Control Systems I
Recitation 09

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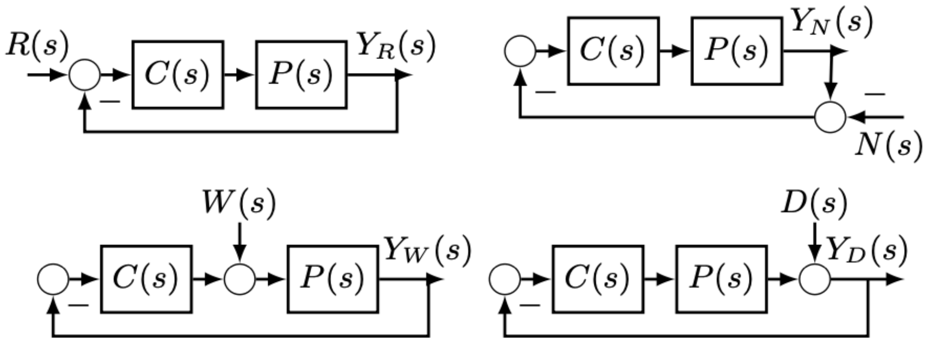
Last Week

System Analysis - Different Transfer Functions



- **Transfer Functions for different Input-Output pairs**

$$Y_R(s) = \frac{P(s)C(s)}{1+P(s)C(s)} \cdot R(s), \quad Y_N(s) = \frac{P(s)C(s)}{1+P(s)C(s)} \cdot N(s)$$

$$Y_W(s) = \frac{P(s)}{1+P(s)C(s)} \cdot W(s), \quad Y_D(s) = \frac{1}{1+P(s)C(s)} \cdot D(s)$$

  - Complementary Sensitivity: $T(s) = \frac{Y(s)}{N(s)} = \frac{Y(s)}{R(s)} = \frac{L(s)}{1+L(s)} = \frac{C(s)G(s)}{1+C(s)G(s)}$
  - Sensitivity: $S(s) = \frac{E(s)}{R(s)} = \frac{Y(s)}{D(s)} = \frac{1}{1+L(s)} = \frac{1}{1+C(s)G(s)}$

- Linearity allows us to: $Y(s) = Y_R(s) + Y_N(s) + Y_W(s) + Y_D(s)$

$$Y(s) = S(s) \cdot [D(s) + P(s) \cdot W(s)] + T(s) \cdot [R(s) + N(s)]$$

- Learnings:
  - System is stable iff $\frac{1}{1+L(s)}$ is stable
  - Conflicting Goals: $S(s) + T(s) = 1$

Last Week

System Analysis - Steady State Error

- Sensitivity describes the relation between $r(t) \rightarrow e(t)$: $E(s) = S(s)R(s) = \frac{1}{1+L(s)}R(s)$, ideally $e(t) \rightarrow 0$
- Steady State error for a step input $R(s) = \frac{1}{s} \rightarrow E(s) = \frac{1}{s}\frac{1}{1+L(s)}$
  - $e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{1}{1+L(s)} = \frac{1}{1+L(0)}$
- Steady State error for ramps $r(t) = \frac{1}{q!}t^q \rightarrow R(s) = \frac{1}{s^{q+1}}$
  - $e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{1}{s^q}\frac{1}{1+L(s)}$
  - To have 0 ss-error we need at least one integrator more in our system

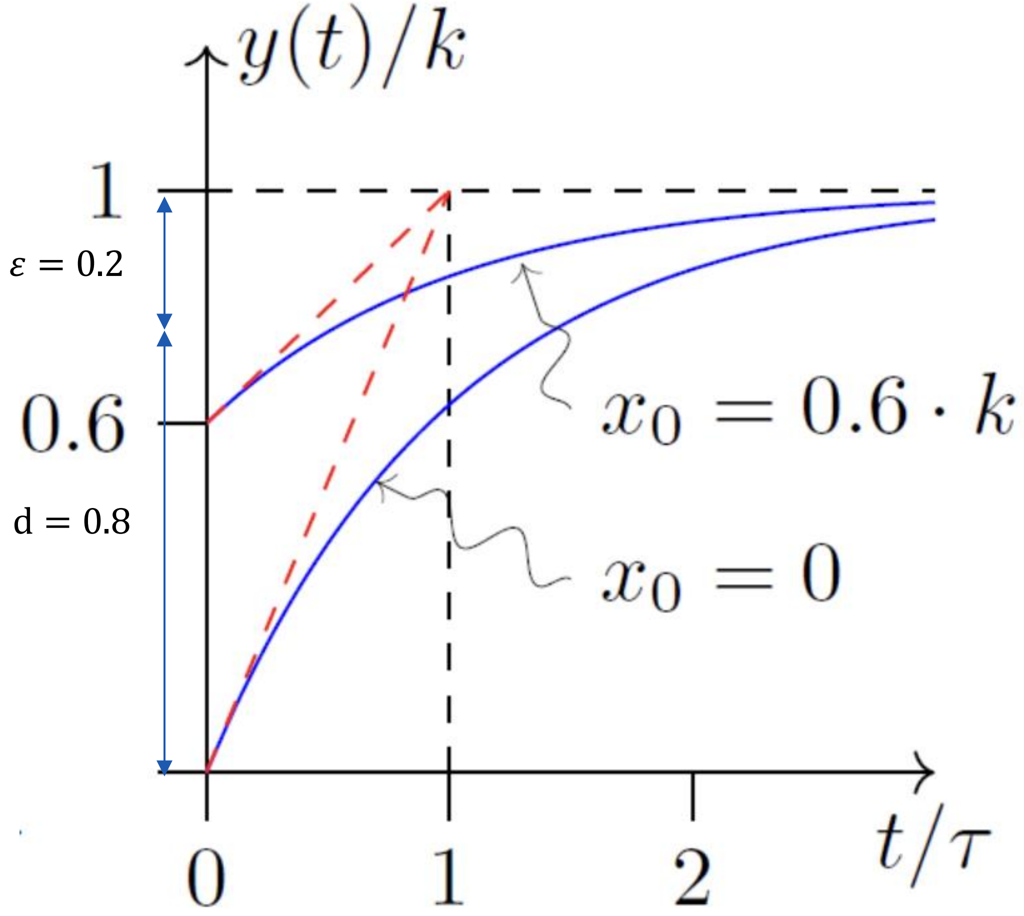
e_{ss}	$q=0$	$q=1$	$q=2$
Type 0	$\frac{1}{1+k_{\text{Bode}}}$	∞	∞
Type 1	0	$\frac{1}{k_{\text{Bode}}}$	∞
Type 2	0	0	$\frac{1}{k_{\text{Bode}}}$

- $L(s) = \frac{k_{bd}}{s^q}\frac{\left(\frac{s}{-z_1}+1\right)\left(\frac{s}{-z_2}+1\right)\ldots\left(\frac{s}{-z_1}+1\right)}{\left(\frac{s}{-p_1}+1\right)\left(\frac{s}{-p_2}+1\right)\ldots\left(\frac{s}{-p_{n-q}}+1\right)} \xrightarrow{s\to 0} L(0) \rightarrow \frac{k_{bd}}{s^q}$
  - q is called type $L(s)$ for $q = 0$ we get $L(0) = k_{bd}$ which is called the DC-gain
  - $q > 0$: $L(0) \rightarrow \infty$
- The same derivations go for disturbances!

Time Response

Step Response: 1 order system

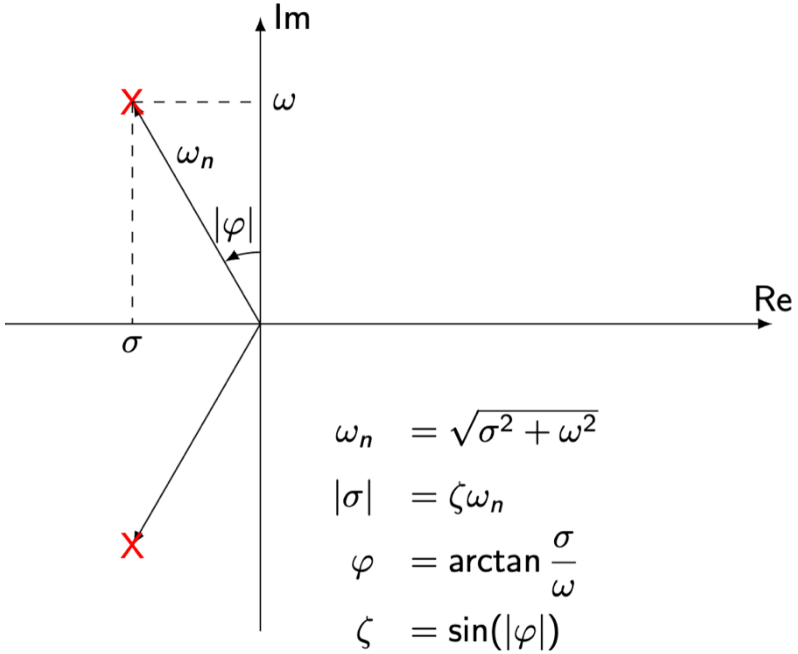
- First order system:
  - $\dot{x}(t) = -\frac{1}{\tau}x(t) + \frac{k}{\tau}u(t),\ y(t) = x(t) \rightarrow G(s) = \frac{k}{\tau s+1}$
- Time response to a step input:
  - $y(t) = x_0 e^{-\frac{t}{\tau}} + k(1 - e^{-\frac{t}{\tau}})$
- What we can see:
  - x_0 determines the start value
  - DC-gain/Steady State: $y_{ss} = k$
  - Evaluating the derivative at $t = 0$:
    - The tangent crosses $y = k$ at $t = \tau$
  - Large τ slow convergence
  - Settling Time: Time it takes to be at d of y_{ss} (beware this is the opposite as in the lecture)
    - $T_d = \tau \ln\left(\frac{1}{1-d}\right) \rightarrow \tau = \frac{T_d}{\ln\left(\frac{1}{1-d}\right)},\ T_d = \tau \ln\left(\frac{1}{\varepsilon}\right) \rightarrow \tau = \frac{T_d}{\ln\left(\frac{1}{\varepsilon}\right)}$
    - $\tau = f(\text{controller parameters})$ tune them to reach desired performance

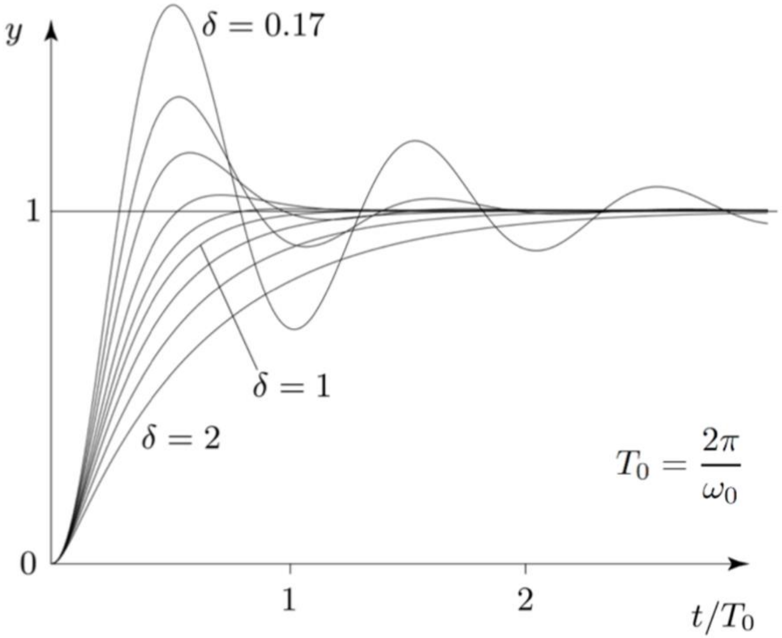


Time Response

Step Response: 2 order system

- Secon order system:
  - $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u(t),\ y(t) = x(t)$
  - $G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n+\omega_n^2}$
- Time response to a step input (depends on ζ): (in Mechanics III)
  - Underdamped ($\zeta < 1$):
    - $y(t) = 1 - \frac{1}{\cos(\varphi)} e^{\sigma t} \cos(\omega t + \varphi)$
    - $\varphi = \arctan\left(\frac{\zeta}{\omega}\right),\ \sigma = \zeta\omega_n,\ \omega = \sqrt{\omega_n^2 - (\zeta\omega_n)^2} = \omega_n\sqrt{1-\zeta^2}$

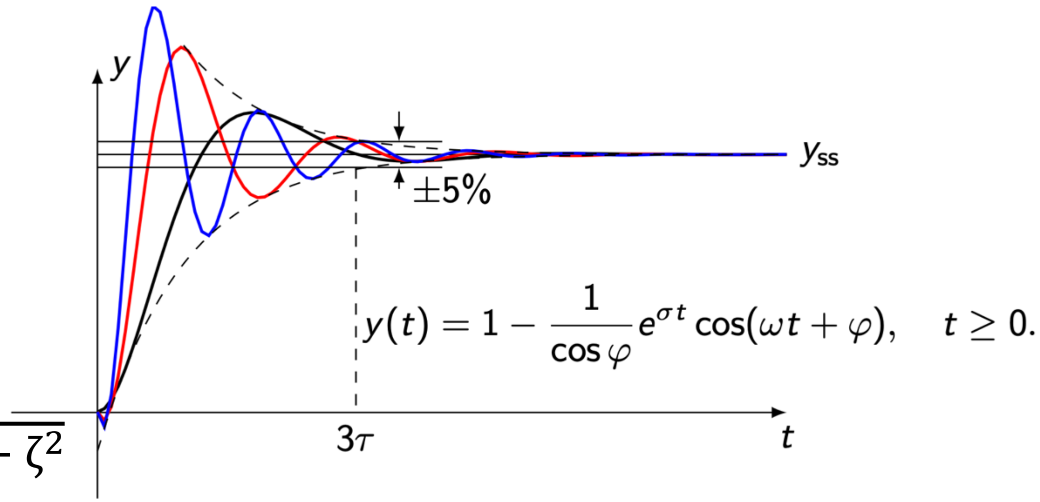




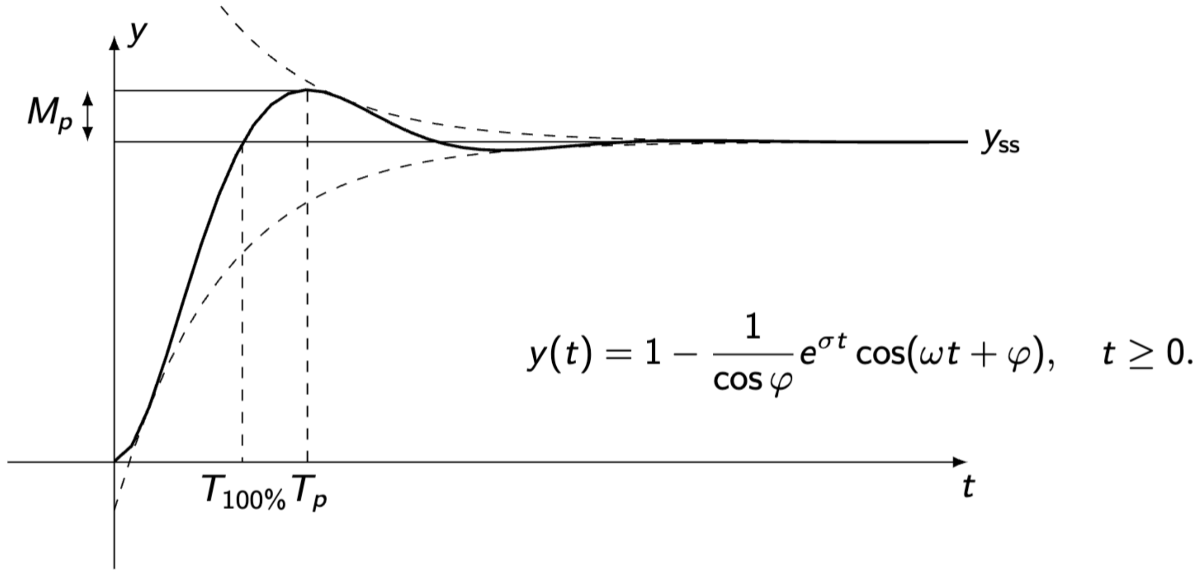
Time Response

Step Response: 2 order system

- Time response to a step input underdamped ($\zeta < 1$):
  - $y(t) = 1 - \frac{1}{\cos(\varphi)} e^{\sigma t} \cos(\omega t + \varphi)$
  - $\varphi = \arctan\left(\frac{\zeta}{\omega}\right), \ \sigma = \zeta\omega_n, \ \omega = \sqrt{\omega_n^2 - (\zeta\omega_n)^2} = \omega_n\sqrt{1-\zeta^2}$



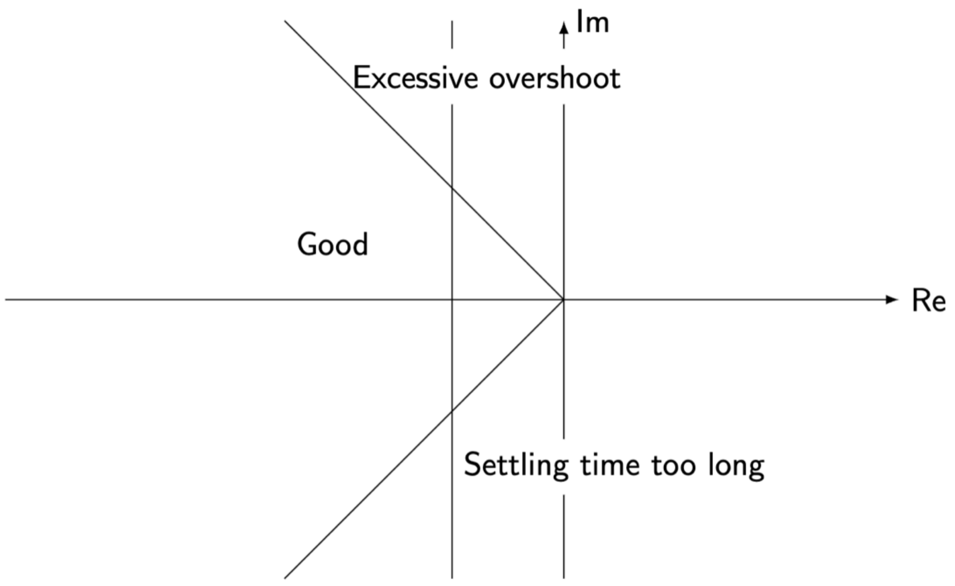
- What can we specify?
  - Settling time: (on exponential envelope)
    - $T_d = \frac{1}{\sigma} \ln\left(\frac{1}{1-d}\right) \rightarrow \sigma = \frac{\ln\left(\frac{1}{1-d}\right)}{T_d}$
  - Time to peak:
    - $T_p = \frac{\pi}{\omega}$
  - Overshoot: (low damping = high overshoot)
    - $M_p = e^{\frac{\sigma\pi}{\omega}} \rightarrow \zeta^2 = \frac{\left(\ln(M_p)\right)^2}{\pi^2 + \left(\ln(M_p)\right)^2}$
  - Rise Time:
    - $T_{100} = \frac{\frac{\pi}{2} - \varphi}{\omega} \approx \frac{\pi}{2\omega_n}$



Time Response

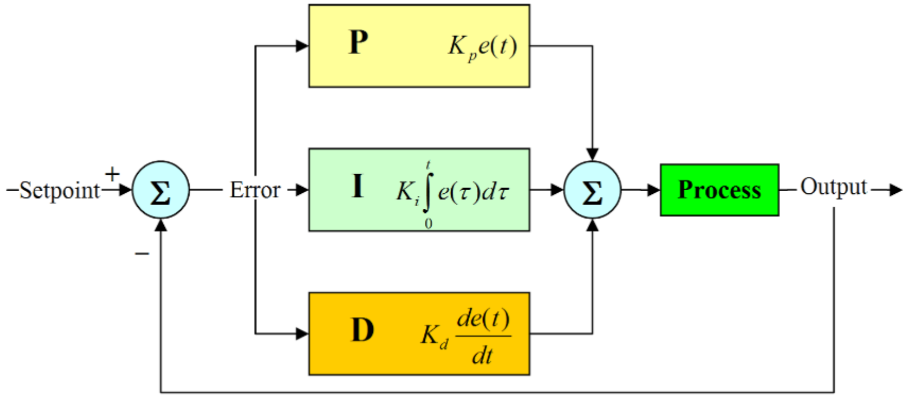
Step Response: Higher Order Systems

- What about higher order systems?
  - Approximate them with a 1 or 2 order system
  - Apply the specifications to the approximation
- What poles to choose?
  - No zeros: Chose the poles with the slowest decay time
  - With zeros: Highest residuals
- In General:
  - We want high real part (fast decay)
  - We want low imaginary part (smaller overshoot)
  - We can see this in the Imaginary Plane



PID Control

Proportional Part



- General Formulation:
  - $u(t) = k_p e(t) + k_d \dot{e}(t) + k_i \int e(t)dt$
  - $U(s) = \left(k_p + k_d s + \frac{k_i}{s}\right) E(s) = C(s)E(s)$
- Proportional Part:
  - Faster response, Lower steady state error
  - Higher sensitivity to noise, Reduces stability margin -> can make system unstable
- Integrator Part:
  - Eliminates steady state error (step input)
  - Introduces oscillation, Reduces stability margin -> can make system unstable
- Derivative Part
  - Reduces Overshooting, Increases Damping, Improves stability margins
  - Very sensitive to noise
  - Not physically realizable (use approximation)

PID Control

PID Design/PID Tuning

- Freestyle:
  - Start with k_p add a bit of k_d to dampen and add k_i to remove ss-error, see what works
- Root Locus in the lecture
  - Recursively changing the values of k_p, k_d, k_i
- Ziegler-Nichols: systematic approach
  - Increase k_p until the system becomes marginally stable (start oscillating without decay)
    - Get k_p^* and $T^* = \frac{\omega^*}{2\pi}$
- Aström-Hägglund: systematic approach
  - Get k_p^* and T^* like Ziegler-Nichols
  - Get $|P(0)|$ using measurements of a step response
- Optimal Design and Stability not guaranteed
  - Real world testing often needed

Regler	k_p	T_i	T_d
P	$0.5 \cdot k_p^*$	$\infty \cdot T^*$	$0 \cdot T^*$
PI	$0.45 \cdot k_p^*$	$0.85 \cdot T^*$	$0 \cdot T^*$
PD	$0.55 \cdot k_p^*$	$\infty \cdot T^*$	$0.15 \cdot T^*$
PID	$0.60 \cdot k_p^*$	$0.50 \cdot T^*$	$0.125 \cdot T^*$

$$\kappa = \frac{1}{|P(0)| \cdot k_p^*}, \qquad x = \alpha_{0,x} \cdot e^{\alpha_{1,x} \cdot \kappa + \alpha_{2,x} \cdot \kappa^2}$$

x	$\mu_{\min} = 0.7$			$\mu_{\min} = 0.5$		
	$\alpha_{0,x}$	$\alpha_{1,x}$	$\alpha_{2,x}$	$\alpha_{0,x}$	$\alpha_{1,x}$	$\alpha_{2,x}$
$\frac{k_p}{k_p^*}$	0.053	2.90	−2.6	0.13	1.9	−1.30
$\frac{T_i}{T^*}$	0.900	−4.40	2.7	0.90	−4.4	2.70

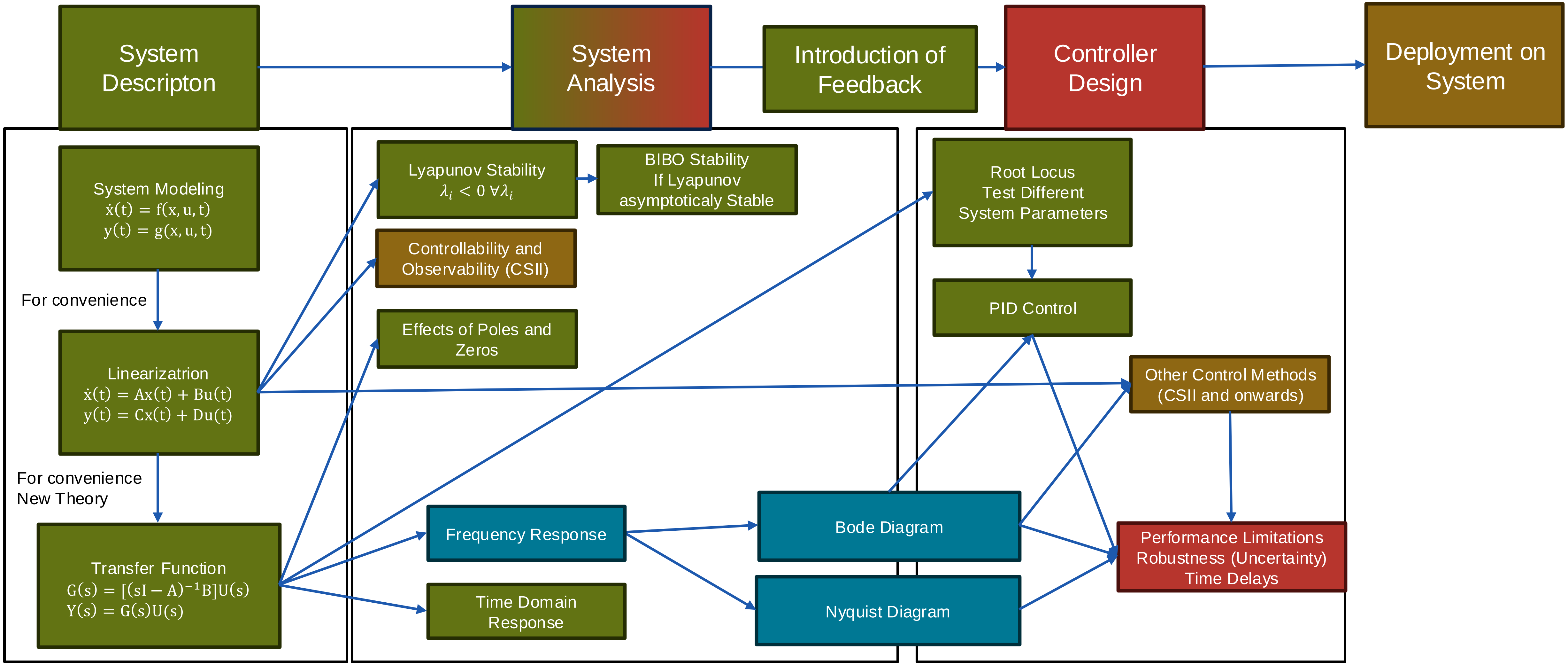
x	$\mu_{\min} = 0.7$			$\mu_{\min} = 0.5$		
	$\alpha_{0,x}$	$\alpha_{1,x}$	$\alpha_{2,x}$	$\alpha_{0,x}$	$\alpha_{1,x}$	$\alpha_{2,x}$
$\frac{k_p}{k_p^*}$	0.33	−0.31	−1.00	0.72	−1.60	1.20
$\frac{T_i}{T^*}$	0.76	−1.60	−0.36	0.59	−1.30	0.38
$\frac{T_d}{T^*}$	0.17	−0.46	−2.10	0.15	−1.40	0.56

Outline

- Frequency Response
  - What?
  - Why?
- Bode Plot
  - What?
  - Example – Reading of a Bode Plot
  - Drawing a Bode Plot
  - Example – Drawing a Bode Plot
  - Bodes Law
- Polar Plot / Nyquist Plot
  - What?
  - Example – Drawing a Nyquist Plot

Conceptual Recap

Classical Control Approach



Frequency Response

What?

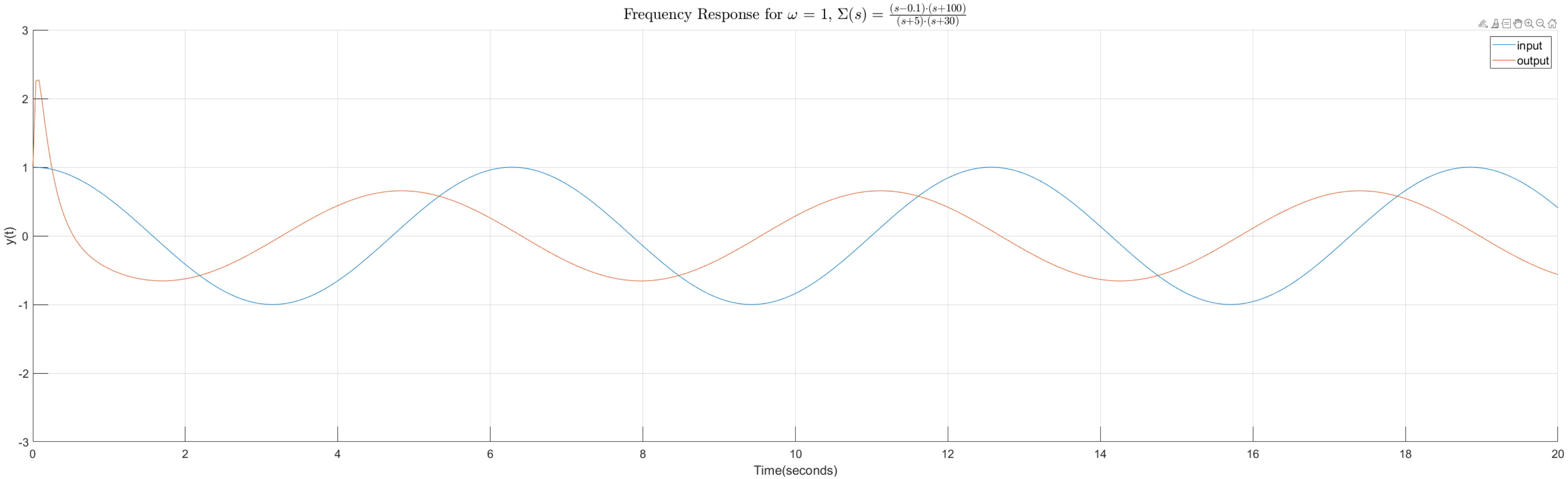
- We want to find the response to a Harmonic Input:
  - $u(t) = \alpha \cdot \cos(\omega \cdot t + \phi)\,, \phi = 0 \text{ in most cases}$
- Reminder General System Response:
  - $y(t) = c \cdot e^{A \cdot t} \cdot x(0) + \int_0^t c \cdot e^{A(t-\rho)} \cdot b \cdot u(\rho) \cdot d\rho + d \cdot u(t) = y_{transient}(t) + y_{\infty}(t)$
- If the system is asymptotically stable: (again if not the following math can still be done)
  - $\lim_{t \to \infty} y_{transient}(t) \to 0 \qquad \Rightarrow \qquad y(t) \to y_{\infty}(t)$
- For a Harmonic Input we get: (see Derivation in my old Script)
  - $y(t) = m(\omega) \cdot \alpha \cdot \cos(\omega \cdot t + \phi + \varphi(\omega))$
  - $m(\omega) = |G(j\omega)| \qquad \varphi(\omega) = \angle G(j\omega)$
- Resulting Response:
  - $y(t) = |G(j\omega)| \cdot \alpha \cdot \cos\big(\omega \cdot t + \phi + \angle G(j\omega)\big)$

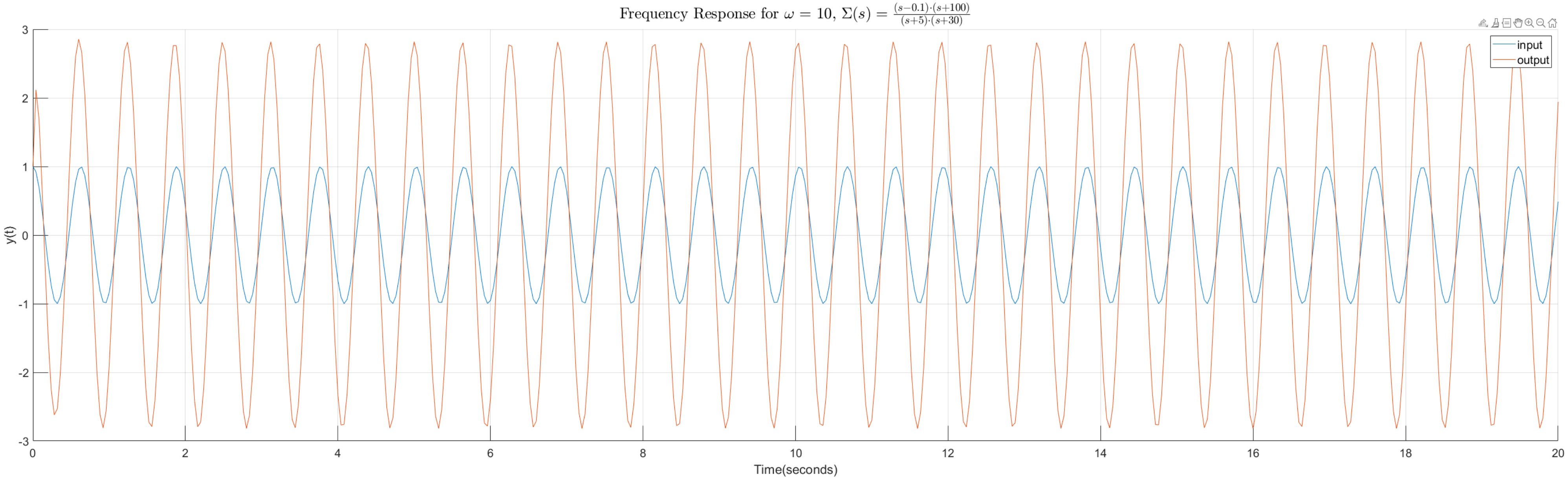
Frequency Response

Why?

- System Response
  - $y(t) = |G(j\omega)| \cdot \alpha \cdot \cos\left(\omega \cdot t + \phi + \angle G(j\omega)\right)$
- What do we see?
  - The system oscillates with the same frequency
  - The amplitude is **frequency dependant**
  - The phase shift is **frequency dependant**
- We can measure this response on a physical system (if stable)
- We can analyse the system behaviour and robustness using this response
- We can in general combine infinite harmonic inputs to model any input
  - Fourier/Laplace Transform
- How can we plot $|G(j\omega)|$ and $\angle G(j\omega)$?
  - Bode Plot
  - Polar / Nyquist Plot

Frequency Response





Bode Plot

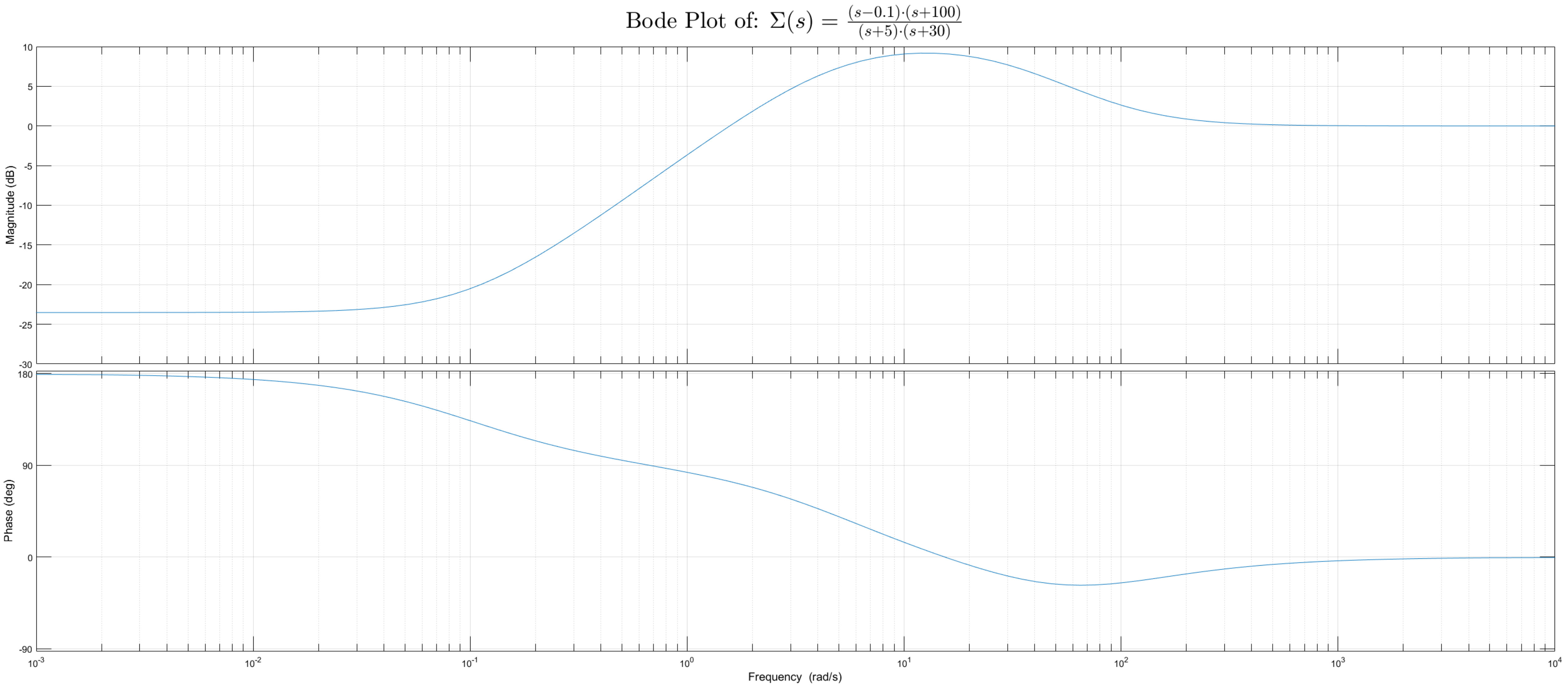
What?

- Two separate frequency **ex**plicit plots for both $|G(j\omega)|$ and $\angle G(j\omega)$
- Magnitude Plot $|G(j\omega)|$:
  - Logarithmic ω axis and dB(decibel) $|G(j\omega)|$
  - Decibel:
    - $|G(j\omega)|_{dB} = 20 \cdot \log_{10}|G(j\omega)|$
    - $|G(j\omega)| = 10^{\frac{|\Sigma(j\omega)|_{dB}}{20}}$
    - Caution when reading of a plot (convert if necessary)
  - $|G(j\omega)| = \sqrt{\mathrm{Re}\big(G(j\omega)\big)^2 + \mathrm{Im}\big(G(j\omega)\big)^2}$
- Phase Plot $\angle G(j\omega)$:
  - Logarithmic ω axis and linear $\angle G(j\omega)$ (in degrees)
  - $\angle G(j\omega) = \mathrm{arctan2}\left(\frac{\mathrm{Im}(G(j\omega))}{\mathrm{Re}(G(j\omega))}\right)$

Dezimalskala	Dezibelskala
100	40
10	20
5	13.97...
2	6.02...
1	0
$1/\sqrt{2}$	-3.0103
0.1	-20
0.01	-40
0	-Inf

$$\mathrm{atan2}(y, x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0, \\ \frac{\pi}{2} - \arctan\left(\frac{x}{y}\right) & \text{if } y > 0, \\ -\frac{\pi}{2} - \arctan\left(\frac{x}{y}\right) & \text{if } y < 0, \\ \arctan\left(\frac{y}{x}\right) \pm \pi & \text{if } x < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

Bode Plot of a system



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Bode Plot

Example

- Read off the values from the bode plot on the previous slide

Frequency $\omega \left[\frac{rad}{s}\right]$	$0\left[\frac{rad}{s}\right]$	$1\left[\frac{rad}{s}\right]$	$4\left[\frac{rad}{s}\right]$	$10\left[\frac{rad}{s}\right]$	$60\left[\frac{rad}{s}\right]$
Magnitude $\lvert G(j\omega)\rvert_{dB}$					
Magnitude $\lvert G(j\omega)\rvert$					
Phase $\angle G(j\omega)$					

Bode Plot

Drawing a Bode Plot

- Using Logarithms is very convenient, we can combine different systems
  - Total System: $G(s) = G_1(s) \cdot G_2(s) \cdot \ldots \cdot G_n(s)$
  - Amplitude in decibel: $|\Sigma(s)|_{dB} = |\Sigma_1(s)|_{dB} + |\Sigma_2(s)|_{dB}$
  - Phase: $\angle\Sigma(s) = \angle\Sigma_1(s) + \angle\Sigma_2(s)$
- When drawing combine the effects of poles and zeros of the sub-systems (addition)
  - The effect is at the position of the pole/zero
  - At the pole/zero the phase shift is approx 50% done
  - For multiplicity $k > 1$, the change is multiplied by k

Type	Magnitude Change	Phase Change
Stable Pole	-20 dB/dec	-90°
Unstable Pole	-20 dB/dec	+90°
Minimumphase zero	+20 dB/dec	+90°
Non-minimumphase zero	+20 dB/dec	-90°
Time Delay	0 dB/dec	$-\omega \cdot T$

Bode Plots

Standard Elements – there are a bunch

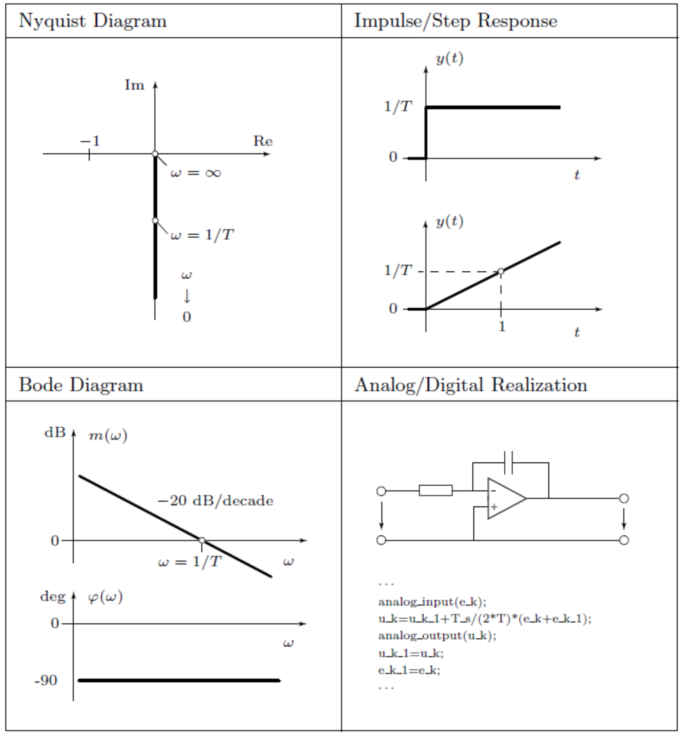
A.1 Integrator Element

Element Acronym: I

Transfer Function: $\Sigma(s) = \frac{1}{T \cdot s}$

Poles/Zeros: $\pi_1 = 0,\ \zeta_1 = \infty$

Internal Description: $\frac{d}{dt}x(t) = \frac{1}{T} \cdot u(t)$
$y(t) = x(t)$



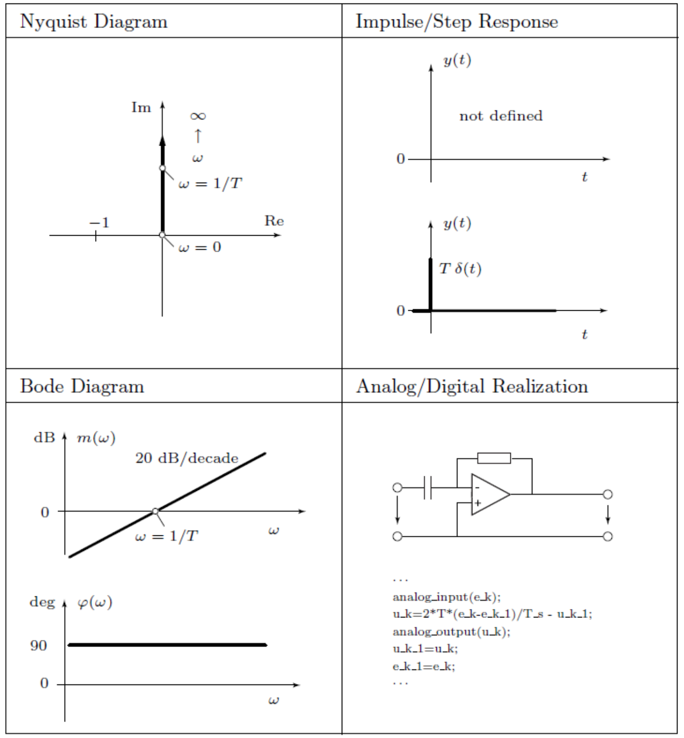
A.2 Differentiator Element

Element Acronym: D

Transfer Function: $\Sigma(s) = T \cdot s$

Poles/Zeros: $\pi_1 = \infty,\ \zeta_1 = 0$

Internal Description: $y(t) = T \cdot \frac{d}{dt}u(t)$



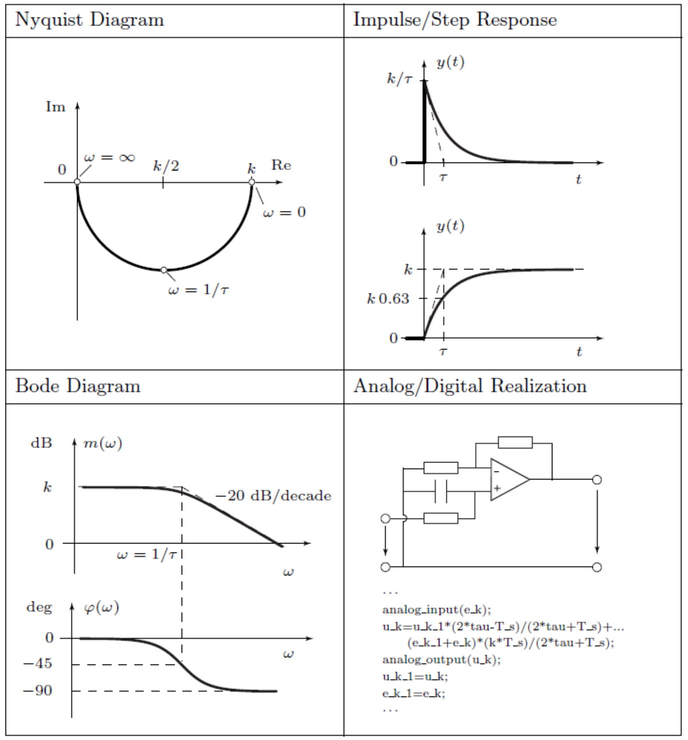
A.3 First-Order Element

Element Acronym: LP-1

Transfer Function: $\Sigma(s) = \frac{k}{\tau \cdot s + 1}$

Poles/Zeros: $\pi_1 = -\frac{1}{\tau},\ \zeta_1 = \infty$

Internal Description: $\frac{d}{dt}x(t) = -\frac{1}{\tau} \cdot x(t) + \frac{1}{\tau} \cdot u(t)$
$y(t) = k \cdot x(t)$



Bode Plots

Standard Elements – there are a bunch

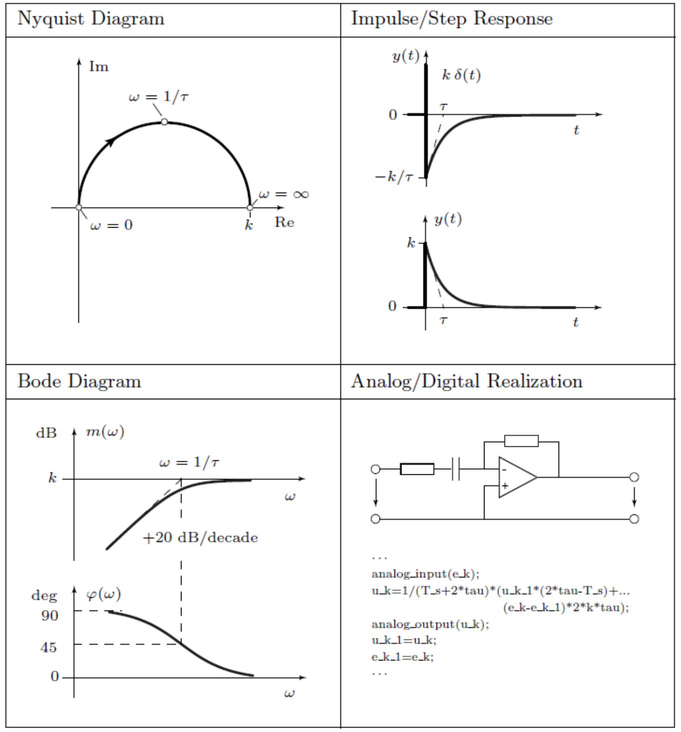
A.4 Realizable Derivative Element

Element Acronym: HP-1

Transfer Function: $\Sigma(s) = k \cdot \frac{\tau \cdot s}{\tau \cdot s + 1} = k \cdot \left(1 - \frac{1}{\tau \cdot s + 1}\right)$

Poles/Zeros: $\pi_1 = -\frac{1}{\tau}$, $\zeta_1 = 0$

Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -\frac{1}{\tau} \cdot x(t) + \frac{1}{\tau} \cdot u(t)$
$y(t) = -k \cdot x(t) + k \cdot u(t)$



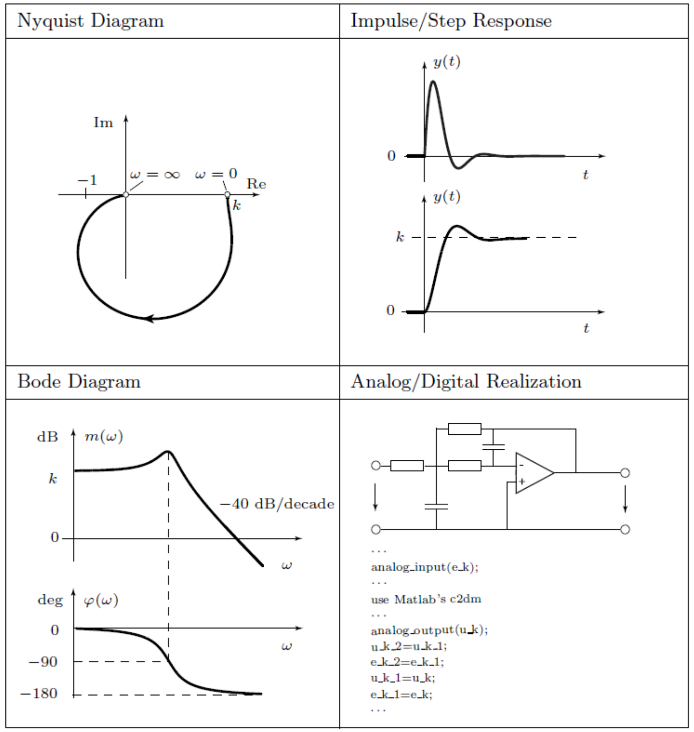
A.5 Second-Order Element

Element Acronym: LP-2

Transfer Function: $\Sigma(s) = k \cdot \frac{\omega_0^2}{s^2 + 2 \cdot \delta \cdot \omega_0 \cdot s + \omega_0^2}$

Poles/Zeros: $\pi_{1,2} = -w_0 \cdot \delta \pm w_0 \sqrt{\delta^2 - 1}$, $\zeta_{1,2} = \infty$

Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = x_2(t)$,
$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = -\omega_0^2 \cdot x_1(t) - 2 \cdot \delta \cdot \omega_0 \cdot x_2(t) + \omega_0^2 \cdot u(t)$
$y(t) = k \cdot x_1(t)$



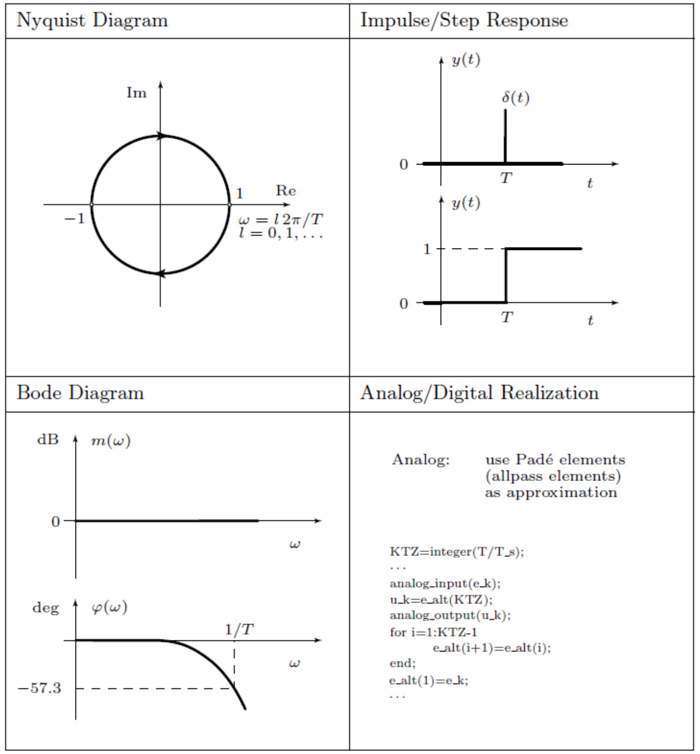
A.10 Delay Element

Element Acronym: -

Transfer Function: $\Sigma(s) = \mathrm{e}^{-s \cdot T}$

Poles/Zeros: not a real-rational element

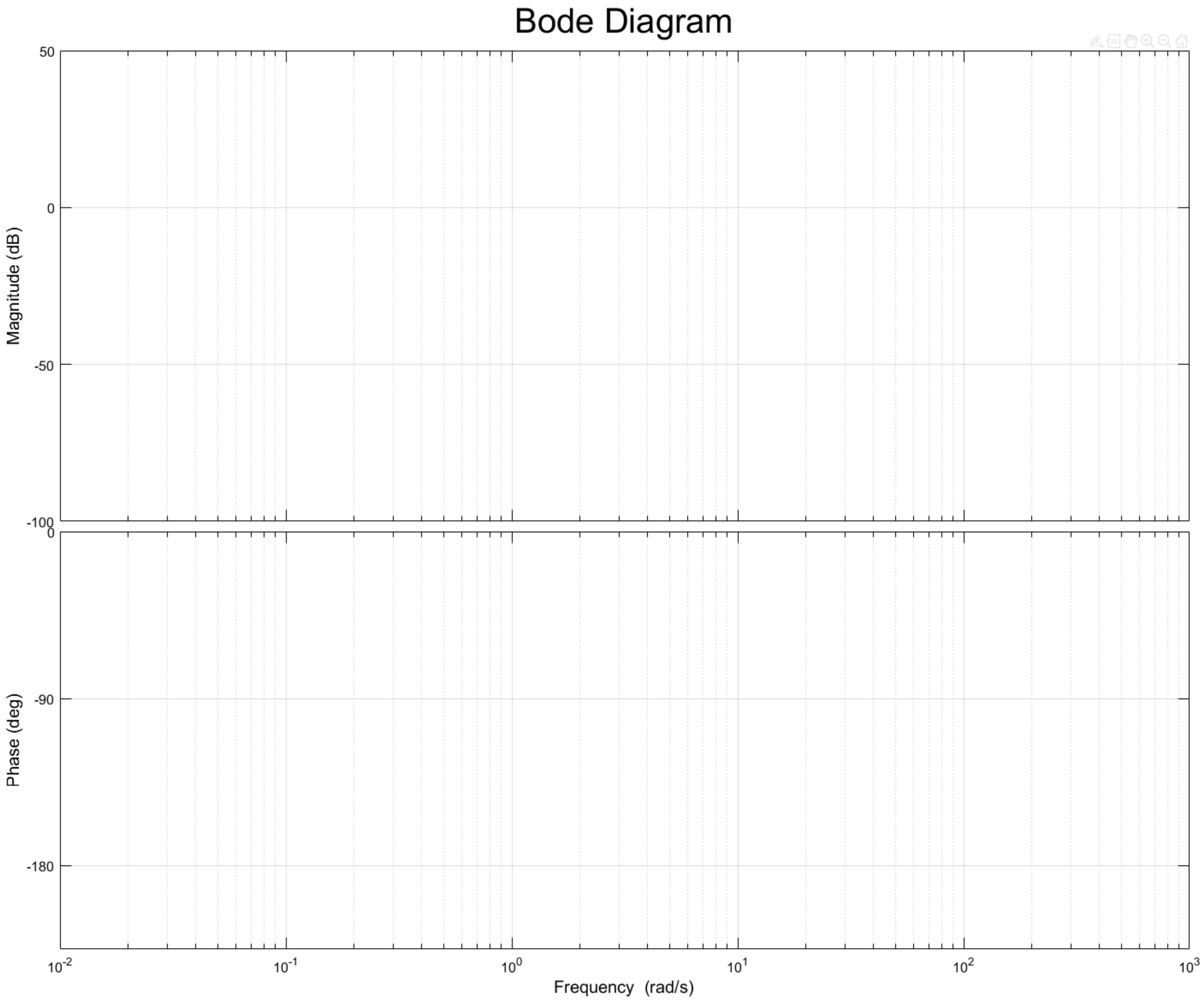
Internal Description: $y(t) = u(t - T)$



Bode Plot

Example - Drawing

- System 1:
  - $G_1(s) = \frac{1}{s \cdot (s+1)}$
- Pole 1: $\omega_{\pi_1} = 0$
- Pole 2: $\omega_{\pi_2} = 1$
- No zeros



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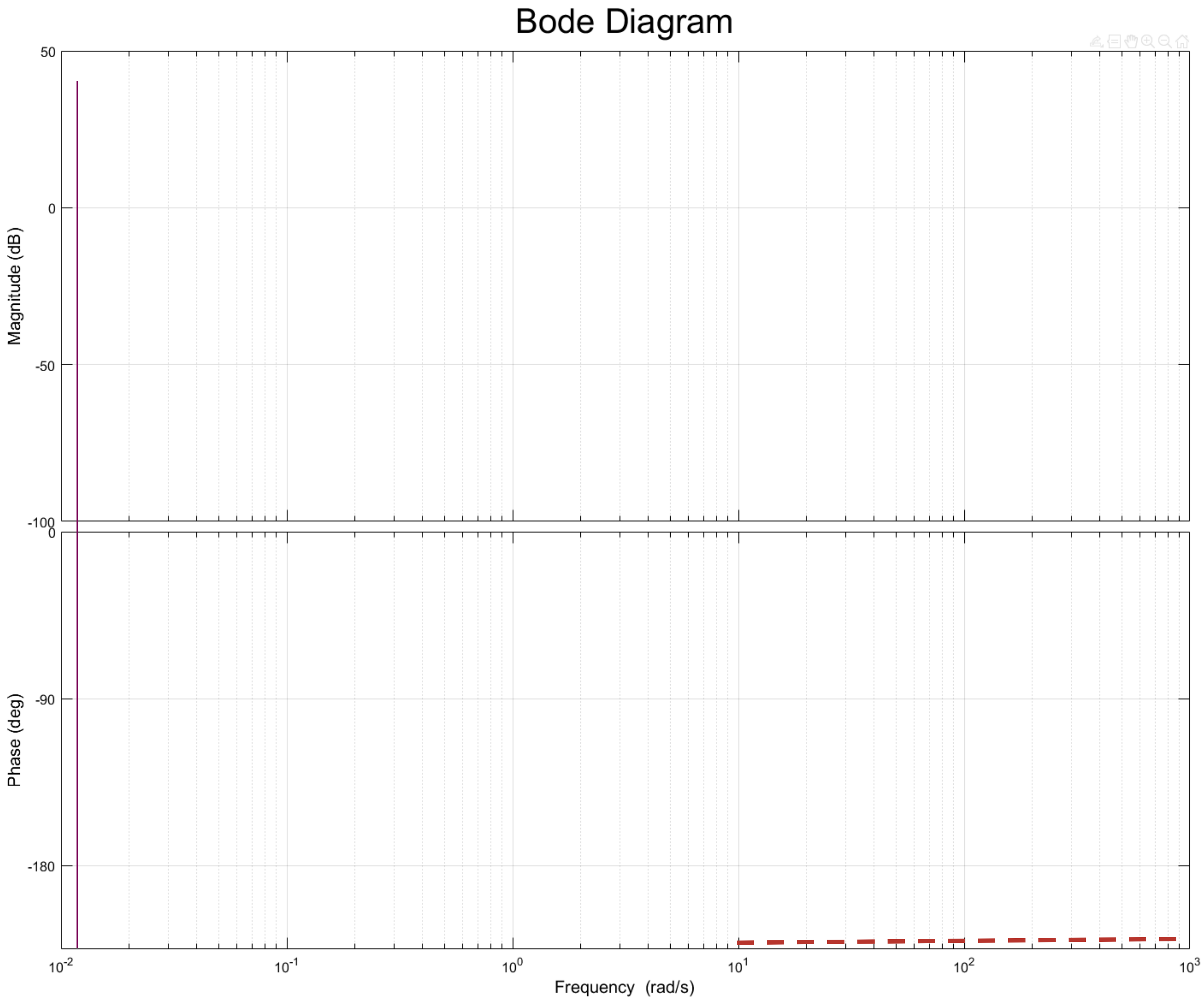
Bode Plot

Example 2

- System 2:
  - $G_2(s) = \frac{5000 \cdot s}{(s+0.2)\cdot(s+10)\cdot(s-30)} = (83.3333 \cdot s) \cdot \frac{1}{(5 \cdot s+1)} \cdot \frac{1}{(0.1 \cdot s+1)} \cdot \frac{1}{(0.03 \cdot s-1)}$
- Pole 1: $\omega_{\pi_1} = 0.2$
- Pole 2: $\omega_{\pi_2} = 10$
- Pole 3: $\omega_{\pi_3} = 30$
- Zero 1: $\omega_{\zeta_1}=0$
- Non-stable zero Phase shifts from -180° to -90°

$$G_2(s) = (83.3333 \cdot s) \cdot \frac{1}{(5 \cdot s+1)} \cdot \frac{1}{(0.1 \cdot s+1)} \cdot \frac{1}{(0.03 \cdot s-1)}$$

Pole 1: $\omega_{\pi_1} = 0.2$
Pole 2: $\omega_{\pi_2} = 10$
Pole 3: $\omega_{\pi_3} = 30$
Zero 1: $\omega_{\zeta_1} = 0$



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Bode Plot

Bodes - Law

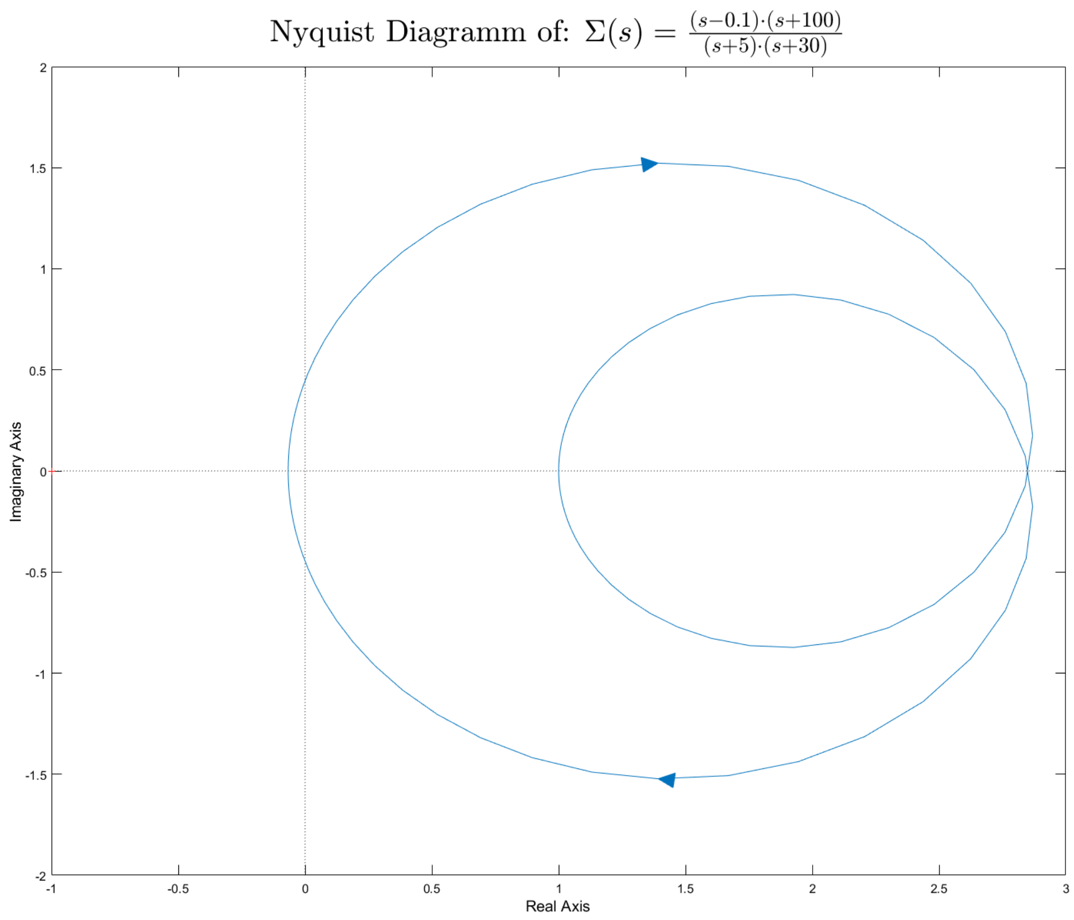
- Phase and Amplitude are not independent
  - $|G(j\omega)|_{dB} = 20 \frac{dB}{dec} \cdot \kappa \quad \Rightarrow \quad \angle G(j\omega) \approx \kappa \cdot \frac{\pi}{2}$
- System: $\Sigma(s) = \frac{b_m \cdot s^m + \dots + b_1 \cdot s + b_0}{s^q \cdot (s^{n-q} + a_{n-k-1} \cdot s^{n-k-1} + \dots + a_1 \cdot s + a_0)}$
  - Relative degree: $r = n - m$
  - System Type: $q =$ number of integrators

- We further have:
  - For $\omega \to \infty$: $\frac{\partial |G(j\omega)|_{dB}}{\partial \log_{10}(\omega)} = -r \cdot 20\text{ dB}$, with $r = n - m$ being the relative degree
  - For $\omega \to 0$: $\angle G(j\omega = 0) = \begin{cases} -q \cdot \frac{\pi}{2}, & \text{for sign}\left(\frac{b_0}{a_0}\right) > 0 \\ -\pi - q \cdot \frac{\pi}{2}, & \text{for sign}\left(\frac{b_0}{a_0}\right) < 0 \end{cases}$

Polar / Nyquist Plot

What?

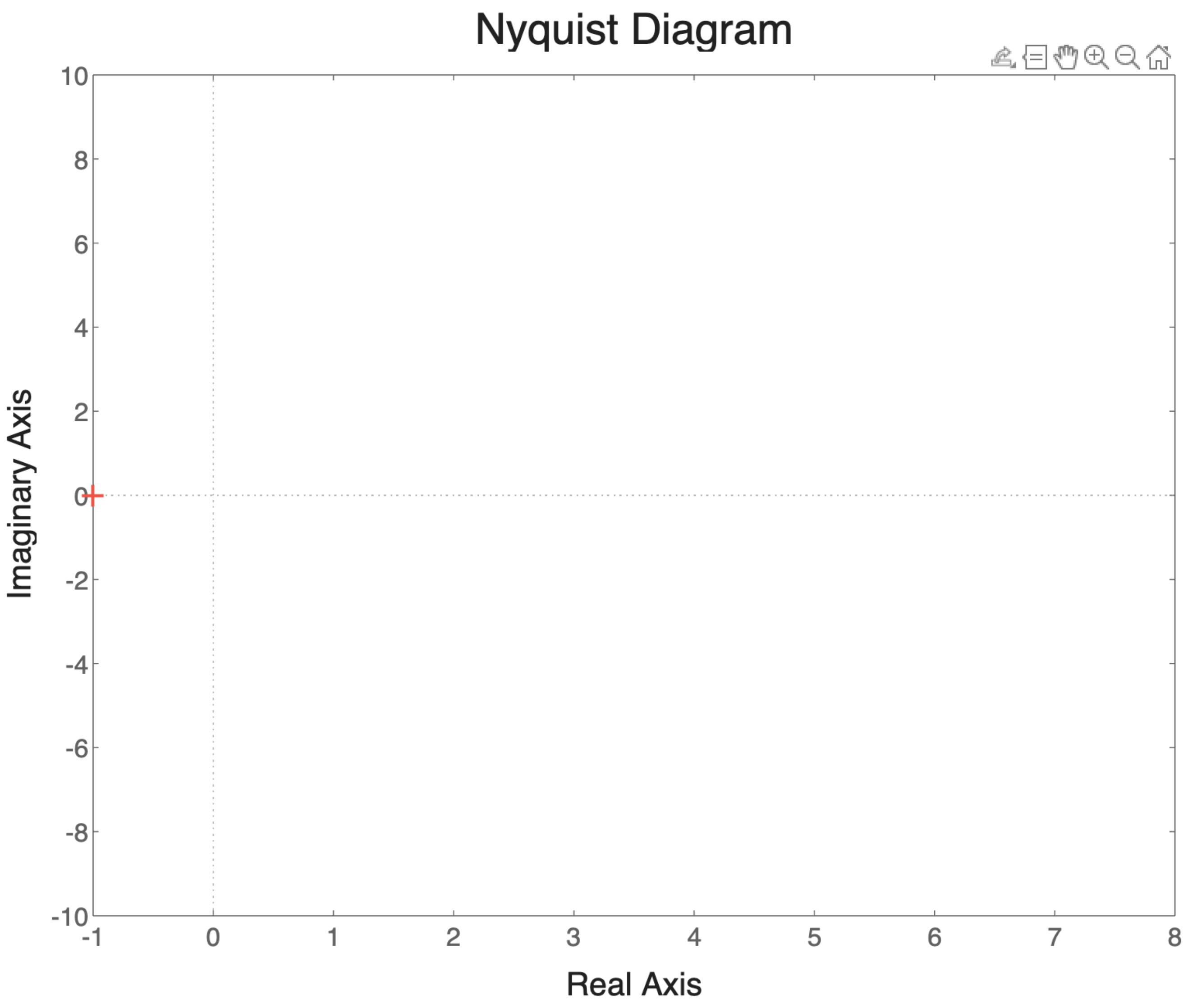
- $|G(s)|$ and $\angle G(s)$ drawn in the complex plane.
  - From $-\infty < \omega < \infty$
- The values are now frequency **implicit**
- Drawing usually using Python or Matlab
- Sketching
  - Look at the extremes $\omega \to 0, \omega \to \infty$
  - Read values of Bode plot
  - Needs to be qualitatively correct
- We mostly want to know where
  - $|G(j\omega)| = 1,\ \text{and}\ \angle G(j\omega) = -180°$
  - System stable iff $\frac{1}{1+L(s)}$ is stable
  - $L(s) = -1$ not allowed!!!



Polar / Nyquist Plot

Example – Drawing a Nyquist Plot

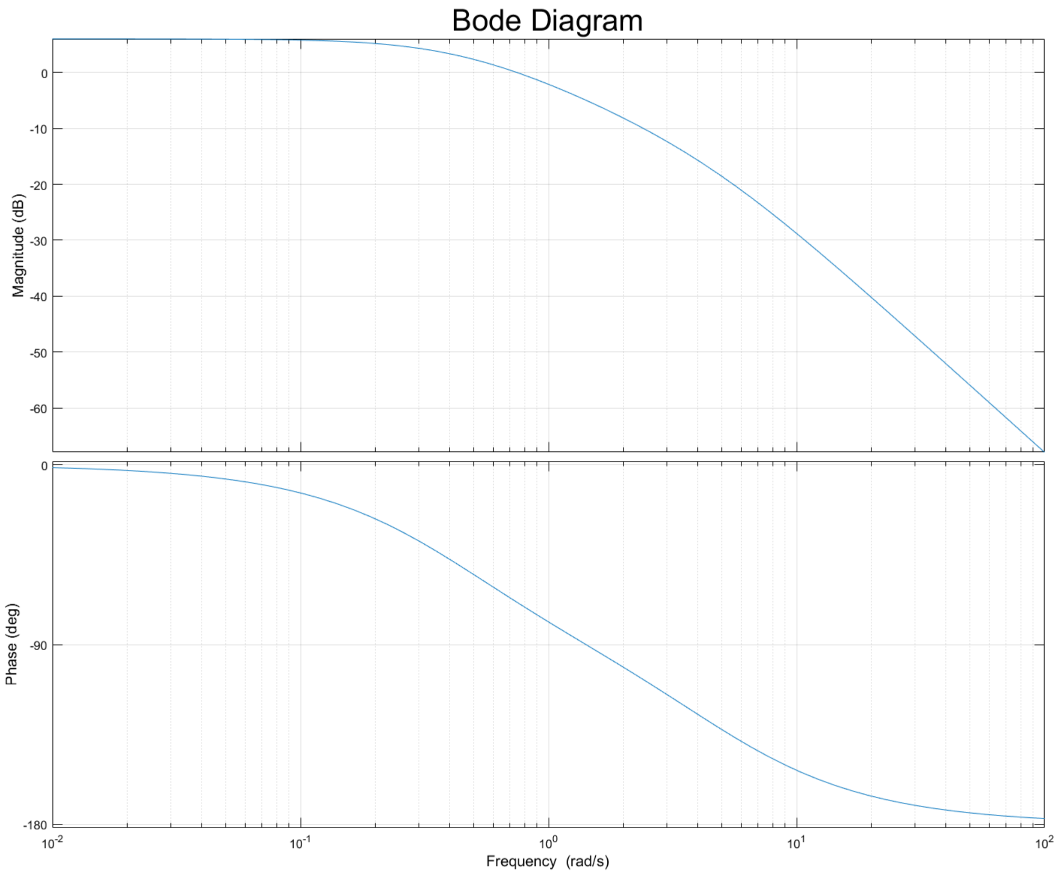
- Draw the Nyquist Plot for:
  - $G(s) = \frac{5(s-0.5)}{s(s+5)}$
- $\omega \to 0^+$:
  - $|G(j\omega)| \to \infty$
  - $\angle G(j\omega) = \begin{cases} -q \cdot \frac{\pi}{2}, & \text{for sign}\left(\frac{b_0}{a_0}\right) > 0 \\ -\pi - q \cdot \frac{\pi}{2}, & \text{for sign}\left(\frac{b_0}{a_0}\right) < 0 \end{cases} = -\frac{3}{2}\pi$
- $\omega \to \infty$:
  - $|G(j\omega)| \to 0$
  - $\angle G(j\omega) \approx \angle \frac{1}{s} = -\frac{\pi}{2}$

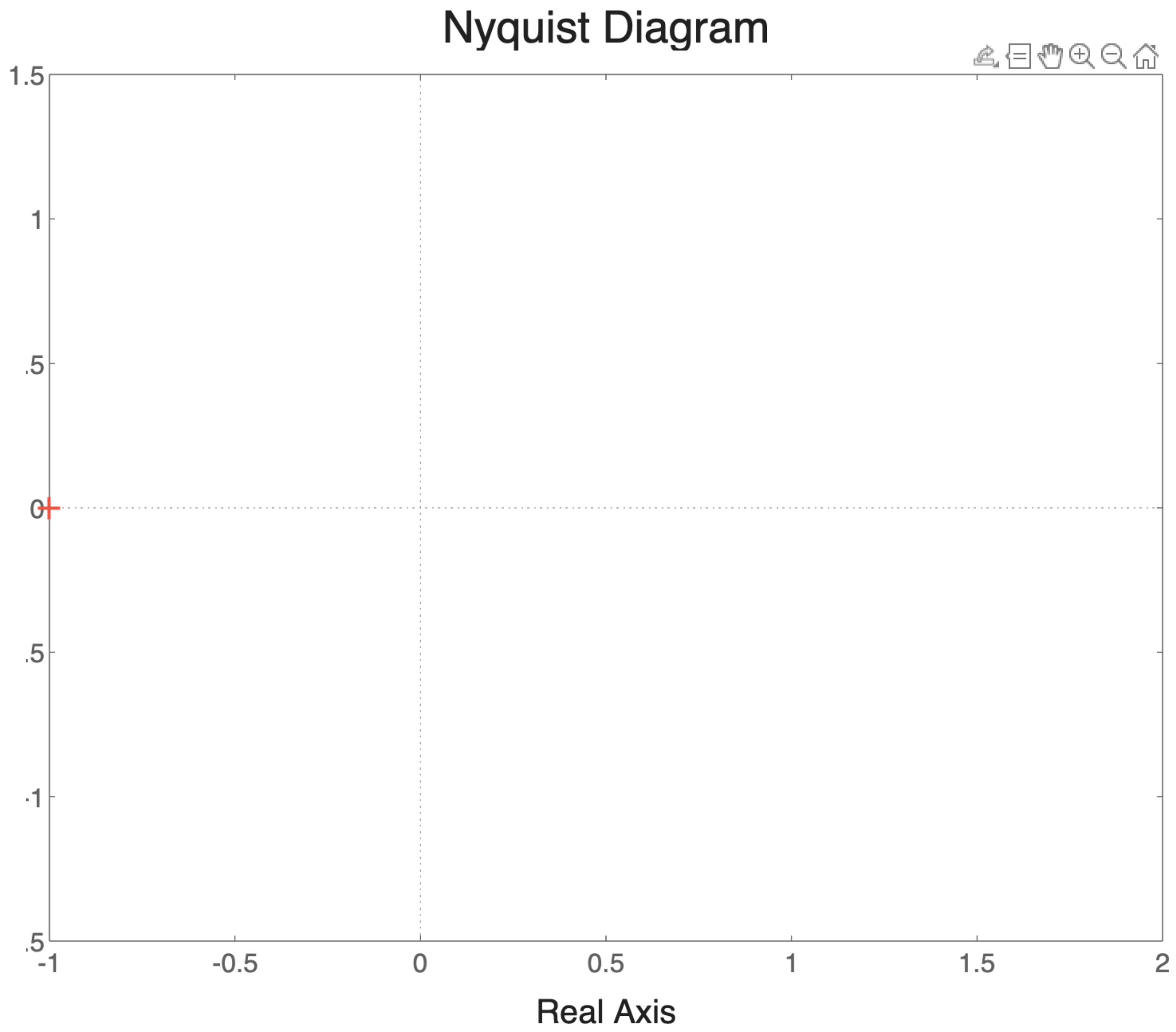


Polar / Nyquist Plot

Example – Drawing a Nyquist Plot

- Draw the Nyquist Plot for the System with the following Bode Plot





Bode and Nyquist Plots

Summary

- Graphical representation of $|G(s)|$ and $\angle G(s)$
- Bode Plot:
  - frequency **ex**plicit
  - Logarithmic, decibel and linear axis scale
  - Quantitive analysis
- Nyquist Plot:
  - frequency **im**plicit
  - Linear axis scale
  - Qualitative analysis
- Why though?
  - Determine system properties from plots (stability, DC-gain,…)
  - Analyse system with controller
  - Determine robustness of the system

Exercise 08

What to do?

- 1:
  - Do two
- 2:
  - Do all
- 3:
  - Not nessecary
- 4:
  - Not nessecary
- 5:
  - Do two distinct once