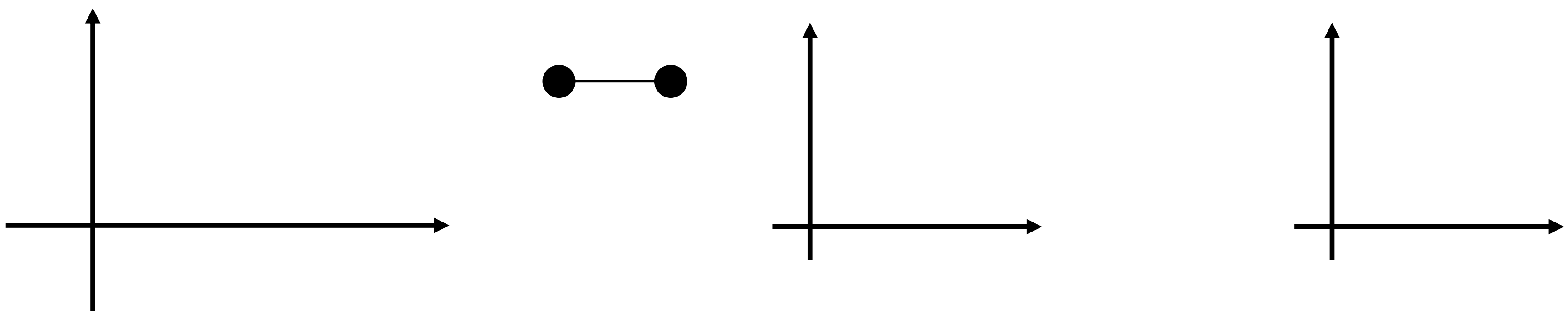
RT-I Vorlesung 7

1.11.2019

Laplace Transformation

$$\mathcal{L}\left\{x\right\} = X = \int_0^\infty x(t) \cdot \mathrm{e}^{-s \cdot t}\,\mathrm{d}t$$

$x(t)\colon\ \mathbb{R} \to \mathbb{R}$ $\qquad\qquad$ $X(s)\colon\ \mathbb{R} + j\mathbb{R} \to \mathbb{R} + j\,\mathbb{R}$



linearity	:	$\mathcal{L}\left\{a \cdot x_1(t) + b \cdot x_2(t)\right\} = a \cdot X_1(s) + b \cdot X_2(s)$
similarity	:	$\mathcal{L}\left\{\frac{1}{a} \cdot x\left(\frac{t}{a}\right)\right\} = X(s \cdot a)$
shift	:	$\mathcal{L}\left\{x(t-T)\right\} = \mathrm{e}^{-T \cdot s} \cdot X(s)$
damping	:	$\mathcal{L}\left\{x(t) \cdot \mathrm{e}^{a \cdot t}\right\} = X(s-a)$
derivative t	:	$\boxed{\mathcal{L}\left\{\frac{\mathrm{d}}{\mathrm{d}t} x(t)\right\} = s \cdot X(s) - x(0)}$
derivative s	:	$\mathcal{L}\left\{t \cdot x(t)\right\} = -\frac{\mathrm{d}}{\mathrm{d}s} X(s)$
integration t	:	$\mathcal{L}\left\{\int_0^t x(\tau)\,\mathrm{d}\tau\right\} = \frac{1}{s} \cdot X(s)$
integration s	:	$\mathcal{L}\left\{\frac{1}{t} \cdot x(t)\right\} = \int_s^\infty X(\sigma)\,\mathrm{d}\sigma$
convolution t	:	$\mathcal{L}\left\{x_1(t) * x_2(t)\right\} = X_1(s) \cdot X_2(s)$
convolution s	:	$\mathcal{L}\left\{x_1(t) \cdot x_2(t)\right\} = X_1(s) * X_2(s)$
initial value	:	$\lim\limits_{t \to 0_+} x(t) = \lim\limits_{s \to \infty} s \cdot X(s)$
final value	:	$\lim\limits_{t \to \infty} x(t) = \lim\limits_{s \to 0_+} s \cdot X(s)$

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = A \cdot x(t) + b \cdot u(t) \qquad\qquad X(s) = (s\,I - A)^{-1} \cdot b \cdot U(s)$$

$$y(t) = c \cdot x(t) \quad \circ\!\!-\!\!\bullet \quad Y(s) = c \cdot (s\,I - A)^{-1} \cdot b \cdot U(s)$$

$$\Sigma(s) = \frac{c \cdot \mathrm{Adj}(s\,I - A) \cdot b}{\det(sI - A)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$= b_m \frac{\prod_{j=1}^{m}(s - \zeta_j)}{\prod_{i=1}^{n}(s - \pi_i)}$$

	TD	FD
SP	$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = A\,x(t) + b\,u(t)$ $y(t) = c\,x(t) + d\,u(t)$	$Y(s) = \left(c\,(s\,I - A)^{-1}\,b + d\right)U(s)$
I/O	$y^{(n)}(t) + \cdots + a_1 y^{(1)}(t) + a_0 y(t)$ $= b_m u^{(m)}(t) + \cdots + b_0 u(t)$	$Y(s) = \frac{b_m s^m + \cdots + b_0}{s^n + \cdots + a_1 s^1 + a_0} U(s)$

Cramer's Rule

$$M^{-1} = \mathrm{Adj}(M) / \det(M)$$

$$[\mathrm{Adj}(M)]_{j,i} = (-1)^{i+j} \cdot \det\big(\hat{M}(i,j)\big)$$

$$\hat{M}(i,j) = \begin{pmatrix} m_{11} & m_{12} & \cdots & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & \cdots & m_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{n1} & m_{n2} & \cdots & \cdots & m_{nn} \end{pmatrix}$$

i-te Zeile

j-te Kolonne

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = A \cdot x(t) + b \cdot u(t), \qquad x(0) = x_0$$

$$y(t) = c \cdot x(t)$$

$$\Sigma(s) = \frac{c \cdot \mathrm{Adj}(s\,I - A) \cdot b}{\det(sI - A)}$$

$$= b_m \frac{\prod_{j=1}^{m}(s - \zeta_j)}{\prod_{i=1}^{n}(s - \pi_i)}$$

QC: Are the eigenvalues λ_i of A equal to the poles π_i of $\Sigma(s)$?

Quick Check: For a system defined by its internal description

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} +1 & 3 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Compute its transfer function $\Sigma(s)$.

Is the system minimal?

Eigenvalues and poles?

Is the system stabilizable?

Hint:

$$\mathrm{Adj}\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$$

20.09. Lektion 1 – Einführung

Modellierung

27.09. Lektion 2 – Modellbildung
4.10. Lektion 3 – Systemdarstellung, Normierung, Linearisierung

Systemanalyse im Zeitbereich

11.10. Lektion 4 – Analyse I, allg. Lösung, Systeme erster Ordnung, Stabilität
18.10. Lektion 5 – Analyse II, Zustandsraum, Steuerbarkeit/Beobachtbarkeit

Systemanalyse im Frequenzbereich

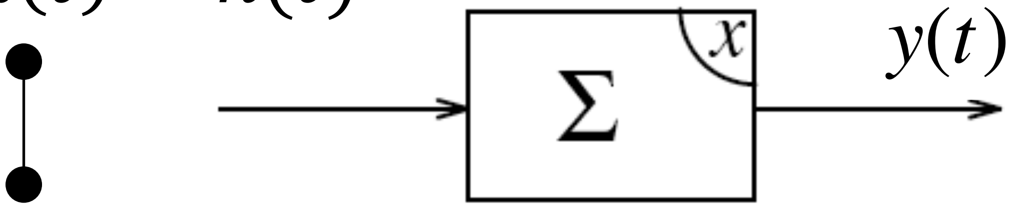
25.10. Lektion 6 – Laplace I, Übertragungsfunktionen
1.11. Lektion 7 – Laplace II, Lösung, Pole/Nullstellen, BIBO-Stabilität
8.11. Lektion 8 – Frequenzgänge (RH hält VL)
15.11. Lektion 9 – Systemidentifikation, Modellunsicherheiten
22.11. Lektion 10 – Analyse geschlossener Regelkreise
29.11. Lektion 11 – Randbedingungen

Reglerauslegung

6.12. Lektion 12 – Spezifikationen geregelter Systeme
13.12. Lektion 13 – Reglerentwurf I, PID (RH hält VL)
20.12. Lektion 14 – Reglerentwurf II, „loop shaping“

The "DC-Gain" of a System

$u(t) = h(t)$



$y(t) = ?\ \ for\ \ \lim t \to \infty$

$U(s) = 1/s$

$$\text{final value} \quad : \lim_{t \to \infty} f(t) = \lim_{s \to 0_+} s \cdot F(s)$$

$$y(\infty) = \lim_{s \to 0_+} s \cdot Y(s) = \lim_{s \to 0_+} s \cdot \Sigma(s) \cdot U(s)$$

$$= \lim_{s \to 0_+} s \cdot \Sigma(s) \cdot \frac{1}{s} = \lim_{s \to 0_+} \Sigma(s) = \Sigma(0)$$

QC:

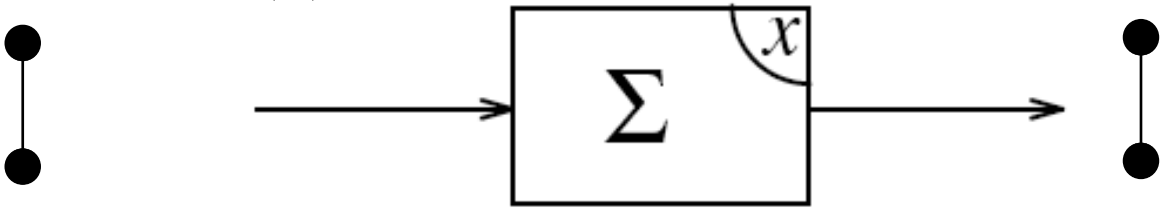
DC gain of $\Sigma(s) = \dfrac{2s+3}{s^2+3s+2}$

DC gain of $\Sigma(s) = \dfrac{s+3}{-s+7}$

The Impulse Response of a System

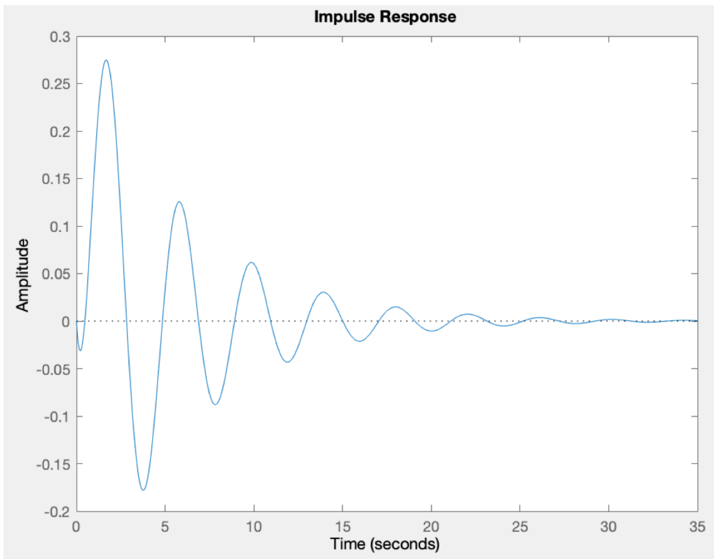
$u(t) = \delta(t)$

$y(t) = \sigma(t)$

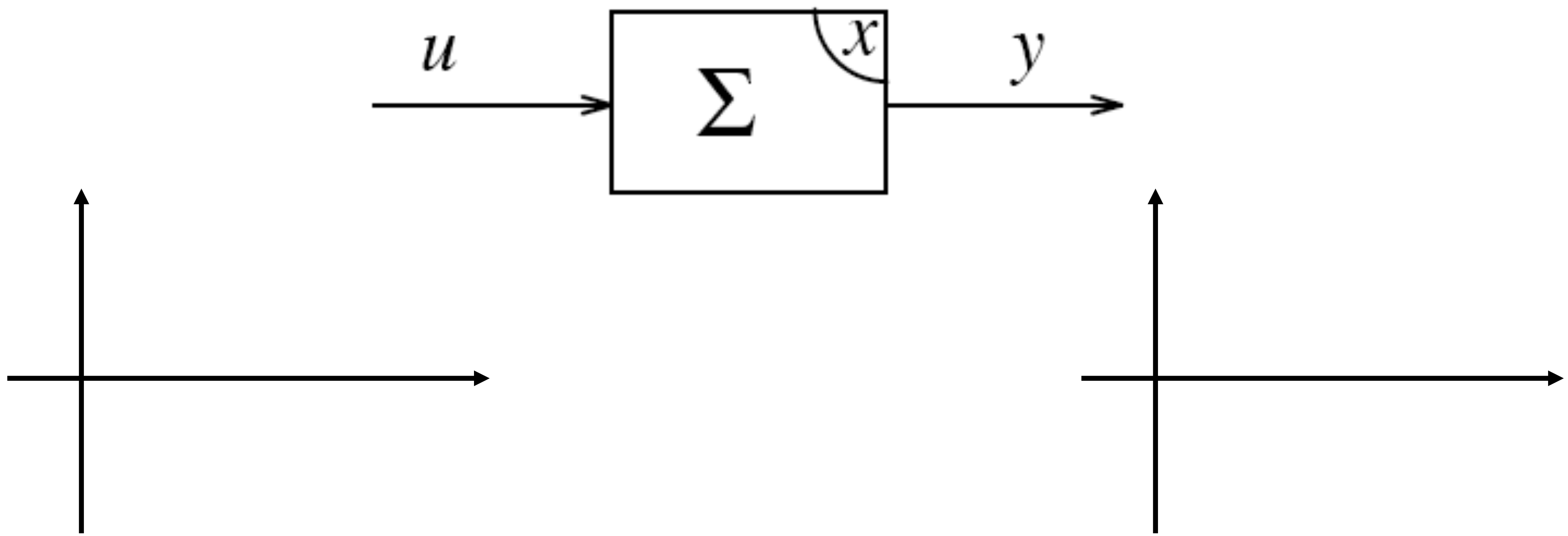


$U(s) = 1$

$Y(s) = \Sigma(s)$

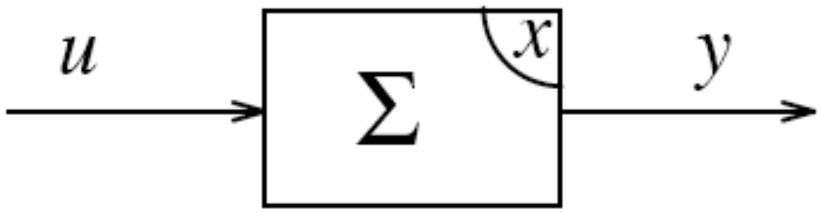


BIBO Stability



Test for BIBO stability: $\int_0^\infty |\sigma(t)| dt < \infty$

BIBO Stability



$$\Sigma(s) = b_m \frac{\prod_{j=1}^{m}(s-\zeta_j)}{\prod_{i=1}^{n}(s-\pi_i)}$$

Main result (valid for LTI I/O systems):

BIBO stable iff all poles π_i of $\Sigma(s)$ have negative real parts; not BIBO stable in all other cases.

BIBO Stability

Quick Check: *Is the first-order system*

$$\frac{d}{dt}x(t) = u(t), \quad y(t) = x(t)$$

asymptotically stable, Lyapunov stable, or unstable?

What about its BIBO stability?

Lyapunov vs. BIBO Stability

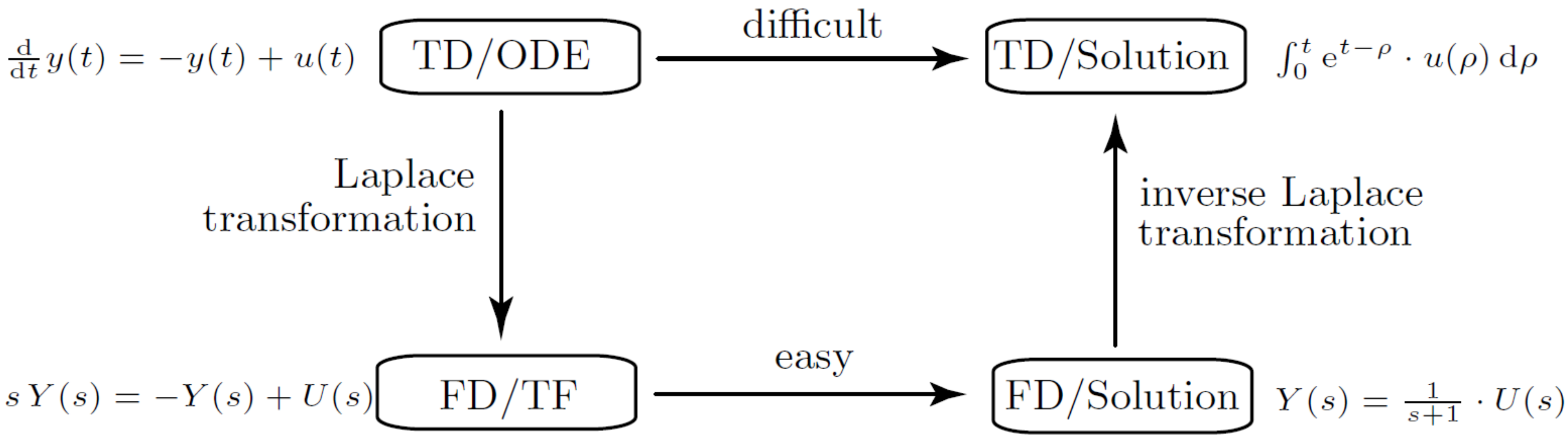
For a system that is completely controllable and observable:

asymptotically stable	→	BIBO stable
asymptotically stable	←	BIBO stable
Lyap. stable or unstable	→	not BIBO stable
Lyap. stable or unstable	←	not BIBO stable

For a system that is *not* completely controllable and observable:

asymptotically stable	→	BIBO stable
?	←	BIBO stable
Lyap. stable or unstable	→	?
Lyap. stable or unstable	←	not BIBO stable

Inverse Laplace Transform



The output of a LTI dynamic system in the frequency-domain is

$$Y(s) = \Sigma(s) \cdot U(s) = \frac{\xi(s)}{\gamma(s)}$$

How can we find $y(t)$?

Inverse Laplace Transform

The rational function $Y(s)$

$$Y(s) = \frac{\xi(s)}{\gamma(s)} = \frac{\xi(s)}{(s-\pi_1)^{\phi_1} \cdot (s-\pi_2)^{\phi_2} \cdot \ldots \cdot (s-\pi_p)^{\phi_p}}$$

can always be written as a partial-fraction expansion

$$Y(s) = \sum_{i=1}^{p} \sum_{k=1}^{\phi_i} \frac{\rho_{i,k}}{(s-\pi_i)^k} \qquad \rho_{i,k} \in \mathbb{C}$$

where the residuals $\rho_{i,k}$ are found using

$$\rho_{i,k} = \lim_{s \to \pi_i} \frac{1}{(\phi_i - k)!} \cdot \left(\frac{\mathrm{d}^{(\phi_i - k)}}{\mathrm{d}s^{(\phi_i - k)}} \left(Y(s) \cdot (s-\pi_i)^{\phi_i} \right) \right)$$

The time-domain solution $y(t)$ is then found to be

$$y(t) = \sum_{i=1}^{p} \sum_{k=1}^{\phi_i} \frac{\rho_{i,k}}{(k-1)!} \cdot t^{k-1} \cdot \mathrm{e}^{\pi_i \cdot t} \cdot h(t)$$

Illustration:

$$Y(s) = \frac{s+2}{(s^2+2s+5)(s+4)} = \frac{-2/13}{s+4} + \frac{4-7j}{52(s+1-2j)} + \frac{4+7j}{52(s+1+2j)}$$

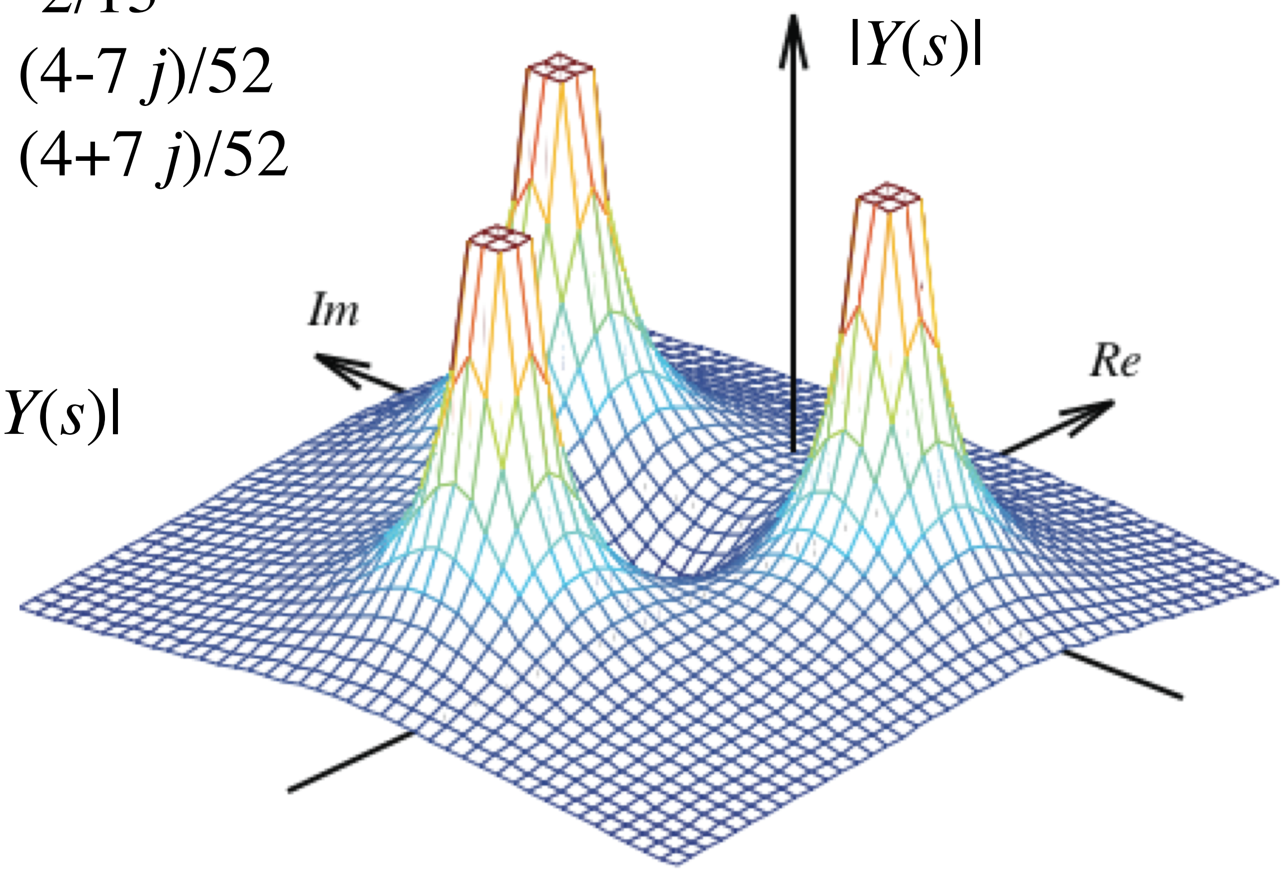
$\zeta_1 = -2$

$\pi_1 = -4$ $\rho_1 = -2/13$

$\pi_2 = -1 + 2\,j$ $\rho_2 = (4-7\,j)/52$

$\pi_3 = -1 - 2\,j$ $\rho_3 = (4+7\,j)/52$

Map: $s \longrightarrow |Y(s)|$



Example

$$y^{(3)}(t) + 7 \cdot y^{(2)}(t) + 15 \cdot y^{(1)}(t) + 9 \cdot y(t) = u^{(1)}(t) + 2 \cdot u(t)$$

$$u(t) = \sin(t) \cdot h(t)$$

$$Y(s) = \frac{s+2}{s^3 + 7 \cdot s^2 + 15 \cdot s + 9} \cdot \frac{1}{s^2+1} = \frac{(s+2)}{(s+1) \cdot (s+3)^2 \cdot (s+j) \cdot (s-j)}$$

$$\rho_{i,k} = \lim_{s \to \pi_i} \frac{1}{(\phi_i - k)!} \cdot \left(\frac{\mathrm{d}^{(\phi_i - k)}}{\mathrm{d}s^{(\phi_i - k)}} \left(Y(s) \cdot (s - \pi_i)^{\phi_i} \right) \right)$$

$$\pi_1 = -1, \quad \phi_1 = 1, \quad \rho_{11} = \frac{1}{8}$$

$$\pi_2 = -3, \quad \phi_2 = 2, \quad \rho_{21} = \frac{1}{200}, \; \rho_{22} = \frac{1}{20}$$

$$\pi_3 = -j, \quad \phi_3 = 1, \quad \rho_{31} = \frac{-13 + j9}{200}$$

$$\pi_4 = +j, \quad \phi_4 = 1, \quad \rho_{41} = \frac{-13 - j9}{200}$$

$$\pi_1 = -1, \quad \phi_1 = 1, \quad \rho_{11} = \frac{1}{8}$$

$$\pi_2 = -3, \quad \phi_2 = 2, \quad \rho_{21} = \frac{1}{200}, \; \rho_{22} = \frac{1}{20}$$

$$\pi_3 = -j, \quad \phi_3 = 1, \quad \rho_{31} = \frac{-13 + j9}{200}$$

$$\pi_4 = +j, \quad \phi_4 = 1, \quad \rho_{41} = \frac{-13 - j9}{200}$$

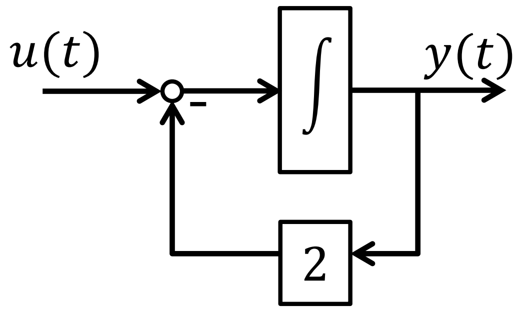
$$y(t) = \sum_{i=1}^{p} \sum_{k=1}^{\phi_i} \frac{\rho_{i,k}}{(k-1)!} \cdot t^{k-1} \cdot \mathrm{e}^{\pi_i \cdot t} \cdot h(t)$$

Remarks:

- BIBO stability “proof”
- Real coefficients always yield complex conjugate residuals
- There are only first and second-order “linear factors” (elementary systems)

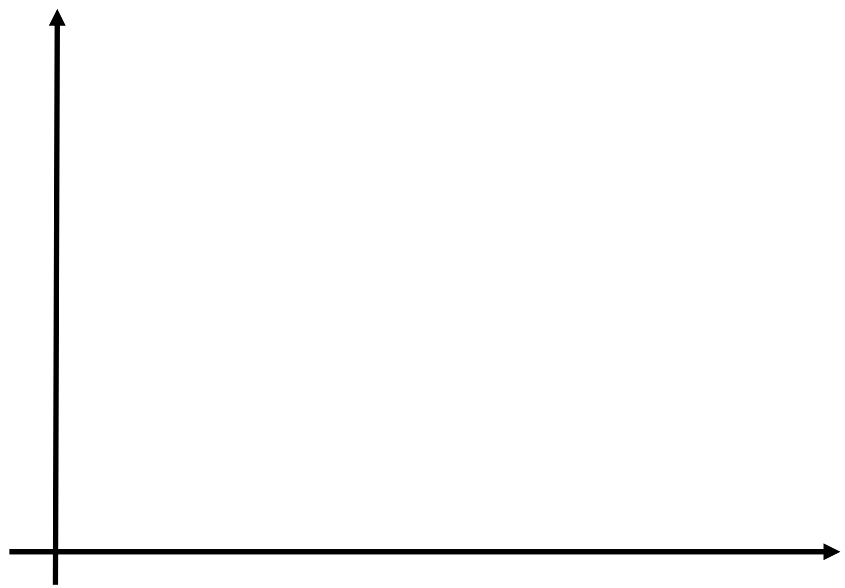
Pole of 1st Order System

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -2 \cdot x(t) + u(t)$$
$$y(t) = x(t)$$



$$\Sigma(s) = \frac{1}{s+2} \quad \Longrightarrow \quad \pi = -2$$

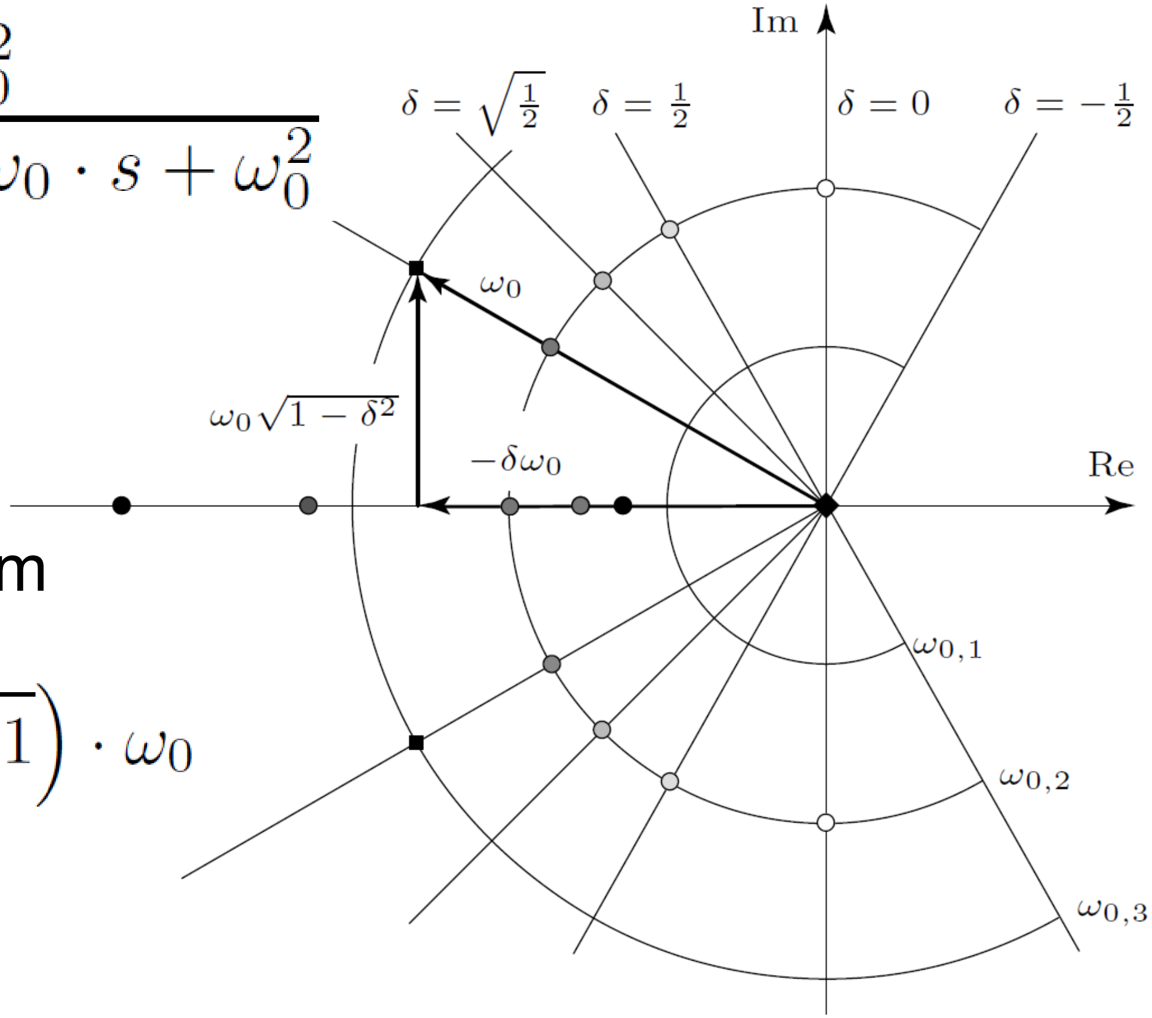
$$\sigma(t) = h(t) \cdot e^{-2 \cdot t}$$



Poles of 2nd-Order Systems

Simples case 2nd-order system with no zeros and gain 1

$$\Sigma(s) = \frac{\omega_0^2}{s^2 + 2 \cdot \delta \cdot \omega_0 \cdot s + \omega_0^2}$$



The poles of this system are at

$$\pi_{1,2} = \left(-\delta \pm \sqrt{\delta^2 - 1}\right) \cdot \omega_0$$

Step Response

Use the partial-fraction expansion approach (details see book)

for the case $0 < \delta < 1$

$$y(t) = h(t) \cdot \left(1 + \mathrm{e}^{\delta_0 \cdot t} \cdot \left(\frac{\delta_0}{\omega_1} \cdot \sin(\omega_1 \cdot t) - \cos(\omega_1 \cdot t)\right)\right)$$

for the case $1 < \delta$

$$y(t) = h(t) \cdot \left(1 + \mathrm{e}^{\delta_0 \cdot t} \cdot \left(\frac{\delta_0}{\omega_2} \cdot \sinh(\omega_2 \cdot t) - \cosh(\omega_2 \cdot t)\right)\right)$$

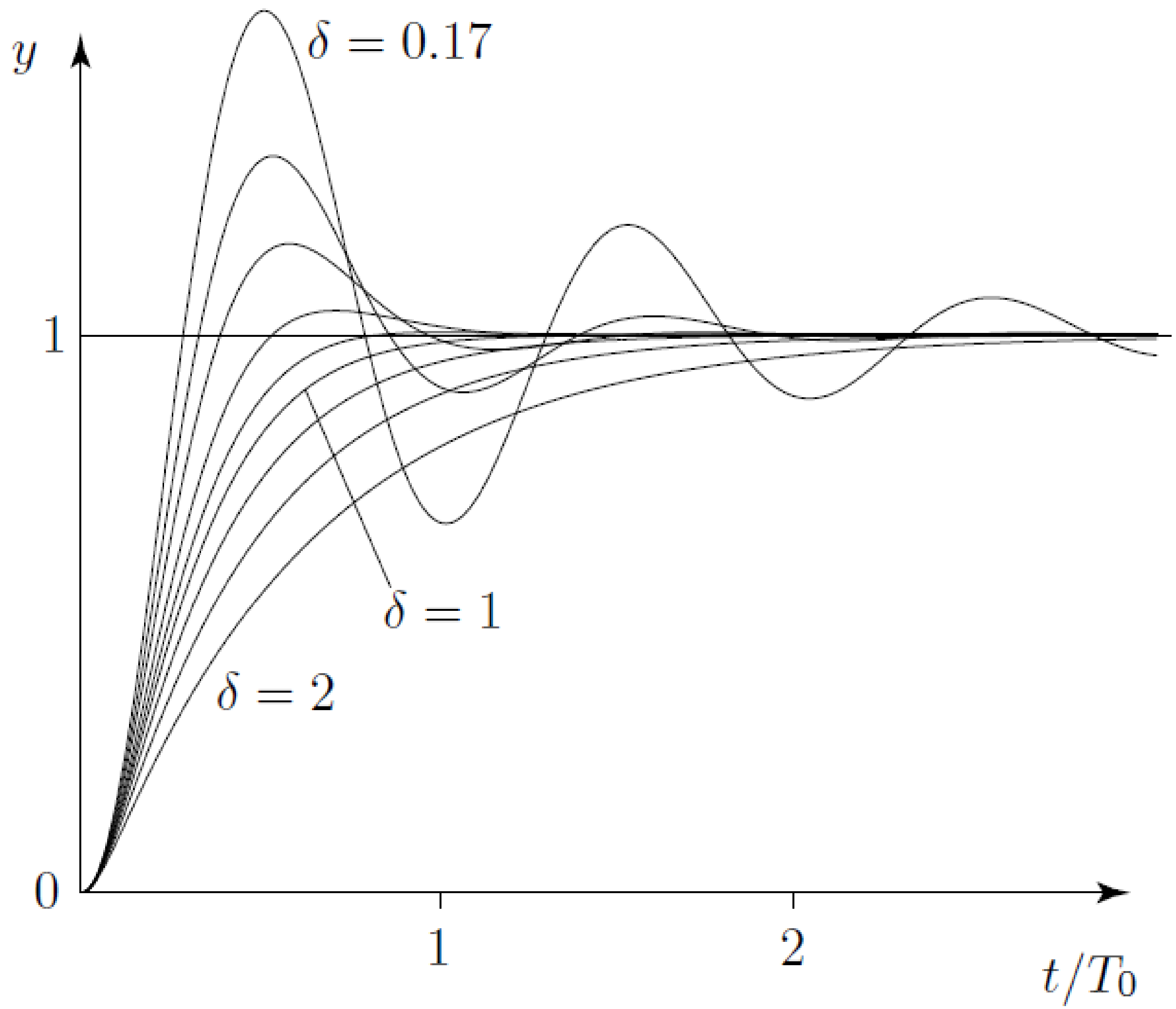
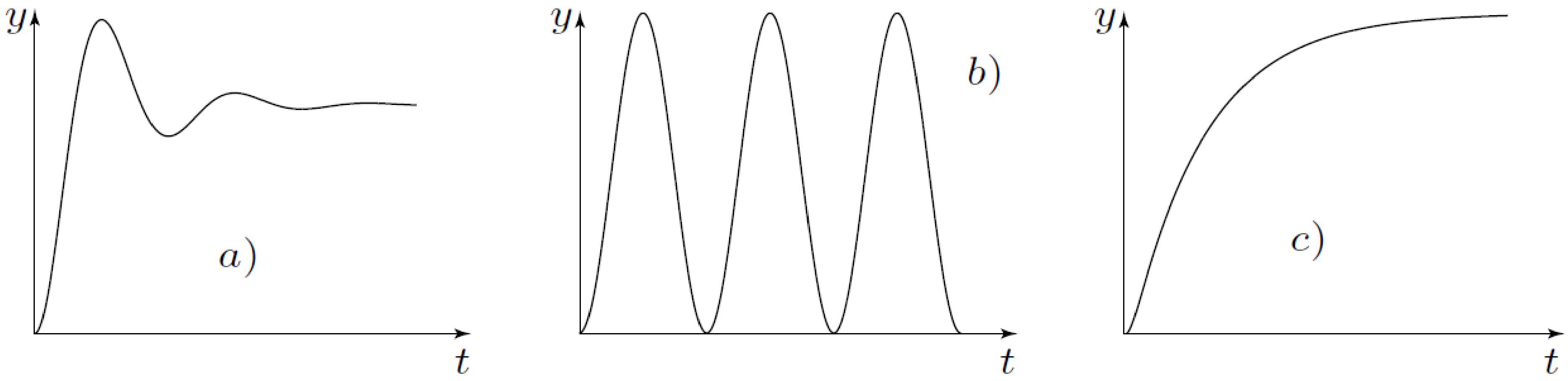


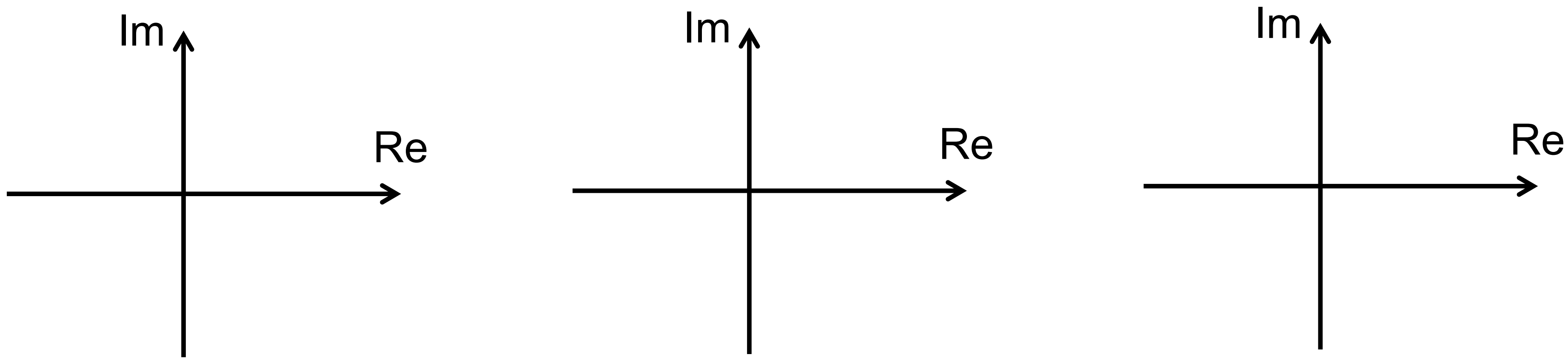
Figure 7.2. Step response of the second-order system (7.12) with normalized time t/T_0 and various damping ratios ($\delta \in \{0.17,\ 0.34,\ 0.5,\ 1/\sqrt{2},\ \sqrt{3}/2,\ 1,\ 1.2,\ 1.5,\ 2\}$).

Poles and Time-Domain Behavior of 2nd-Order Systems

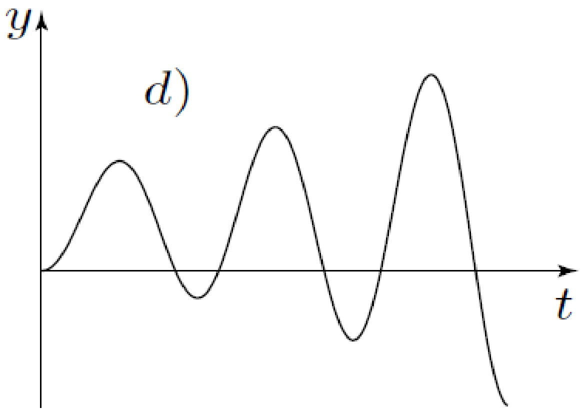
- The system output has a zero gradient at $t = 0$ and asymptotically reaches its final value $y(\infty) = 1$.
- The system outputs overshoots for $0 < \delta < 1$ and increases monotonically for $\delta > 1$.
- The system output scales with the system's natural frequency.
- For $\delta > 1$ the step response is dominated by that pole which is closer by the origin.
- More properties later (lecture 12)

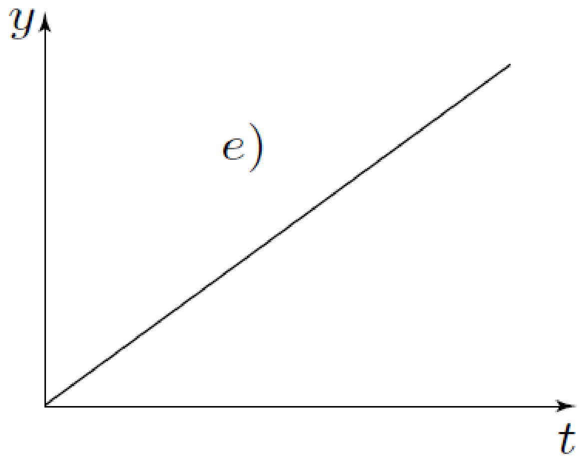
Where are the Poles?

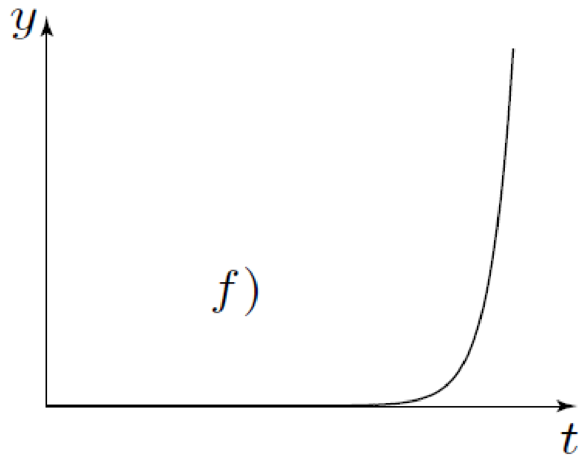


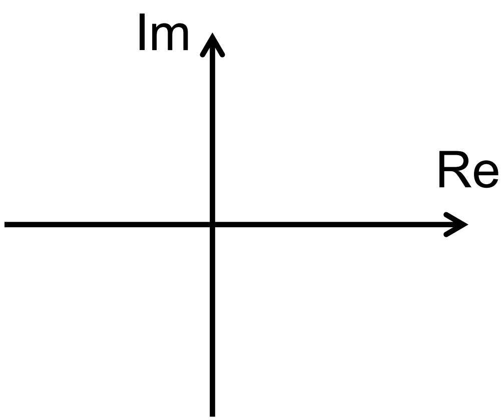


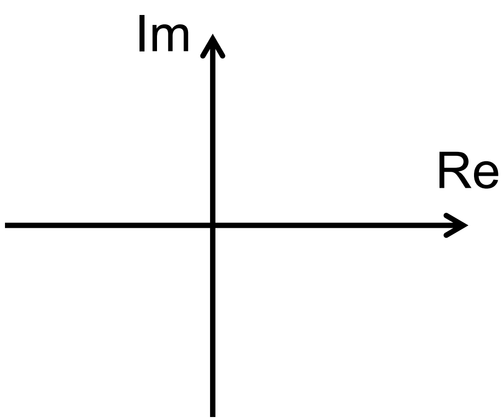
Where are the Poles?

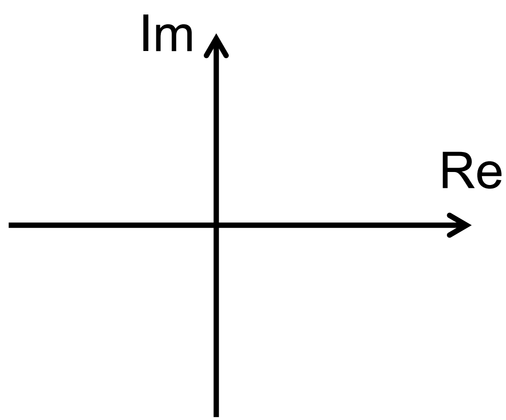












Zeros

$$\Sigma(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}, \quad a_i, b_j \in \mathbb{R}$$

the zeros ζ_i are those frequencies s at which

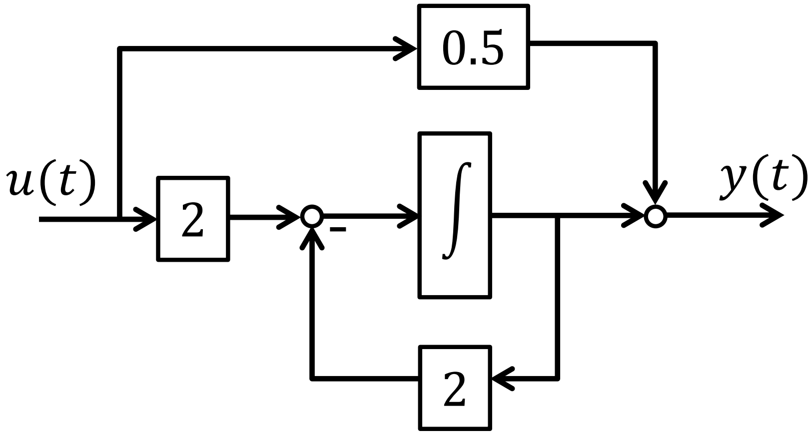
$$\Sigma(s) \text{ is zero}$$

The zeros ζ_i are those frequencies for which a non-zero input $u^*(t)$ and initial conditions $x^*(0)$ exist that produce a zero output $y^*(t)=0$ (details see book).

If the real part of all zeros ζ_i is negative then the system is *minimum phase*. If the real part of at least one zero is positive the system is *non-minimum phase.*

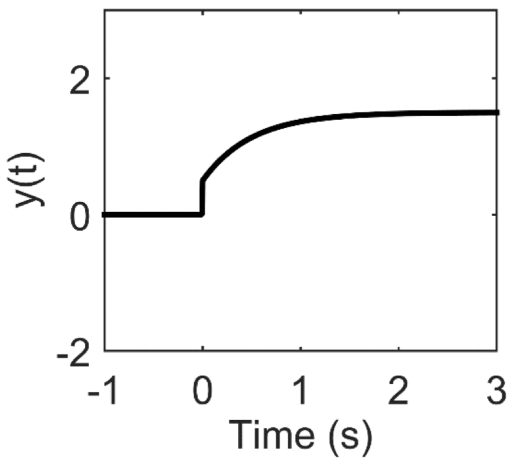
Zero of 1st-Order System

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -2 \cdot x(t) + 2 \cdot u(t)$$
$$y(t) = x(t) + 0.5 \cdot u(t)$$

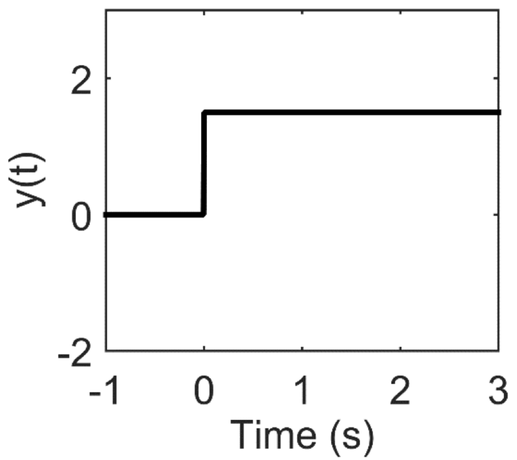


Zeros of 1st-Order Systems

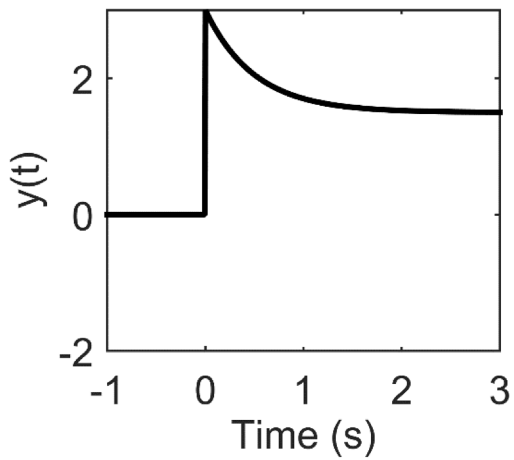
$$\Sigma(s) = \frac{0.5s + 3}{s + 2}$$



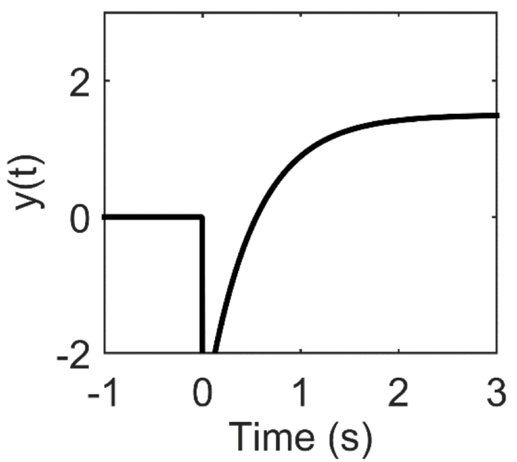
$$\Sigma(s) = \frac{1.5s + 3}{s + 2}$$



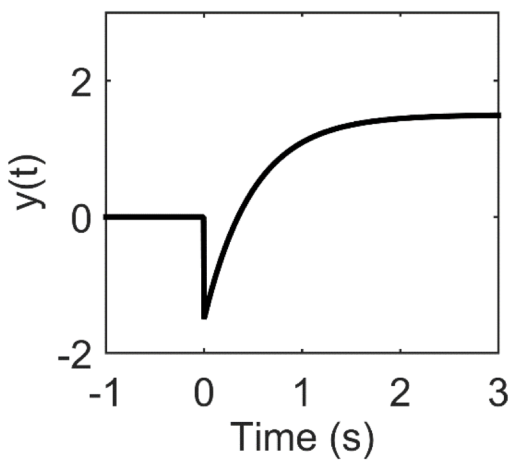
$$\Sigma(s) = \frac{3s + 3}{s + 2}$$



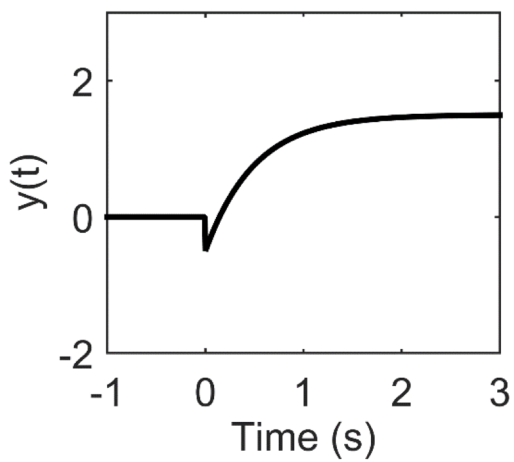
$$\Sigma(s) = \frac{-3s + 3}{s + 2}$$



$$\Sigma(s) = \frac{-1.5s + 3}{s + 2}$$

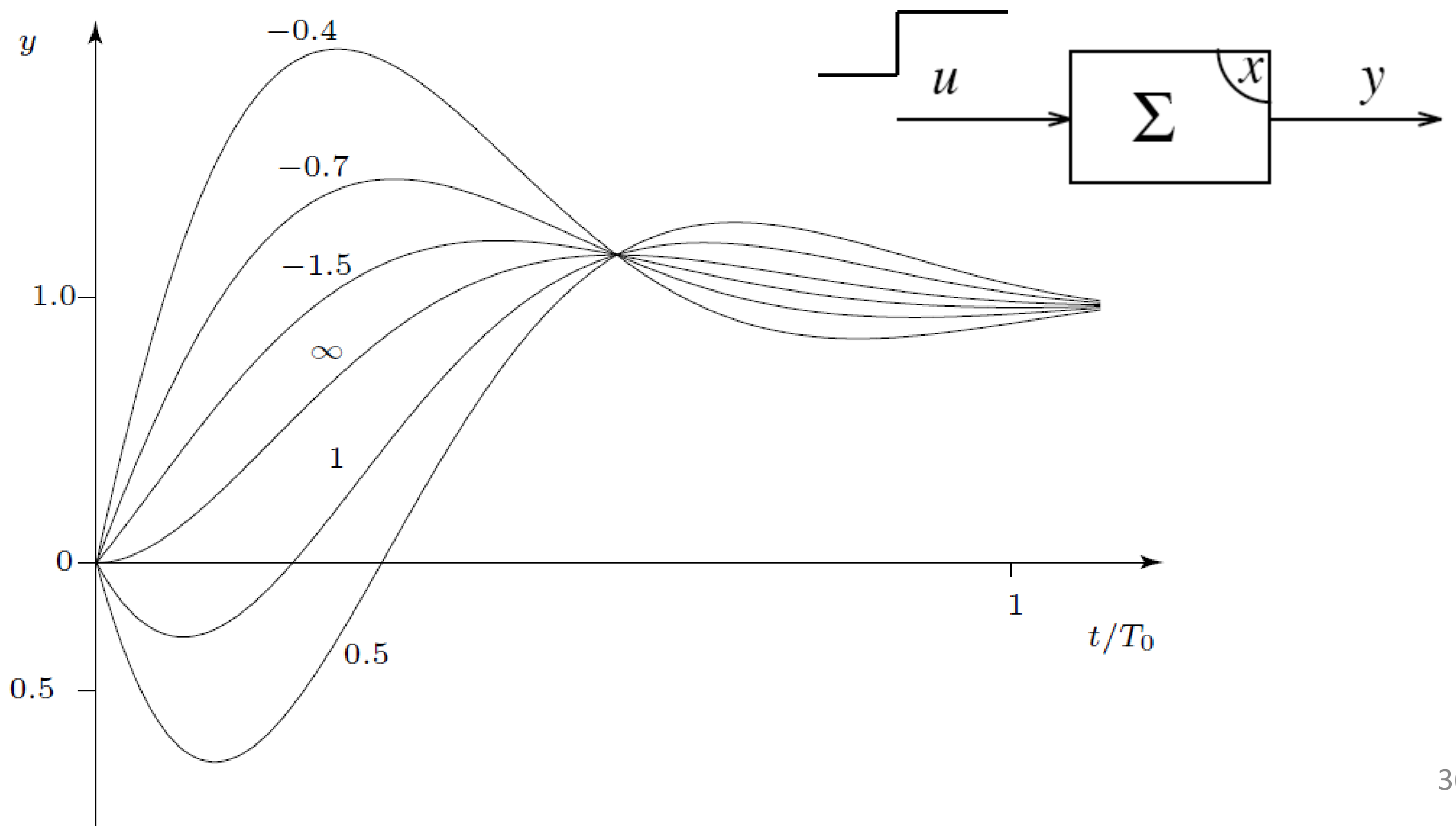


$$\Sigma(s) = \frac{-0.5s + 3}{s + 2}$$



Zeros and Time-Domain Behavior of 2nd-Order Systems

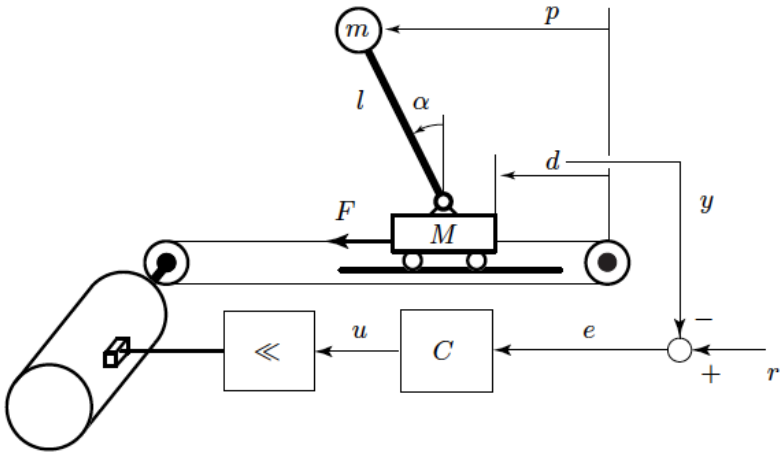
$$\Sigma(s) = \frac{(-s/\zeta + 1) \cdot \omega_0^2}{s^2 + 2 \cdot \delta \cdot \omega_0 \cdot s + \omega_0^2} \qquad \delta = 0.5$$



Zeros and Time-Domain Behavior of Higher-Order Systems

- A zero close to a system pole will reduce the influence of this pole.
- In the limit case $\zeta_i = \pi_i$, the residual becomes zero and the system order appears to be reduced by 1.
- The closer to the origin a zero is, the more important is its influence. This influence manifests itself in an increasing overshoot of the step response.
- A nonminimumphase zero poses an important limitation on feedback control.
- In contrast to system instability, nonminimumphase zeros can often be shifted by a different sensor configuration.

Transfer Functions Inverted Pendulum



cart position sensing : $c_1 = [\,1 \;\; 0 \;\; 0 \;\; 0\,]$

stick angle sensing : $c_2 = [\,0 \;\; 0 \;\; 1 \;\; 0\,]$

tip position sensing : $c_3 = [\,1 \;\; 0 \;\; l \;\; 0\,]$

$$\Sigma_1(s) = \frac{l \cdot s^2 - g}{s^2 \cdot (l \cdot M \cdot s^2 - g \cdot (m + M))}$$

$$\Sigma_2(s) = \frac{-s^2}{s^2 \cdot (l \cdot M \cdot s^2 - g \cdot (m + M))}$$

$$\Sigma_3(s) = \frac{-g}{s^2 \cdot (l \cdot M \cdot s^2 - g \cdot (m + M))}$$