Control Systems I
Recitation 05

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Last Week

Time Response

$$\frac{d}{dt}x(t) = A \cdot x(t) + B \cdot u(t)$$
$$y(t) = C \cdot x(t) + D \cdot u(t)$$

- From the derivation we found:
  - $\mathbf{y(t) = C \cdot e^{At} \cdot x_0 + C \cdot \int_0^t e^{A(t-\tau)} Bu(t) d\tau + D \cdot u(t)}$
  - Initial Condition Response:
    - $y_{IC}(t) = C \cdot e^{At} \cdot x_0$
    - Describes how the system behaves naturally
  - Force Response:
    - $y_F(t) = C \cdot \int_0^t e^{A(t-\tau)} Bu(t) d\tau$
    - How the system reacts to the input
  - Feedthrough:
    - $y_{FF}(t) = D \cdot u(t)$
    - Direct effect of the input (usually 0)

Last Week

Time Response

$$\frac{d}{dt}x(t) = A \cdot x(t) + B \cdot u(t)$$
$$y(t) = C \cdot x(t) + D \cdot u(t)$$

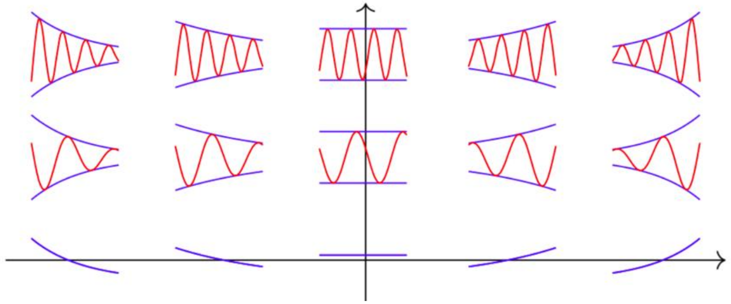
- From the derivation we found:
  - $\mathbf{y(t) = C \cdot e^{\mathit{At}} \cdot x_0 + C \cdot \int_0^t e^{A(t-\tau)} Bu(t) d\tau + D \cdot u(t)}$
- Matrix Exponential:
  - $\mathrm{e}^{\mathrm{At}} = \sum_{\mathrm{k}=0}^{\infty} \frac{(\mathrm{At})^{\mathrm{k}}}{\mathrm{k}!} = \mathbb{I} + \mathrm{At} + \frac{1}{2}(\mathrm{At})^2 + \cdots + \frac{1}{\mathrm{k}!}(\mathrm{At})^{\mathrm{k}} + \cdots$
  - $\mathrm{e}^{\mathrm{At}} = \begin{bmatrix} e^{A_{11}t} & 0 \\ 0 & e^{A_{22}t} \end{bmatrix}$, if A is diagonal
  - Jordan Form:
    - $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \rightarrow e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} e^{\lambda t}$
    - $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \rightarrow e^{At} = \begin{bmatrix} 1 & t & \frac{1}{2!}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} e^{\lambda t}$
- Keep in mind in practice we never calculate this by hand

Last Week

Similarity Transform

- Define an invertible matrix T $\left(x(t) = T \cdot \tilde{x}(t) \rightarrow \tilde{x}(t) = T^{-1}x(t)\right)$:
  - This results in the system:
  - $\begin{matrix} \frac{d}{dt}\tilde{x}(t) = \tilde{A} \cdot \tilde{x}(t) + \tilde{B} \cdot u(t) \\ y(t) = \tilde{C} \cdot \tilde{x}(t) + \tilde{D}\, u(t) \end{matrix}, with\ \tilde{A} = T^{-1}AT,\ \tilde{B} = T^{-1}B, \tilde{C} = CT$
- **The system stays the same!**
- Most often we use the Eigenvalue Transform:
  - $T = V$ and $\tilde{A} = \Lambda,\ \ \tilde{B} = V^{-1}B,\ \tilde{C} = CV,\ x = V\tilde{x} \rightarrow \tilde{x} = V^{-1}x$
  - with $V = [v_1, v_2, \dots, v_n]$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$
  - v_i and λ_i being the i'th Eigenvector and Eigenvalue.
- For the output we have:
  - $y(t) = \tilde{C} \cdot e^{\tilde{A}t} \cdot \tilde{x}_0 + \tilde{C} \cdot \int_0^t e^{\tilde{A}(t-\tau)}\tilde{B}u(t)d\tau + D \cdot u(t)$
  - $y(t) = C \cdot V \cdot e^{\Lambda t} \cdot V^{-1}x_0 + C \cdot V \cdot \int_0^t e^{\Lambda(t-\tau)}V^{-1}Bu(t)d\tau + D \cdot u(t)$

Last Week



Stability

- 1. Calcualte the Eigenvalues of the A matrix. $\det(A - I\lambda) = 0$
- 2. Lyapunov:
  - Lyapunov Stable: $\|x_0\| < \epsilon, \text{and } u = 0 \rightarrow \|x(t)\| < \delta, \quad \forall t \geq 0$
    - A system is Lyapunov stable if $\mathrm{Re}(\lambda_i) \leq 0, \forall \lambda_i$
    - All Eigenvalues of A have real-part less or equal to zero
  - Lyapunov Asymptotically Stable: $\|x_0\| < \epsilon, \text{and } u = 0 \rightarrow \lim_{t \rightarrow \infty} \|x(t)\| = 0$
    - A system is Lyapunov asymptotically stable if $\mathrm{Re}(\lambda_i) < 0, \forall \lambda_i$
    - All Eigenvalues of A have real-part less than zero
  - Lyapunov Unstable (Neither Stable nor asymptotically stable)
    - A system is Lyapunov unstable stable if $\mathrm{Re}(\lambda_i) > 0, \text{ for any } \lambda_i$
- 3. BIBO:
  - BIBO Stable: $\|u(t)\| < \epsilon \quad \forall t \geq 0, \text{and } x_0 = 0 \ \rightarrow \|y(t)\| < \delta, \quad \forall t \geq 0$
    - A minimal LTI system is BIBO stable if it is Lyapunov Asymptotically Stable

Last Week

Stability

- This notion of stability only holds for the linear system!!!
- For the non-linear system we have:
  - If Lyapunov unstable -> Non-linear system is unstable
  - If Lyapunov asymptotically stable -> Non-linear system is stable
  - If Lyapunov system is stable -> nothing can be said (non-linear systems theory is needed)

Outline

- Time Response – Exponential Functions
  - What? Why? How?
  - Connection to the Transfer Function

- Transfer Function
  - What? Why?
  - Properties of the Transfer Function
  - Transfer Function -> State Space
  - Example

- Laplace Transform
  - What? Why?
  - What the heeeeeeell?
  - Laplace<->Transfer Function

Time Response – Exponential Functions

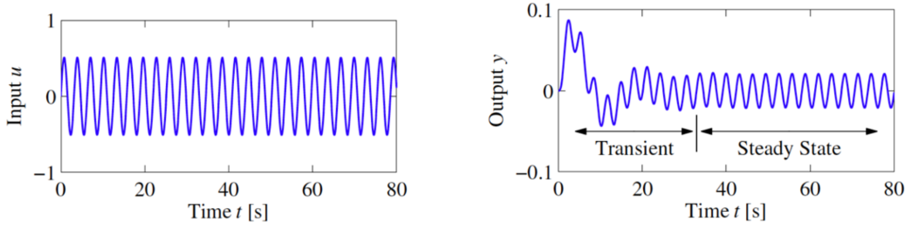
What? Why?

- What?
  - Solution to $y(t) = C \cdot e^{At} \cdot x_0 + C \cdot \int_0^t e^{A(t-\tau)} Bu(t) d\tau + D \cdot u(t)$:
  - For zero input we saw what happens (Eigenvalues -> Stability analysis)
  - For general inputs?
    - We use exponential inputs $u(t) = e^{st}, s = \sigma + j \cdot \omega \in \mathbb{C}$
      - Real number: $u(t) = e^{\sigma t}$
      - Imaginary: $u(t) = e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$
      - Complex: $u(t) = e^{\sigma t} e^{j\omega t} + e^{\sigma t} e^{-j\omega t} = 2e^{\sigma t}\cos(\omega t)$
- Why?
  - Output can be computed easily
  - Any function can be expressed as a sum of exponentials -> *Laplace Transform*

Time Response – Exponential Functions

How?

- Solution to $y(t) = C \cdot e^{At} \cdot x_0 + C \cdot \int_0^t e^{A(t-\tau)} Bu(t) d\tau + D \cdot u(t),\ u(t) = e^{st}$:
  - $y(t) = C \cdot e^{At} \cdot x_0 + C \cdot \int_0^t e^{A(t-\tau)} Be^{s\tau} d\tau + D \cdot e^{st}$
  - $y(t) = C \cdot e^{At} \cdot x_0 + Ce^{At} \int_0^t e^{(sI-A)\tau} Bd\tau + D \cdot e^{st}$
  - $y(t) = C \cdot e^{At} \cdot x_0 + Ce^{At}(sI - A)^{-1}\left(e^{(sI-A)t} - I\right) + D \cdot e^{st}$
  - $\mathbf{y(t) = Ce^{At}\left[x_0 - (sI - A)^{-1}B\right] + \left[C(sI - A)^{-1}B + D\right]e^{st}}$



- The first part $Ce^{At}[x_0 - (sI - A)^{-1}B]$ converges to 0 if the system is asymptotical stable
- Steady State Output (what we want) is given by:
  - $y(t) = G(s)e^{st}, s \in \mathbb{C}$
  - $G(s)$ is the *Transfer Function*

- *Note: If the system is not stable then the output will diverge, and the steady state discussion will become obsolete (there is not steady state). However, the mathematical validity of the Transfer Function and its system describing properties remains valid!*

Transfer Function

What? Why?

- What?
  - Relates the steady state output of the system as function of the exponential input signal:
  - $y(t) = G(s)e^{st}, s \in \mathbb{C}$
  - $\mathrm{G(s) = C(sI - A)^{-1}B + D = C\frac{adj(sI-A)}{det(sI-A)}B + D}$
  - Matrix inverse:
    - $M^{-1} = \frac{adj(M)}{\det(M)}$,
    - 2x2: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, 3x3:

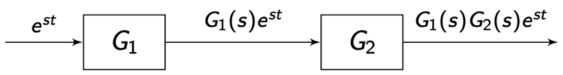
$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{\begin{bmatrix} ei-fh & hc-ib & bf-ce \\ gf-di & ai-gc & dc-af \\ dh-ge & gb-ah & ae-db \end{bmatrix}}{|\mathbf{A}|} = \frac{\begin{bmatrix} ei-fh & hc-ib & bf-ce \\ gf-di & ai-gc & dc-af \\ dh-ge & gb-ah & ae-db \end{bmatrix}}{aei+bfg+cdh-gec-hfa-idb}$$

- Why?
  - Systems can be viewed as Multiplication (see Block Diagrams)
  - Basics for System Analysis Tools

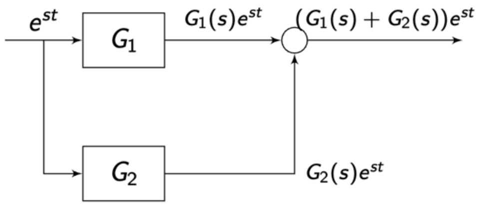
▶ Serial interconnection:

$$G(s) = G_2(s)G_1(s).$$



▶ Parallel interconnection:

$$G(s) = G_1(s) + G_2(s).$$



Transfer Function

Properties of the Transfer Function

- Transfer Function in a General case:
  - $$G(s) = C(sI - A)^{-1}B + D = C\frac{\mathrm{adj}(sI-A)}{\det(sI-A)}B + D = \frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\dots+b_0}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\dots+a_0} + d$$
- Properties:
  - **Transfer Function allows us to determine BIBO stability:**
    - Denominator of $G(s)$ is the characteristic polynomial of A $\det(sI - A)$ (Eigenvalue calculation)
    - I.e. the **poles** of $G(s)$ (roots of $\det(sI - A)$) are the eigenvalues of the system
    - Main result: Systems with poles on the imaginary axis are not BIBO stable
  - **Minimal Realization:**
    - When constructing the Transfer Function unnecessary states will be removed
    - Can use the transfer function to derive the minimal realization of the system

Transfer Function

State Space -> Transfer Function

- General case:
  - $G(s) = C(sI - A)^{-1}B + D = C\frac{\mathrm{adj}(sI-A)}{\det(sI-A)}B + D$
  - $G(s) = \frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\cdots+b_0}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\cdots+a_0} + d$

- If A is diagonal with eigenvalues λ_i:
  - $A = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$
  - $G(s) = \frac{c_1 b_1}{s-\lambda_1} + \frac{c_2 b_2}{s-\lambda_2} + \cdots + \frac{c_n b_n}{s-\lambda_n} + d$

Transfer Function

State Space <- Transfer Function

- If in partial fraction form:
  - $G(s) = \frac{p_1}{s-\lambda_1} + \frac{p_2}{s-\lambda_2} + \cdots + \frac{p_n}{s-\lambda_n} + d$
  - $A = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}, B = \begin{bmatrix} \sqrt{p_1} \\ \sqrt{p_2} \\ \cdots \\ \sqrt{p_n} \end{bmatrix}, C = [\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n}], D = d$
- General case: (Controllable Canonical Form)
  - $G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0} + d$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & \ddots & & 1 \\ -a_0 & -a_1 & \ldots & & & -a_{n-1} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & \ldots & b_{n-1} \end{bmatrix}, \qquad D = [d];$$

Transfer Function

Example

- Given the system:
  - $A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$
- Calculate the Transfer Function:
  - $G(s) = C(sI - A)^{-1}B$
  - $(sI - A) = \begin{bmatrix} s+2 & 0 \\ 0 & s-1 \end{bmatrix}, (sI - A)^{-1} = \frac{1}{(s+2)(s-1)} \begin{bmatrix} s-1 & 0 \\ 0 & s+2 \end{bmatrix}$
  - $G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{(s+2)(s-1)} \begin{bmatrix} s-1 & 0 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{2(s-1)}{(s+2)s-1)}$
  - But wait we can reduce the system (this was not the minimal realisation):
  - $G(s) = \frac{2}{(s+2)}$
  - This also makes intuitive sense when we look at the state space (for larger systems or in different form this is not the case)

Laplace Transform

What? Why?

- Why?
  - We said every signal can be computed as a sum of exponential inputs:
    - $u(t) = \sum_i U_i e^{s_i t}$, because of linearity $y(t) = \sum_i G(s_i) U_i \, e^{s_i t}$
  - This is not entirely true (we need an integral for this to be true for every function):
    - $u(t) = K \int U(s) e^{st} ds \ \rightarrow y(t) = K \int G(s) U(s) e^{st} ds$
    - We use the Laplace Transform to define the $U(s)$
- What?
  - Given a function $u(t)$ the Laplace Transform $U(s)$ is defined by:
    - $\mathcal{L}\{u(t)\} = U(s) = \int_0^{\infty} u(t) e^{-st} dt \, , s \in \mathbb{C}$
  - The inverse Laplace Transform is defined by:
    - $\mathcal{L}^{-1}\{U(s)\} = u(t) = \frac{1}{2\pi j} \lim_{\omega \rightarrow \infty} \int_{\sigma - j\omega}^{\sigma + j\omega} U(s) e^{st} ds$
  - Inverse Laplace Transform allows us to write every signal as an infinite sum of exponentials

Laplace Transform

Laws

- We usually don't calculate this by hand but use Laplace Tables

Linearität	:	$\mathcal{L}\{a \cdot x_1(t) + b \cdot x_2(t)\} = a \cdot X_1(s) + b \cdot X_2(s)$	
Ähnlichkeit	:	$\mathcal{L}\{\frac{1}{a} \cdot x(\frac{t}{a})\} = X(s \cdot a)$	
Verschiebung	:	$\mathcal{L}\{x(t-T)\} = e^{-T \cdot s} \cdot X(s)$	
Dämpfung	:	$\mathcal{L}\{x(t) \cdot e^{a \cdot t}\} = X(s-a)$	
Ableitung t	:	$\mathcal{L}\{\frac{\mathrm{d}}{\mathrm{d}t}x(t)\} = s \cdot X(s) - x(0)$	(1)
n-te Abl. t	:	$\mathcal{L}\{\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n}\} = s^n \cdot X(s) \ \left(\frac{\mathrm{d}^k x(t=0)}{\mathrm{d}t^k} = 0 \ \forall k\right)$	(2)
Ableitung s	:	$\mathcal{L}\{t \cdot x(t)\} = -\frac{\mathrm{d}}{\mathrm{d}s}X(s)$	
Integration t	:	$\mathcal{L}\{\int_0^t x(\tau)\mathrm{d}\tau\} = \frac{1}{s} \cdot X(s)$	
Integration s	:	$\mathcal{L}\{\frac{1}{t} \cdot x(t)\} = \int_s^\infty X(\sigma)\mathrm{d}\sigma$	
Faltung t	:	$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s) \cdot X_2(s)^1$	
Faltung s	:	$\mathcal{L}\{x_1(t) \cdot x_2(t)\} = X_1(s) * X_2(s)$	
Anfangswert	:	$\lim_{t \to 0_+} x(t) = \lim_{s \to \infty} s \cdot X(s)$	(3)
Endwert	:	$\lim_{t \to \infty} x(t) = \lim_{s \to 0_+} s \cdot X(s)$	(4)

$x(t)$	$X(s)$
$\delta(t)$	1
$h(t)$	$\frac{1}{s}$
$h(t) \cdot t^n \cdot e^{\alpha \cdot t}$	$\frac{n!}{(s-\alpha)^{n+1}}$
$h(t) \cdot \sin(\omega \cdot t)$	$\frac{\omega}{s^2+\omega^2}$
$h(t) \cdot \cos(\omega \cdot t)$	$\frac{s}{s^2+\omega^2}$
$h(t) \cdot \sinh(\omega \cdot t)$	$\frac{\omega}{s^2-\omega^2}$
$h(t) \cdot \cosh(\omega \cdot t)$	$\frac{s}{s^2-\omega^2}$

Laplace Transform

What the heeeeeell?

- The Laplace Transform, lifts the system from the time domain into the frequency domain:
- The time domain "describes how a signal is composed of sinusoidal with exponential decay"
  - Extension of the Fourier Series/Transform
- Check this video: https://www.youtube.com/watch?v=n2y7n6jw5d0&ab_channel=ZachStar

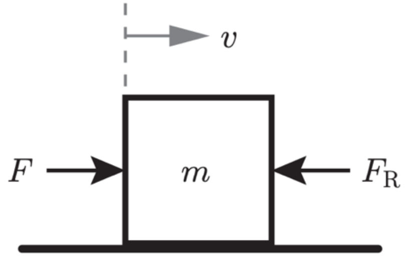
Laplace Transform

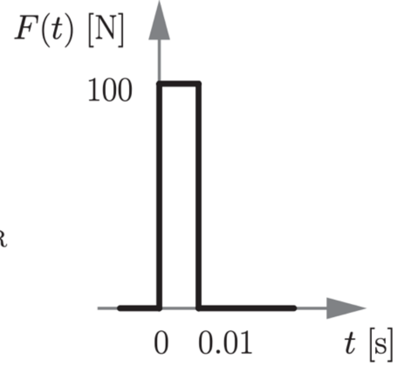
Laplace<->Transfer Function

- You don’t really need to understand the Laplace Transform and Frequency Domain, but you need to need to use the math of it.
- From Analysis III remember: $\mathcal{L}\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s)$
  - $\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{x(t)} = \mathrm{Ax(t)} + \mathrm{Bu(t)}$
  - $\mathrm{y(t)} = \mathrm{Cx(t)} + \mathrm{Du(t)}$
  - $\mathrm{sX(s)} = \mathrm{AX(s)} + \mathrm{BU(s)} \rightarrow \mathrm{X(s)} = (\mathrm{sI} - \mathrm{A})^{-1} BU(s)$
  - $\mathrm{Y(s)} = \mathrm{CX(s)} + \mathrm{DU(s)}$
  - $\boldsymbol{Y(s) = C(\mathrm{s}\mathbf{I} - A)^{-1} BU(s) + DU(s) = [(\mathrm{s}\mathbf{I} - A)^{-1}B + D]\, U(s)}$
  - **We again get the Transfer Function!**
    - **Laplace Transform of the State Space Results in the Transfer Function**
- We can compute the output as: (using Tables)
  - $y_{ss}(t) = \mathcal{L}^{-1}\{\mathrm{Y(s)}\}$

Laplace Transform

Example (Old exam question)



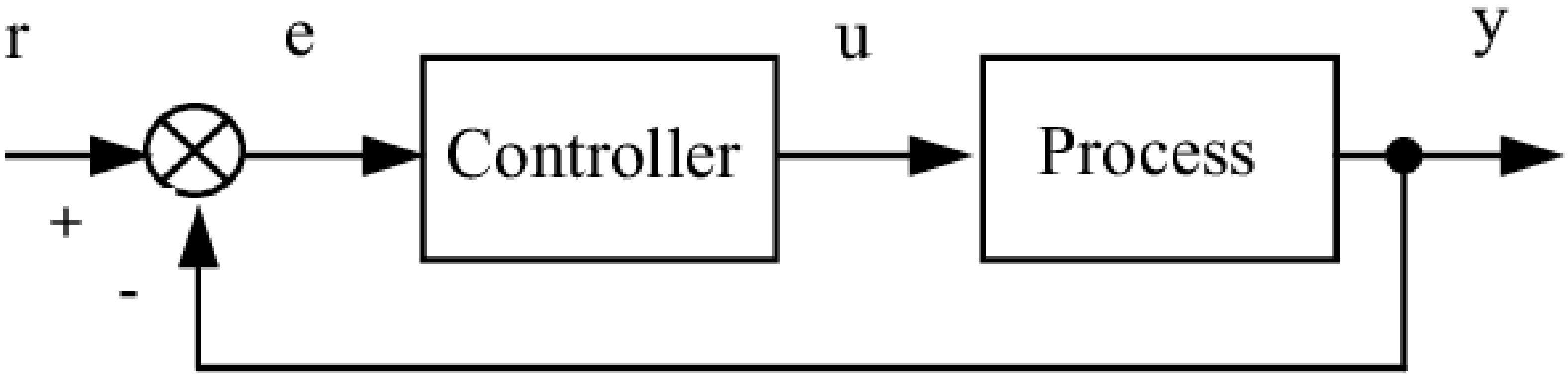


- We have:
  - $\mathrm{m}\frac{dv}{dt} + Bv = F(t), F(t) = \delta(t)(dirac\ delta = impulse), B = 20, m = 10$
- Calculate how far the block moves (steady state output of the position):
  - Augment the system to incorporate the position: $\frac{dv}{dt} = -\frac{B}{m}v + \frac{1}{m}F(t), \frac{dx}{dt} = v$
    - $A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{10} \end{bmatrix}, C = [1, 0], D = 0$
  - Calculate the Laplace Transform: $Y(s) = G(s)U(s) = \mathrm{C(sI - A)^{-1}BU(s)}$
    - $\mathrm{G(s)} = \mathrm{C(sI - A)}^{-1}B = [1, 0]\begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} = [1, 0]\frac{\begin{bmatrix} s+2 & 1 \\ 0 & s \end{bmatrix}^{-1}}{s(s+2)}\begin{bmatrix} 0 \\ \frac{1}{10} \end{bmatrix} = \cdots = \frac{1}{10}\frac{1}{s(s+2)}$
    - $U(s) = 1$ (see Table)
    - $Y(s) = \frac{1}{10}\frac{1}{s(s+2)}$
  - Calculate the steady state output
    - $y_\infty(t \to \infty) = \lim_{s \to 0} sY(s) = s\frac{1}{10}\frac{1}{s(s+2)} = \frac{1}{10}\frac{1}{2} = 0.05m$

Dynamic Compensator

Main Idea

- When designing a Controller, we basically come up with another Transfer Function:
  - $G(s) = \frac{c_{n-1}s^{n-1}+c_{n-2}s^{n-2}+\cdots+c_0}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\cdots+a_0} + k$
  - But here the input is the error, and the output is the control input?



- Will be relevant later on, so don’t worry about it now!

Summary

What is actually relevant!

- Transfer Function describes the steady state output to a exponential input:
  - $y_{ss}(\mathrm{t}) = [\mathrm{C}(\mathrm{sI}-\mathrm{A})^{-1}\mathrm{B}+\mathrm{D}]\mathrm{e}^{\mathrm{st}} = G(s)e^{st}$
  - **Transfer Function allows us to determine BIBO stability:**
    - The **poles** of $\mathrm{G}(\mathrm{s})$ (roots of $\det(\mathrm{sI}-\mathrm{A})$) are the eigenvalues of the system
  - **Minimal Realization:**
    - When constructing the Transfer Function unnecessary states will be removed
  - **Know how to Convert between State Space and Transfer Functions!**
- Laplace Transform helps us to compute the steady state output for exponential inputs:
  - $\mathcal{L}\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s)$
  - $Y(s) = C(sI-A)^{-1}BU(s) + DU(s) = [(sI-A)^{-1}B + D]\,U(s)$
  - $y_{ss}(t) = \mathcal{L}^{-1}\{\mathrm{Y}(\mathrm{s})\}$ (use tables for this)
- For now, ignore Dynamic Compensator