# Control Systems I
# Recitation 08

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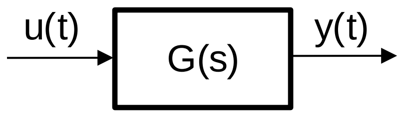
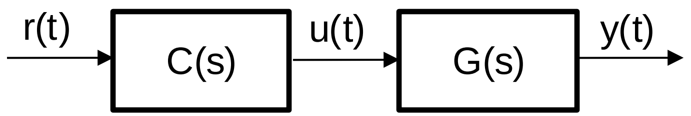
# Controllers and Feedback

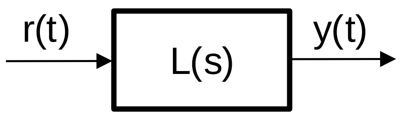
## Introduction

- Until now we only looked at the system: $u(t) \rightarrow y(t)$
  - Stability and Analysis until now only for this

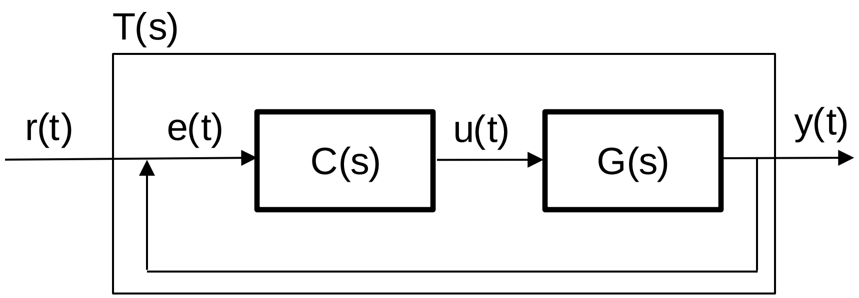



- We can introduce a Controller (no feedback yet):
  - **Open Loop** System: $r(t) \rightarrow y(t)$
    - $L(s) = C(s)G(s)$



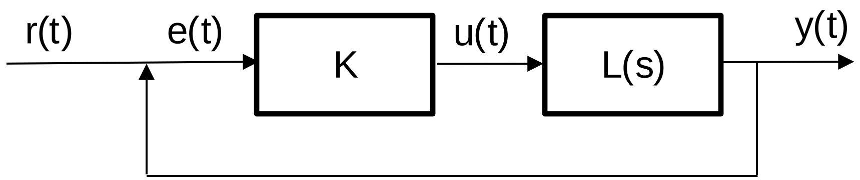




- We can Introduce Feedback:
  - **Closed Loop** System: $r(t) \rightarrow y(t)$
    - $T(s) = \frac{L(s)}{1+L(s)} = \frac{C(s)G(s)}{1+C(s)G(s)}$
  - Changes the system behaviour dynamically
    - Unstable –> stable
    - Stable -> “More” stable (quicker or less oscillation)
    - Stable -> unstable



# Root Locus

What?

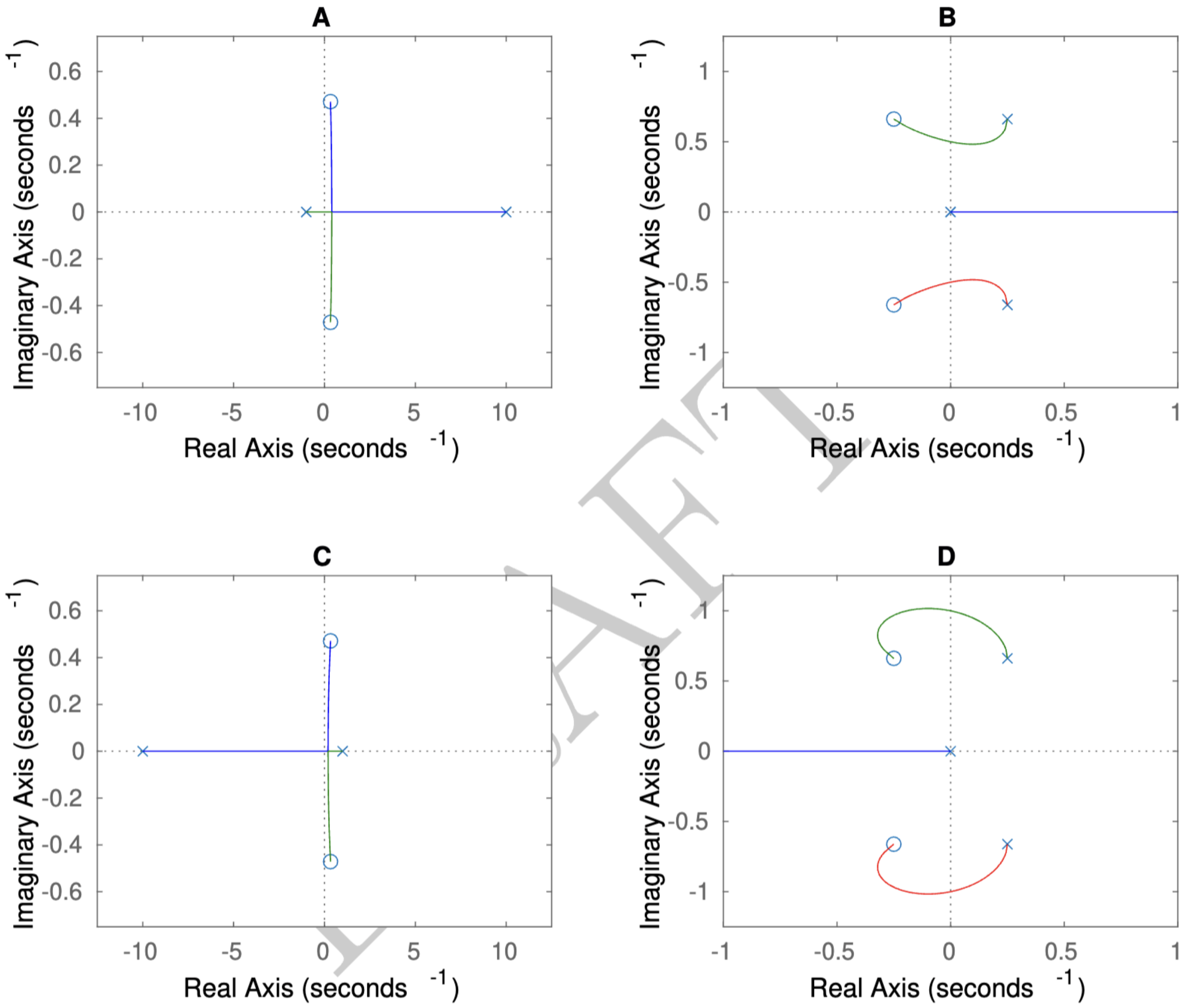



- Add proportional controller to the open loop system:
  - $kL(s) = k\frac{N(s)}{D(s)} \rightarrow T(s) = \frac{kL(s)}{1+kL(s)} = \frac{kN(s)}{D(s)+kN(s)}$
  - $kL(s) = k\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$
- Root Locus = Graphical analysis of the closed loop system for different values of $k$ or any other parameter
  - Plot the position of the zeros and poles for all possible $k$
- Using only the **open loop system** to analyse the **closed loop system**!
  - We only need to know $L(s)$!!!
  - Get quick (qualitative) info about the system response
- Different extremes:
  - $k = 0$: Poles of $T(s) \rightarrow$ Poles of $L(s)$
  - $k \rightarrow \infty$: Poles of $T(s) \rightarrow$ Zeros of $L(s)$
    - This also explains why we should avoid non-minimum phase zeros
    - Since degree of N(s) is smaller than D(s) the "excess" poles go to ∞

# Root Locus

## Extracting Information

- Get Open Loop Transfer Function:
  - Zeros are circles
  - Poles are crosses
  - $L(s) = \frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$
- Stability:
  - Check for poles with positive real part
  - Zeros with positive real part mean that the system becomes unstable if k to large
- Damping and Oscillation (poles and zeros):
  - Imaginary parts -> Oscillation
  - Negative real part -> Damping

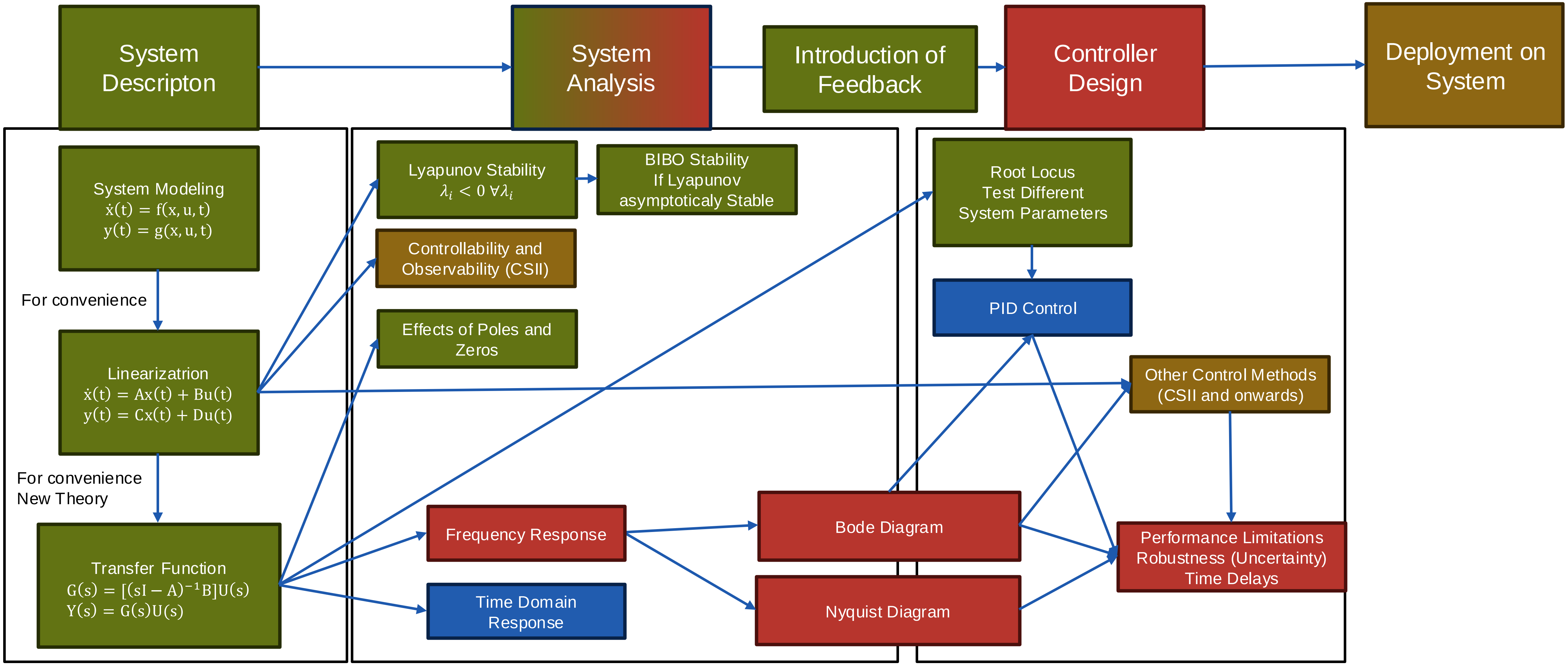


# Outline

- System Analysis
  - What is there to check?
  - Different Transfer Functions
  - Sidenote on Total System Behaviour
- Time Response
  - Steady State Error
  - Example
  - Step Response: 1 order system
  - Step Response: 2 order system
- PID Controller
  - Proportional Part
  - Integral Part
  - Derivative Part
  - PID Tuning

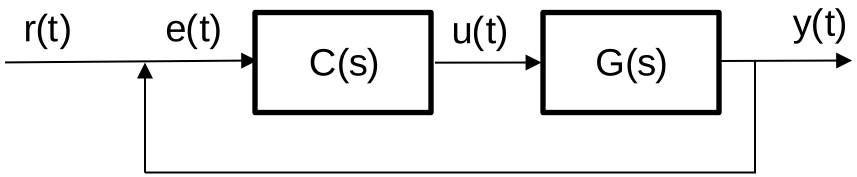
# Conceptual Recap

## Classical Control Approach



# System Analysis

What is there to check?

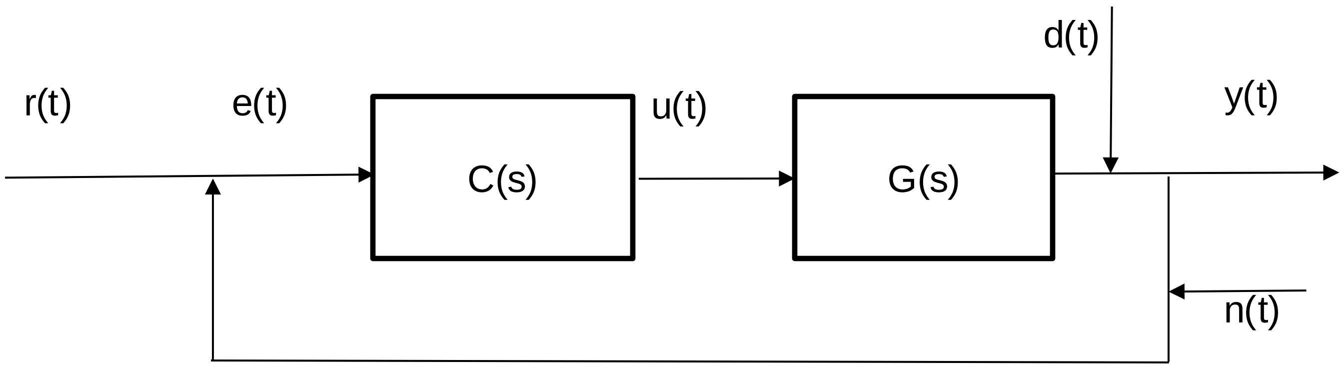



- Qualitative Analysis: (already seen)
  - Stability Analysis: Lyapunov, BIBO, Root Locus
- Quantitative Analysis: (today)
  - Steady State behaviour
    - Do we converge to the desired value?
  - Speed of Convergence
    - How fast do we converge?
  - Overshoot
    - Do we have oscillation and if so, how big are these?
- How about robustness? (next week)
  - Modelling errors
    - If the nominal system is stable is the real system also stable?
    - How big of an error can the system compensate for?
  - Noise/Disturbance Rejection
    - If we have disturbances/noise how good can the closed loop system handle these?

# System Analysis

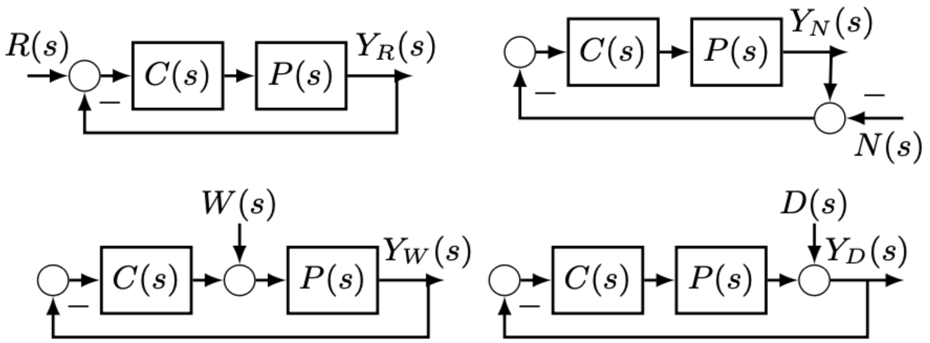
## Different Transfer Functions

- We saw the Closed Loop System:
  - **Complementary Sensitivity:** $\mathrm{r(t) \rightarrow y(t)}$ or $\mathrm{n(t) \rightarrow y(t)}$
    - $\mathrm{T(s)} = \frac{\mathrm{Y(s)}}{\mathrm{N(s)}} = \frac{\mathrm{Y(s)}}{\mathrm{R(s)}} = \frac{\mathrm{L(s)}}{\mathrm{1+L(s)}} = \frac{\mathrm{C(s)G(s)}}{\mathrm{1+C(s)G(s)}}$
- There are more:
  - **Sensitivity:** $\mathrm{r(t) \rightarrow e(t)}$ or $\mathrm{d(t) \rightarrow y(t)}$
    - $\mathrm{S(s)} = \frac{\mathrm{E(s)}}{\mathrm{R(s)}} = \frac{\mathrm{Y(s)}}{\mathrm{D(s)}} = \frac{1}{\mathrm{1+L(s)}} = \frac{1}{\mathrm{1+C(s)G(s)}}$
- We also see that $S(s) + \mathrm{T(s)} = 1$
  - We can either track the reference perfect or reject noise perfect



# System Analysis

Sidenote on Total System Behaviour




- We can do this for all possible inputs to the system:
- **We get:**

$$Y_R(s) = \frac{P(s)C(s)}{1+P(s)C(s)} \cdot R(s), \quad Y_N(s) = \frac{P(s)C(s)}{1+P(s)C(s)} \cdot N(s)$$

$$Y_W(s) = \frac{P(s)}{1+P(s)C(s)} \cdot W(s), \quad Y_D(s) = \frac{1}{1+P(s)C(s)} \cdot D(s)$$

- Linearity allows us to: $Y(s) = Y_R(s) + Y_N(s) + Y_W(s) + Y_D(s)$
  - We thus get: $T(s) = \frac{L(s)}{1+L(s)}, S(s) = \frac{1}{1+L(s)} \rightarrow$

$$Y(s) = S(s) \cdot [D(s) + P(s) \cdot W(s)] + T(s) \cdot [R(s) + N(s)]$$

- Learnings:
  - System is stable iff $\frac{1}{1+L(s)}$ is stable

# Time Response

## Steady State Error

- Error: $e(t) = r(t) - y(t)$ should ideally go to zero $\lim_{t\to\infty} e(t) = 0$
- Sensitivity describes the relation between $r(t) \rightarrow e(t)$: $E(s) = S(s)R(s) = \frac{1}{1+L(s)} R(s)$
- For a step input $R(s) = \frac{1}{s} \rightarrow E(s) = \frac{1}{s}\frac{1}{1+L(s)}$
  - Steady state behaviour:
    - $e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{1}{1+L(s)} = \frac{1}{1+L(0)}$
  - How can we make sure we have no steady state error?
    - $L(0) \rightarrow \infty \rightarrow$ need a pole at $s = 0$ (integrator)
    - If the closed loop system is stable and we have an integrator we have 0 steady state error
- How do we get $L(0)$?
  - $L(s) = \frac{k_{bd}}{s^q}\frac{\left(\frac{s}{-z_1}+1\right)\left(\frac{s}{-z_2}+1\right)\ldots\left(\frac{s}{-z_1}+1\right)}{\left(\frac{s}{-p_1}+1\right)\left(\frac{s}{-p_2}+1\right)\ldots\left(\frac{s}{-p_{n-q}}+1\right)} \xrightarrow{s\to 0} L(0) \rightarrow \frac{k_{bd}}{s^q}$
  - $q$ is called type $L(s)$ for $q = 0$ we get $L(0) = k_{bd}$ which is called the DC-gain
  - $q > 0$: $L(0) \rightarrow \infty$

# Time Response

## Steady State Error

- Steady State error for a step input $R(s) = \frac{1}{s} \rightarrow E(s) = \frac{1}{s}\frac{1}{1+L(s)}$
  - $e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{1}{1+L(s)} = \frac{1}{1+L(0)}$
- What about ramps $r(t) = \frac{1}{q!} t^q \rightarrow R(s) = \frac{1}{s^{q+1}}$?
  - $e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{1}{s^q}\frac{1}{1+L(s)}$
  - To have 0 ss-error we need at least one integrator more in our system

| $e_{ss}$ | $q=0$ | $q=1$ | $q=2$ |
|---|---|---|---|
| Type 0 | $\frac{1}{1+k_{\text{Bode}}}$ | $\infty$ | $\infty$ |
| Type 1 | 0 | $\frac{1}{k_{\text{Bode}}}$ | $\infty$ |
| Type 2 | 0 | 0 | $\frac{1}{k_{\text{Bode}}}$ |

- The same derivations go for disturbances!

# Time Response

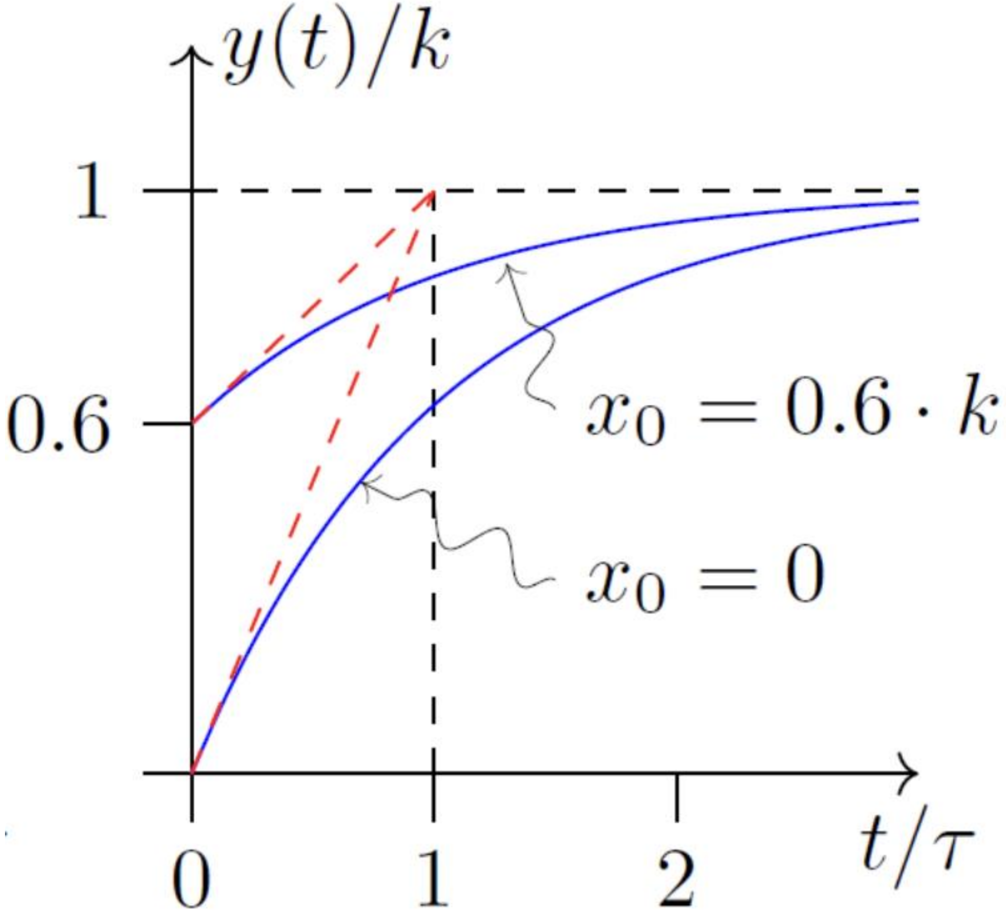
## Steady State Error - Example

- Given the system: $G(s) = \frac{s+3}{s^3+5s+3}$ and a Controller $C(s) = 3 + 6s$
- What is the steady state error for a step input a linear ramp?
  - 
  - 
  - Step input:
  - Impulse input:
    - 
    - 
  - Propose a controller that we get a zero ss error:
    - 

# Time Response

Step Response: 2 order system

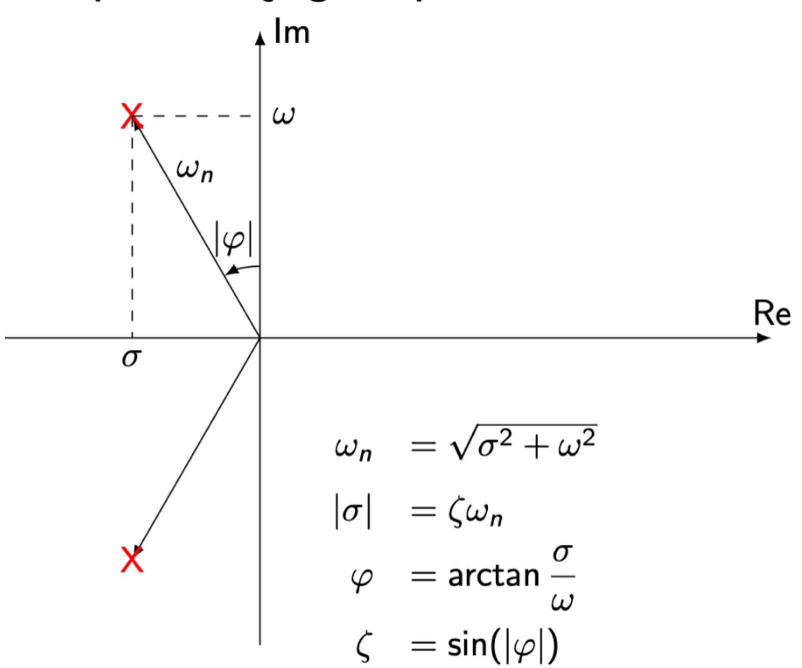
- First order system:
  - $\dot{x}(t) = -\frac{1}{\tau}x(t) + \frac{k}{\tau}u(t),\ y(t) = x(t)$
  - $G(s) = \frac{k}{\tau s+1}$
- Time response to a step input:
  - $y(t) = x_0 e^{-\frac{t}{\tau}} + k(1 - e^{-\frac{t}{\tau}})$
- What we can see:
  - $x_0$ determines the start value
  - DC-gain/Steady State: $y_{ss} = k$
  - Evaluating the derivative at $t = 0$:
    - The tangent crosses $y = k$ at $t = \tau$
  - Large $\tau$ slow convergence
  - Settling Time: Time it takes to be at $d\%$ of $y_{ss}$ (beware this is the opposite as in the lecture)
    - $T_d = \tau \ln\left(\frac{100}{1-d}\right) \rightarrow \tau = \frac{T_d}{\ln\left(\frac{100}{1-d}\right)}$

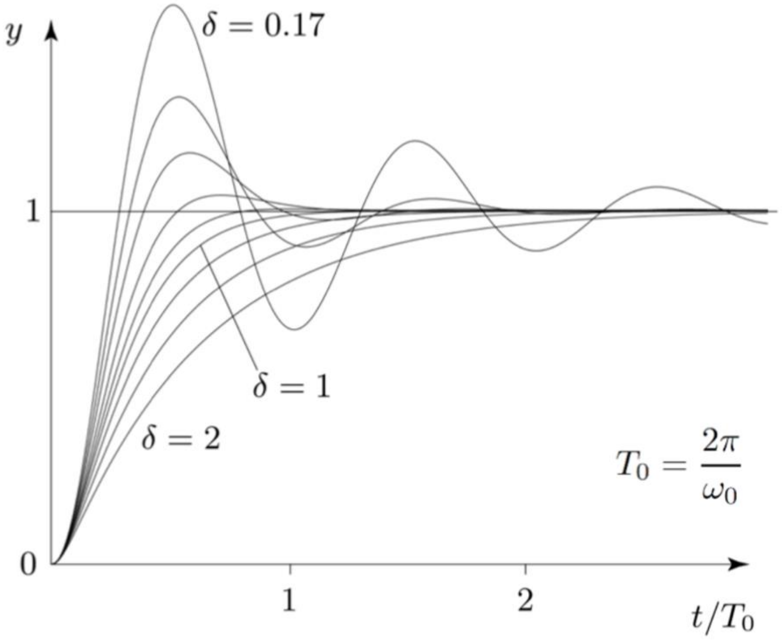


# Time Response

## Step Response: 2 order system

- Secon order system:
  - $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u(t),\ \ y(t) = x(t)$
  - $G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n+\omega_n^2}$
- Time response to a step input (depends on ζ): (in Mechanics III)
  - Underdamped ($\zeta < 1$):
    - $y(t) = 1 - \frac{1}{\cos(\varphi)} e^{\sigma t} \cos(\omega t + \varphi)$
    - $\varphi = \arctan\left(\frac{\zeta}{\omega}\right),\ \ \sigma = \zeta\omega_n,\ \ \omega = \sqrt{\omega_n^2 - (\zeta\omega_n)^2} = \omega_n\sqrt{1-\zeta^2}$
  - Overdamped ($\zeta > 1$): We get $\cosh()$ and $\sinh()$ terms (not relevant for you)

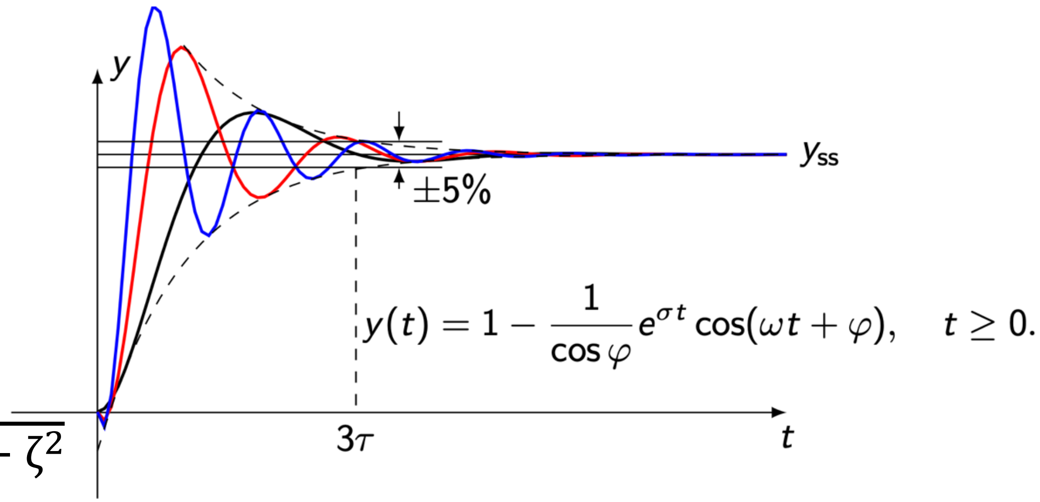


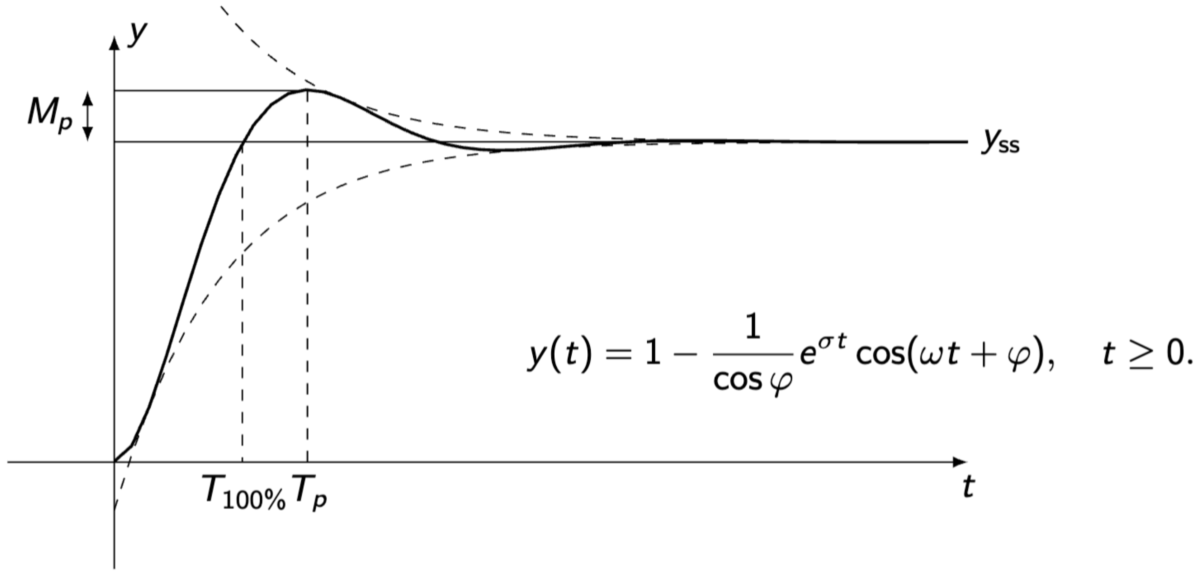
# Time Response

## Step Response: 2 order system

- Time response to a step input underdamped ($\zeta < 1$):
  - $y(t) = 1 - \frac{1}{\cos(\varphi)} e^{\sigma t} \cos(\omega t + \varphi)$
  - $\varphi = \arctan\left(\frac{\zeta}{\omega}\right), \ \sigma = \zeta\omega_n, \ \omega = \sqrt{\omega_n^2 - (\zeta\omega_n)^2} = \omega_n\sqrt{1-\zeta^2}$



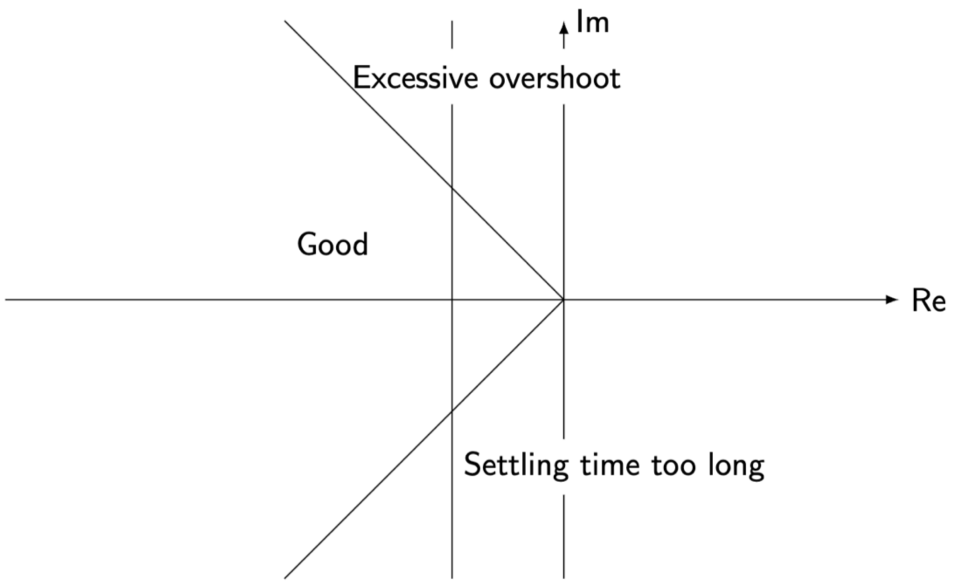

- What can we specify?
  - Settling time: (on exponential envelope)
    - $T_d = \frac{1}{\sigma}\ln\left(\frac{100}{1-d}\right) \rightarrow \sigma = \frac{\ln\left(\frac{100}{1-d}\right)}{T_d}$
  - Time to peak:
    - $T_p = \frac{\pi}{\omega}$
  - Overshoot: (low damping = high overshoot)
    - $M_p = e^{\frac{\sigma\pi}{\omega}} \rightarrow \zeta^2 = \frac{\left(\ln(M_p)\right)^2}{\pi^2 + \left(\ln(M_p)\right)^2}$
  - Rise Time:
    - $T_{100} = \frac{\frac{\pi}{2} - \varphi}{\omega} \approx \frac{\pi}{2\omega_n}$



# Time Response

## Step Response: Higher Order Systems

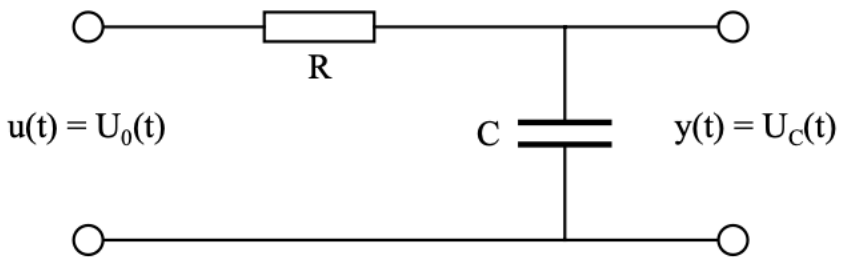
- What about higher order systems?
  - Approximate them with a 1 or 2 order system
  - Apply the specifications to the approximation
- What poles to choose?
  - No zeros: Chose the poles with the slowest decay time
  - With zeros: Highest residuals
- In General:
  - We want high real part (fast decay)
  - We want low imaginary part (smaller overshoot)
  - We can see this in the Imaginary Plane

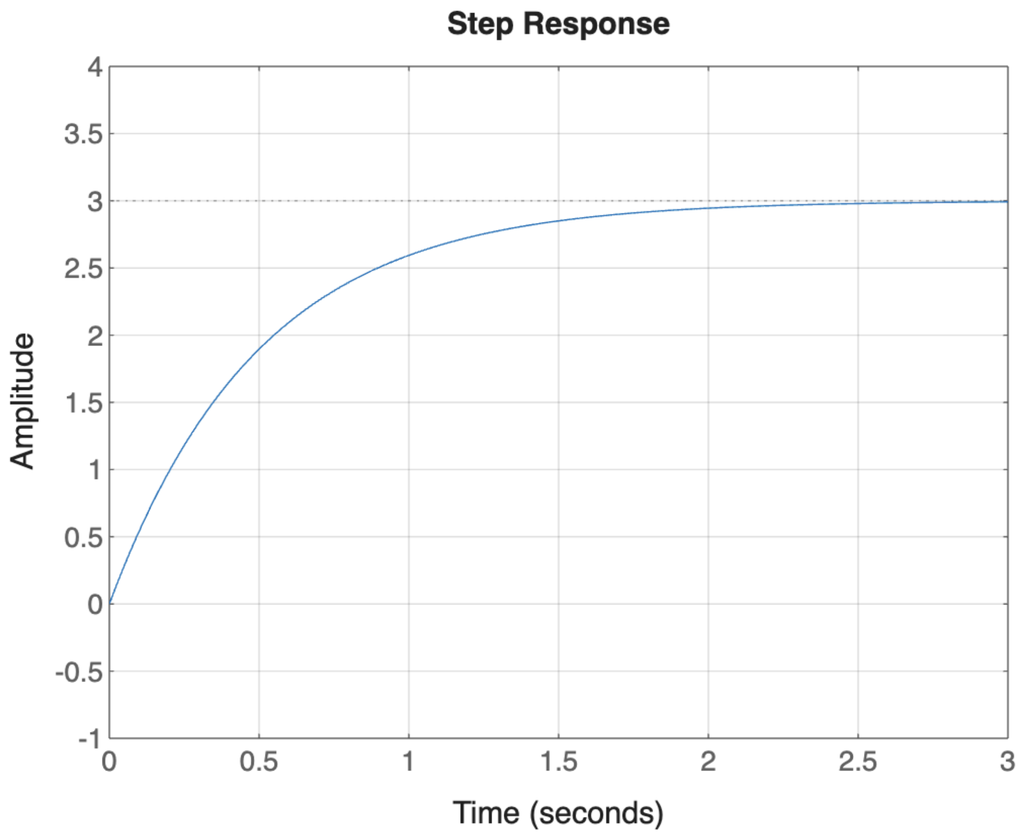


# Time Response

## Step Response: Example

- We have a electrical system that can be described by:
  - $u(t) = RC\dot{y}(t) + y(t), R = 1$
- We got the plot of a step input voltage:
  - $u(t) = a \cdot h(t)$
- What is the value of $C$?
  -
  -
    -
- What are the value of $a$?
  -
- If we choose $R = 2$ what does $C$ need to be for an identical system behaviour?
  -



# PID Control

## Proportional Part

- General Formulation:
  - $u(t) = k_p e(t) + k_d \dot{e}(t) + k_i \int e(t)dt$
  - $U(s) = \left(k_p + k_d s + \frac{k_i}{s}\right) E(s) = C(s)E(s)$
- We get: $T(s)$: $r(t) \rightarrow y(t)$ or $n(t) \rightarrow y(t)$, $S(s)$: $r(t) \rightarrow e(t)$ or $d(t) \rightarrow y(t)$
  - $T(s) = \frac{\left(k_p + k_d s + \frac{k_i}{s}\right)G(s)}{1+\left(k_p + k_d s + \frac{k_i}{s}\right)G(s)}$, $S(s) = \frac{1}{1+\left(k_p + k_d s + \frac{k_i}{s}\right)G(s)}$
- Proportional Part:
  - $T(s) = \frac{(k_p)G(s)}{1+(k_p)G(s)}$, $S(s) = \frac{1}{1+(k_p)G(s)}$
  - High $k_p$: $T(s) \rightarrow 1, S(s) \rightarrow 0$
    - Faster response
    - Lower steady state error
    - Higher sensitivity to noise

# PID Control

## Integral Part

- General Formulation:
  - $u(t) = k_p e(t) + k_d \dot{e}(t) + k_i \int e(t)dt$
  - $U(s) = \left(k_p + k_d s + \frac{k_i}{s}\right) E(s) = C(s)E(s)$
- Proportional and Integrator Part:
  - As seen before integrator part reduces steady state error to 0
  - High $k_i$:
    - System starts to oscillate
      - Integrator fills up and needs to be emptied on the other side
    - Still sensitive to noise

# PID Control

## Derivative Part

- General Formulation:
  - $u(t) = k_p e(t) + k_d \dot{e}(t) + k_i \int e(t) dt$
  - $U(s) = \left(k_p + k_d s + \frac{k_i}{s}\right) E(s) = C(s)E(s)$
- Proportional, Derivative and Integrator Part:
  - High $k_d$:
    - Steady state error not affected
    - Less oscillation (more damping)
      - Slows down the system
    - Sensitivity to noise increases
      - High frequency noise derivatives are large

# PID Control

## Summary

- Proportional Part:
  - Faster response
  - Lower steady state error
  - Higher sensitivity to noise
  - Reduces stability margin -> can make system unstable
- Integrator Part:
  - Eliminates steady state error (step input)
  - Introduces oscillation
  - Reduces stability margin -> can make system unstable
- Derivative Part
  - Reduces Overshooting, Increases Damping
  - Improves stability margins
  - Very sensitive to noise
  - Not physically realizable (use approximation)

# PID Control

## PID Design/PID Tuning

- Freestyle:
  - Start with $k_p$ add a bit of $k_d$ to dampen and add $k_i$ to remove ss-error, see what works
- Root Locus in the lecture
  - Recursively changing the values of $k_p, k_d, k_i$
- Ziegler-Nichols: systematic approach
  - Increase $k_p$ until the system becomes marginally stable (start oscillating without decay)
    - Get $k_p^*$ and $T^* = \frac{\omega^*}{2\pi}$
- Aström-Hägglund: systematic approach
  - Get $k_p^*$ and $T^*$ like Ziegler-Nichols
  - Get $|P(0)|$ using measurements of a step response
- Optimal Design and Stability not guaranteed
  - Real world testing often needed

| Regler | $k_p$ | $T_i$ | $T_d$ |
|---|---|---|---|
| P | $0.5 \cdot k_p^*$ | $\infty \cdot T^*$ | $0 \cdot T^*$ |
| PI | $0.45 \cdot k_p^*$ | $0.85 \cdot T^*$ | $0 \cdot T^*$ |
| PD | $0.55 \cdot k_p^*$ | $\infty \cdot T^*$ | $0.15 \cdot T^*$ |
| PID | $0.60 \cdot k_p^*$ | $0.50 \cdot T^*$ | $0.125 \cdot T^*$ |

$$\kappa = \frac{1}{|P(0)| \cdot k_p^*}, \quad x = \alpha_{0,x} \cdot e^{\alpha_{1,x} \cdot \kappa + \alpha_{2,x} \cdot \kappa^2}$$

| | $\mu_{\min} = 0.7$ | | | $\mu_{\min} = 0.5$ | | |
|---|---|---|---|---|---|---|
| $x$ | $\alpha_{0,x}$ | $\alpha_{1,x}$ | $\alpha_{2,x}$ | $\alpha_{0,x}$ | $\alpha_{1,x}$ | $\alpha_{2,x}$ |
| $\frac{k_p}{k_p^*}$ | 0.053 | 2.90 | −2.6 | 0.13 | 1.9 | −1.30 |
| $\frac{T_i}{T^*}$ | 0.900 | −4.40 | 2.7 | 0.90 | −4.4 | 2.70 |

| | $\mu_{\min} = 0.7$ | | | $\mu_{\min} = 0.5$ | | |
|---|---|---|---|---|---|---|
| $x$ | $\alpha_{0,x}$ | $\alpha_{1,x}$ | $\alpha_{2,x}$ | $\alpha_{0,x}$ | $\alpha_{1,x}$ | $\alpha_{2,x}$ |
| $\frac{k_p}{k_p^*}$ | 0.33 | −0.31 | −1.00 | 0.72 | −1.60 | 1.20 |
| $\frac{T_i}{T^*}$ | 0.76 | −1.60 | −0.36 | 0.59 | −1.30 | 0.38 |
| $\frac{T_d}{T^*}$ | 0.17 | −0.46 | −2.10 | 0.15 | −1.40 | 0.56 |

# Exercise 07

What to do?

- 1:
  - A) do
  - B) look at solution and think about the answer
  - C) look at solution and think about the answer
  - D) optional
  - E) play around a bit
- 2:
  - A) optional
  - B) do, need solution of A)
  - C) do, cumbersome but good practice
  - D) do
- 3:
  - A) do
  - B) do
  - C) optional