ETH zürich

Control Systems I
Recitation 08

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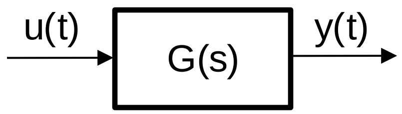
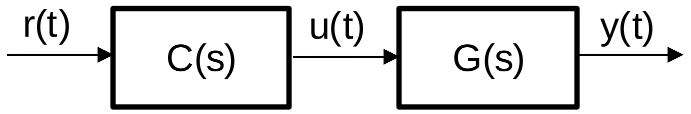
Controllers and Feedback

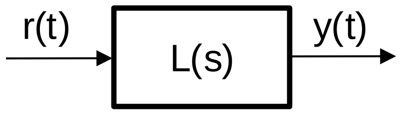
Introduction

- Until now we only looked at the system: $u(t) \rightarrow y(t)$
  - Stability and Analysis until now only for this

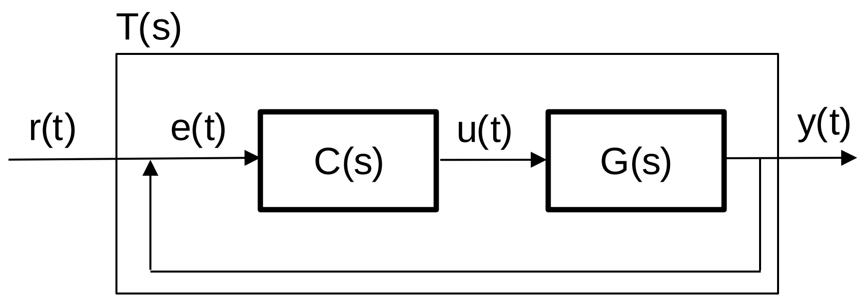


- We can introduce a Controller (no feedback yet):
  - **Open Loop** System: $r(t) \rightarrow y(t)$
    - $L(s) = C(s)G(s)$



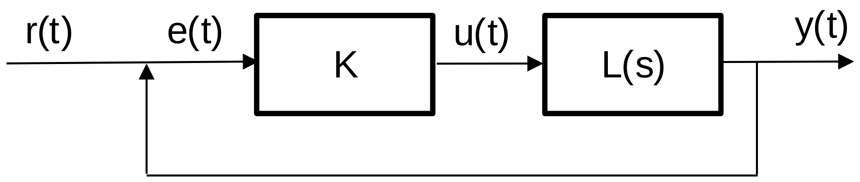


- We can Introduce Feedback:
  - **Closed Loop** System: $r(t) \rightarrow y(t)$
    - $T(s) = \frac{L(s)}{1+L(s)} = \frac{C(s)G(s)}{1+C(s)G(s)}$
  - Changes the system behaviour dynamically
    - Unstable –> stable
    - Stable -> “More” stable (quicker or less oscillation)
    - Stable -> unstable



Root Locus

What?

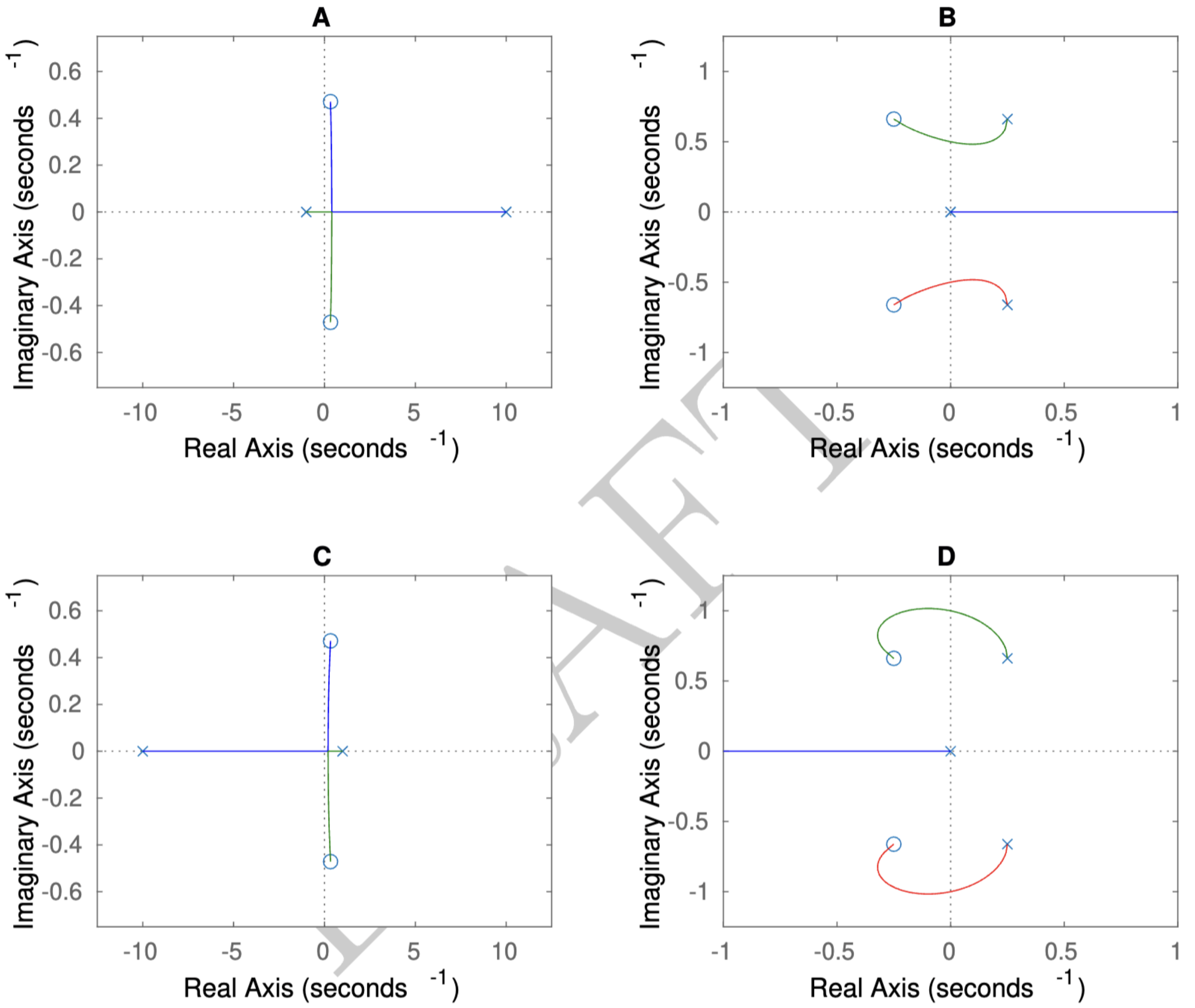


- Add proportional controller to the open loop system:
  - $kL(s) = k\frac{N(s)}{D(s)} \rightarrow T(s) = \frac{kL(s)}{1+kL(s)} = \frac{kN(s)}{D(s)+kN(s)}$
  - $kL(s) = k\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$
- Root Locus = Graphical analysis of the closed loop system for different values of k or any other parameter
  - Plot the position of the zeros and poles for all possible k
- Using only the **open loop system** to analyse the **closed loop system**!
  - We only need to know $L(s)$!!!
  - Get quick (qualitative) info about the system response
- Different extremes:
  - $k = 0$: Poles of $T(s)$ → Poles of $L(s)$
  - $k \rightarrow \infty$: Poles of $T(s)$ → Zeros of $L(s)$
    - This also explains why we should avoid non-minimum phase zeros
    - Since degree of N(s) is smaller than D(s) the ”excess” poles go to ∞

Root Locus

Extracting Information

- Get Open Loop Transfer Function:
  - Zeros are circles
  - Poles are crosses
  - $L(s) = \frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$
- Stability:
  - Check for poles with positive real part
  - Zeros with positive real part mean that the system becomes unstable if k to large
- Damping and Oscillation (poles and zeros):
  - Imaginary parts -> Oscillation
  - Negative real part -> Damping

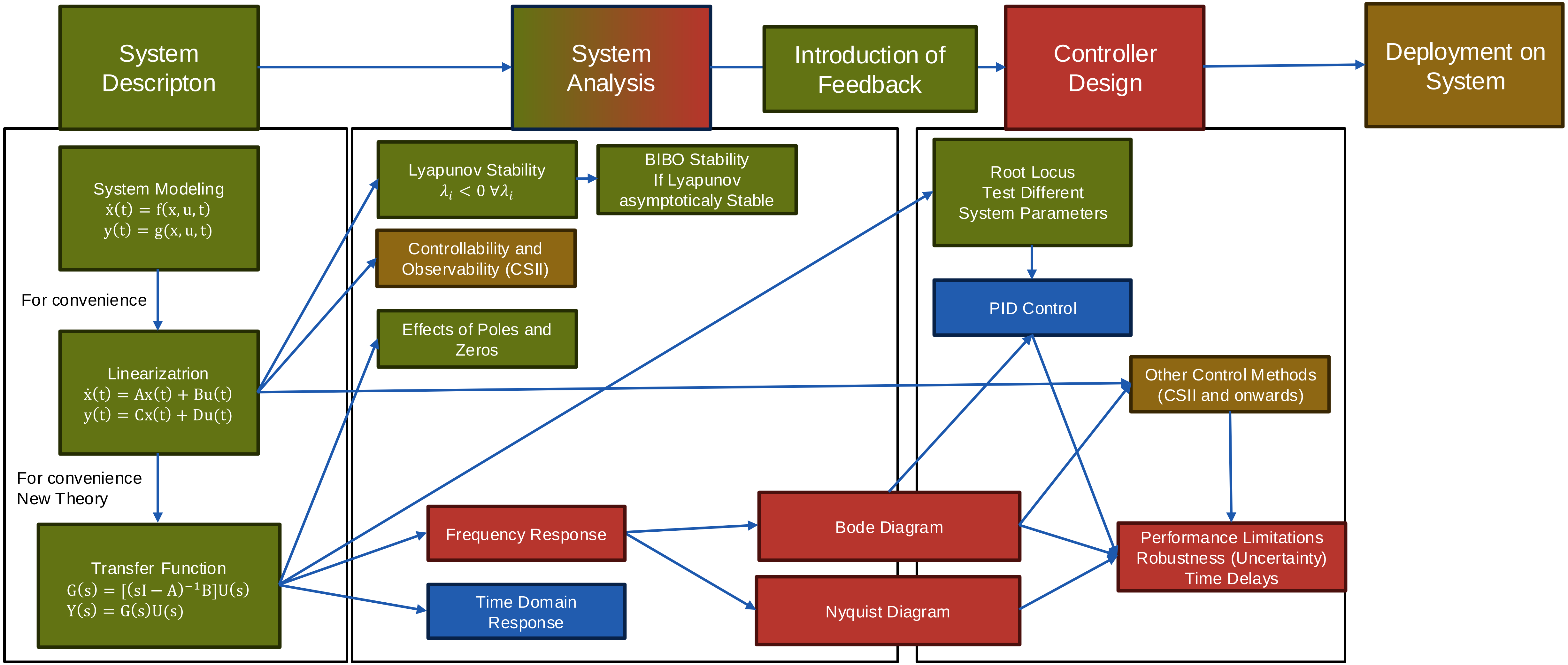


Outline

- System Analysis
  - What is there to check?
  - Different Transfer Functions
  - Sidenote on Total System Behaviour
- Time Response
  - Steady State Error
  - Example
  - Step Response: 1 order system
  - Step Response: 2 order system
- PID Controller
  - Proportional Part
  - Integral Part
  - Derivative Part
  - PID Tuning

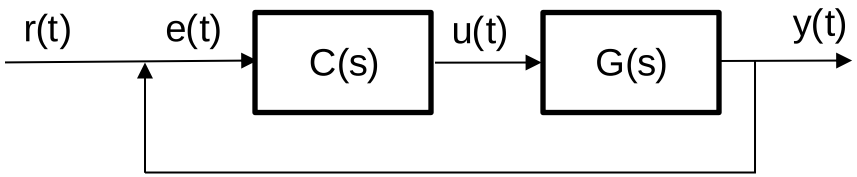
Conceptual Recap

Classical Control Approach



System Analysis

What is there to check?

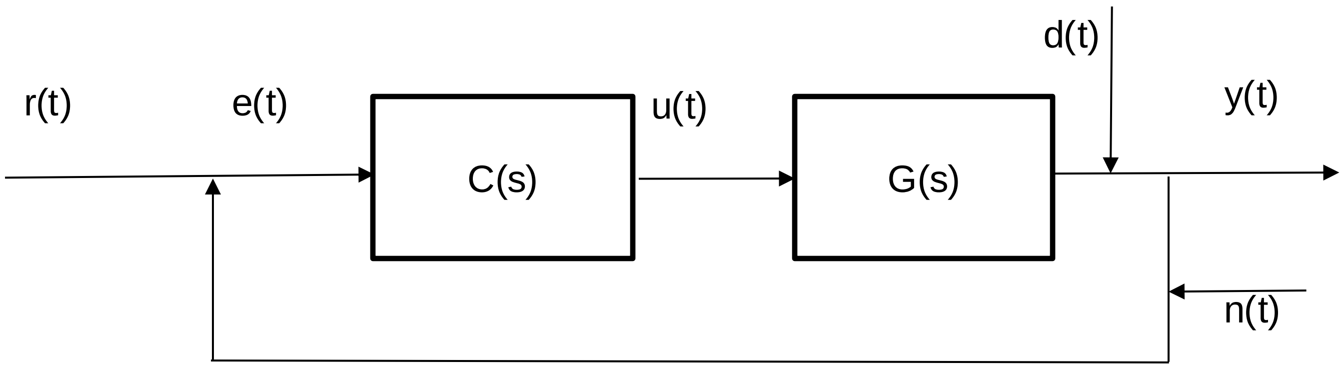


- Qualitative Analysis: (already seen)
  - Stability Analysis: Lyapunov, BIBO, Root Locus
- Quantitative Analysis: (today)
  - Steady State behaviour
    - Do we converge to the desired value?
  - Speed of Convergence
    - How fast do we converge?
  - Overshoot
    - Do we have oscillation and if so, how big are these?
- How about robustness? (next week)
  - Modelling errors
    - If the nominal system is stable is the real system also stable?
    - How big of an error can the system compensate for?
  - Noise/Disturbance Rejection
    - If we have disturbances/noise how good can the closed loop system handle these?

System Analysis

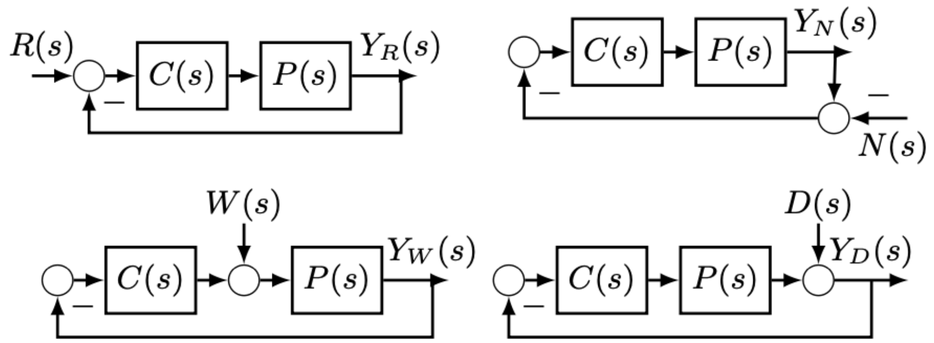
Different Transfer Functions

- We saw the Closed Loop System:
  - **Complementary Sensitivity:** $\mathrm{r(t)} \rightarrow \mathrm{y(t)}$ or $\mathrm{n(t)} \rightarrow \mathrm{y(t)}$
    - $\mathrm{T(s)} = \frac{\mathrm{Y(s)}}{\mathrm{N(s)}} = \frac{\mathrm{Y(s)}}{\mathrm{R(s)}} = \frac{\mathrm{L(s)}}{1+\mathrm{L(s)}} = \frac{\mathrm{C(s)G(s)}}{1+\mathrm{C(s)G(s)}}$
- There are more:
  - **Sensitivity:** $\mathrm{r(t)} \rightarrow \mathrm{e(t)}$ or $\mathrm{d(t)} \rightarrow \mathrm{y(t)}$
    - $\mathrm{S(s)} = \frac{\mathrm{E(s)}}{\mathrm{R(s)}} = \frac{\mathrm{Y(s)}}{\mathrm{D(s)}} = \frac{1}{1+\mathrm{L(s)}} = \frac{1}{1+\mathrm{C(s)G(s)}}$
- We also see that $S(s) + \mathrm{T(s)} = 1$
  - We can either track the reference perfect or reject noise perfect



System Analysis

Sidenote on Total System Behaviour



- We can do this for all possible inputs to the system:
- **We get:**

$$Y_R(s) = \frac{P(s)C(s)}{1+P(s)C(s)} \cdot R(s), \quad Y_N(s) = \frac{P(s)C(s)}{1+P(s)C(s)} \cdot N(s)$$

$$Y_W(s) = \frac{P(s)}{1+P(s)C(s)} \cdot W(s), \quad Y_D(s) = \frac{1}{1+P(s)C(s)} \cdot D(s)$$

- Linearity allows us to: $Y(s) = Y_R(s) + Y_N(s) + Y_W(s) + Y_D(s)$
  - We thus get: $\mathrm{T}(s) = \frac{\mathrm{L}(s)}{1+\mathrm{L}(s)}, S(s) = \frac{1}{1+\mathrm{L}(s)} \rightarrow$

$$Y(s) = S(s) \cdot [D(s) + P(s) \cdot W(s)] + T(s) \cdot [R(s) + N(s)]$$

- Learnings:
  - System is stable iff $\frac{1}{1+\mathrm{L}(s)}$ is stable

Time Response

Steady State Error

- Error: $e(t) = r(t) - y(t)$ should ideally go to zero $\lim_{t\to\infty} e(t) = 0$
- Sensitivity describes the relation between $r(t) \to e(t)$: $E(s) = S(s)R(s) = \frac{1}{1+L(s)} R(s)$
- For a step input $R(s) = \frac{1}{s} \to E(s) = \frac{1}{s}\frac{1}{1+L(s)}$
  - Steady state behaviour:
    - $e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{1}{1+L(s)} = \frac{1}{1+L(0)}$
  - How can we make sure we have no steady state error?
    - $L(0) \to \infty \to$ need a pole at $s = 0$ (integrator)
    - If the closed loop system is stable and we have an integrator we have 0 steady state error
- How do we get $L(0)$?
  - $L(s) = \frac{k_{bd}}{s^q}\frac{\left(\frac{s}{-z_1}+1\right)\left(\frac{s}{-z_2}+1\right)\ldots\left(\frac{s}{-z_1}+1\right)}{\left(\frac{s}{-p_1}+1\right)\left(\frac{s}{-p_2}+1\right)\ldots\left(\frac{s}{-p_{n-q}}+1\right)} \xrightarrow{s\to 0} L(0) \to \frac{k_{bd}}{s^q}$
  - q is called type $L(s)$ for $q = 0$ we get $L(0) = k_{bd}$ which is called the DC-gain
  - $q > 0$: $L(0) \to \infty$

Time Response

Steady State Error

- Steady State error for a step input $R(s)=\frac{1}{s} \rightarrow E(s)=\frac{1}{s}\frac{1}{1+L(s)}$
  - $e_{ss}=\lim_{t\to\infty} e(t)=\lim_{s\to 0} sE(s)=\lim_{s\to 0}\frac{1}{1+L(s)}=\frac{1}{1+L(0)}$
- What about ramps $r(t)=\frac{1}{q!}t^q \rightarrow R(s)=\frac{1}{s^{q+1}}$?
  - $e_{ss}=\lim_{t\to\infty} e(t)=\lim_{s\to 0} sE(s)=\lim_{s\to 0}\frac{1}{s^q}\frac{1}{1+L(s)}$
  - To have 0 ss-error we need at least one integrator more in our system

e_{ss}	$q=0$	$q=1$	$q=2$
Type 0	$\frac{1}{1+k_{\text{Bode}}}$	∞	∞
Type 1	0	$\frac{1}{k_{\text{Bode}}}$	∞
Type 2	0	0	$\frac{1}{k_{\text{Bode}}}$

- The same derivations go for disturbances!

Time Response

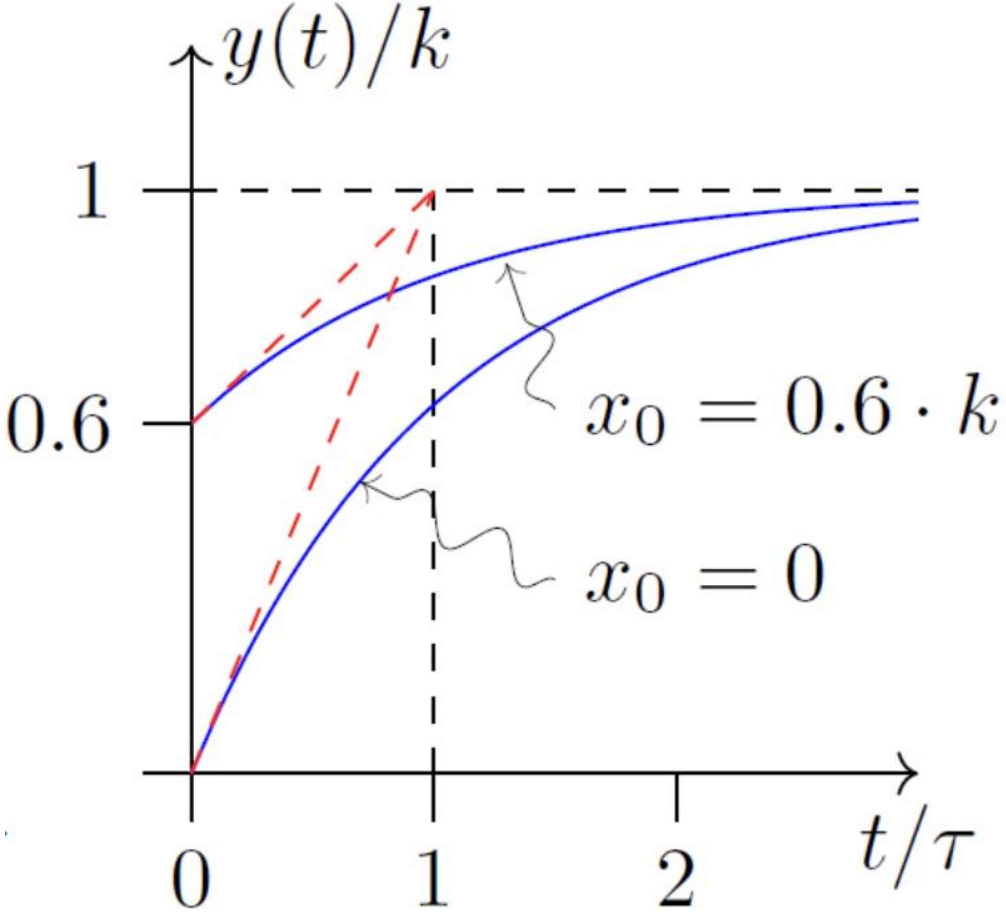
Steady State Error

- Given the system: $G(s) = \frac{s+3}{s^3+5s+3}$ and a Controller $C(s) = 3 + 6s$
- What is the steady state error for a step input a linear ramp?
  - $L(s) = \frac{s+3}{s^3+5s+3}(3 + 6s)$
  - $L(0) = \frac{3}{3}(3) = 3$
  - Step input: $e_{ss} = \lim_{s\to 0} sE(s) = \frac{1}{1+L(0)} = \frac{1}{1+3} = \frac{1}{2}$
  - Impulse input:
    - $e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{1}{s}\frac{1}{1+L(s)} = \lim_{s\to 0} \frac{1}{s}\frac{1}{1+\frac{s+3}{s^3+5s+3}(3+6s)}$
    - $e_{ss} = \lim_{s\to 0} \frac{1}{s}\frac{s^3+5s+3}{s^3+5s+(s+3)(3+6s)} \to \infty$
  - Propose a controller that we get a zero ss error:
    - $C(s) = 3 + 6s + \frac{3}{s^2}$

Time Response

Step Response: 2 order system

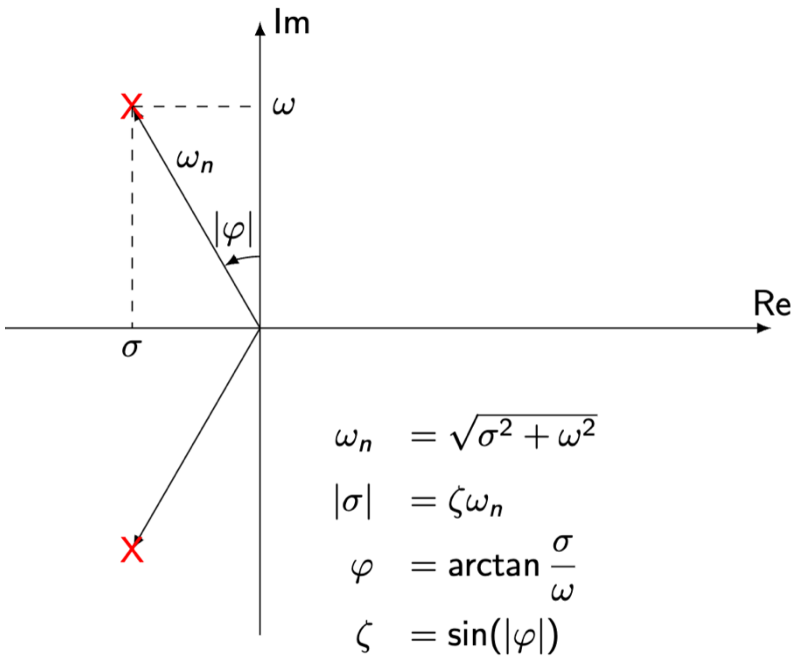
- First order system:
  - $\dot{x}(t) = -\frac{1}{\tau}x(t) + \frac{k}{\tau}u(t),\; y(t) = x(t)$
  - $G(s) = \frac{k}{\tau s+1}$
- Time response to a step input:
  - $y(t) = x_0 e^{-\frac{t}{\tau}} + k(1 - e^{-\frac{t}{\tau}})$
- What we can see:
  - x_0 determines the start value
  - DC-gain/Steady State: $y_{ss} = k$
  - Evaluating the derivative at $t = 0$:
    - The tangent crosses $y = k$ at $t = \tau$
  - Large τ slow convergence
  - Settling Time: Time it takes to be at $d\%$ of y_{ss} (beware this is the opposite as in the lecture)
    - $T_d = \tau \ln\left(\frac{100}{1-d}\right) \rightarrow \tau = \frac{T_d}{\ln\left(\frac{100}{1-d}\right)}$

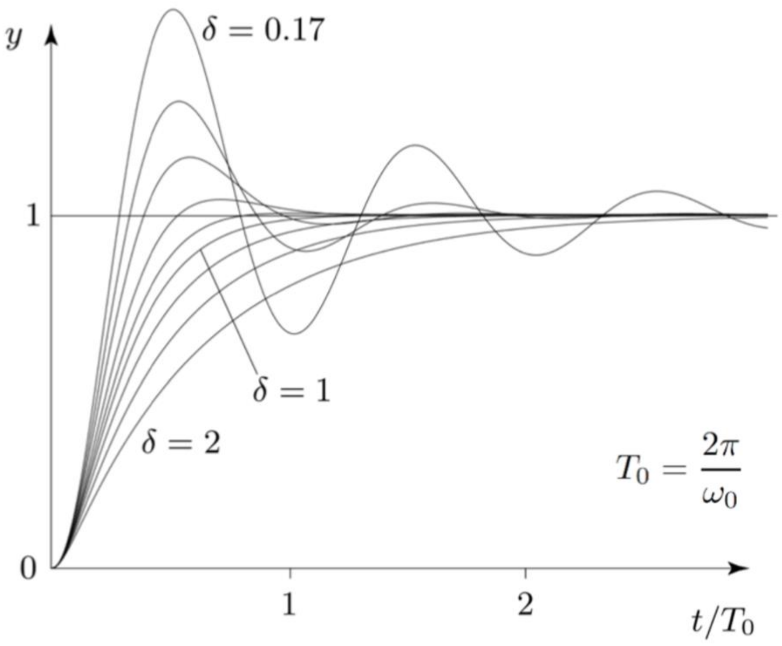


Time Response

Step Response: 2 order system

- Secon order system:
  - $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u(t),\ y(t) = x(t)$
  - $G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n+\omega_n^2}$
- Time response to a step input (depends on ζ): (in Mechanics III)
  - Underdamped ($\zeta < 1$):
    - $y(t) = 1 - \frac{1}{\cos(\varphi)} e^{\sigma t} \cos(\omega t + \varphi)$
    - $\varphi = \arctan\left(\frac{\zeta}{\omega}\right),\ \sigma = \zeta\omega_n,\ \omega = \sqrt{\omega_n^2 - (\zeta\omega_n)^2} = \omega_n\sqrt{1-\zeta^2}$
  - Overdamped ($\zeta > 1$): We get $\cosh()$ and $\sinh()$ terms (not relevant for you)





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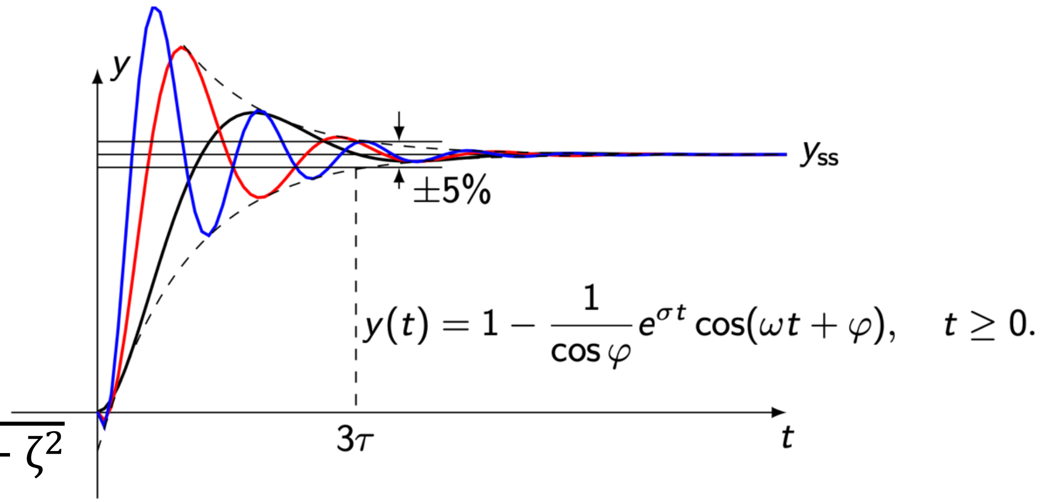
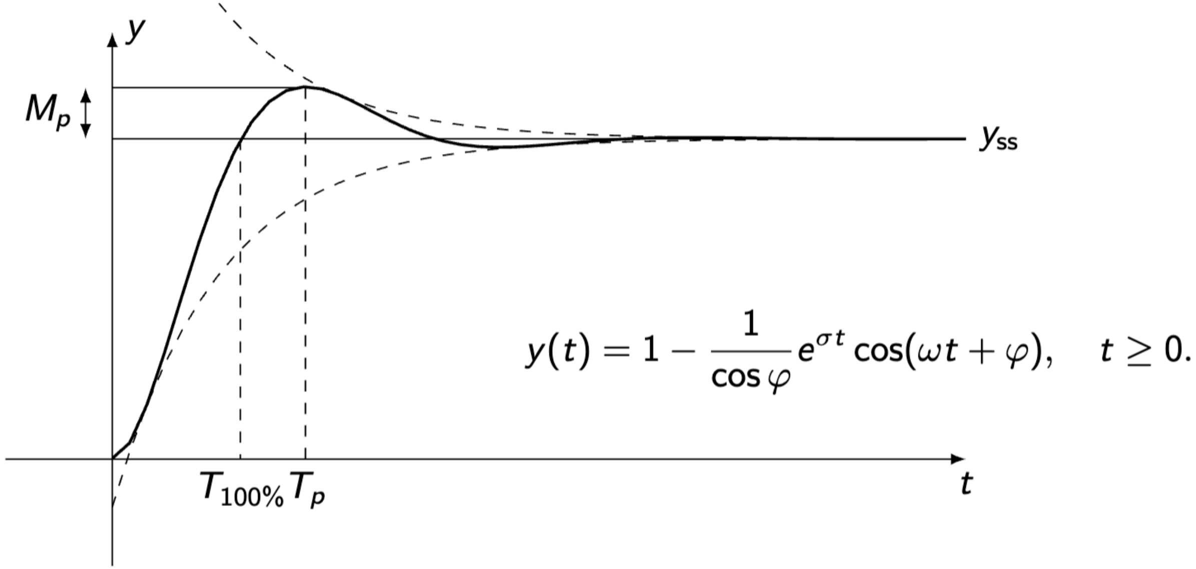
Time Response

Step Response: 2 order system

- Time response to a step input underdamped ($\zeta < 1$):
  - $y(t) = 1 - \frac{1}{\cos(\varphi)} e^{\sigma t} \cos(\omega t + \varphi)$
  - $\varphi = \arctan\left(\frac{\zeta}{\omega}\right)$, $\sigma = \zeta\omega_n$, $\omega = \sqrt{\omega_n^2 - (\zeta\omega_n)^2} = \omega_n\sqrt{1-\zeta^2}$



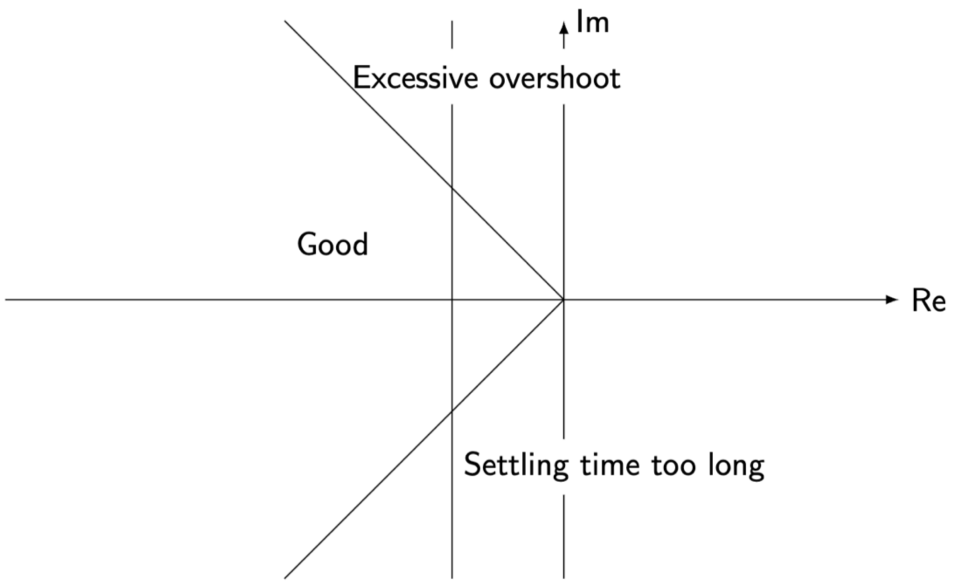
- What can we specify?
  - Settling time: (on exponential envelope)
    - $T_d = \frac{1}{\sigma} \ln\left(\frac{100}{1-d}\right) \rightarrow \sigma = \frac{\ln\left(\frac{100}{1-d}\right)}{T_d}$
  - Time to peak:
    - $T_p = \frac{\pi}{\omega}$
  - Overshoot: (low damping = high overshoot)
    - $M_p = e^{\frac{\sigma\pi}{\omega}} \rightarrow \zeta^2 = \frac{\left(\ln(M_p)\right)^2}{\pi^2 + \left(\ln(M_p)\right)^2}$
  - Rise Time:
    - $T_{100} = \frac{\frac{\pi}{2} - \varphi}{\omega} \approx \frac{\pi}{2\omega_n}$



Time Response

Step Response: Higher Order Systems

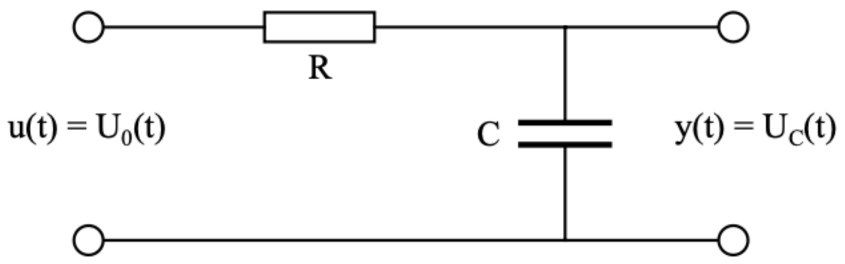
- What about higher order systems?
  - Approximate them with a 1 or 2 order system
  - Apply the specifications to the approximation
- What poles to choose?
  - No zeros: Chose the poles with the slowest decay time
  - With zeros: Highest residuals
- In General:
  - We want high real part (fast decay)
  - We want low imaginary part (smaller overshoot)
  - We can see this in the Imaginary Plane

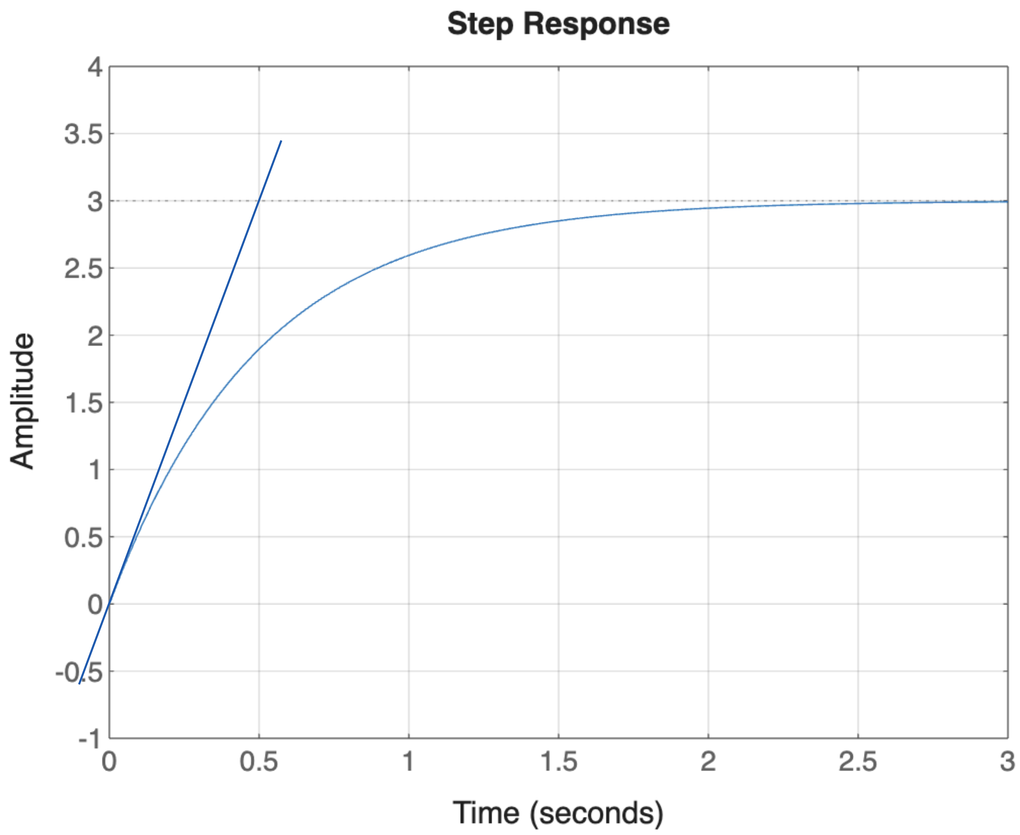


Time Response

Step Response: Example

- We have a electrical system that can be described by:
  - $u(t) = RC\dot{y}(t) + y(t), R = 1$
- We got the plot of a step input voltage:
  - $u(t) = a \cdot h(t)$
- What is the value of C?
  - $\dot{y}(t) = -\frac{1}{C}y(t) + \frac{a}{C}h(t), \tau = C, k = a$
  - τ: where $y(t)$ crosses $u(t)$
    - $\tau = 0.5 \rightarrow C = 0.5F$
- What are the value of a?
  - $y_{ss} = k = a = 3$
- If we choose $R = 2$ what does C need to be for an identical system behaviour?
  - $RC = 0.5 \rightarrow 2C = 0.5 \rightarrow C = 0.25$





PID Control

Proportional Part

- General Formulation:
  - $u(t) = k_p e(t) + k_d \dot{e}(t) + k_i \int e(t)dt$
  - $U(s) = \left(k_p + k_d s + \frac{k_i}{s}\right) E(s) = C(s)E(s)$
- We get: $T(s)\colon\ r(t) \to y(t) \text{ or } n(t) \to y(t),\ \ S(s)\colon r(t) \to e(t) \text{ or } d(t) \to y(t)$
  - $T(s) = \frac{\left(k_p + k_d s + \frac{k_i}{s}\right)G(s)}{1+\left(k_p + k_d s + \frac{k_i}{s}\right)G(s)},\ \ S(s) = \frac{1}{1+\left(k_p + k_d s + \frac{k_i}{s}\right)G(s)}$
- Proportional Part:
  - $T(s) = \frac{\left(k_p\right)G(s)}{1+\left(k_p\right)G(s)},\ \ S(s) = \frac{1}{1+\left(k_p\right)G(s)}$
  - High k_p: $T(s) \to 1, S(s) \to 0$
    - Faster response
    - Lower steady state error
    - Higher sensitivity to noise

PID Control

Integral Part

- General Formulation:
  - $u(t) = k_p e(t) + k_d \dot{e}(t) + k_i \int e(t) dt$
  - $U(s) = \left(k_p + k_d s + \frac{k_i}{s}\right) E(s) = C(s)E(s)$
- Proportional and Integrator Part:
  - As seen before integrator part reduces steady state error to 0
  - High k_i:
    - System starts to oscillate
      - Integrator fills up and needs to be emptied on the other side
    - Still sensitive to noise

PID Control

Derivative Part

- General Formulation:
  - $u(t) = k_p e(t) + k_d \dot{e}(t) + k_i \int e(t)dt$
  - $U(s) = \left(k_p + k_d s + \frac{k_i}{s}\right) E(s) = C(s)E(s)$
- Proportional, Derivative and Integrator Part:
  - High k_d:
    - Steady state error not affected
    - Less oscillation (more damping)
      - Slows down the system
    - Sensitivity to noise increases
      - High frequency noise derivatives are large

PID Control

Summary

- Proportional Part:
  - Faster response
  - Lower steady state error
  - Higher sensitivity to noise
  - Reduces stability margin -> can make system unstable
- Integrator Part:
  - Eliminates steady state error (step input)
  - Introduces oscillation
  - Reduces stability margin -> can make system unstable
- Derivative Part
  - Reduces Overshooting, Increases Damping
  - Improves stability margins
  - Very sensitive to noise
  - Not physically realizable (use approximation)

PID Control

PID Design/PID Tuning

- Freestyle:
  - Start with k_p add a bit of k_d to dampen and add k_i to remove ss-error, see what works
- Root Locus in the lecture
  - Recursively changing the values of k_p, k_d, k_i
- Ziegler-Nichols: systematic approach
  - Increase k_p until the system becomes marginally stable (start oscillating without decay)
    - Get k_p^* and $T^* = \frac{\omega^*}{2\pi}$
- Aström-Hägglund: systematic approach
  - Get k_p^* and T^* like Ziegler-Nichols
  - Get $|P(0)|$ using measurements of a step response
- Optimal Design and Stability not guaranteed
  - Real world testing often needed

Regler	k_p	T_i	T_d
P	$0.5 \cdot k_p^*$	$\infty \cdot T^*$	$0 \cdot T^*$
PI	$0.45 \cdot k_p^*$	$0.85 \cdot T^*$	$0 \cdot T^*$
PD	$0.55 \cdot k_p^*$	$\infty \cdot T^*$	$0.15 \cdot T^*$
PID	$0.60 \cdot k_p^*$	$0.50 \cdot T^*$	$0.125 \cdot T^*$

$$\kappa = \frac{1}{|P(0)| \cdot k_p^*}, \quad x = \alpha_{0,x} \cdot e^{\alpha_{1,x} \cdot \kappa + \alpha_{2,x} \cdot \kappa^2}$$

	$\mu_{\min} = 0.7$			$\mu_{\min} = 0.5$		
x	$\alpha_{0,x}$	$\alpha_{1,x}$	$\alpha_{2,x}$	$\alpha_{0,x}$	$\alpha_{1,x}$	$\alpha_{2,x}$
$\frac{k_p}{k_p^*}$	0.053	2.90	-2.6	0.13	1.9	-1.30
$\frac{T_i}{T^*}$	0.900	-4.40	2.7	0.90	-4.4	2.70

	$\mu_{\min} = 0.7$			$\mu_{\min} = 0.5$		
x	$\alpha_{0,x}$	$\alpha_{1,x}$	$\alpha_{2,x}$	$\alpha_{0,x}$	$\alpha_{1,x}$	$\alpha_{2,x}$
$\frac{k_p}{k_p^*}$	0.33	-0.31	-1.00	0.72	-1.60	1.20
$\frac{T_i}{T^*}$	0.76	-1.60	-0.36	0.59	-1.30	0.38
$\frac{T_d}{T^*}$	0.17	-0.46	-2.10	0.15	-1.40	0.56

Exercise 07

What to do?

- 1:
  - A) do
  - B) look at solution and think about the answer
  - C) look at solution and think about the answer
  - D) optional
  - E) play around a bit
- 2:
  - A) optional
  - B) do, need solution of A)
  - C) do, cumbersome but good practice
  - D) do
- 3:
  - A) do
  - B) do
  - C) optional