Control Systems I
Recitation 06

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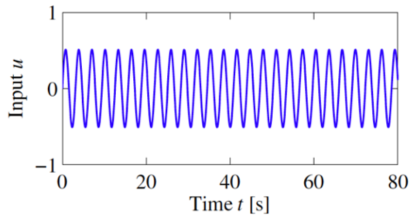
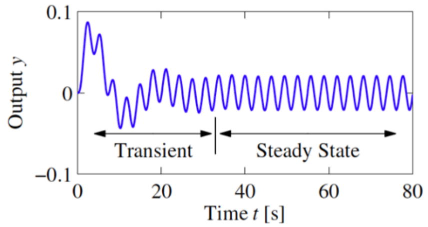
Last Week





Time Response for Exponential Inputs -> Transfer Function

- Solution to $y(t) = C \cdot e^{At} \cdot x_0 + C \cdot \int_0^t e^{A(t-\tau)} Bu(t) d\tau + D \cdot u(t),\;\; u(t) = e^{st}$:
  - $\mathbf{y(t) = Ce^{At}\left[x_0 - (sI - A)^{-1}B\right] + \left[C(sI - A)^{-1}B + D\right]e^{st}}$
- The first part $Ce^{At}[x_0 - (sI - A)^{-1}B]$ converges to 0 if the system is asymptotical stable
- Steady State Output (what we want) is given by:
  - $y(t) = G(s)e^{st}, s \in \mathbb{C}$
  - $G(s)$ is the **Transfer Function**
  - $G(s) = C(sI - A)^{-1}B + D = C\frac{\operatorname{adj}(sI-A)}{\det(sI-A)}B + D$
  - Matrix inverse:
    - $M^{-1} = \frac{\operatorname{adj}(M)}{\det(M)}$,
    - 2x2: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, 3x3:

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{\begin{bmatrix} ei-fh & hc-ib & bf-ce \\ gf-di & ai-gc & dc-af \\ dh-ge & gb-ah & ae-db \end{bmatrix}}{|\mathbf{A}|} = \frac{\begin{bmatrix} ei-fh & hc-ib & bf-ce \\ gf-di & ai-gc & dc-af \\ dh-ge & gb-ah & ae-db \end{bmatrix}}{aei+bfg+cdh-gec-hfa-idb}$$

Last Week

Transfer Function

- Transfer Function in a General case:
  - $G(s) = C(sI - A)^{-1}B + D = C\frac{\mathrm{adj}(sI-A)}{\det(sI-A)}B + D = \frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\cdots+b_0}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\cdots+a_0} + d$
- Properties:
  - **Transfer Function allows us to determine BIBO stability:**
    - Denominator of $G(s)$ is the characteristic polynomial of A $\det(sI - A)$ (Eigenvalue calculation)
    - I.e. the **poles** of $G(s)$ (roots of $\det(sI - A)$) are the eigenvalues of the system
    - Main result: Systems with poles on the imaginary axis are not BIBO stable
  - **Minimal Realization:**
    - When constructing the Transfer Function unnecessary states will be removed
    - Can use the transfer function to derive the minimal realization of the system
  - More on this today

Last Week

State Space <-> Transfer Function

- If in partial fraction form:
- $G(s) = \frac{p_1}{s-\lambda_1} + \frac{p_2}{s-\lambda_2} + \cdots + \frac{p_n}{s-\lambda_n} + d$
- $A = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}, B = \begin{bmatrix} \sqrt{p_1} \\ \sqrt{p_2} \\ \cdots \\ \sqrt{p_n} \end{bmatrix}, C = \left[\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n}\right], D = d$
- General case: (Controllable Canonical Form)
- $G(s) = \frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\cdots+b_0}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\cdots+a_0} + d$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & \ddots & & 1 \\ -a_0 & -a_1 & \ldots & & & -a_{n-1} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & \ldots & b_{n-1} \end{bmatrix}, \qquad D = [d];$$

- If A is diagonal with eigenvalues λ_i:
  - $A = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$
  - $G(s) = \frac{c_1 b_1}{s-\lambda_1} + \frac{c_2 b_2}{s-\lambda_2} + \cdots + \frac{c_n b_n}{s-\lambda_n} + d$
- General case:
  - $G(s) = C(sI - A)^{-1}B + D = C\frac{\mathrm{adj}(sI-A)}{\det(sI-A)}B + D$
  - $G(s) = \frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\cdots+b_0}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\cdots+a_0} + d$

Last Week

Laplace Transform

- What?
  - Given a function $\mathrm{u(t)}$ the Laplace Transform $\mathrm{U(s)}$ is defined by: (Use Tables)
    - $\mathcal{L}\{\mathrm{u(t)}\} = \mathrm{U(s)} = \int_0^\infty \mathrm{u(t)e^{-st}dt}, \mathrm{s} \in \mathbb{C}$
  - The inverse Laplace Transform is defined by: (Use Tables)
    - $\mathcal{L}^{-1}\{\mathrm{U(s)}\} = \mathrm{u(t)} = \frac{1}{2\pi \mathrm{j}} \lim_{\omega\to\infty} \int_{\sigma-\mathrm{j}\omega}^{\sigma+\mathrm{j}\omega} \mathrm{U(s)e^{st}ds}$
  - Inverse Laplace Transform allows us to write every signal as an infinite sum of exponentials
- Using: $\mathcal{L}\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s)$
  - $\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{x(t)} = \mathrm{Ax(t)} + \mathrm{Bu(t)} \rightarrow \mathrm{sX(s)} = \mathrm{AX(s)} + \mathrm{BU(s)} \rightarrow \mathrm{X(s)} = (\mathrm{sI} - \mathrm{A})^{-1} BU(s)$
  - $\mathrm{y(t)} = \mathrm{Cx(t)} + \mathrm{Du(t)} \rightarrow \mathrm{Y(s)} = \mathrm{CX(s)} + \mathrm{DU(s)}$
  - $\boldsymbol{Y(s) = C(\mathrm{s}\mathbf{I} - A)^{-1}BU(s) + DU(s) = [(\mathrm{s}\mathbf{I} - A)^{-1}B + D]\, U(s)}$
  - **Laplace Transform of the State Space Results in the Transfer Function**
- We can compute the output as: (using Tables)
  - $y_{ss}(t) = \mathcal{L}^{-1}\{\mathrm{Y(s)}\}$

Last Week

What was relevant!

- Transfer Function describes the steady state output to a exponential input:
  - $y_{ss}(t) = [C(sI - A)^{-1}B + D]e^{st} = G(s)e^{st}$
  - **Transfer Function allows us to determine BIBO stability:**
    - The **poles** of $G(s)$ (roots of $\det(sI - A)$) are the eigenvalues of the system
  - **Minimal Realization:**
    - When constructing the Transfer Function unnecessary states will be removed
  - **Know how to Convert between State Space and Transfer Functions!**
- Laplace Transform helps us to compute the steady state output for exponential inputs:
  - $\mathcal{L}\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s)$
  - $Y(s) = C(sI - A)^{-1}BU(s) + DU(s) = [(sI - A)^{-1}B + D]\,U(s)$
  - $y_{ss}(t) = \mathcal{L}^{-1}\{Y(s)\}$ (use tables for this)

Outline

- Transfer Function continued - Different Forms of the Transfer Function
  - Why?

- Steady State Response
  - General Input
  - Sinusoidal Input
  - Step Input
  - Impulse Input

- Transfer Function continued – Effects of Poles and Zeros
  - Example

Transfer Function

Different Forms

- Transfer Function in a general form: (note some of the a_i, b_i values can be zero)
  - $G(s) = \frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\dots+b_0}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\dots+a_0} + d = \frac{N(s)}{D(s)} + d$
- We can rewrite the Transfer Function in different ways (purely mathematical rewriting):
  - Partial Fraction Expansion:
    - $G(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \frac{r_3}{s-p_3} + \cdots + r_0$
  - Root Locus Form:
    - $G(s) = \frac{k_{rl}}{s^q} \frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_{n-q})}$
  - Bode Form:
    - $G(s) = \frac{k_{bd}}{s^q} \frac{\left(\frac{s}{-z_1}+1\right)\left(\frac{s}{-z_2}+1\right)\dots\left(\frac{s}{-z_1}+1\right)}{\left(\frac{s}{-p_1}+1\right)\left(\frac{s}{-p_2}+1\right)\dots\left(\frac{s}{-p_{n-q}}+1\right)}$

- p_i are the poles of the system (can be imaginary)
  - "instabilities" of the system
- r_i are the residuals
  - Describe the system response of each pole
- z_i are the zeros of the transfer function
  - "blind spots" of the system

Transfer Function

Different Forms - Why?

- Rewriting can make the interpretation easier:
  - Direct access to the poles / zeros of a system
  - Direct access to the residuals of the system
  - Direct access to relevant quantities (becomes important in the following weeks)
  - Allows for graphical interpretation of the Transfer Function
- Use computers to do this (MatLab, SymPy, …):
  - See Notebook for a tool
- For the residuals there is a trick (Cover up method):
  - Non-repeating pole:
    - $r_i = \lim_{s \to p_i} (s - p_i) G(s)$
  - Repeating pole (m)
    - $r_i = \frac{1}{(m-1)!} \lim_{s \to p_i} \frac{d^{m-1}}{ds^{m-1}} \left((s - p_i) G(s) \right)$

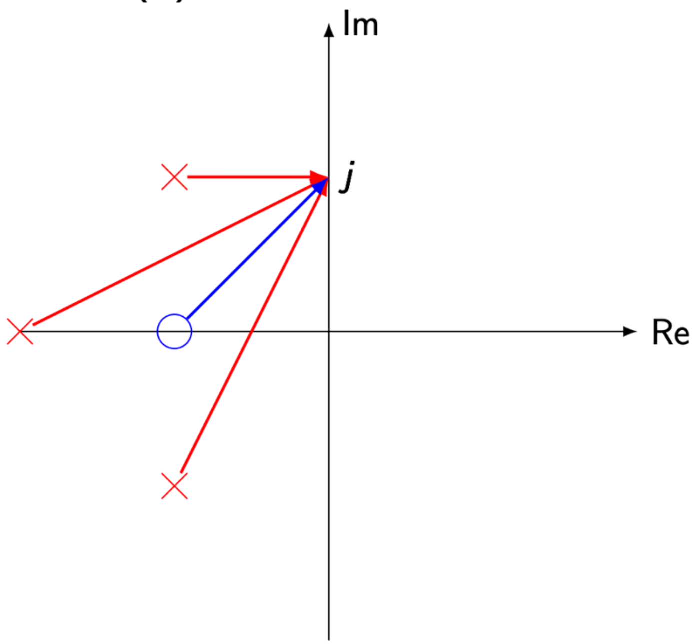
Steady State Response

General Input and Sinusoidal Input

- We already looked at this last week:
  - Given the Transfer Function
    - $Y(s) = C(sI - A)^{-1}BU(s) + DU(s) = [(sI - A)^{-1}B + D]\, U(s)$
  - We can compute the output as: (using Tables)
    - $y_{ss}(t) = \mathcal{L}^{-1}\{Y(s)\}$
    - Usually using:
      - $y_{ss}(t \to \infty) = \lim_{s \to 0} sY(s)$
- For sinusoidal Inputs $u(t) = \sin(\omega t)\,,\ s = j\omega$
  - $y_{ss}(t) = |G(j\omega)| \sin\big(t + \angle G(j\omega)\big)$
  - We can even do this graphically

Steady State Response

Sinusoidal Input - Graphical Calculation

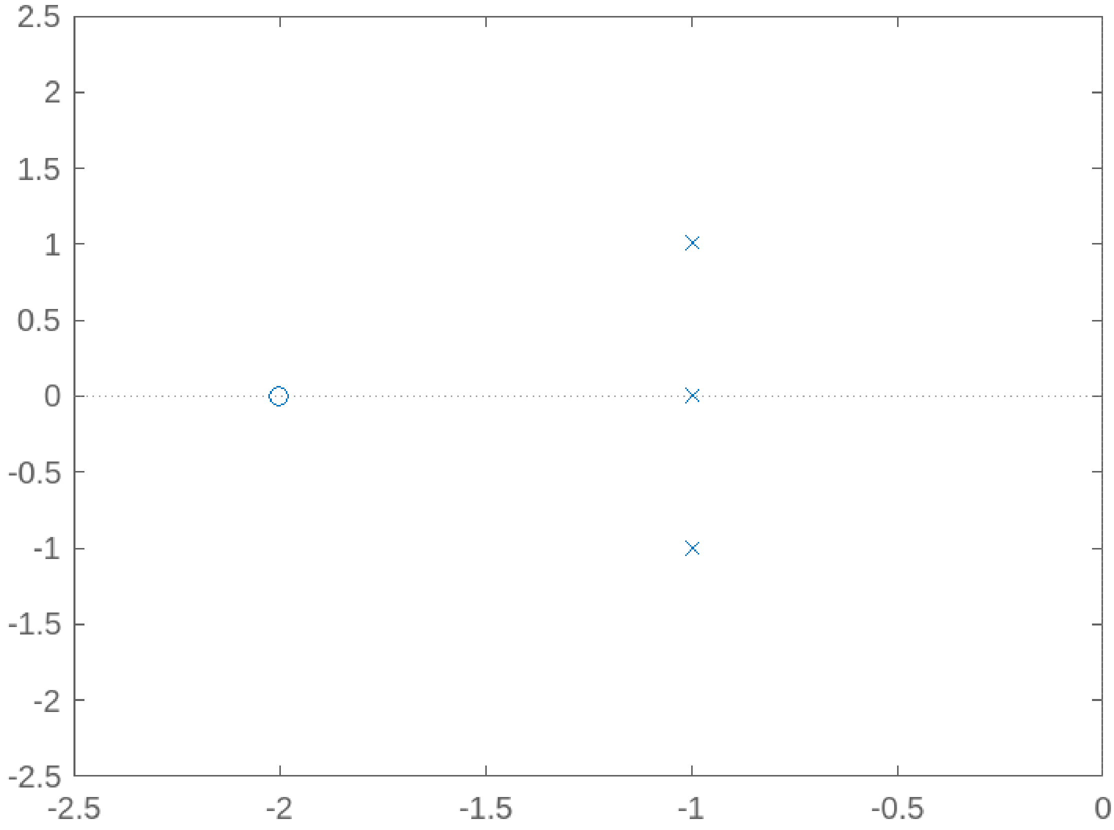


- For sinusoidal Inputs $u(t) = \sin(\omega t)$, $s = j\omega$
  - $y_{ss}(t) = |G(j\omega)| \sin\left(t + \angle G(j\omega)\right)$
  - Having the Root Locus Form $G(s) = \frac{k_{rl}}{s^q} \frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_{n-q})}$ we get:
    - $\rightarrow |G(j\omega)| = \frac{|k_l|}{|\omega^q|} \frac{|(j\omega-z_1)||(j\omega-z_2)|\ldots|(j\omega-z_m)|}{|(j\omega-p_1)||(j\omega-p_2)|\ldots|(j\omega-p_m)|}$
    - Where $|(j\omega - z_1)|$ is the distance from the zero/pole to $j\omega$
    - $\rightarrow \angle G(j\omega) = \angle(j\omega - z_1) + \angle(j\omega - z_2) + \cdots - \angle(j\omega)^q - \angle(j\omega - p_1) - \angle(j\omega - p_2) + \cdots$
    - Where$\angle(j\omega - z_1)$ are the angles of the vector from the zeros/poles to $j\omega$ to the real axis

Steady State Response

Sinusoidal Input - Graphical Calculation - Example

- Given the pole zero plot of $G(s)$
  - Circles are zeros crosses are poles
- Calculate the steady state response for $u(t) = \sin(2t)$
  - $\text{atan}(2) + \text{atan}(3) = 135°$

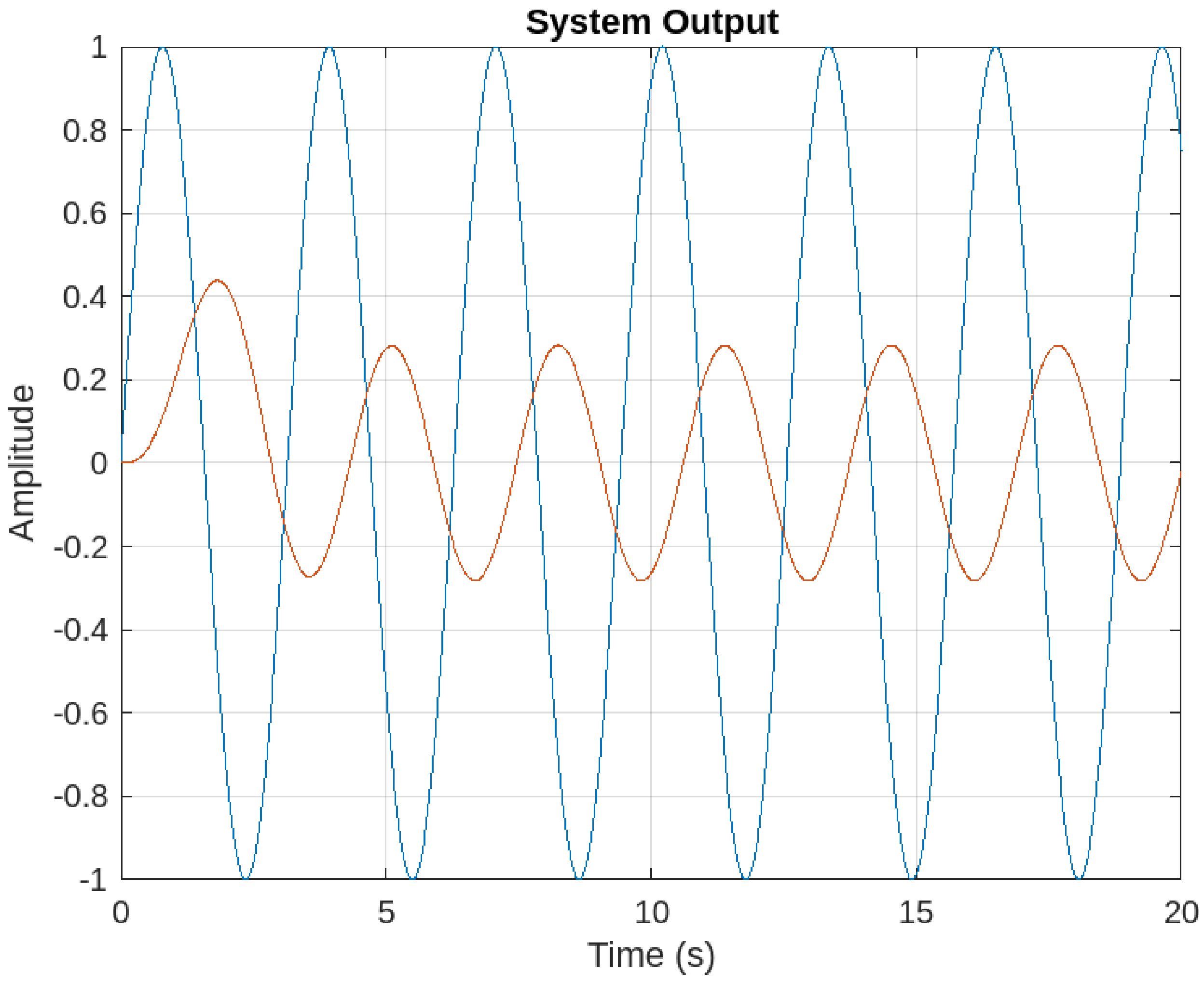


- Solution:
  - We have $s = j2$
  - $y_{ss}(t) = |G(j\omega)| \sin\big(t + \angle G(j\omega)\big)$
  - $|G(j\omega)| = \frac{|(j2+2)|}{|(j2+1)||(j2+1-j)|\ldots|(j2+1+j)|} = \frac{\sqrt{2^2+2^2}}{\sqrt{2^2+1^2}\sqrt{1^1+1^1}\sqrt{1^2+3^2}} = \frac{\sqrt{8}}{\sqrt{5}\sqrt{2}\sqrt{10}} = \sqrt{\frac{8}{2*3*10}} = \frac{2}{\sqrt{50}} \approx 0.2828$
  - $\angle G(j\omega) = \angle(j2+2) - \angle(j2+1) - \angle(j2+1+j) - \angle(j2+1-j)$
  - $\angle G(j\omega) = 45° - \text{atan}(2) - 45° - \text{atan}(3) = -135°$
  - $y_{ss}(t) = 0.2828 \sin(2t - 135°)$

Steady State Response

Sinusoidal Input - Graphical Calculation - Example

- Solution:
  - $|G(j\omega)| = \frac{2}{\sqrt{50}} \approx 0.2828$
  - $\angle G(j\omega) = -135°$



Steady State Response

Step Input

- For step Inputs $u(t) = h(t)$, with $h(t)$ being the Heaviside function (impulse modelling)
  - $h(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$
  - For $D = 0, x(0) = 0, u(t) = h(t)$ the output becomes:
    - $y_{step}(t) = \int_0^t Ce^{A(t-\tau)}B\, d\tau = -CA^{-1}B + CA^{-1}e^{At}B$
  - For stable systems:
    - $y_{ss}(t) = -CA^{-1}B = \text{const}$
  - First order systems (scalar):
    - $y_{step}(t) = y_{ss}(t)(1 - e^{at})$
    - More on this in lecture 8 or 9
  - In general, we need to use Laplace:
    - $y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{G(s)U(s)\}, U(s) = \frac{1}{s}$

Steady State Response

Step Input – Example (Old exam question)

- Given the system $G(s) = \frac{s-4}{(s+3)(s+2)}$ calculate the output $y(t)$ given $u(t) = h(t)$:
- Hint: First calculate $Y(s)$, then the residuals and then do the inverse
- Solution:
  - $Y(s) = G(s)U(s) = \frac{s-4}{(s+3)(s+2)}\frac{1}{s}, U(s) = \frac{1}{s}$
  - Poles and residuals:
    - $s_1 = -3: \quad r_1 = \lim_{s\to-3}(s+3)\frac{s-4}{(s+3)(s+2)s} = \lim_{s\to-3}\frac{s-4}{(s+2)s} = \lim_{s\to-3}\frac{-3-4}{(-3+2)(-3)} = -\frac{7}{3}$
    - $s_2 = -2: \quad r_2 = \lim_{s\to-2}(s+2)\frac{s-4}{(s+3)(s+2)s} = \lim_{s\to-2}\frac{s-4}{(s+3)s} = \lim_{s\to-2}\frac{-2-4}{(-2+3)(-2)} = 3$
    - $s_2 = 0: \quad r_3 = \lim_{s\to0} s\frac{s-4}{(s+3)(s+2)s} = \lim_{s\to0}\frac{s-4}{(s+3)(s+2)} = \lim_{s\to0}\frac{-4}{(3)(2)} = -\frac{2}{3}$
  - $Y(s) = -\frac{7}{3}\frac{1}{s+3} + \frac{3}{s+2} - \frac{2}{3}\frac{1}{s}$
  - $y(t) = -\frac{7}{3}e^{-3t} + 3e^{-2t} - \frac{2}{3}$

Steady State Response

Impulse Input

- For impulse Inputs $u(t) = \delta(t)$, with $\delta(t)$ being the Dirac delta function (impulse modelling)
  - $\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & \text{else} \end{cases}, \int_{-\infty}^{\infty} \delta(t) = 1, \int_{-\infty}^{\infty} f(t)\delta(t) = f(0)$
  - For $D = 0, x(0) = 0, u(t) = k\delta(t)$ the output becomes:
    - $y_{imp}(t) = \int_0^t Ce^{A(t-\tau)}B\, k\delta(t)d\tau = kCe^{At}B$
  - The impulse response is basically the same as the initial condition response with $x(0) = B$
    - In other words: the impulse response describes the system (see Signal and Systems)
  - If we have the Transfer Function in Partial Fraction $G(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \frac{r_3}{s-p_3} + \cdots + r_0$:
    - $y(t) = r_1e^{p_1t} + r_2e^{p_2t} + \ldots + r_ne^{p_nt}$
    - Every pole contributes to the system response (weighted by the residual)
    - If pole is imaginary, we get a sinusoidal (Notebook example)
    - The zeros effect the residuals

- How do poles and zeros explain the behaviour of the system?

Transfer Function Continued

Effects of Poles and **Zeros**

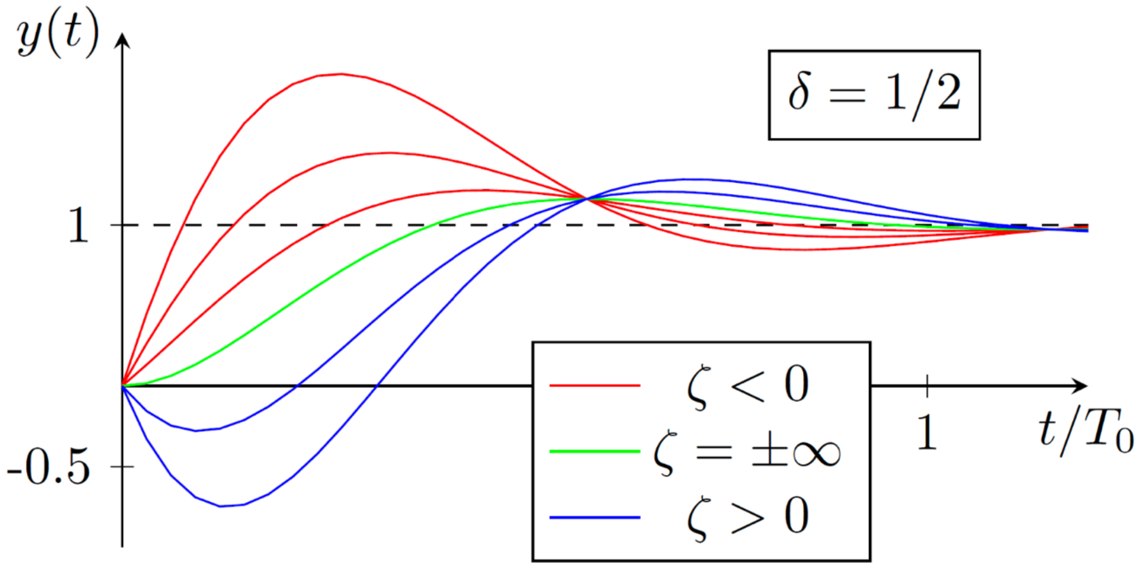
- What poles do is clear by now:
  - $G(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \frac{r_3}{s-p_3} + \cdots + r_0 \ \rightarrow \ y(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \ldots + r_n e^{p_n t}$
  - Every pole contributes to the system response (weighted by the residual)
  - If pole is imaginary, we get a sinusoidal (Notebook example)

- What is the effect of the zeros? $G(s) = \frac{k_{rl}}{s^q} \frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_{n-q})}$
  - A zero close to the pole weakens its effect (see lecture)
  - If a zero is equal to the pole we get a **pole-zero cancellation**
    - This is what happened when we got the minimal realization of a system
    - This means this state is not reachable from the input or we can't see it in the output
      - Controllability and Observability will be looked at in CSII
    - If the pole is stable no worries
    - If the pole is unstable, we have a problem! -> change your system design

Transfer Function Continued

Effects of Poles and **Zeros**

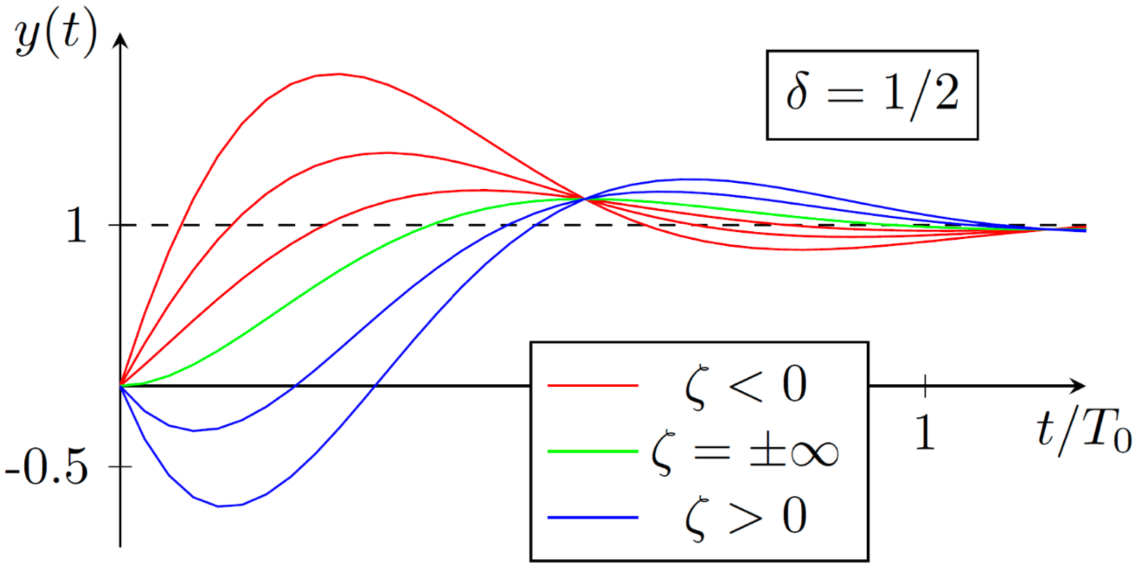
- Zeros are derivative terms (Poles are integrals: $\frac{d}{dt}y(t) = u(t) \rightarrow sY(s) = U(s) \rightarrow Y(s) = \frac{U(s)}{s}$)
  - $y(t) = \frac{d}{dt}u(t) \rightarrow Y(s) = sU(s)$
    - Note a zero by itself is not possible, as this would be an acausal system
  - If a system can be written as $G(s) = \left(\frac{s}{-z} + 1\right)\tilde{G}(s) = \tilde{G}(s) + \frac{s}{-z}\tilde{G}(s)$, the output will be:
    - $y(t) = \tilde{y}(t) + \frac{1}{-z}\dot{\tilde{y}}(t)$, where we use that $\tilde{G}(s) \rightarrow \tilde{y}(t)$
  - When plotting the step response for a system with different zeros we get:



Transfer Function Continued

Effects of Poles and **Zeros**

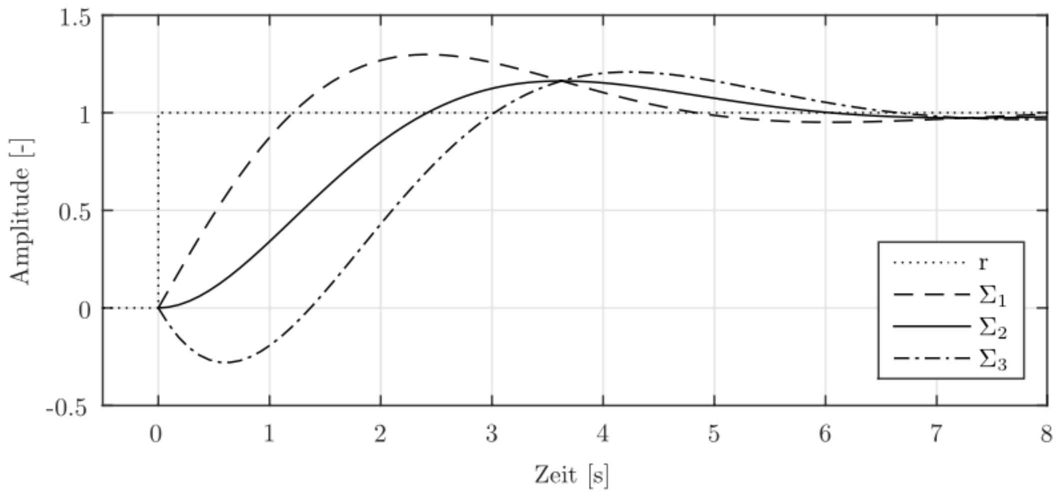
- What we learn:
  - Zeros introduce a non-zero derivative from the start $\dot{y}(0) \neq 0$
  - Larger zeros have a smaller influence -> Smaller zeros have larger effects
  - If zero is negative, we “go in the right direction”
  - If zero is positive we go into the wrong direction -> **Non-minimum phase zero (bad)**
  - Zeros are the result of the matrices B and C -> we can tune them to get a desired zero



Transfer Function Continued

Effects of Poles and **Zeros** – Example (Old exam question)

- For the systems $G_i(s) = \frac{a_i s+1}{s^2+s+1}$ we have the following step response plots: $a_i \propto -\frac{1}{z_i}$, z_i=zero



- Given the step response plots what can be said about the parameters a_i?

☐ $a_1 < a_2 < a_3$

☐ $a_2 < a_1 < a_3$

■ $a_3 < a_2 < a_1$

☐ $a_1 < a_3 < a_2$

- a_3 is non minimum phase thus has the largest zero ($a_3 < 0$)
- a_2 has a no derivative at $t = 0$ thus the zero is very large ($a_2 = 0$)
  - In other words, we don't have a s term in the nominator
- a_1 has a derivative at $t = 0$ and is minimum phase ($a_1 > 0$)

Summary

Whats relevant?

- Know the different forms and what we can use them form
  - Conversion less important since often a lot of work
  - Use residual trick where needed
- Know how to compute the steady state response for different signals -> Laplace
  - We have solutions for some important functions (remember superposition)
- Behaviour of zeros
  - A zero close to the pole weakens its effect (see lecture)
  - If a zero is equal to the pole we get a **pole-zero cancellation**
    - Pole zero cancellation results in minimal realisation
    - Only okay if pole is stable, else its a PROBLEM
  - Zeros behave like a derivative
    - Larger zeros have a smaller influence -> Smaller zeros have larger effects
    - If zero is negative, we “go in the right direction”
    - If zero is positive we go into the wrong direction -> **Non-minimum phase zero (bad)**