Control Systems I
Recitation 03

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Last Week

System Modelling

- 1. Identify system boundaries
  - Inputs, Outputs, State, Parameters
- 2. Write down differential equations:
  - $\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{storage} = \sum \mathrm{infows} - \sum \mathrm{outflows}$
  - $\sum F = ma = m\ddot{x}, \rightarrow \sum T = I\ddot{\theta}$
- 3. Formulate in standard form
  - $\frac{\mathrm{d}}{\mathrm{dt}}x(t) = f(x(t), u(t))$
  - $y(t) = h(x(t), u(t))$
- 4. Normalize (not relevant)
- **5. Linearize (today)**

Last Week

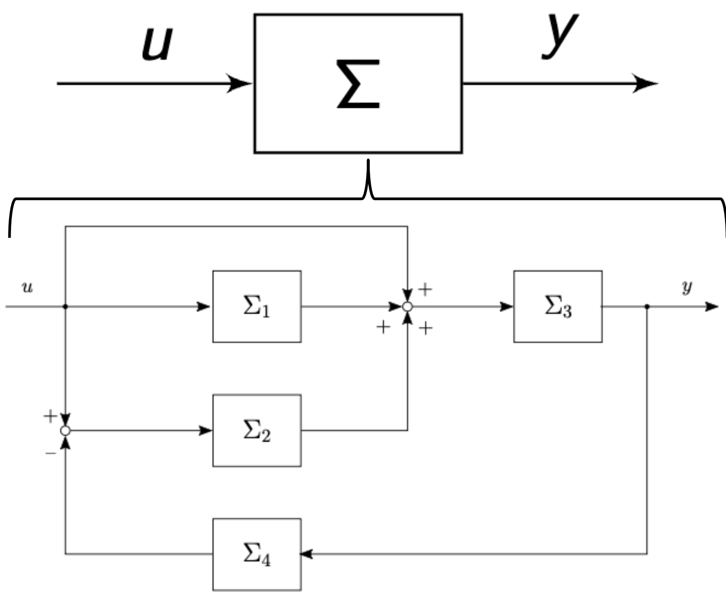
Output Calculation

- Almost impossible to find a closed form solution $y(t) = g(u(t), x_0)\ \forall\ u(t), x_0$
- Rely on discretization and approximations:
  - $\dot{x} = f\big(x(t), u(t)\big) \approx \frac{x(t+\Delta t) - x(t)}{\Delta t}$
  - $\mathbf{x}(\mathbf{t} + \boldsymbol{\Delta}\mathbf{t}) \approx \mathbf{x}(\mathbf{t}) + \boldsymbol{\Delta}\mathbf{t} \cdot \mathbf{f}\big(\mathbf{x}(\mathbf{t}), \mathbf{u}(\mathbf{t})\big)$
  - Solve this at every time step to get a simulation

Last Week

Block Diagrams and Block Diagram Algebra

- Combine large systems to one block
- *1. Pick and Place different interconnections*
  - Serial: $\Sigma_2\Sigma_1$, Parallel: $\Sigma_1 + \Sigma_2$, Negative Feedback: $(I + \Sigma_1\Sigma_2)^{-1}\Sigma_1$
- **2. Start from the end and work yourself to the start:**
  - 1. Define every output of a block as function of its immediate input
    - $y_i = \Sigma_i \, u_i$
  - 2. Define each input as function of other systems output (if applicable)
    - $u_i = f(y_j) \forall \, j$
    - 3. Solve system of equation
    - Best to start from the back and work your way to the beginning



Outline

- System Classification
  - What? Why? How?
  - Example

- Linearization and LTI
  - What? Why? How?
  - Example

System Classification

What? and Why?

- What? -> Determine/Compare system characteristics:

Linear $\rightarrow$ Superposition possible $\Sigma(\alpha \cdot u_1 + \beta \cdot u_2) = \alpha \cdot \Sigma(u_1) + \beta \cdot \Sigma(u_2)$ $y(t) = \alpha \cdot u(t), \quad y(t) = \frac{d}{dt}u(t), \quad y(t) = \int_0^t u(\tau)d\tau$	**Non-Linear** $\Sigma(\alpha \cdot u_1 + \beta \cdot u_2) \neq \alpha \cdot \Sigma(u_1) + \beta \cdot \Sigma(u_2)$ $y(t) = \alpha \cdot u(t) + \beta, \quad y(t) = \sin(u(t))$
Causal = *not* dependant on *future* values $y(t) = u(t-\tau) \quad \forall \tau \geq 0, \quad y(t) = \int_{-\infty}^{0} u(\tau)d\tau$	**Non-Causal** = dependant on *future* values $y(t) = u(t+5), \quad \int_{-\infty}^{t+1} u(\tau)d\tau$
Static = dependant on current values only $y(t) = 3 \cdot u(t), \quad y(t) = \sqrt{u(t)}$	**Dynamic** = also dependant on past values. $y(t) = \int_0^t u(\tau)d\tau, \quad y(t) = u(t-\tau) \quad \forall \tau \neq 0$
Time invariant $y(t) = \frac{d}{dt}u(t), \quad y(t) = 3 \cdot u(t)$	**Time varying** = parameters are time dependant $y(t) = sin(t) \cdot u(t), \quad y(t) = u(t) + t$
State $x(t)$ Vector $x(t) \in \mathbb{R}^n$ of values at t that fully describe the system. Past and future!	**Dimension** Minimal number of variables, n, to fully describe the system.

- Why? -> Know the assumptions/limitations of the system/model -> applicable theories

System Classification

How?

- Linear:
  - Check if $\tilde{u}(t) = au_1(t) + bu_2(t) \rightarrow \tilde{y}(t) = ay_1(t) + by_2(t)$: yes –> linear, no -> non-linear
  - Or check if there is a non-linear function present (be careful of affine functions $y(t) = au(t) + b$
- Causal:
  - In if there is a $\mathrm{f(t+a), a > 0}$ or $\mathrm{f(bt), b > 1}$ -> non-causal.
  - If only $\mathrm{f(t-a), a \geq 0}$ -> causal
- Dynamic:
  - If $\mathrm{y(t)}$ is a direct function of $\mathrm{u(t)}$or $\mathrm{y(t)} \rightarrow \mathrm{y(t) = g(u(t), y(t))}$-> static
  - Else dynamic, e.g. $\mathrm{y(t) = g\big(u(t \pm a)\big)}$ or $\mathrm{y(t)} = \frac{\mathrm{d}}{\mathrm{dt}}\left[\mathrm{f\big(y(t)\big) + g\big(u(t)\big)}\right]$
- Time varying:
  - Is t directly in there -> $\mathrm{f\big(\mathbf{t}, y(t), u(t)\big)}$
- State and Dimension:
  - From modelling and state reduction

System Classification

Example

- $y(t) = L\sin\big(u(t-3)\big), u(t), y(t) \in \mathbb{R}^2$
  - Linearity: -> Non-Linear
    - $\tilde{u}(t) = au_1(t) + bu_2(t) \ \rightarrow \tilde{y}(t) = L\sin\big(au_1(t) + bu_2(t)\big) \neq aL\sin\big(u_1(t)\big) + bL\sin\big(u_2(t)\big)$
    - Or see that $\sin\big(u(t)\big)$ is a non-linear function
  - Causality: -> Causal
    - We see only dependant on past values of $u(t)$
  - Static/Dynamic: -> Dynamic
    - We see $y(t)$ is dependent on past inputs of $u(t)$, namely at $t-3$
  - Time Varying: -> Time invariant
    - We see that there is not explicit dependency on time
  - Dimension: -> 2
    - Given by the state of the system

System Classification

Example

- $\frac{\mathrm{d}}{dt}\mathrm{y}(\mathrm{t}) = \mathrm{t} \cdot \mathrm{y}(\mathrm{t}) + \mathrm{u}(\mathrm{t}+1)$
  - Linearity: -> Linear
    - We see there are only linear (not affine) functions
    - Showing that this is linear is a bit more tricky (see PS02 – 1f)
  - Causality: -> Non-Causal
    - We see $\mathrm{y}(\mathrm{t})$ is dependent on future values of $\mathrm{u}(\mathrm{t})$, namely $\mathrm{t}+1$
  - Static/Dynamic: -> Dynamic
    - We see $\mathrm{y}(\mathrm{t})$ is dependent on past inputs of $\mathrm{y}(\mathrm{t})$ through the derivative
  - Time Varying: -> Time variant
    - We see that there is an explicit time dependency
  - Dimension: -> 1
    - Given by the state of the system

Linearization

What? Why?

- What?
  - Approximate Non-Linear ODE with first order Linear ODE:
    - $\frac{d}{dt}x(t) = f\big(x(t), u(t)\big) \rightarrow \frac{d}{dt}x(t) = Ax(t) + Bu(t)$
    - $y(t) = h\big(x(t), u(t)\big) \rightarrow y(t) = Cx(t) + Du(t)$
  - Around an **equilibrium point**: There can be multiple!!
    - $\frac{d}{dt}x(t) = f\big(x(t), u(t)\big) = 0 \ \rightarrow x_e, u_e \text{ and } y_e$
  - Results in an Linear Time Invariant System (LTI System)
- Why?
  - Way easier math:
    - Superposition
  - Closed form solution possible
  - Valid for operation around equilibrium

Linearization

How?

- 1. Equilibrium determination:
  - $\frac{d}{dt}x(t) = f\big(x(t), u(t)\big) = 0 \;\rightarrow x_e, u_e \text{ and } y_e$
  - Sometimes we can choose x_e, u_e and y_e
- 2. Approximate Non-Linear ODE with Linear ODE:
  - Looking at points near equilibrium point:
    - $x_i(t) = x_{i,e} + \delta x_i(t), \text{where } |\delta x_i(t)| \ll 1$
    - $u(t) = u_e + \delta u(t), \text{where } |\delta u(t)| \ll 1$
    - $u(t) = u_e + \delta u(t), \text{where } |\delta u(t)| \ll 1$
  - We get:
    - $\frac{d}{dt}\big(x_e + \delta x(t)\big) = \frac{d}{dt}\delta x(t) = f(x_e + \delta x(t), u_e + \delta u(t))$
    - $y_e + \delta y(t) = g(x_e + \delta x(t), u_e + \delta u(t))$

Linearization

How?

- We get:
  - $\frac{\mathrm{d}}{\mathrm{dt}}\left(\mathrm{x_e} + \delta\mathrm{x}(\mathrm{t})\right) = \frac{\mathrm{d}}{\mathrm{dt}}\delta\mathrm{x}(\mathrm{t}) = \mathrm{f}(\mathrm{x_e} + \delta\mathrm{x}(\mathrm{t}), \mathrm{u_e} + \delta\mathrm{u}(\mathrm{t}))$
  - $\mathrm{y_e} + \delta\mathrm{y}(\mathrm{t}) = \mathrm{g}(\mathrm{x_e} + \delta\mathrm{x}(\mathrm{t}), \mathrm{u_e} + \delta\mathrm{u}(\mathrm{t}))$
- Using Taylor Expansion:

$$f\left(x_e + \delta x(t), u_e + \delta u(t)\right) = \underbrace{f(x_e, u_e)}_{=0} + \left.\frac{\partial f}{\partial x}\right|_{x=x_e, u=u_e} \cdot \delta x(t) + \left.\frac{\partial f}{\partial u}\right|_{x=x_e, u=u_e} \cdot \delta u(t) + \mathcal{O}(2)$$

$$g\left(x_e + \delta x(t), u_e + \delta u(t)\right) = \underbrace{g(x_e, u_e)}_{=y_e} + \left.\frac{\partial g}{\partial x}\right|_{x=x_e, u=u_e} \cdot \delta x(t) + \left.\frac{\partial g}{\partial u}\right|_{x=x_e, u=u_e} \cdot \delta u(t) + \mathcal{O}(2)$$

- This results in: y_e gets cancelled
  - $\frac{\mathrm{d}}{\mathrm{dt}}\delta\mathrm{x}(\mathrm{t}) = \mathrm{A}\,\delta\mathrm{x}(\mathrm{t}) + \mathrm{B}\,\delta\mathrm{u}(\mathrm{t})$
  - $\delta\mathrm{y}(\mathrm{t}) = \mathrm{C}\,\delta\mathrm{x}(\mathrm{t}) + \mathrm{D}\delta\mathrm{u}(\mathrm{t})$

Linearization

How?

- This results in:
  - $\frac{d}{dt}\delta x(t) = A\,\delta x(t) + B\,\delta u(t)$
  - $\delta y(t) = C\,\delta x(t) + D\delta u(t)$
- With:

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{(x_e,u_e)} \in \mathbb{R}^{n\times n} \qquad C = \left.\frac{\partial g(x,u)}{\partial x}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial x_1} & \cdots & \frac{\partial g_p}{\partial x_n} \end{bmatrix}\right|_{(x_e,u_e)} \in \mathbb{R}^{p\times n}$$

$$B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}\right|_{(x_e,u_e)} \in \mathbb{R}^{n\times m} \qquad D = \left.\frac{\partial g(x,u)}{\partial u}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \cdots & \frac{\partial g_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial u_1} & \cdots & \frac{\partial g_p}{\partial u_m} \end{bmatrix}\right|_{(x_e,u_e)} \in \mathbb{R}^{p\times m}$$

Linearization

How?

- For convenience we remove the deviation $\mathrm{x(t)} = \delta\mathrm{x(t)}$
- And we the the LTI state-space model: notice A, B, C and D are constant

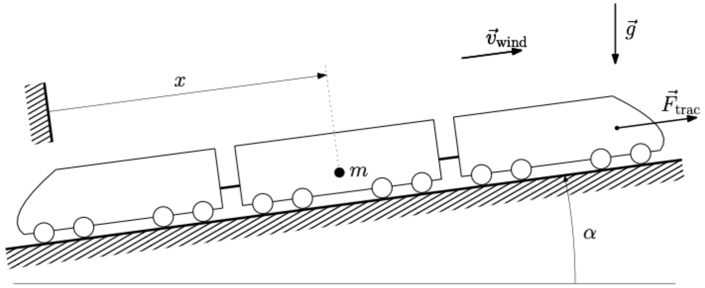
$$\boxed{\begin{aligned}\frac{d}{dt}x(t) &= A \cdot x(t) + B \cdot u(t) \\ y(t) &= C \cdot x(t) + D \cdot u(t)\end{aligned}}$$

$$A = \frac{\partial f(x,u)}{\partial x}\bigg|_{(x_e,u_e)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\Bigg|_{(x_e,u_e)} \in \mathbb{R}^{n \times n} \qquad C = \frac{\partial g(x,u)}{\partial x}\bigg|_{(x_e,u_e)} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial x_1} & \cdots & \frac{\partial g_p}{\partial x_n} \end{bmatrix}\Bigg|_{(x_e,u_e)} \in \mathbb{R}^{p \times n}$$

$$B = \frac{\partial f(x,u)}{\partial u}\bigg|_{(x_e,u_e)} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}\Bigg|_{(x_e,u_e)} \in \mathbb{R}^{n \times m} \qquad D = \frac{\partial g(x,u)}{\partial u}\bigg|_{(x_e,u_e)} = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \cdots & \frac{\partial g_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial u_1} & \cdots & \frac{\partial g_p}{\partial u_m} \end{bmatrix}\Bigg|_{(x_e,u_e)} \in \mathbb{R}^{p \times m}$$

- **Remember this is always a deviation from the equilibrium point:**
  - **If you want the actual value, you must add the equilibrium value**
  - If from the simulation we have $\mathrm{x(t = 3) = 5}$, then the actual value is $\mathrm{x_{actual}(t = 3) = 5 + x_e}$

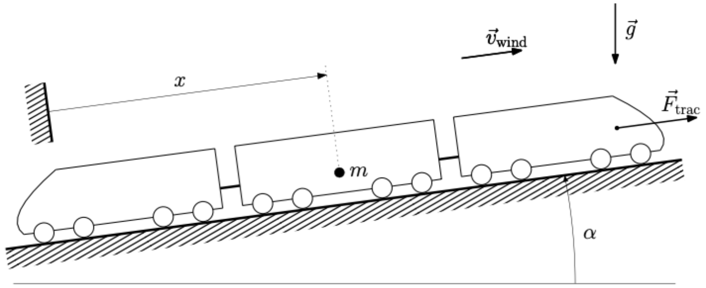
Linearization



Example - Train Driving up a Hill ($\mathrm{v_{wind}} = \mathrm{v_{wind}}(\mathrm{t}), \alpha = \mathrm{const}$)

- We have: ($\mathrm{v_{wind}} = \mathrm{const}$)
  - $\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{x}(\mathbf{t}) = \begin{bmatrix} \mathrm{x}_2 \\ \frac{1}{\mathrm{m}}\left[\mathrm{u}(\mathrm{t}) - \frac{1}{2}\rho c_{\mathrm{w}} \mathrm{A}(\mathrm{x}_2(\mathrm{t}) - v_{wind})^2 \mathrm{sign}(\mathrm{x}_2(\mathrm{t}) - v_{wind}) - \sin(\alpha)\,\mathrm{mg}\right] \end{bmatrix} = \mathrm{f}(\mathbf{x}(\mathrm{t}), \mathrm{u}(\mathrm{t}))$
  - $\mathrm{y}(\mathrm{t}) = \mathrm{x}_1(\mathrm{t})$
- 1. Get equilibrium point:
  - $\mathrm{f}(\mathbf{x_e}, \mathrm{u_e}) = \begin{bmatrix} \mathrm{x}_2 \\ \frac{1}{\mathrm{m}}\left[\mathrm{u}(\mathrm{t}) - \frac{1}{2}\rho c_{\mathrm{w}} \mathrm{A}(\mathrm{x}_2(\mathrm{t}) - v_{wind})^2 \mathrm{sign}(\mathrm{x}_2(\mathrm{t}) - v_{wind}) - \sin(\alpha)\,\mathrm{mg}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
  - $x_{1,e} = \beta \in \mathbb{R} \rightarrow \mathrm{y_e} = \beta$
  - $x_{2,e} = 0\ and\ u_e = -\frac{1}{2}\rho c_{\mathrm{w}} \mathrm{A}(v_{wind})^2 \mathrm{sign}(v_{wind}) + \sin(\alpha)\,\mathrm{mg}$

Linearization



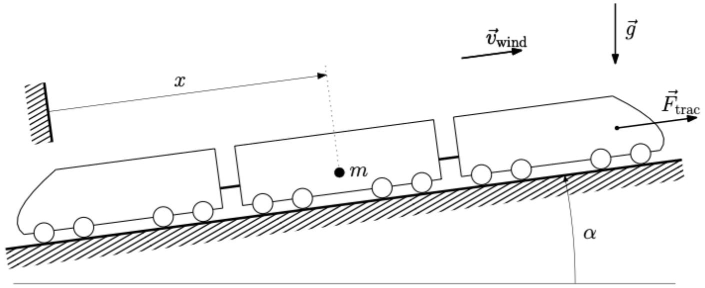
Example - Train Driving up a Hill ($v_{\text{wind}} = v_{\text{wind}}(t), \alpha = \text{const}$)

- 1. Get equilibrium point:
  - $f(\mathbf{x_e}, u_e) = \begin{bmatrix} x_2 \\ \frac{1}{m}\left[u(t) - \frac{1}{2}\rho c_w A (x_2(t) - v_{\text{wind}})^2 \text{sign}(x_2(t) - v_{\text{wind}}) - \sin(\alpha)\, mg\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
  - $x_{1,e} = \beta \in \mathbb{R} \rightarrow y_e = \beta$
  - $x_{2,e} = 0 \text{ and } u_e = -\frac{1}{2}\rho c_w A (v_{\text{wind}})^2 \text{sign}(v_{\text{wind}}) + \sin(\alpha)\, mg$

- 2. Calculate Matrix:
  - $A = \begin{bmatrix} \frac{\partial}{\partial x_1} x_2 & \frac{\partial}{\partial x_2} x_2 \\ \frac{\partial}{\partial x_1} f_2 & \frac{\partial}{\partial x_2} f_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \rho c_w A (x_{2,e} - v_{\text{wind}}) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\rho c_w A v_{\text{wind}} \end{bmatrix}$
  - $B = \begin{bmatrix} \frac{\partial}{\partial u} x_2 \\ \frac{\partial}{\partial u} \frac{1}{m}\left[u(t) - \frac{1}{2}\rho c_w A (x_2(t) - v_{\text{wind}})^2 \text{sign}(x_2(t) - v_{\text{wind}}) - \sin(\alpha)\, mg\right] \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$
  - $C = \begin{bmatrix} \frac{\partial}{\partial x_1} x_1 & \frac{\partial}{\partial x_2} f_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$
  - $D = \begin{bmatrix} \frac{\partial}{\partial u} x_1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

Linearization

Example - Train Driving up a Hill ($v_{wind} = v_{wind}(t), \alpha = const$)



- 3. Write down LTI system:
  - $\frac{d}{dt}x(t) = \begin{bmatrix} 0 & 1 \\ 0 & -\rho c_w A v_{wind} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$
  - $y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0\, u(t)$