Control Systems I
Recitation 07

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Last Week

Different Forms

- Transfer Functions:
  - $G(s) = \frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\cdots+b_0}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\cdots+a_0} + d = \frac{N(s)}{D(s)} + d$
  - Partial Fraction Expansion:
    - $G(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \frac{r_3}{s-p_3} + \cdots + r_0$
  - Root Locus Form:
    - $G(s) = \frac{k_{rl}}{s^q} \frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_{n-q})}$
  - Bode Form:
    - $G(s) = \frac{k_{bd}}{s^q} \frac{\left(\frac{s}{-z_1}+1\right)\left(\frac{s}{-z_2}+1\right)\ldots\left(\frac{s}{-z_1}+1\right)}{\left(\frac{s}{-p_1}+1\right)\left(\frac{s}{-p_2}+1\right)\ldots\left(\frac{s}{-p_{n-q}}+1\right)}$

- p_i are the poles of the system (can be imaginary)
  - "instabilities" of the system
- r_i are the residuals
  - Describe the system response of each pole
- z_i are the zeros of the transfer function
  - "blind spots" of the system

- Non-repeating pole:
  - $r_i = \lim_{s \to p_i} (s - p_i)G(s)$
- Repeating pole (m)
  - $r_i = \frac{1}{(m-1)!} \lim_{s \to p_i} \frac{d^{m-1}}{ds^{m-1}} \left((s - p_i)G(s)\right)$

Last Week

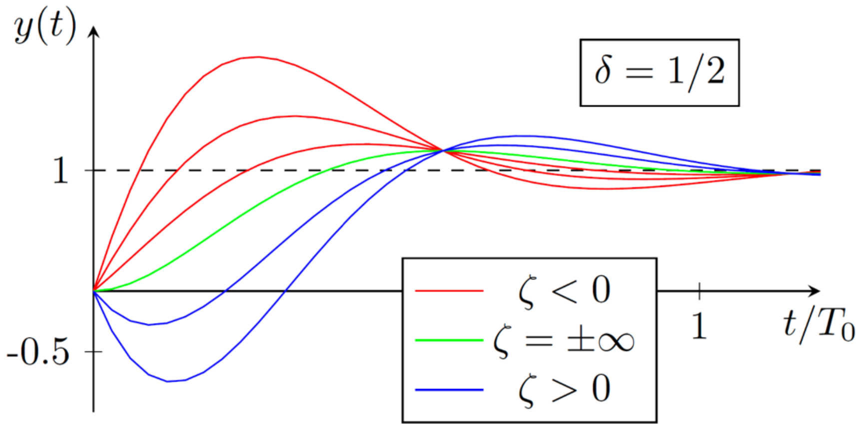
(Steady) State Response for Different Input Signals

- General Case (Use Tables)
  - $Y(s) = [(sI - A)^{-1}B + D]\,U(s) = G(s)U(s) \rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\}, \quad y_{ss}(t \rightarrow \infty) = \lim_{s \rightarrow 0} sY(s)$
- For sinusoidal Inputs $u(t) = \sin(\omega t)\,,\ s = j\omega$
  - $y_{ss}(t) = |G(j\omega)| \sin\big(t + \angle G(j\omega)\big)$ (We can even do this graphically)
- Step Input $u(t) = h(t)$
  - $y_{step}(t) = -CA^{-1}B + CA^{-1}e^{At}B \rightarrow y_{ss}(t) = -CA^{-1}B = \text{const}$
    - $y_{step}(t) = y_{ss}(t)(1 - e^{at})$
    - $y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{G(s)U(s)\}, U(s) = \frac{1}{s}$
- Impulse Inputs $u(t) = \delta(t)$
  - $y_{imp}(t) = \int_0^t Ce^{A(t-\tau)}B\, k\delta(t)d\tau = kCe^{At}B$
  - If we have the Transfer Function in Partial Fraction $G(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \frac{r_3}{s-p_3} + \cdots + r_0$:
    - $y(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \ldots + r_n e^{p_n t}$

Last Week

Effects of Zeros

- What is the effect of the zeros? $G(s) = \frac{k_{rl}}{s^q} \frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_{n-q})}$
  - A zero close to the pole weakens its effect (see lecture)
  - If a zero is equal to the pole we get a **pole-zero cancellation** (minimal realization)
    - If the pole is stable no worries
    - If the pole is unstable, we have a problem! -> change your system design (B&C)
  - Zeros introduce a non-zero derivative from the start $\dot{y}(0) \neq 0$
  - Larger zeros have a smaller influence -> Smaller zeros have larger effects
  - If zero is negative, we “go in the right direction” (pole-zero cancellation doable)
  - If zero is positive we go into the wrong direction -> **Non-minimum phase zero (bad)**



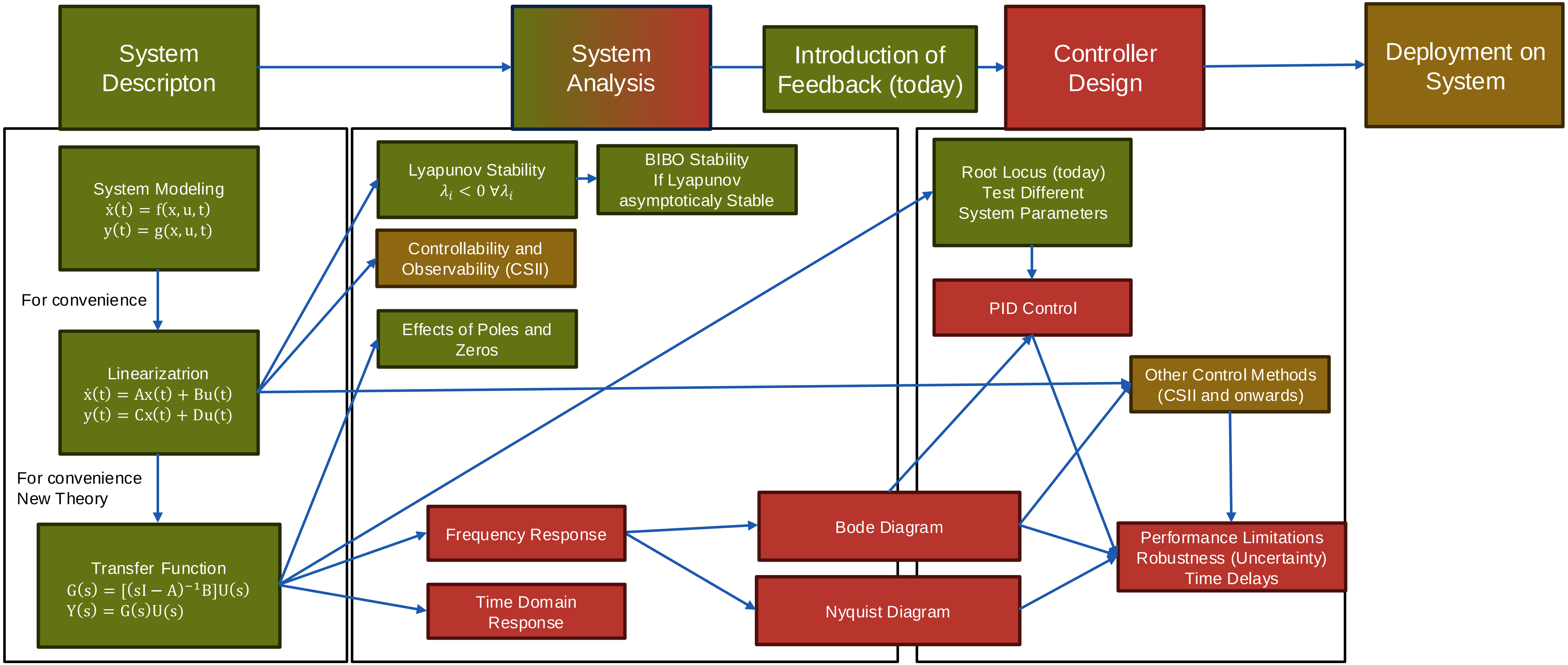
Outline

- Conceptual Recap
  - Classical Control Approach

- Controllers and Feedback
  - Introduction
- Root Locus
  - What?
  - Some Analysis Upfront
  - Drawing a Root Locus Curve
  - Extracting Information from a Root Locus Plot
  - Example

Conceptual Recap

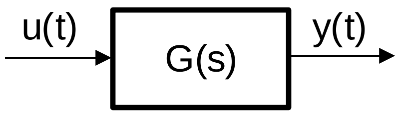
Classical Control Approach



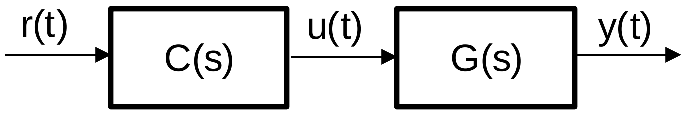
Controllers and Feedback

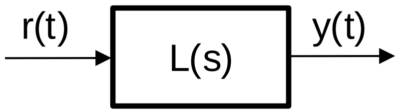
Introduction

- Until now we only looked at the system: $u(t) \rightarrow y(t)$
  - Stability and Analysis until now only for this

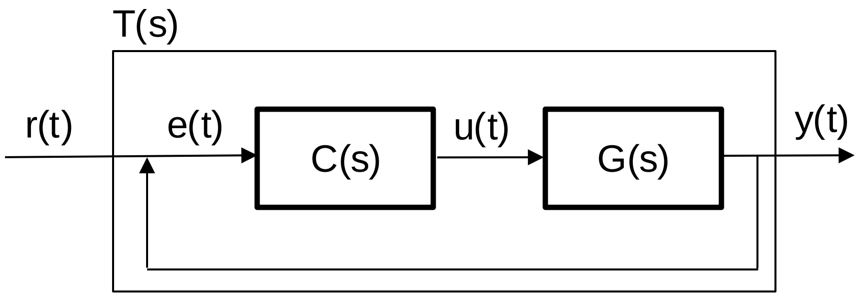


- We can introduce a Controller (no feedback yet):
  - **Open Loop** System: $r(t) \rightarrow y(t)$
    - $L(s) = C(s)G(s)$



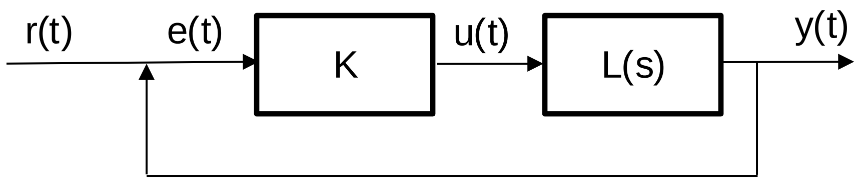


- We can Introduce Feedback:
  - Closed Loop System: $r(t) \rightarrow y(t)$
    - $T(s) = \frac{L(s)}{1+L(s)} = \frac{C(s)G(s)}{1+C(s)G(s)}$
  - Changes the system behaviour dynamically
    - Unstable –> stable
    - Stable -> “More” stable (quicker or less oscillation)
    - Stable -> unstable



Root Locus

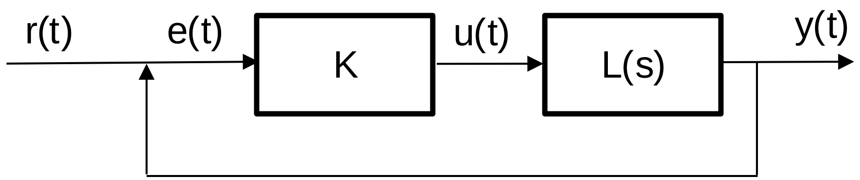
What?



- Add proportional controller to the open loop system:
  - $kL(s) = k\frac{N(s)}{D(s)} \rightarrow T(s) = \frac{kL(s)}{1+kL(s)} = \frac{kN(s)}{D(s)+kN(s)}$
  - $kL(s) = k\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$
- We want to analyse the poles of $T(s)$ for different k
- *Sidenotes:*
  - *Poles are symmetric about the real axis (either on it or complex conjugates)*
  - *The degree of* $D(s) + kN(s)$ *is the same as* $D(s) \rightarrow \#OL - poles = \#CL - poles$
- Root Locus = Graphical analysis of the closed loop system for different values of k or any other parameter
  - Plot the position of the zeros and poles for all possible k
- Using only the **open loop system** to analyse the **closed loop system**!
  - We only need to know $L(s)$!!!
  - Get quick (qualitative) info about the system response

Root Locus

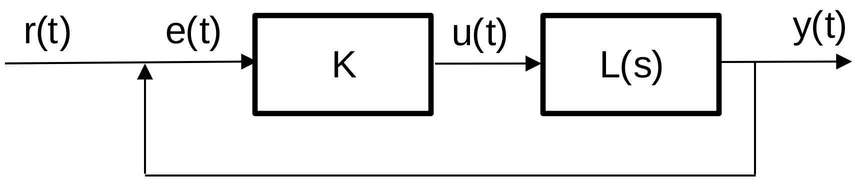
Some Analysis Upfront



- Add proportional controller to the open loop system:
  - $\mathrm{kL(s)} = \mathrm{k}\frac{\mathrm{N(s)}}{\mathrm{D(s)}} \rightarrow \mathrm{T(s)} = \frac{\mathrm{kL(s)}}{1+\mathrm{kL(s)}} = \frac{\mathrm{kN(s)}}{\mathrm{D(s)+kN(s)}}$
  - $\mathrm{kL(s)} = \mathrm{k}\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$
- Different extremes:
  - $k = 0$: Poles of $T(s)$ → Poles of $L(s)$
  - $k \rightarrow \infty$: Poles of $T(s)$ → Zeros of $L(s)$
    - This also explains why we should avoid non-minimum phase zeros
    - Since degree of N(s) is smaller than D(s) the "excess" poles go to ∞

Root Locus

Some Analysis Upfront



- The poles of $T(s)$ define the system behaviour
  - $D(s) + kN(s) = 0$
- We can rewrite this:
  - $\frac{N(s)}{D(s)} = -\frac{1}{k} = \frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)}$
- To now find a connection between the zeros and poles we analyse the angle and magnitude
  - Angle: $\angle\frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)} = \angle -\frac{1}{k}$
    - $\angle(s-z_1) + \angle(s-z_2) + \cdots - \angle(s-p_1) - \angle(s-p_2) = \begin{cases} 180° + q \cdot 360° & if\ k > 0 \\ 0° + q \cdot 360° & if\ k < 0 \end{cases}$
  - Magnitude: $\frac{|(s-z_1)||(s-z_2)|\dots|(s-z_m)|}{|(s-p_1)||(s-p_2)|\dots|(s-p_n)|} = \left|-\frac{1}{k}\right|$
    - $\frac{|(s-z_1)||(s-z_2)|\dots|(s-z_m)|}{|(s-p_1)||(s-p_2)|\dots|(s-p_n)|} = \frac{1}{|k|}$
  - All possible poles of $T(s)$(closed loop system) must satisfy the above equalities
    - We call all these points (i.e. the resulting curve) the root locus

ETH zürich

Root Locus

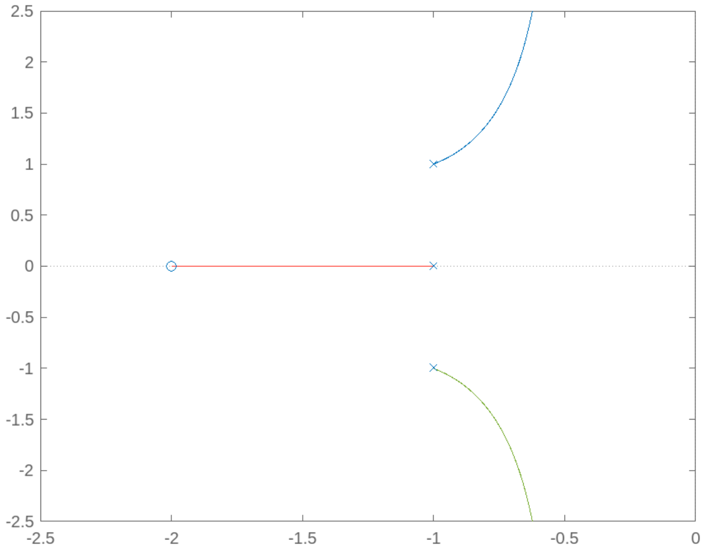
Drawing a Root Locus Curve - Rules

- Starting Points: $k = 0$: Poles of $T(s)$ → Poles of $\mathrm{L}(s)$
- End Points: $k \to \infty$: Poles of $T(s)$ → Zeros of $\mathrm{L}(s)/\infty$
- Root Locus on the Real Axis:
  - All points on the real axis to the **left** of an **odd number of poles/zeros are on the positive k** root locus.
  - All points on the real axis to the **left** of an **even number of poles/zeros (or none) are on the negative k** root locus.
  - Can be derived from the angle criterion
- Asymptotic behaviour for $k \to \infty$:
  - Poles move to zeros
  - Excess Pole “radiate” outwards (m and n are the highest power of N(s) and D(s))
    - $\angle \frac{180° + q \cdot 360°}{n-m}, k > 0$ or $\angle \frac{0° + q \cdot 360°}{n-m}, k < 0$
    - The asymptotes meet at
    - $s = \frac{\sum p_i - \sum z_i}{n-m}$

Root Locus

Drawing a Root Locus Curve - Rules

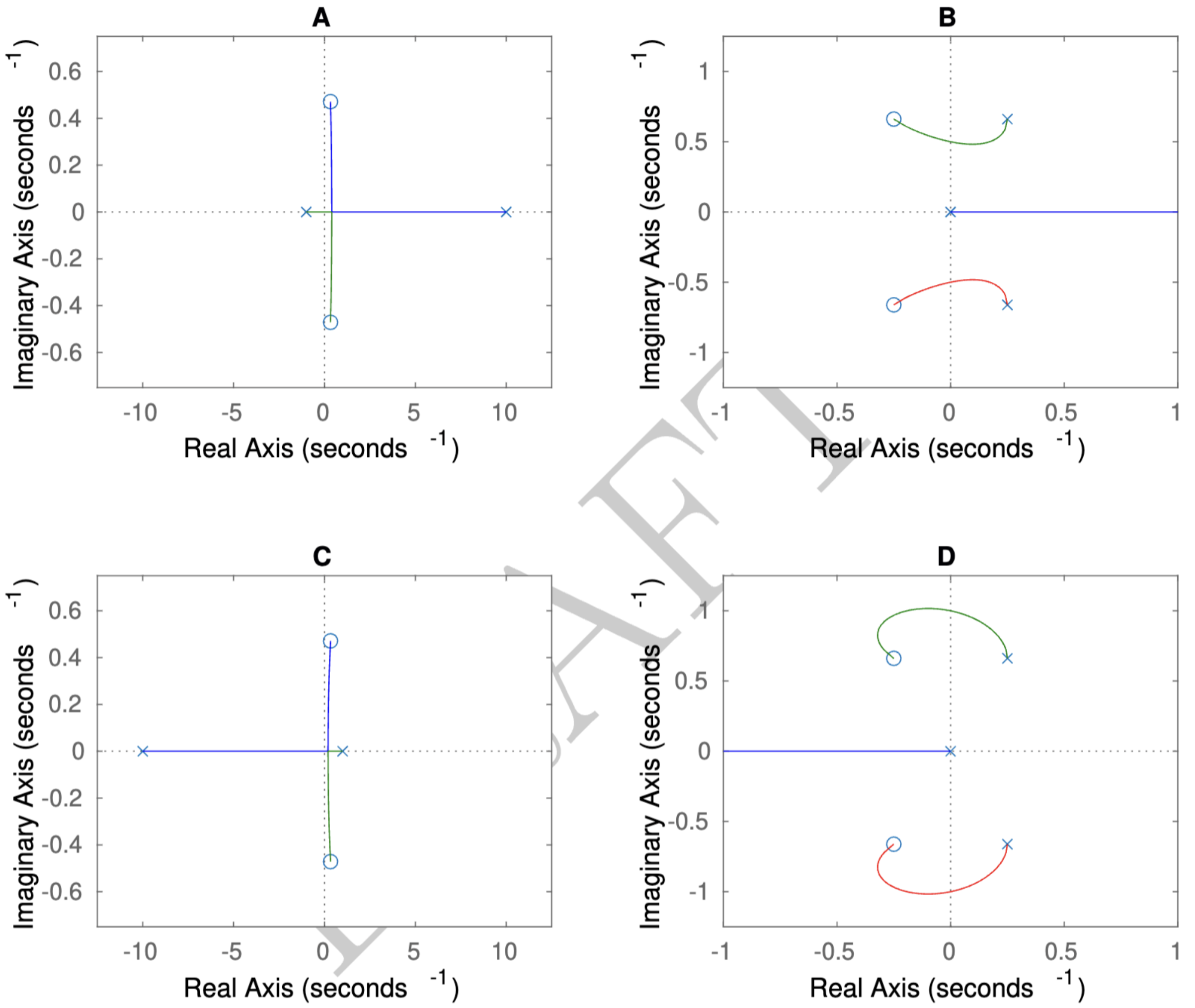
- Break away/Break In points (when the root locus meet or diverge from the real axis)
  - At the break away / break in points it must hold:
    - $N'(s)D(s) = N(s)D'(s)$
- Crossing at Imaginary Axis:
  - Find the value k analytically using $D(s) + kN(s) = 0$
- Many more on the website but not necessary for you
- Usually use MatLab or Python to plot the root locus of a system



Root Locus

Extracting Information

- Get Open Loop Transfer Function:
  - Zeros are circles
  - Poles are crosses
  - $L(s) = \frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$
- Stability:
  - Check for poles with positive real part
  - Zeros with positive real part mean that the system becomes unstable if k to large
- Damping and Oscillation (poles and zeros):
  - Imaginary parts -> Oscillation
  - Negative real part -> Damping



Root Locus

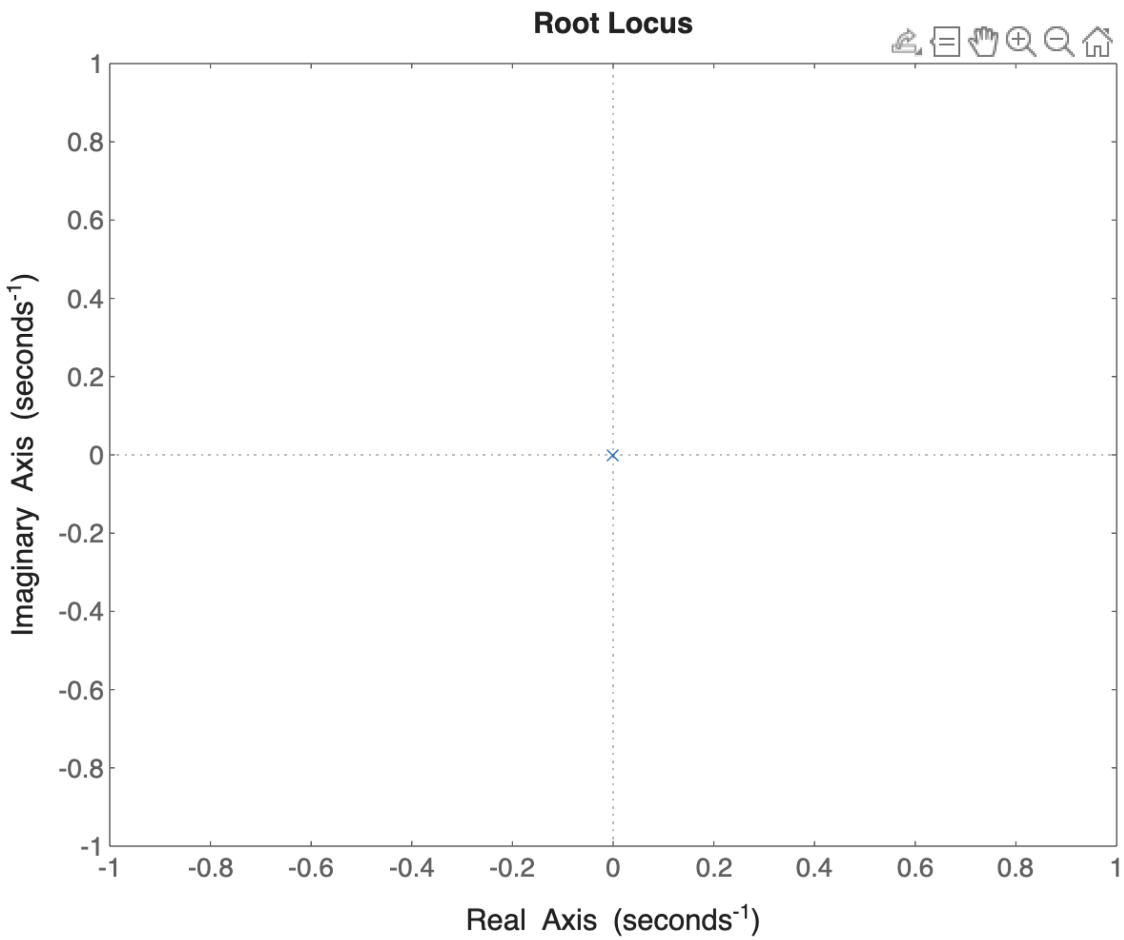
Summary

- We looked only at the proportional Gain:
  - We could vary any parameter
- We can also use other controllers and vary any parameter:
  - **PI** compensator: $C(s) = k_\mathrm{P} + k_\mathrm{I}\frac{1}{s} = k_\mathrm{P}\frac{s + k_\mathrm{I}/k_\mathrm{P}}{s}$,
  - **PD** compensator: $C(s) = k_\mathrm{P} + k_\mathrm{D}s$ (this is an idealized compensator),
  - **PID** compensator: $C(s) = k_\mathrm{P} + k_\mathrm{I}\frac{1}{s} + k_\mathrm{D}s = k_\mathrm{P}\frac{k_\mathrm{I}/k_\mathrm{P} + s + k_\mathrm{D}/k_\mathrm{P}s}{s}$,
  - **Lead** compensator: $\frac{s-z}{s-p}$, with $z < p$,
  - **Lag** compensator: $\frac{s-z}{s-p}$, with $p < z$,
  - Serial combination of several of these...

Root Locus

Example - Together

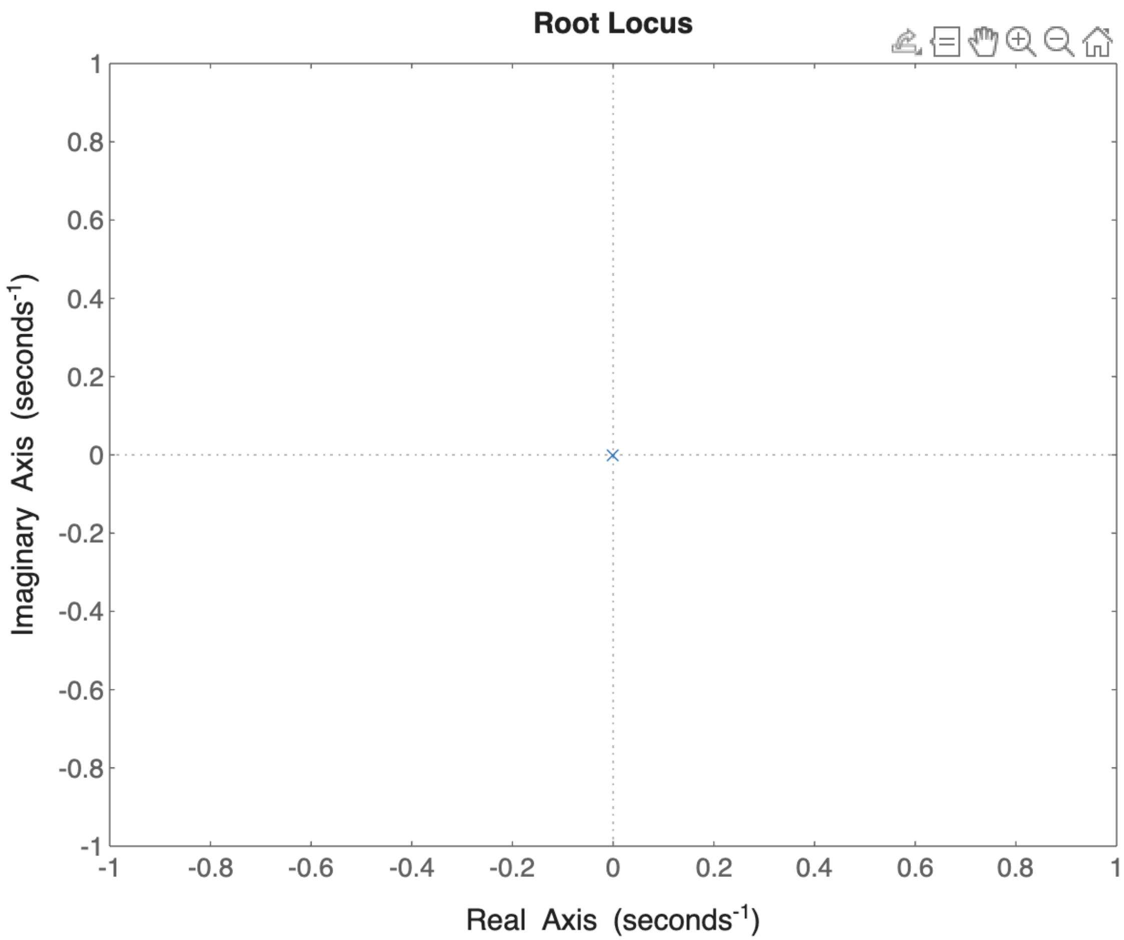
- $L(s) = \frac{s^2+2s+2}{s^2-4s-5} = \frac{(s+1+j)(s+1-j)}{(s+1)(s-5)}$
- Draw Poles and Zeros
- Find Break Away Point
  - $N'(s)D(s) = N(s)D'(s)$
  -
  -
  -
  -
  -
  -
- Connect lines



Root Locus

Example - Drawing

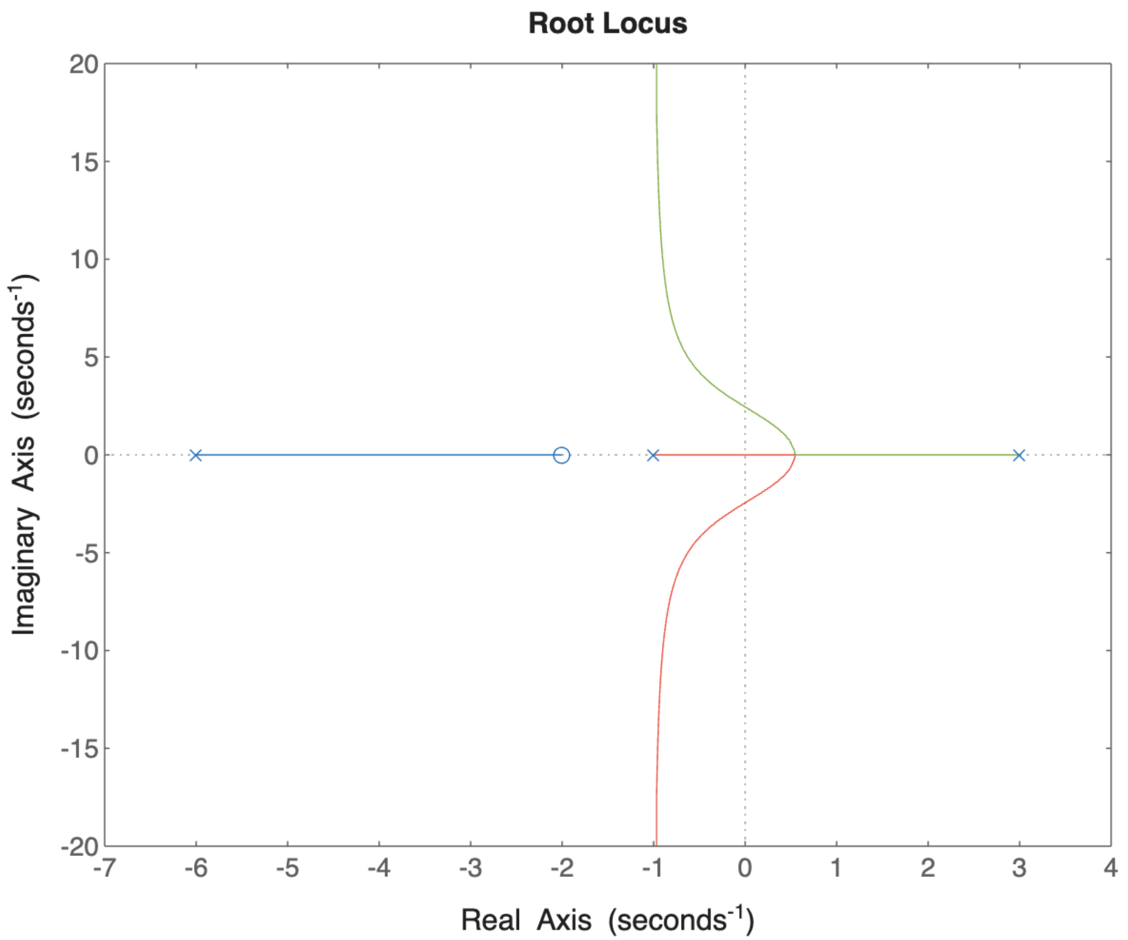
- $L(s) = \frac{s+2}{s^3 - 4s^2 - 5s} = \frac{(s+2)}{s(s+1)(s-5)}$
- Draw Poles and Zeros
- Find Asymptotic Behaviour:
  - $n - m = 2$
  - $\angle$
  - $s =$
- Could determine Break Away Point (not necessary)
- Connect lines



Root Locus

Example - Analysis

- *Open Loop Transfer Function:*
  - $L(s) = \frac{(s+2)}{(s-3)(s+6)(s+1)}$
- *Is the open loop system stable?*
  - ❑ *Yes*
  - ❑ *No*
- *Is there a k s.t. the closed loop system is stable?*
  - ❑ *Yes*
  - ❑ *No*
- *Is there a k s.t. the closed loop system is stable and has no overshoot/oscillation?*
  - ❑ *Yes*
  - ❑ *No*
- *See Notebook*



Root Locus

Example – Exam Style question

- Assign the plots to the corresponding Transfer Functions:
- $F_1 = \frac{s^2+3s+5}{(s+2)*(s-3)*s}, F_2 = \frac{s^2+3s+5}{(s+2)*(s-3)*s^3}$
- $F_3 = \frac{s^2+3s+5}{(s+2)*(s-3)*(s+1)^6}, F_4 = \frac{s^2+3s+5}{(s+2)*(s-3)*(s+1)}$
- Solution:
- $A \rightarrow$
- $B \rightarrow$
- $C \rightarrow$
- $D \rightarrow$