# Control Systems I
# Recitation 04

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# Last Week

## System Classification

| | |
|---|---|
| **Linear** $\rightarrow$ Superposition possible<br>$\Sigma(\alpha \cdot u_1 + \beta \cdot u_2) = \alpha \cdot \Sigma(u_1) + \beta \cdot \Sigma(u_2)$<br>$y(t) = \alpha \cdot u(t), \quad y(t) = \frac{d}{dt}u(t), \quad y(t) = \int_0^t u(\tau)d\tau$ | **Non-Linear**<br>$\Sigma(\alpha \cdot u_1 + \beta \cdot u_2) \neq \alpha \cdot \Sigma(u_1) + \beta \cdot \Sigma(u_2)$<br>$y(t) = \alpha \cdot u(t) + \beta, \quad y(t) = \sin(u(t))$ |
| **Causal** = *not* dependant on *future* values<br>$y(t) = u(t-\tau) \quad \forall \tau \geq 0, \quad y(t) = \int_{-\infty}^0 u(\tau)d\tau$ | **Non-Causal** = dependant on *future* values<br>$y(t) = u(t+5), \quad \int_{-\infty}^{t+1} u(\tau)d\tau$ |
| **Static** = dependant on current values only<br>$y(t) = 3 \cdot u(t), \quad y(t) = \sqrt{u(t)}$ | **Dynamic** = also dependant on past values.<br>$y(t) = \int_0^t u(\tau)d\tau, \quad y(t) = u(t-\tau) \quad \forall \tau \neq 0$ |
| **Time invariant**<br>$y(t) = \frac{d}{dt}u(t), \quad y(t) = 3 \cdot u(t)$ | **Time varying** = parameters are time dependant<br>$y(t) = sin(t) \cdot u(t), \quad y(t) = u(t) + t$ |
| **State** $x(t)$<br>Vector $x(t) \in \mathbb{R}^n$ of values at $t$ that fully describe the system. Past and future! | **Dimension**<br>Minimal number of variables, $n$, to fully describe the system. |

# Last Week

## LTI-Systems

- LTI state-space model: notice A, B, C and D are constant

$$\frac{d}{dt}x(t) = A \cdot x(t) + B \cdot u(t)$$
$$y(t) = C \cdot x(t) + D \cdot u(t)$$

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{(x_e,u_e)} \in \mathbb{R}^{n \times n}$$

$$C = \left.\frac{\partial g(x,u)}{\partial x}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial x_1} & \cdots & \frac{\partial g_p}{\partial x_n} \end{bmatrix}\right|_{(x_e,u_e)} \in \mathbb{R}^{p \times n}$$

$$B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}\right|_{(x_e,u_e)} \in \mathbb{R}^{n \times m}$$

$$D = \left.\frac{\partial g(x,u)}{\partial u}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \cdots & \frac{\partial g_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial u_1} & \cdots & \frac{\partial g_p}{\partial u_m} \end{bmatrix}\right|_{(x_e,u_e)} \in \mathbb{R}^{p \times m}$$

- **Remember this is always a deviation from the equilibrium point:**

# Outline

- Time Response
  - What? Why? How?
  - Example

- Similarity Transform
  - What? Why? How?
  - Example
  - Connection to Stability

- Stability
  - Definition
  - How?
  - Linear and Non-Linear Systems

# Time Response

## What? Why?

- What?
  - Find a solution to the LTI system

$$\frac{d}{dt}x(t) = A \cdot x(t) + B \cdot u(t)$$
$$y(t) = C \cdot x(t) + D \cdot u(t)$$

  - $y(t) = S(u(t), x_0, A, B, C, D)$

- Why?
  - Test different Inputs (as last week)
  - Understand/Derive the notion of Stability
  - Groundwork for Frequency Domain calculations

# Time Response

$$\frac{d}{dt}x(t) = A \cdot x(t) + B \cdot u(t)$$
$$y(t) = C \cdot x(t) + D \cdot u(t)$$

How?

- We can use superposition(linearity) and look at two cases:
  - 1. No input and any initial condition: $u_{IC}(t) = 0, x_{0,IC} = x_0$
  - 2. Any input and zero initial condition: $u_F(t) = u(t)$, $x_{0,F} = 0$
  - They satisfy $x_0 = x_{0,IC} + x_{0,F}$ and $u(t) = u_{IC}(t) + u_F(t)$ thus we have $y(t) = y_{IC}(t) + y_F(t)$
- Looking at the zero input we have (Initial Condition Response):
  - $\frac{d}{dt}x_{IC}(t) = A \cdot x_{IC}(t)$
  - Easy to solve: $x_{IC}(t) = e^{At}x_0$
- Looking at the zero initial condition we have (Forced Response):
  - $\frac{d}{dt}x_F(t) = A \cdot x_F(t) + B \cdot u(t)$
  - Not so easy put can be shown that: $x_F(t) = \int_0^t e^{A(t-\tau)}Bu(t)d\tau$

# Time Response

How?

$$\frac{d}{dt}x(t) = A \cdot x(t) + B \cdot u(t)$$
$$y(t) = C \cdot x(t) + D \cdot u(t)$$

- Using the definition of $y(t)$ and the previous solutions we get:
  - $\mathbf{y(t) = C \cdot e^{At} \cdot x_0 + C \cdot \int_0^t e^{A(t-\tau)} Bu(t) d\tau + D \cdot u(t)}$
  - Initial Condition Response:
    - $y_{IC}(t) = C \cdot e^{At} \cdot x_0$
    - Describes how the system behaves naturally
  - Force Response:
    - $y_F(t) = C \cdot \int_0^t e^{A(t-\tau)} Bu(t) d\tau$
    - How the system reacts to the input
  - Feedthrough:
    - $y_{FF}(t) = D \cdot u(t)$
    - Direct effect of the input (usually 0)

# Time Response

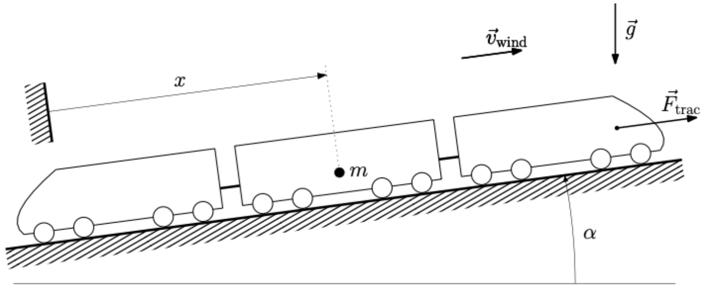
$$\frac{d}{dt}x(t) = A \cdot x(t) + B \cdot u(t)$$
$$y(t) = C \cdot x(t) + D \cdot u(t)$$

How?

- We have:
  - $\mathbf{y(t) = C \cdot e^{\mathit{At}} \cdot x_0 + C \cdot \int_0^t e^{A(t-\tau)} Bu(t) d\tau + D \cdot u(t)}$
- Matrix Exponential:
  - $e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = \mathbb{I} + At + \frac{1}{2}(At)^2 + \cdots + \frac{1}{k!}(At)^k + \cdots$
  - $e^{At} = \begin{bmatrix} e^{A_{11}t} & 0 \\ 0 & e^{A_{22}t} \end{bmatrix}$, if A is diagonal
  - Jordan Form:
    - $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \rightarrow e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} e^{\lambda t}$
    - $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \rightarrow e^{At} = \begin{bmatrix} 1 & t & \frac{1}{2!}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} e^{\lambda t}$
  - See Similarity Transform to get these matrices
- Keep in mind in practice we never calculate this by hand (useful for theory)

# Time Response

Example - Train Driving up a Hill ($v_{wind} = v_{wind}(t), \alpha = const$)




- We got the LTI system with $u(t) = u_0 = const., x_0 = [0,0]^T$:
  - $\frac{d}{dt}x(t) = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{m}\rho c_w A v_{wind} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t) = \begin{bmatrix} 0 & 1 \\ 0 & -L \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u(t)$
  - $y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0\, u(t)$
  - Using: $y(t) = C \cdot e^{At} \cdot x_0 + C \cdot \int_0^t e^{A(t-\tau)} B u(t) d\tau + D \cdot u(t)$
- We have: $e^{At} = \begin{bmatrix} 1 & -\frac{e^{-Lt}-1}{L} \\ 0 & e^{-Lt} \end{bmatrix}$
  - $y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\frac{e^{-Lt}-1}{L} \\ 0 & e^{-Lt} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \int_0^t \begin{bmatrix} 1 & -\frac{e^{-L(t-\tau)}-1}{L} \\ 0 & e^{-L(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} u_0 d\tau$
  - $y(t) = 0 + \int_0^t -\frac{e^{-L(t-\tau)}-1}{L} b u_0 d\tau = -\frac{bu_0}{L} \int_0^t e^{-L(t-\tau)} - 1 d\tau = \frac{b}{L^2}\left(e^{-Lt} - 1 + Lt\right)$
  - $y(t) = \frac{b}{L^2}\left(e^{-Lt} - 1 + Lt\right)$
  - If we would have chosen $C = [0,1]$ (only velocity as output)
  - $y(t) = \frac{b}{L}\left(1 - e^{-Lt}\right)$

# Similarity Transform

What? Why?

- What?
  - Rewrite the state of our system $\tilde{x}(t) = Kx(t)$

$$\frac{d}{dt}x(t) = A \cdot x(t) + B \cdot u(t)$$
$$y(t) = C \cdot x(t) + D \cdot u(t)$$

  - Not so relevant for exam but good to know/useful trick
- Why?
  - Sometimes easier system to work with
  - Get minimal realisation
  - **Diagonalize the system**
    - Matrix Exponential

# Similarity Transform

$$\frac{d}{dt}x(t) = A \cdot x(t) + B \cdot u(t)$$
$$y(t) = C \cdot x(t) + D \cdot u(t)$$

How?

- Define an invertible matrix $T$:
  - $x(t) = T \cdot \tilde{x}(t) \rightarrow \tilde{x}(t) = T^{-1}x(t)$
  - Our system then becomes:
  - $$\begin{aligned} T\frac{d}{dt}\tilde{x}(t) &= A \cdot T \cdot \tilde{x}(t) + B \cdot u(t) \\ y(t) &= C \cdot T \cdot \tilde{x}(t) + D\, u(t) \end{aligned} \Longrightarrow \begin{aligned} \frac{d}{dt}\tilde{x}(t) &= T^{-1} \cdot A \cdot T \cdot \tilde{x}(t) + T^{-1} \cdot B \cdot u(t) \\ y(t) &= C \cdot T \cdot \tilde{x}(t) + D\, u(t) \end{aligned}$$
- This results in the system:
  - $$\begin{aligned} \frac{d}{dt}\tilde{x}(t) &= \tilde{A} \cdot \tilde{x}(t) + \tilde{B} \cdot u(t) \\ y(t) &= \tilde{C} \cdot \tilde{x}(t) + \tilde{D}\, u(t) \end{aligned}, with\ \tilde{A} = T^{-1}AT,\ \tilde{B} = T^{-1}B, \tilde{C} = CT$$
- It can be shown that these systems are the same

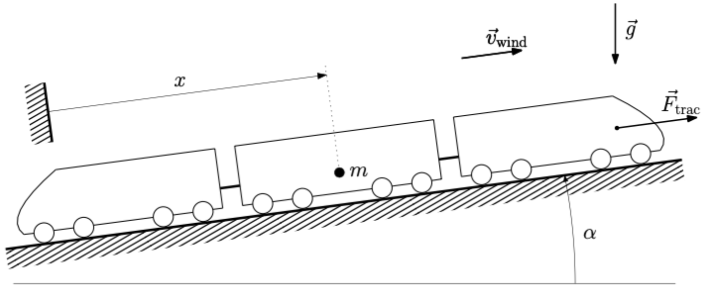
# Similarity Transform

How?

- Most often we use Eigenvalue Transform (Eigendecomposition):
  - Recall Eigenvalues: $Av_i = \lambda_i v_i$
    - Put that into a matrix: V is the matrix of Eigenvectors
      - $AV = A[v_1, v_2, \dots, v_n] = [\lambda_1 v_1, \lambda_2 v_2, \dots, \lambda_n v_n] = V\Lambda, \quad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$
      - $V^{-1}AV = \Lambda$
  - This looks very familiar:
    - $\begin{aligned} \frac{d}{dt}\tilde{x}(t) &= \tilde{A} \cdot \tilde{x}(t) + \tilde{B} \cdot u(t) \\ y(t) &= \tilde{C} \cdot \tilde{x}(t) + \tilde{D}\, u(t) \end{aligned}$, with $\tilde{A} = T^{-1}AT$, $\tilde{B} = T^{-1}B, \tilde{C} = CT$
- We have $T = V$ and $\tilde{A} = \Lambda, \tilde{B} = V^{-1}B, \tilde{C} = CV, x = V\tilde{x} \rightarrow \tilde{x} = V^{-1}x$
  - with $V = [v_1, v_2, \dots, v_n]$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$
- For the output we have:
  - $y(t) = \tilde{C} \cdot e^{\tilde{A}t} \cdot \tilde{x}_0 + \tilde{C} \cdot \int_0^t e^{\tilde{A}(t-\tau)} \tilde{B} u(t) d\tau + D \cdot u(t)$
  - $y(t) = C \cdot V \cdot e^{\Lambda t} \cdot V^{-1} x_0 + C \cdot V \cdot \int_0^t e^{\Lambda(t-\tau)} V^{-1} B u(t) d\tau + D \cdot u(t)$

# Similarity Transform

Example - Train Driving up a Hill ($v_{wind} = v_{wind}(t), \alpha = const$)




- Recall the system with $u(t) = u_0 = const., x_0 = [0, 0]^T$:
  - $\frac{d}{dt}x(t) = \begin{bmatrix} 0 & 1 \\ 0 & -L \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix}u(t)$
  - $y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0\, u(t)$
- Get eigenvalues and Eigenvectors of A:
  - $V = \begin{bmatrix} 1 & -\frac{1}{L} \\ 0 & 1 \end{bmatrix}, \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & -L \end{bmatrix}, V^{-1} = \begin{bmatrix} 1 & \frac{1}{L} \\ 0 & 1 \end{bmatrix}$
  - $y(t) = C \cdot V \cdot e^{\Lambda t} \cdot V^{-1}x_0 + C \cdot V \cdot \int_0^t e^{\Lambda(t-\tau)}V^{-1}Bu(t)d\tau + D \cdot u(t)$
  - $y(t) = 0 + \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{L} \\ 0 & 1 \end{bmatrix}\int_0^t \begin{bmatrix} 1 & 0 \\ 0 & e^{-L(t-\tau)} \end{bmatrix} d\tau \begin{bmatrix} 1 & \frac{1}{L} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ b \end{bmatrix}u_0$
  - $y(t) = 0 + \begin{bmatrix} 1 & -\frac{1}{L} \end{bmatrix}\begin{bmatrix} t & 0 \\ 0 & \frac{1-e^{-Lt}}{L} \end{bmatrix}\begin{bmatrix} \frac{1}{L}b \\ b \end{bmatrix}u_0 = \frac{b}{L^2}\left(e^{-Lt} - 1 + Lt\right)$
  - We see same as before!!

# Similarity Transform

## Connection to Stability

- We had the Eigenvalue Transform (Eigendecomposition):
  - $\widetilde{\mathrm{A}} = \Lambda, \widetilde{\mathrm{B}} = \mathrm{V}^{-1}B, \tilde{C} = CV, x = V\tilde{x} \rightarrow \tilde{x} = V^{-1}x$
  - with $\mathrm{V} = [\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_\mathrm{n}]$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_\mathrm{n})$
  - $\mathrm{y(t)} = \mathrm{C} \cdot \mathrm{V} \cdot \mathrm{e}^{\Lambda \mathrm{t}} \cdot V^{-1} x_0 + \mathrm{C} \cdot V \cdot \int_0^\mathrm{t} \mathrm{e}^{\Lambda(\mathrm{t}-\tau)} \mathrm{V}^{-1} \mathrm{Bu(t)d\tau} + \mathrm{D} \cdot \mathrm{u(t)}$
- For a system with no input (natural response):
  - If only real Eigenvalues:
    - $\mathrm{e}^{\Lambda \mathrm{t}} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$
    - some sort of exponential decay/growth
  - In the case of imaginary Eigenvalues/Eigenvectors: $\lambda = \sigma + j\omega$
    - $\mathrm{e}^{\Lambda \mathrm{t}} = \begin{bmatrix} e^{(\sigma + j\omega)t} & 0 \\ 0 & e^{(\sigma + j\omega)t} \end{bmatrix} = \begin{bmatrix} e^{\sigma t} \sin(\omega t + \phi) & 0 \\ 0 & e^{\sigma t} \sin(-\omega t + \phi) \end{bmatrix}$
    - some sort of exponential decay/growth with oscillation
  - This gives us a hint for Stability
  - Keep in mind this may be some “unreal” state but the qualitative behavior is the same

# Stability

## Definition

- Lyapunov:
  - Lyapunov Stable:
    - For no input and a bounded initial condition the system remains bounded
    - $\|x_0\| < \epsilon, \text{and } u = 0 \rightarrow \|x(t)\| < \delta, \quad \forall t \geq 0$
  - Lyapunov Asymoptotically Stable:
    - For a bounded initial condition and no input the system converges to 0
    - $\|x_0\| < \epsilon, \text{and } u = 0 \rightarrow \lim_{t\to\infty} \|x(t)\| = 0$
  - Lyapunov Unstable: If not stable or asymptotically stable
- BIBO:
  - BIBO (Bounded Input, Bounded Output):
    - For a zero initial condition and bounded input the systems ouput remains bounded
    - $\|u(t)\| < \epsilon \quad \forall t \geq 0, \text{and } x_0 = 0 \rightarrow \|y(t)\| < \delta, \quad \forall t \geq 0$

# Stability

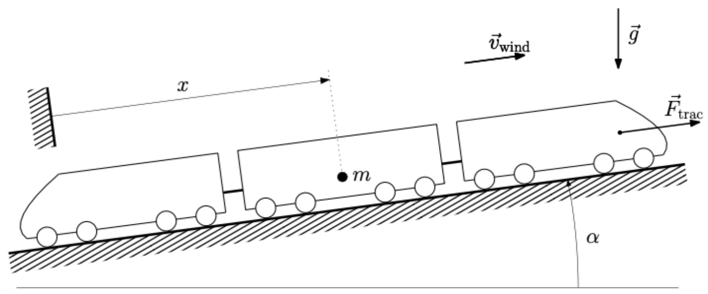
## How?

- We saw that the Eigenvalues tell us a lot about the system!!!
- 1. Calcualte the Eigenvalues of the A matrix. $\det(A - I\lambda) = 0$
- 2. Lyapunov:
  - Lyapunov Stable:
    - A system is Lyapunov stable if $\mathrm{Re}(\lambda_i) \leq 0, \forall \lambda_i$
    - All Eigenvalues of A have real-part less or equal to zero
  - Lyapunov Asymptotically Stable
    - A system is Lyapunov asymptotically stable if $\mathrm{Re}(\lambda_i) < 0, \forall \lambda_i$
    - All Eigenvalues of A have real-part less than zero
  - Lyapunov Unstable
    - A system is Lyapunov unstable stable if $\mathrm{Re}(\lambda_i) > 0,\ \text{for any}\ \lambda_i$
- 3. BIBO:
  - BIBO Stable:
    - A minimal LTI system is BIBO stable if it is Lyapunov Asymptotically Stable

# Stability

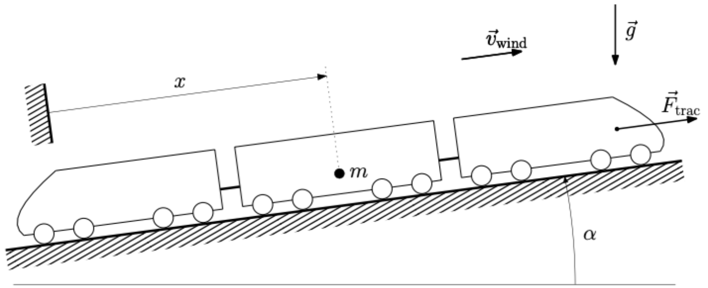
## Linear and Non-Linear systems

- This notion of stability only holds for the linear system!!!
- For the non-linear system we have:
  - If Lyapunov unstable -> Non-linear system is unstable
  - If Lyapunov asymptotically stable -> Non-linear system is stable
  - If Lyapunov system is stable -> nothing can be said (non-linear systems theory is needed)
- For the train example:
  - $A = \begin{bmatrix} 0 & 1 \\ 0 & -L \end{bmatrix}$
  - $\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & -L \end{bmatrix} \rightarrow \lambda_1 = 0, \lambda_2 = -L$
  - System is Lyapunov stable
  - Non-linear system:
    - We can’t know, form the linear system itself.
    - Depends on $\alpha, v_{wind}$ etc.



# Stability

## Example – Old Exam question




- We have a matrix A:
  - $A = \begin{bmatrix} 1 & -2 \\ \alpha & -3 \end{bmatrix}$
  - For what $\alpha$ is the system asymptotically stable?

$$\begin{aligned} \det(\lambda I - A) &= \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ \alpha & -3 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} \lambda - 1 & 2 \\ -\alpha & \lambda + 3 \end{bmatrix}\right) \\ &= (\lambda - 1)(\lambda + 3) + 2\alpha \\ &= \lambda^2 + 2\lambda + (2\alpha - 3) \overset{!}{=} 0 \end{aligned}$$

  - $\lambda_{1,2} = \frac{\left(-2 \pm \sqrt{4 - 4(2\alpha - 3)}\right)}{2} = \frac{\left(-2 \pm \sqrt{4 - 8\alpha + 12}\right)}{2} = \frac{\left(-2 \pm \sqrt{16 - 8\alpha}\right)}{2}$
  - $\mathrm{Re}(\lambda_1) < 0$ for any $\alpha$
  - $\mathrm{Re}(\lambda_2) < 0$:
    - $-2 + \sqrt{16 - 8\alpha} < 0 \rightarrow 16 - 8\alpha < 4 \rightarrow 12 < 8\,\alpha$
  - $\mathrm{Re}(\lambda_2) < 0$: for $\alpha > 1.5$