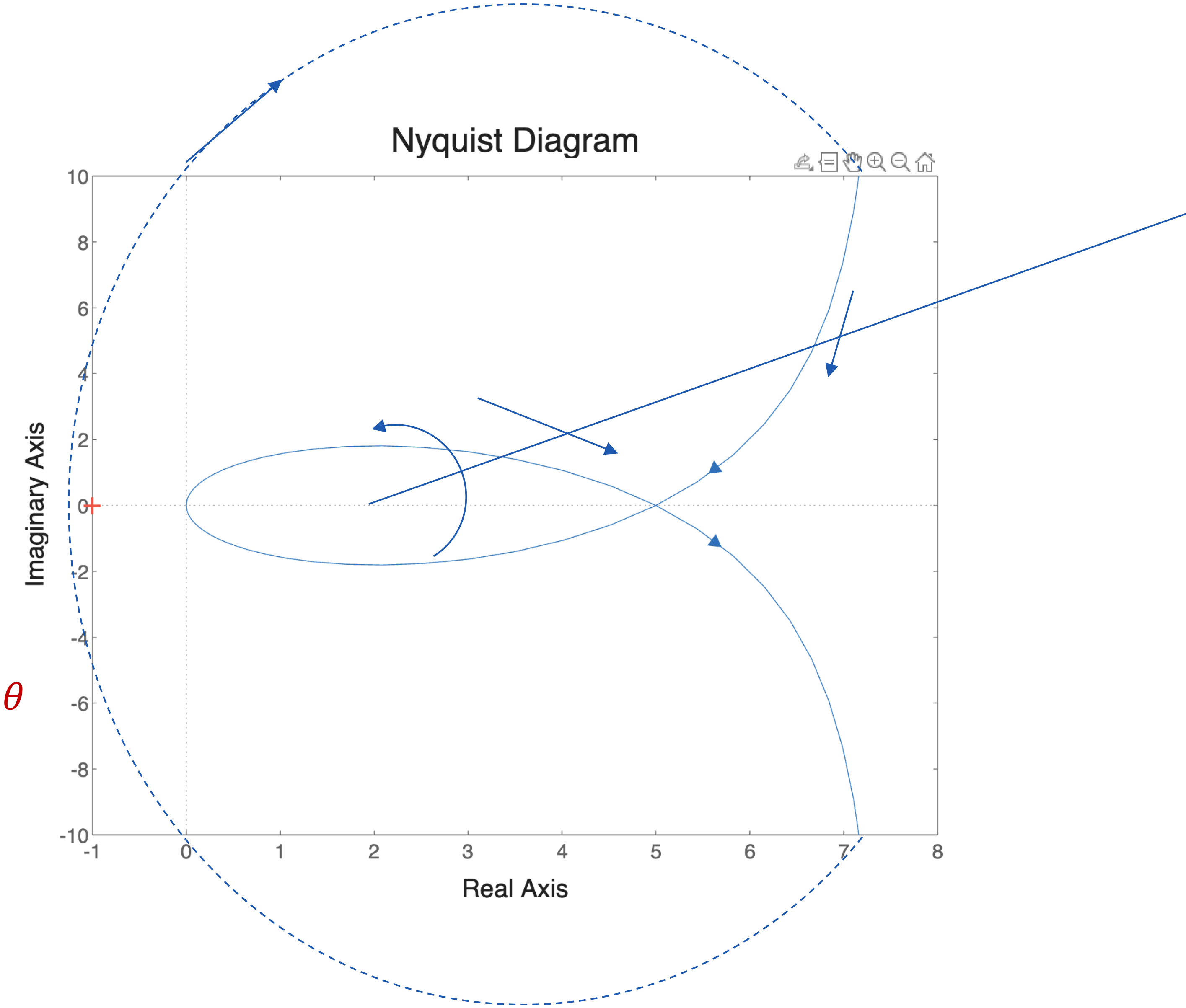
- Draw a line outwards from the point $-\frac{1}{k}$
- Draw the crossings of the Nyquist plot with this line (keep the direction in mind)
- Add the number of crossings (counter-clockwise positive, clockwise negative)
- Example:
  - Encirclements around -1:
    - 0 since 2 CCW and 2 CW
  - Encirclements around 2:
    - -2 since 0 CCW and 2 CW



Last Week

How to count encirclements – Infinity

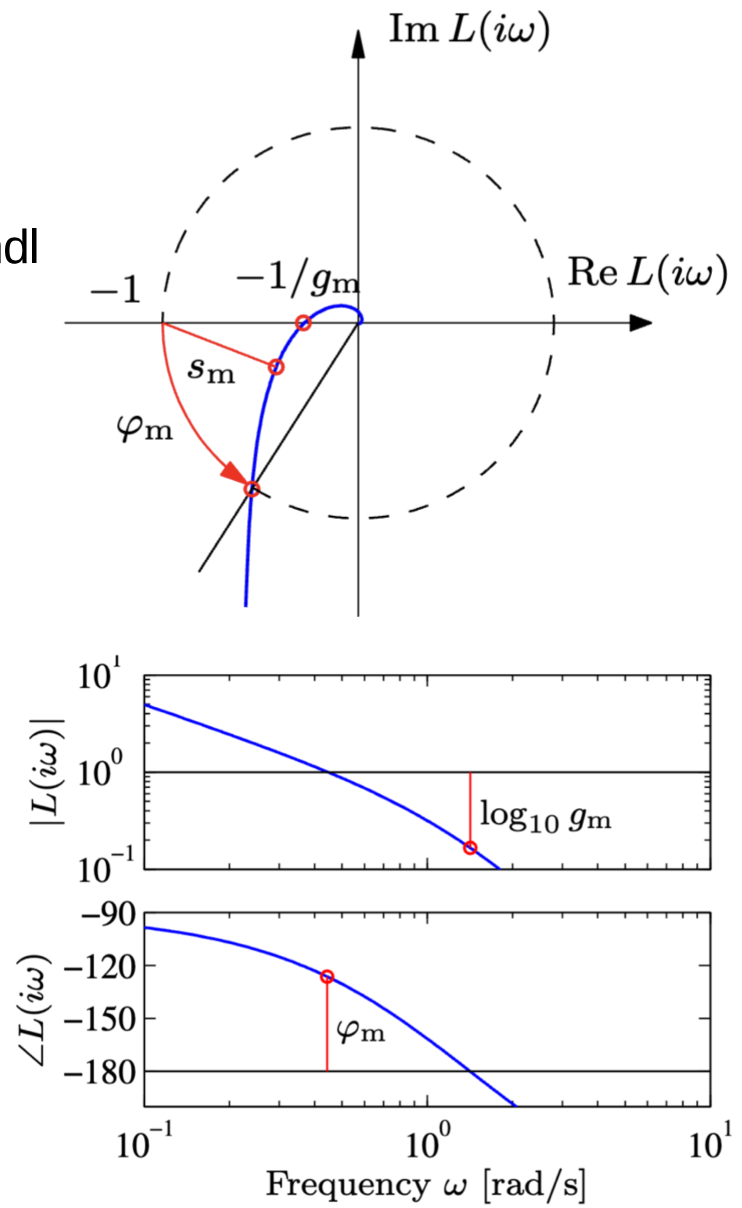
- Take $\lim_{\varepsilon \to 0} \angle L(\varepsilon e^{j\theta}) = f(\theta)$
- Now look at what happens for $\theta: -\frac{\pi}{2} \to \frac{\pi}{2}$
  - $\lim_{\varepsilon \to 0} \angle L(\varepsilon e^{j\theta}): f\left(-\frac{\pi}{2}\right) \to f\left(\frac{\pi}{2}\right)$
- Close the loop accordingly
- Example: $G(s) = \frac{5(s-0.5)}{s(s+5)}$
  - $\lim_{\varepsilon \to 0} \angle L(\varepsilon e^{j\theta}) = \lim_{\varepsilon \to 0} \angle \frac{5(\varepsilon e^{j\theta}-0.5)}{\varepsilon e^{j\theta}(\varepsilon e^{j\theta}+5)}$
  - $= \lim_{\varepsilon \to 0} \angle \frac{5(-0.5)}{\varepsilon e^{j\theta}(5)} = \lim_{\varepsilon \to 0} \angle -\frac{0.5}{\varepsilon} e^{-j\theta} = \pi - \theta$
  - For $\theta: -\frac{\pi}{2} \to \frac{\pi}{2}$:
    - $\lim_{\varepsilon \to 0} \angle L(\varepsilon e^{j\theta}): \frac{3\pi}{2} \to \frac{\pi}{2}$
- Encirclements around 2:
  - 0 since 1 CCW and 1 CW



Stability Margins

What?

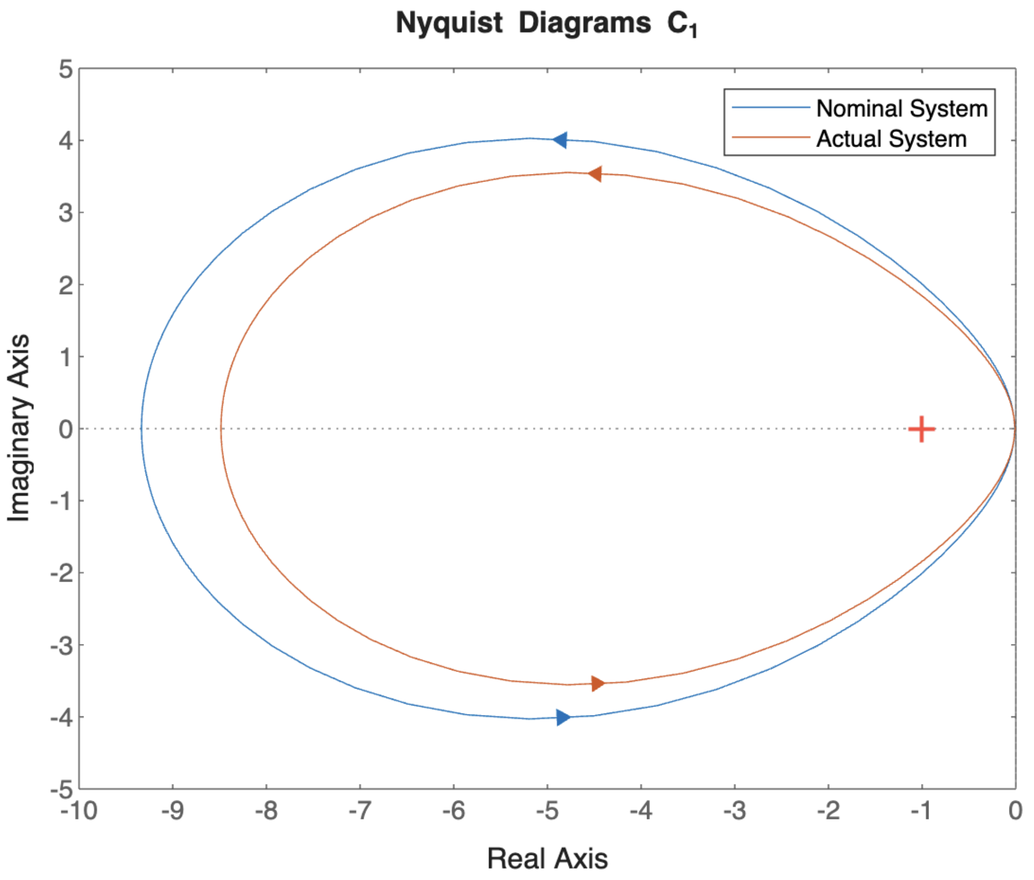
- The stability margins tell us how far away we are from a -1 crossing:
  - Tells us how much modelling errors the closed loop system can handl before going unstable!
- We have:
  - g_m: gain margin
    - How much more can we “blow up” the system
    - At $\angle L(j\omega_g) = -180 \rightarrow g_m = \frac{1}{|L(j\omega_g)|}$
  - φ_m: phase margin
    - How much phase shift/lag can the system handle
    - At $|L(j\omega_c)| = 1 \rightarrow \varphi_m = \angle L(j\omega_g) + 180°$
    - ω_c = cross-over frequency
- We often need to find a trade-off between performance and robustness:
  - We can either have a robust system or a system that performs well

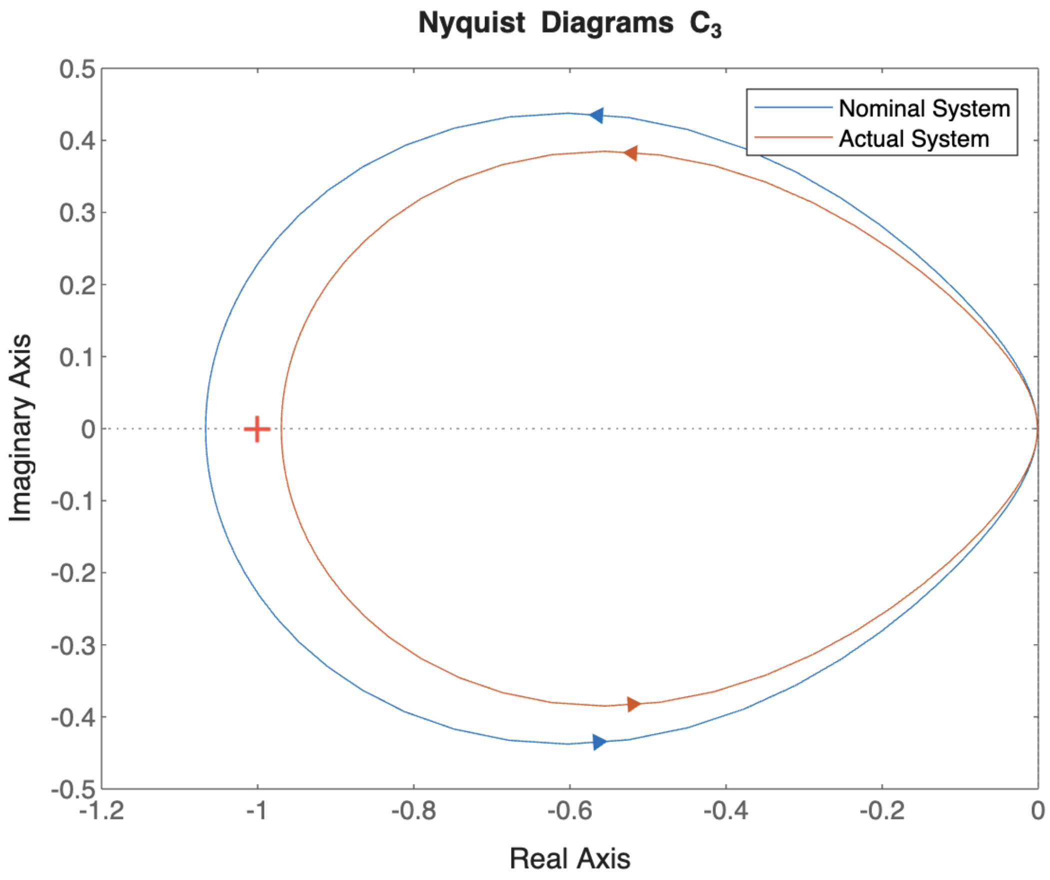


Stability Margins

Example

- We see C_1 can handle this modelling error while C_3 can't.
  - This can also be seen in the Nyquist Plots.



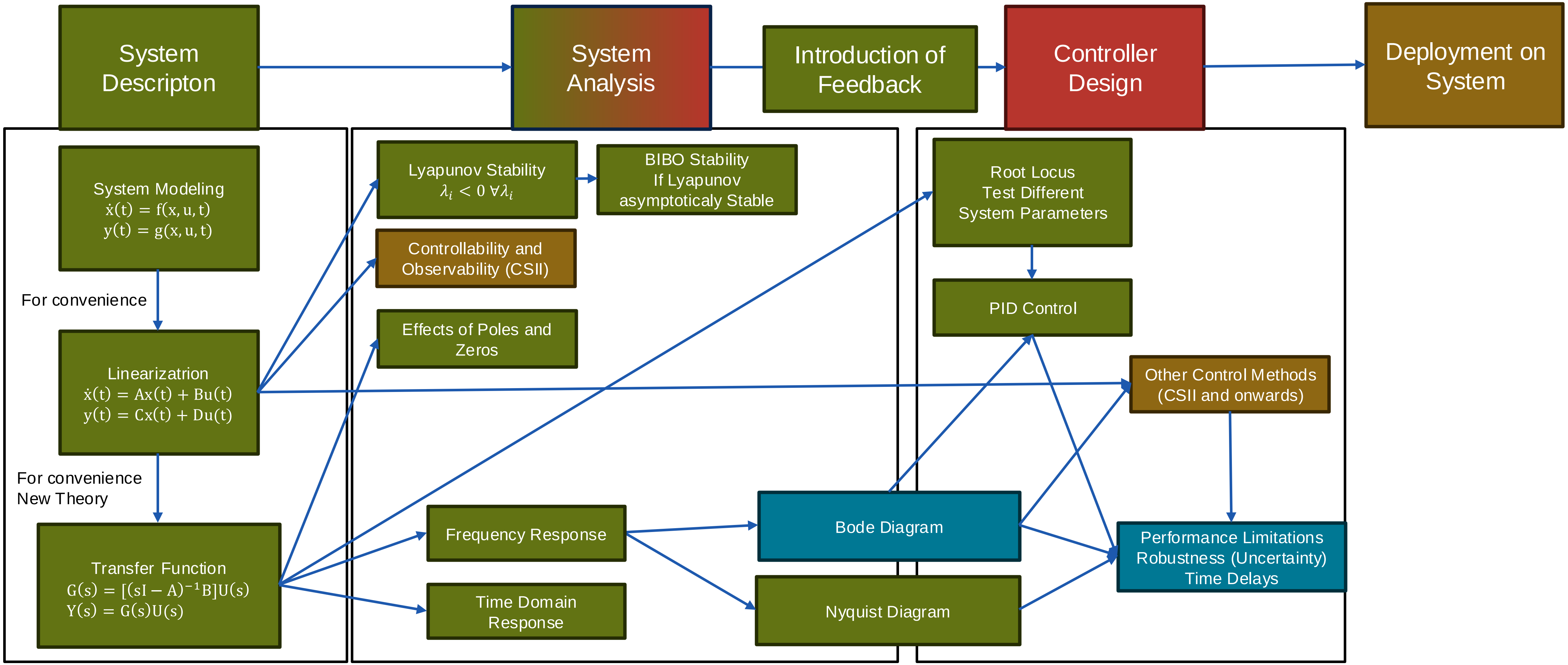


Outline

- Frequency-Domain Specifications
  - Intro to Noise, Disturbances and Commands
  - Context
  - Problem and its Solution
  - Bode Obstacle Plot
- Loop Shaping
  - Why?
  - What?
  - Elements
  - Example?
  - Caveats

Conceptual Recap

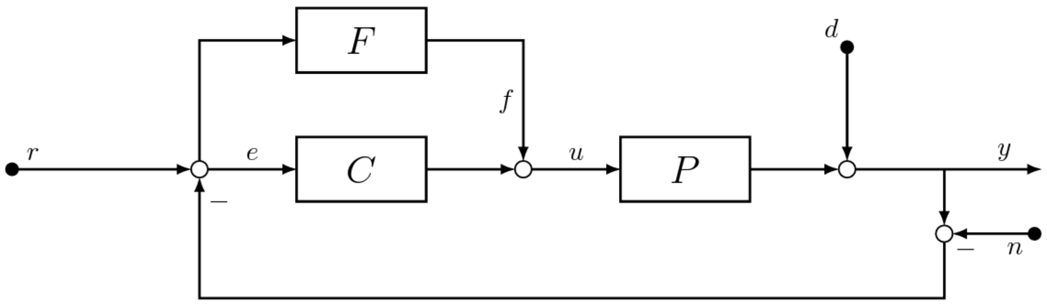
Classical Control Approach

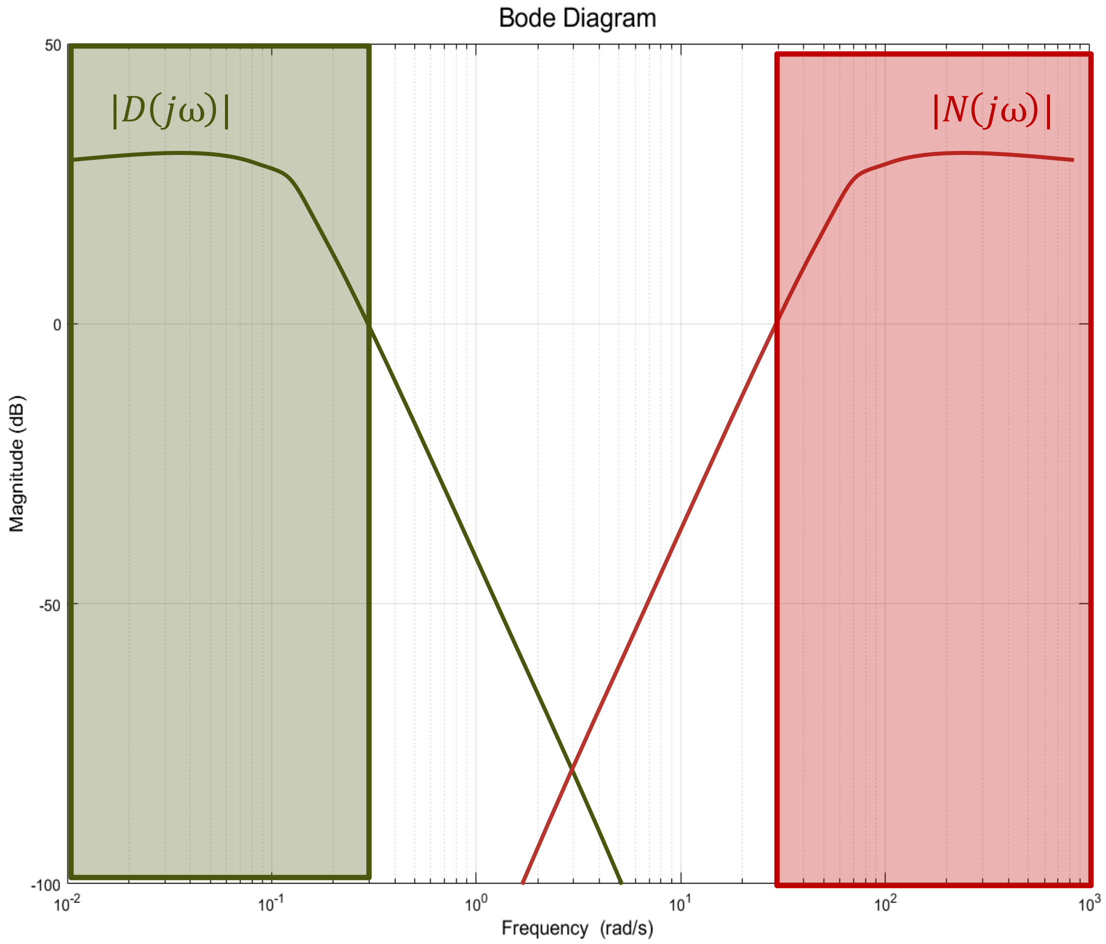


Frequency-Domain Specifications

Intro to Noise, Disturbances and Commands

- Commands r
  - Reference value we want the system to obtain
    - Position, Velocity, Temperature
  - Usually low frequency ($< 10Hz$)
- Disturbances: d
  - Unpredictable signals that effect the system
    - Wind, Friction, Bump in the road
  - Usually low frequency ($< 10Hz$)
- Noise: n
  - Corrupting signal in the measurement
    - Measurement noise
  - Usually high frequency ($> 100Hz$)

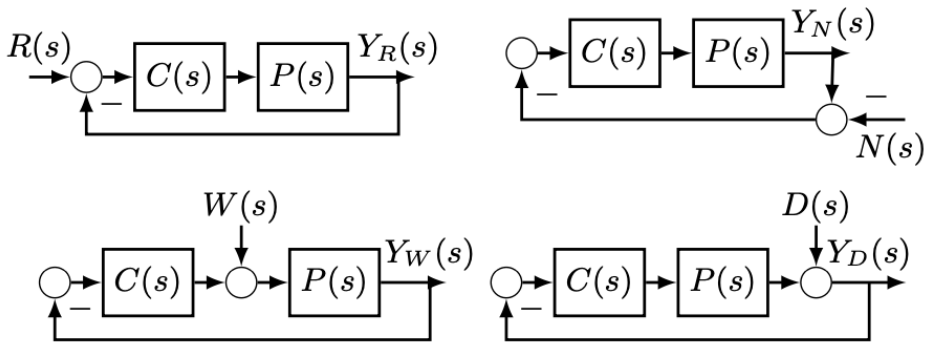




Frequency-Domain Specifications

Context

- Remember what a Controller is used for
  - Stabilize a system (Root Locus, PID, Nyquist)
  - Reach a certain performance criterion (1order and 2 order system specifications, **today**)
  - Robustness (perform well even with **disturbances, noise** or modelling errors)



$$Y_R(s) = \frac{P(s)C(s)}{1+P(s)C(s)} \cdot R(s), \quad Y_N(s) = \frac{P(s)C(s)}{1+P(s)C(s)} \cdot N(s)$$

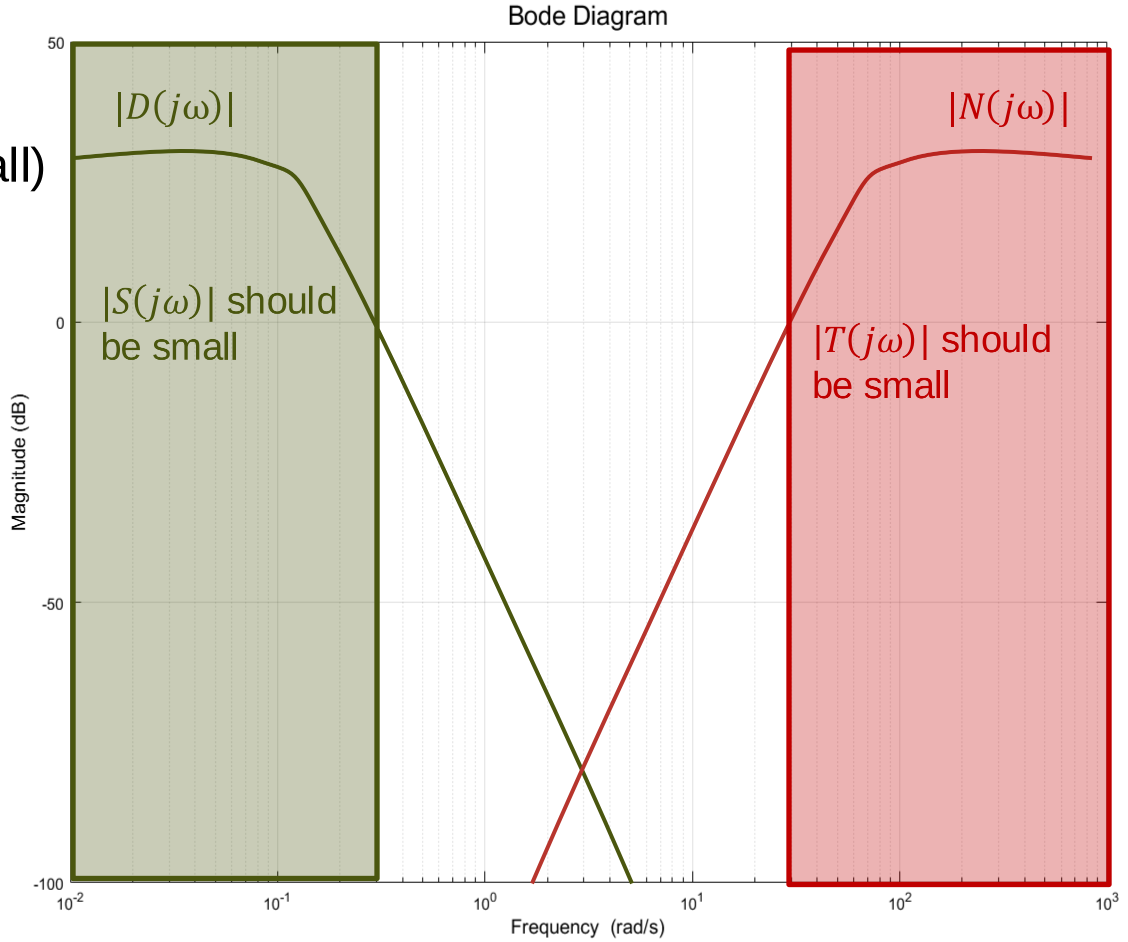
$$Y_W(s) = \frac{P(s)}{1+P(s)C(s)} \cdot W(s), \quad Y_D(s) = \frac{1}{1+P(s)C(s)} \cdot D(s)$$

- We want:
  - $T(s) = \frac{L(s)}{1+L(s)} = \frac{C(s)G(s)}{1+C(s)G(s)}, \; S(s) = \frac{1}{1+L(s)} = \frac{1}{1+C(s)G(s)}$
  - Good Reference Tracking $(R(s) = T(s)Y(s))$
  - **Good Disturbance Rejection** $(D(s) = S(s)Y(s))$
  - **Good Noise Rejection** $(N(s) = T(s)Y(s))$

Frequency-Domain Specifications

Problem and its Solution

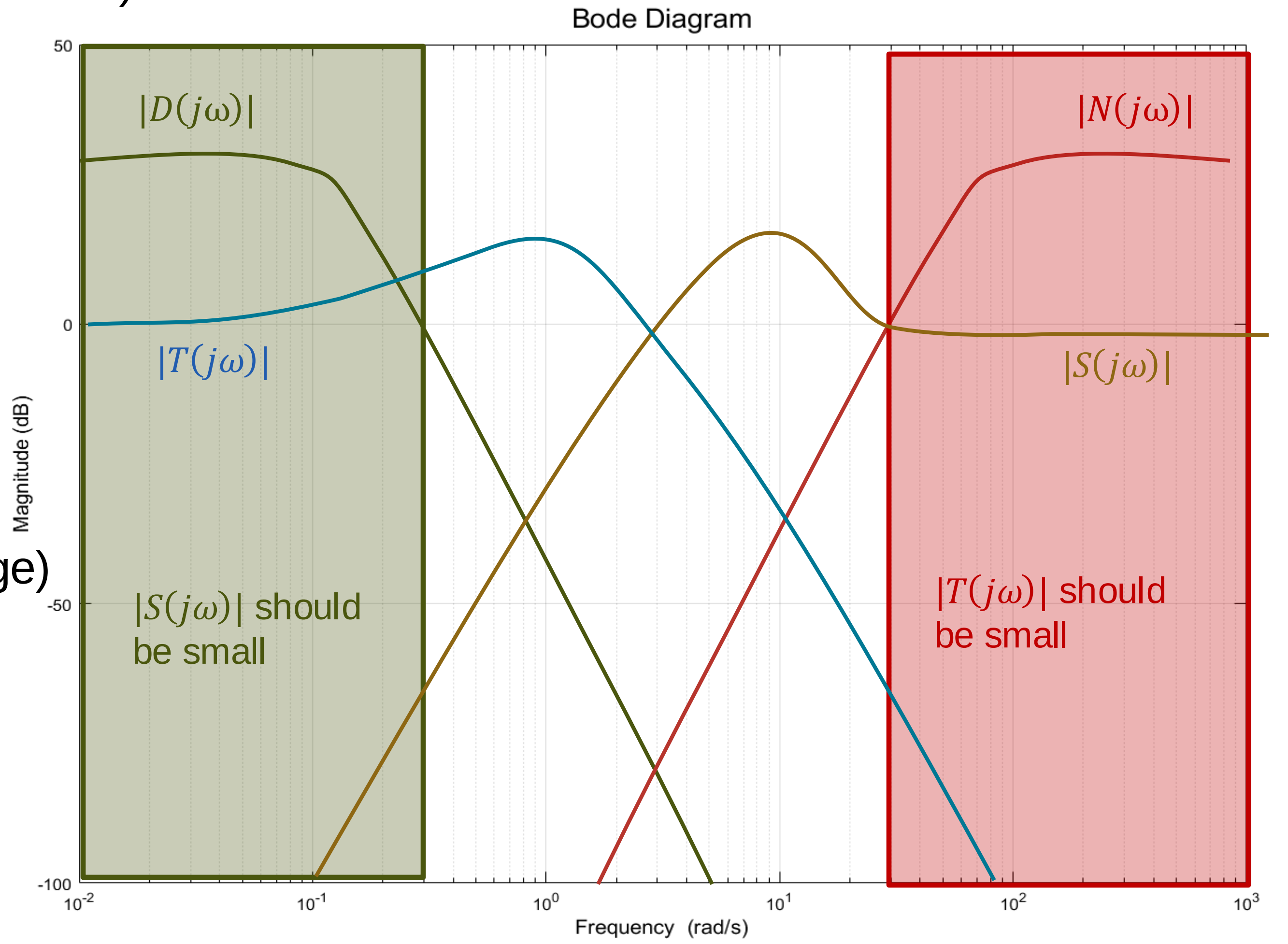
- For good performance/robustness:
  - Reference Tracking ($\mathrm{R(s) = T(s)Y(s)}$, $\mathrm{T(s)}$ large)
  - **Disturbance Rejection** ($\mathrm{D(s) = S(s)Y(s)}$, $\mathrm{S(s)}$ small)
  - **Noise Rejection** ($\mathrm{N(s) = S(s)Y(s)}$, $\mathrm{T(s)}$ small)
- **Problem:** $\mathrm{S(s) + T(s) = 1}$
  - $\mathrm{T(s)}$ and $\mathrm{S(s)}$ can't be small at the same time!!!
- Bodes Integral: (see Notebook)
  - $\int_0^{\infty} \ln|S(j\omega)| d\omega = \pi \cdot \sum_{i=1}^{n_+} \pi_i^+ = const$
  - If we decrease $S(s)$ it gets bigger somewhere else
- BUT noise and disturbances are only present in different frequency regimes so we can split them!



Frequency-Domain Specifications

Problem and its Solution

- **Disturbance Rejection** ($\mathrm{D}(s) = \mathrm{S}(s)\mathrm{Y}(s)$, $\mathrm{S}(s)$ small)
  - $\mathrm{S}(s)$ only needs to be small up to ω_d
  - $\mathrm{T}(s)$ must then be 1 in that region
- **Noise Rejection** ($\mathrm{N}(s) = \mathrm{S}(s)\mathrm{Y}(s)$, $\mathrm{T}(s)$ small)
  - $\mathrm{T}(s)$ only needs to be small after ω_n
  - $S(s)$ must then be 1 in that region
- Reference Tracking ($\mathrm{R}(s) = \mathrm{T}(s)\mathrm{Y}(s)$, $\mathrm{T}(s)$ large)
  - For free from disturbance rejection

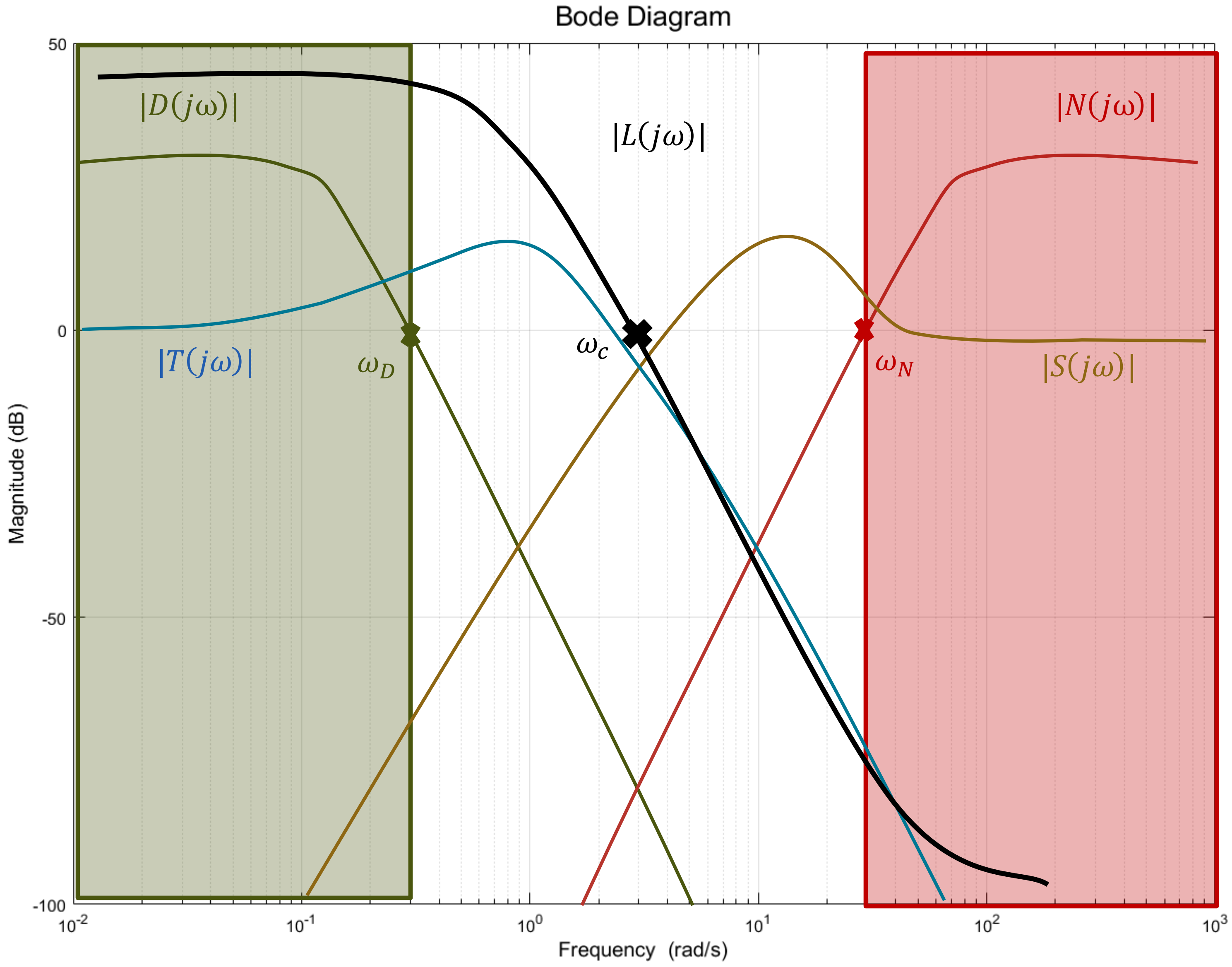


Frequency-Domain Specifications

Problem and its Solution

- For $|S(j\omega)| \ll 1$
  - $\Rightarrow \left|\frac{1}{1+L(j\omega)}\right| \ll 1 \Rightarrow |L(j\omega)| \gg 1$

- For $|T(j\omega)| \ll 1$
  - $\Rightarrow \left|\frac{L(j\omega)}{1+L(j\omega)}\right| \ll 1 \Rightarrow |L(j\omega)| \ll 1$

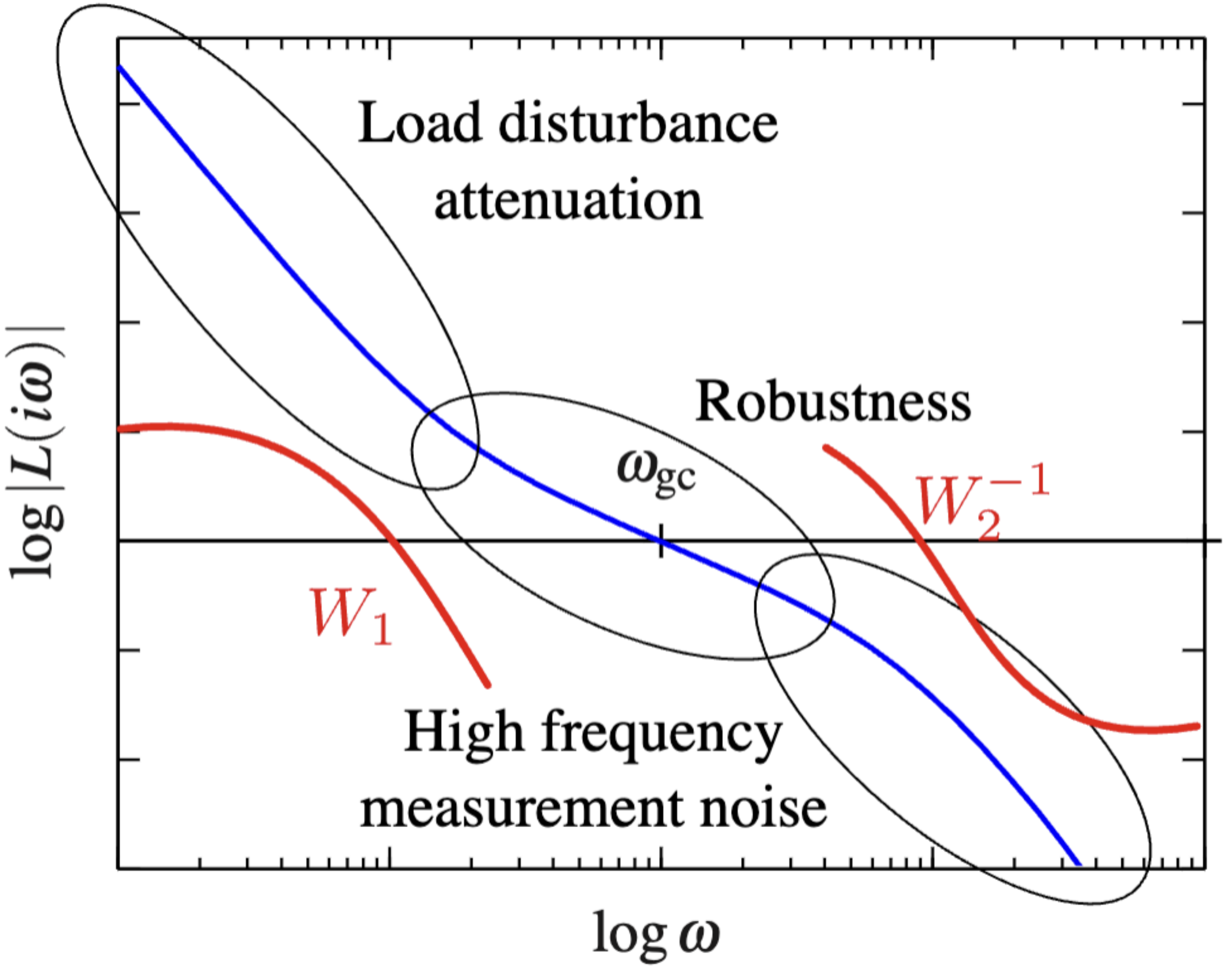


Frequency-Domain Specifications

Bode Obstacle Plot

- For $|S(j\omega)| \ll 1$
  - $\Rightarrow \left|\frac{1}{1+L(j\omega)}\right| \ll 1 \Rightarrow |L(j\omega)| \gg 1$
  - $|L(j\omega)| > |W_1(j\omega)|$

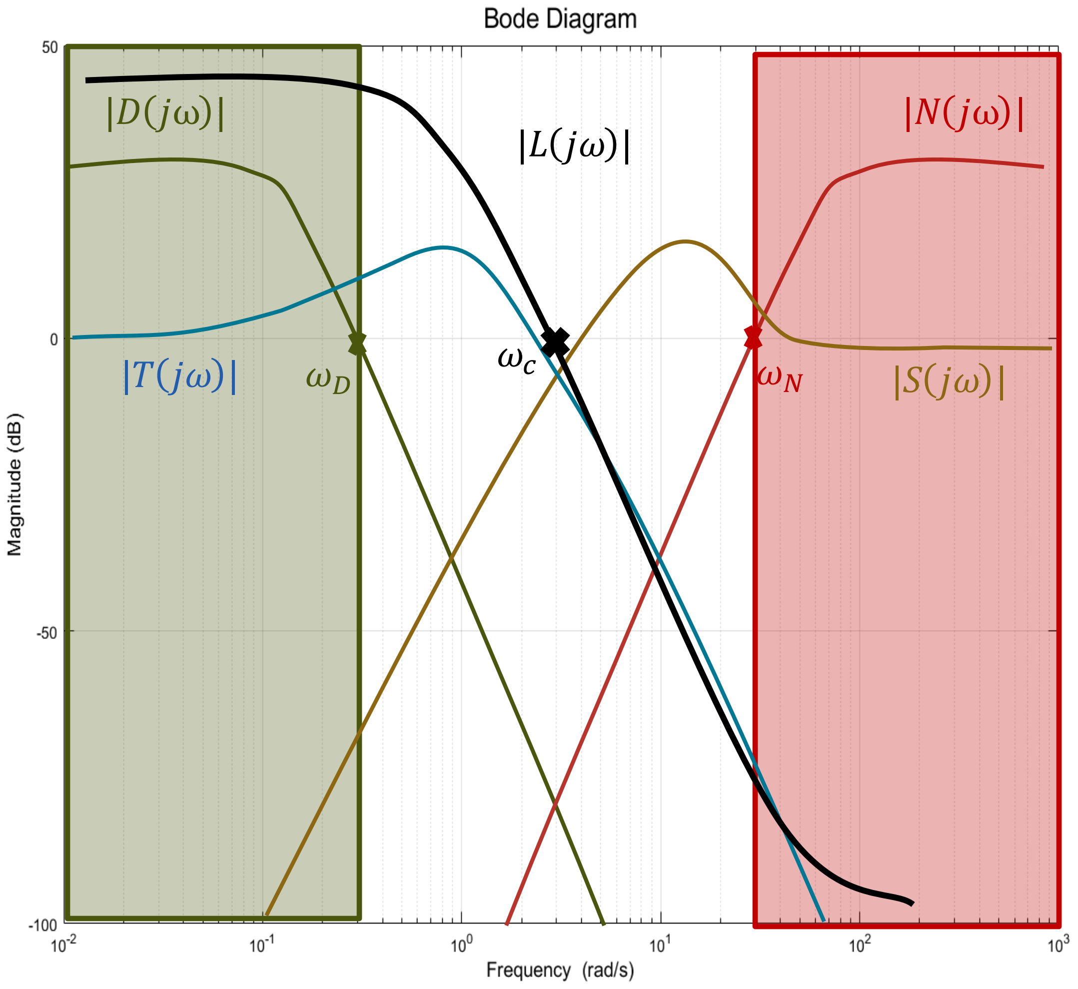
- For $|T(j\omega)| \ll 1$
  - $\Rightarrow \left|\frac{L(j\omega)}{1+L(j\omega)}\right| \ll 1 \Rightarrow |L(j\omega)| \ll 1$
  - $|L(j\omega)| < \left|W_2^{-1}(j\omega)\right|$



Frequency-Domain Specifications

Closed Loop Bandwidth - Completeness

- Closed Loop Bandwidth:
  - Frequency at which the reference is tracked up to a certain value k.
  - We use $k \approx 0.7$ (often also $k = 0.5$)
  - $Y(j\omega_c) = 0.7R(s) \rightarrow T(j\omega_c) = 0.7 \approx 1/\sqrt{2}$
  - $|T(j\omega_c)| = \frac{1}{\sqrt{2}} \rightarrow |L(j\omega_c)| = 1$
  - For this to be true $L(j\omega_c) = -j \rightarrow \angle L(j\omega_c) = -90°$
- Crossover Frequency ω_c= closed loop bandwidth
- We want to maximise the bandwidth/ω_c while still having
  - $\alpha\omega_d < \omega_c < \beta\omega_n$



Loop Shaping

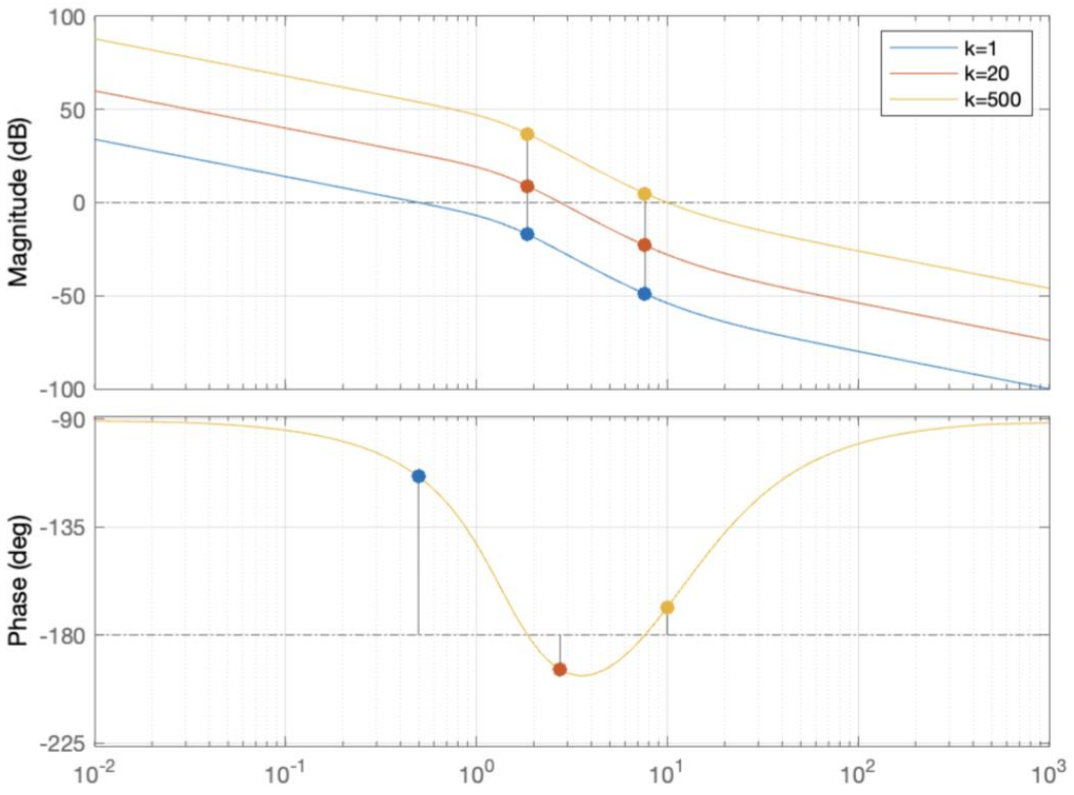
Why?

- We saw some methods to derive controllers:
  - Root Locus, PID
  - Work but are sometimes a bit freestyle
  - Difficult to precisely incorporate requirements
- We want a $C(s)$ such that our closed loop system with $P(s)$ satisfies a list of requirements
- What can we do?
  - Create a $L(s)$ that satisfies all requirements
  - Get $C(s) = \frac{L(s)}{P(s)}$
- This is a bad idea:
  - $C(s)$ can be infeasible
  - $C(s)$ can be overly complex
  - Only works for stable systems

Loop Shaping

What?

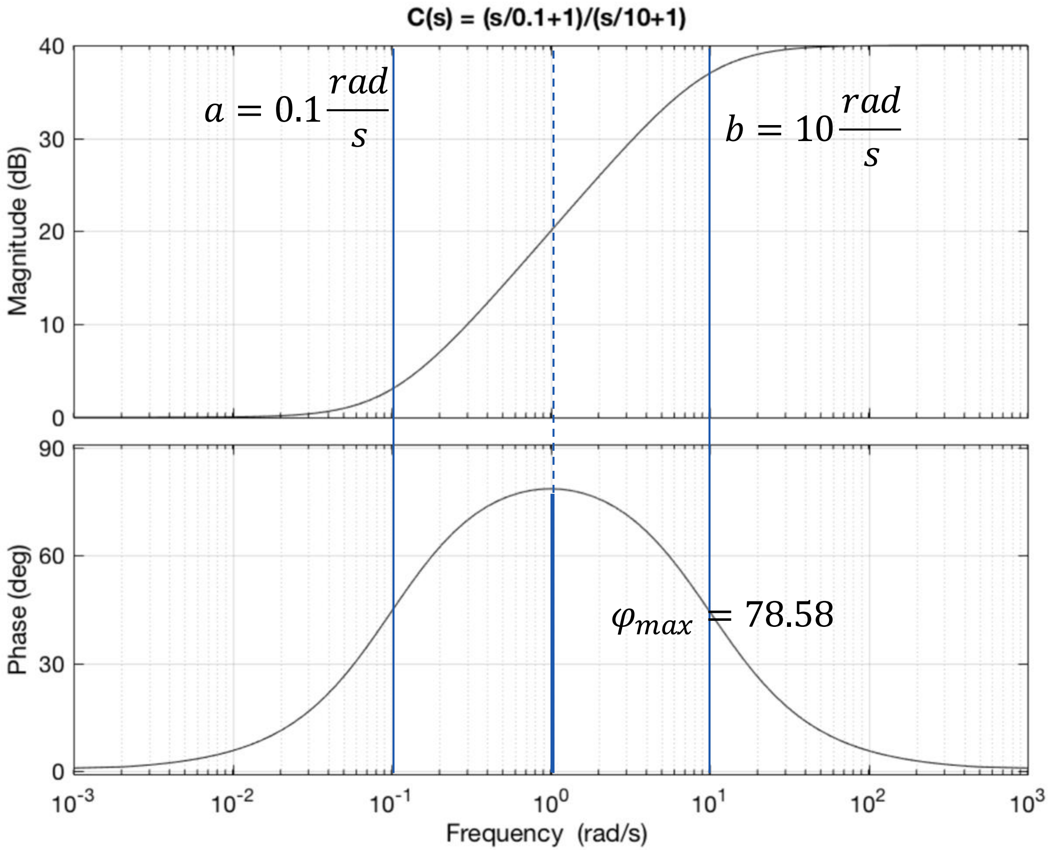
- What can we do better?
  - Use our knowledge and choose a combination of the following elements
  - Gain: k, Integrator: $1/s$, Lead: $\frac{s/a+1}{s/b+1}$ $(0 < \mathrm{a} < \mathrm{b})$, Lag: $\frac{s/a+1}{s/b+1}$ $(0 < \mathrm{b} < \mathrm{a})$
- **Gain: k**
  - Shifts the Magnitude Diagram up or down
  - Phase stays constant
- User Guide:
  - Move system up or down as desired
- **Integrator: $1/s$**
  - Gets rid of steady state error
  - $|L(0)| \to \infty$
- User Guide:
  - Add as many integrators as needed to get rid of the steady state error (beware of phase)



Loop Shaping

Elements

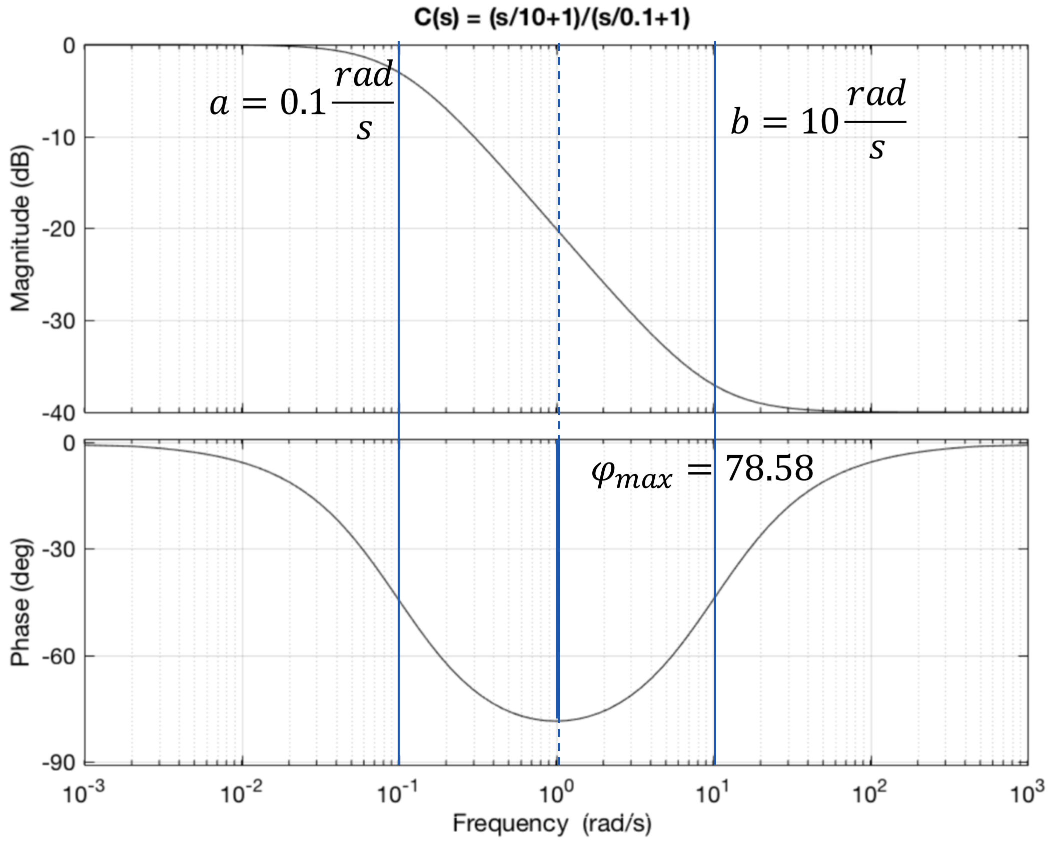
- **Lead:** $\frac{\boldsymbol{s/a+1}}{\boldsymbol{s/b+1}} = \frac{\mathbf{b}}{\mathbf{a}}\frac{\boldsymbol{s+a}}{\boldsymbol{s+b}}(\mathbf{0 < a < b})$, PD-Controller for $b \to \infty$
  - Increases magnitude by b/a at high frequencies
  - Creates a slope of $20\frac{dB}{dec}$ between a and b
  - Increases the phase at $\sqrt{ab}$ (midpoint of $[a, b]$) by:
    - $\varphi_{max} = 2\arctan\left(\sqrt{b/a}\right) - 90°$
- User guide:
  - Used to increase the phase margin
  - Pick desired crossover frequency $\sqrt{ab}$
  - Pick b/a for desired phase shift ($b/a \uparrow \to \varphi \uparrow$)
  - Use a gain k to shift crossover frequency
- Danger:
  - Increases magnitude at high frequencies -> Sensitive to noise



Loop Shaping

Elements

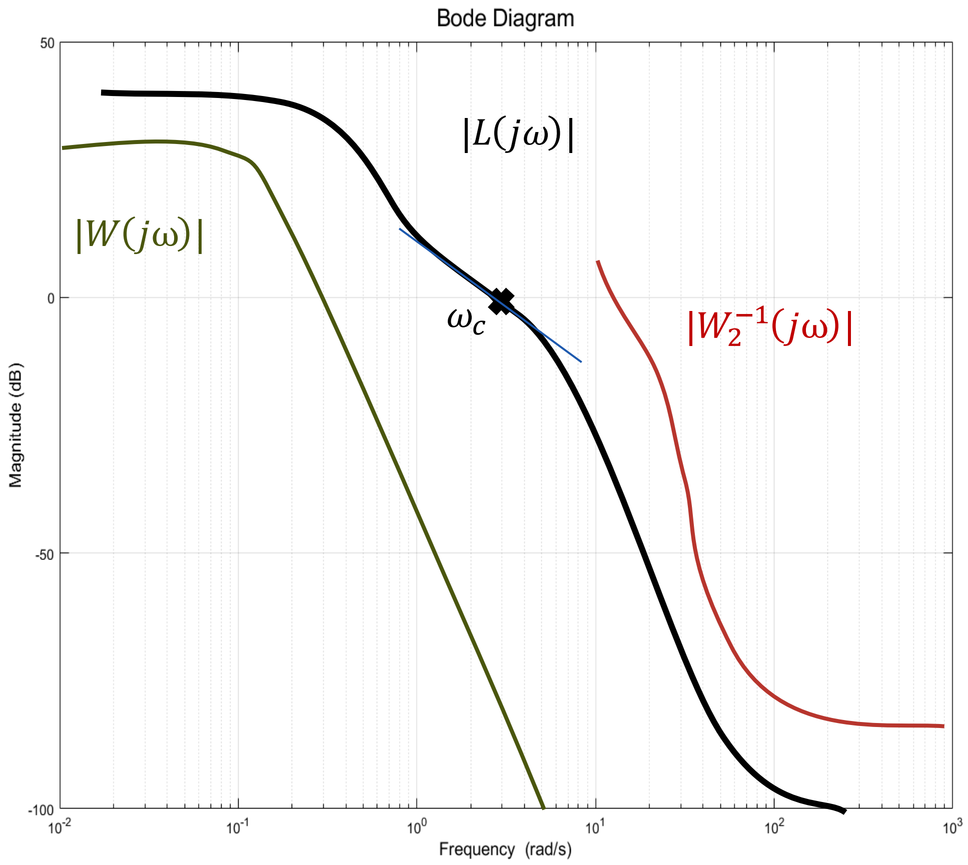
- **Lag:** $\frac{s/a+1}{s/b+1} = \frac{\mathbf{b}}{\mathbf{a}}\frac{s+a}{s+b}(\mathbf{0} < \mathbf{b} < \mathbf{a})$, PI-Controller for $b \to 0$
  - Decreases magnitude by b/a at high frequencies
  - Creates a slope of $-20\frac{dB}{dec}$ between a and b
  - Decreases the phase at $\sqrt{ab}$ (midpoint of $[a, b]$) by:
    - $\varphi_{max} = 2\arctan\left(\sqrt{b/a}\right) - 90°$
- User guide:
  - Used to improve disturbance rejection/ref tracking
  - Pick a small enough to not affect ω_c and φ_{margin}
    - For this $a \ll \omega_c$
- Danger:
  - Phase lag at small frequencies -> reduction of phase margin



Loop Shaping

General Approach

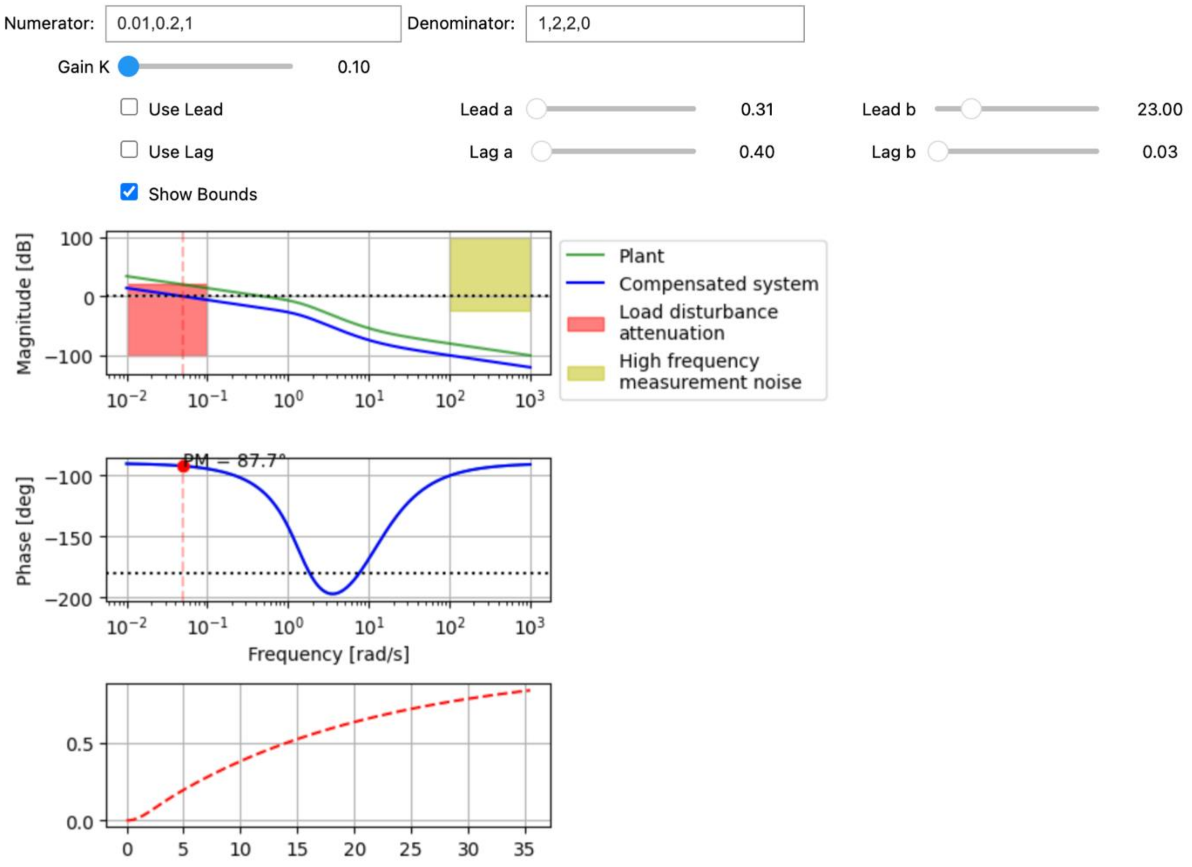
- We want our $L(j\omega)$ to something like in the plot
- Start from the left:
  - Figure out how many integrators are needed in $C(s)$
    - Input and order of the system
  - Set the gain that at low frequencies $|L(j\omega)| > |W_1(j\omega)|$
    - Good ref tracking/disturbance rejection
  - Add lead/lag terms such that at ω_c the slope is $-20\,\frac{dB}{dec}$
    - Good crossover frequency
    - Normalize them to not affect the magnitude
  - Past ω_c, add poles as needed for $|L(j\omega)| < \left|W_2^{-1}(j\omega)\right|$
- PID is just a special case of Loop Shaping
  - $C_{PID} = k \cdot \frac{s/z_1+1}{s+0} \cdot \frac{s/z_2+1}{s/p+1}$



Loop Shaping

Example

- See Notebook Exercise



Loop Shaping

Caveats

- If the system, we have unstable poles or non-minimumphase zeros we can still do Loop Shaping but always check with Nyquist that the system is stable.
- To see the effects of unstable poles/ non-minimumphase zeros we do use a trick:
  - $P(s) \ = \ P_{mp}(s) \, D(s)$
  - $P(s) = \frac{s-z}{s-p} = \frac{s+z}{s+p} \cdot \frac{s+p}{s-p} \cdot \frac{s-z}{s+z}$
  - $P_{mp}(s)$: All mirrors of the unstable poles and zeros
  - $D(s)$: original plant with the inverse of $P_{mp}(s)$
    - $|D(j\omega)| = 1$ (all pass filter) -> only alters the phase of the system
    - Chose the sign of $D(s)$ so the phase is negative
  - Once we controlled the nice system $P_{mp}(s)$, $D(s)$ will fuck up our lag/phase margins

Loop Shaping

Caveats

- Non-minimumphase zero:
  - $D(s) = -\frac{s-z}{s+z} \rightarrow \angle D(j\omega) = -2\arctan\left(\frac{\omega}{z}\right)$
  - At ω_c the system lags $-2\arctan\left(\frac{\omega_c}{z}\right)$
  - Nmp zeros force the system to be slow as max gain and crossover frequency are reduced
    - Slow (small z) nmp zeros are worse than fast (large z) ones

- Unstable pole:
  - $D(s) = \frac{s+zp}{s-p} \rightarrow \angle D(j\omega) = -2\arctan\left(\frac{p}{\omega}\right)$
  - At ω_c the system lags $-2\arctan\left(\frac{p}{\omega_c}\right)$
  - Unstable poles force the system to be faster as min gain and crossover frequency are increase
    - Slow (small p) poles are better than fast (large p) ones
    - Fast system requires strong and fast controllers/ actuators

Performance Limitation

Extra

- Sometimes it is just not possible to get all requirements due to nmp zeros or unstable poles
- Rule of thumb on the crossover frequency limits
  - Nominal: $\max\{10 \cdot \omega_d, 2 \cdot \omega_{p^+}\} < \omega_c < \min\left\{\frac{1}{10} \cdot \omega_n, \frac{1}{2} \cdot \omega_\tau, \frac{1}{2} \cdot \omega_{\zeta^+}\right\}$
  - Conservative: $\max\{10 \cdot \omega_d, 5 \cdot \omega_{p^+}\} < \omega_c < \min\left\{\frac{1}{10} \cdot \omega_n, \frac{1}{5} \cdot \omega_\tau, \frac{1}{5} \cdot \omega_{\zeta^+}\right\}$
  - ω_d and ω_n are the crossover frequencies of the disturbance and noise
  - ω_{p^+} and ω_{z^+} are the unstable poles and nmp zeros
  - $\omega_\tau = \frac{1}{\tau}$ effect of the time delay (next week)
- If ω_c the system can be controlled reasonably (no design specification)

Exercise 10

What to do?

- 1:
  - Do
- 2:
  - Do
- 3:
  - Do
- 4:
  - Skip
- 5:
  - Skip