MPC Summary

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1 Systems Theory

1.1 System Dynamics

1.1.1 Continuous Time

Nonlinear Time-Invariant Continuous Time State Space

$\dot x = g(x,u) \quad x\in\mathbb{R}^n,\ u\in\mathbb{R}^m \quad g:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$

$y = h(x,u) \quad y\in\mathbb{R}^p \quad h:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^p$

LTI Continuous Time State Space

Linearization using Taylor Expansion around operating point:

$$f(x)\approx f(\bar x)+\frac{\partial f}{\partial x^\top}\Big|_{\bar x}(x-\bar x)$$

Resulting system:

$$\dot x=\overbrace{\frac{\partial g}{\partial x^\top}\Big|_{x_s,u_s}}^{A^c\in\mathbb{R}^{n\times n}}\delta x+\overbrace{\frac{\partial g}{\partial u^\top}\Big|_{x_s,u_s}}^{B^c\in\mathbb{R}^{n\times m}}\delta u$$

$$y=\underbrace{\frac{\partial h}{\partial x^\top}\Big|_{x_s,u_s}}_{C\in\mathbb{R}^{p\times n}}\delta x+\underbrace{\frac{\partial h}{\partial u^\top}\Big|_{x_s,u_s}}_{D\in\mathbb{R}^{p\times m}}\delta u$$

$$\dot x=A^cx+B^cu \qquad y=Cx+Du$$

Solution: $e^{A^ct}=\sum_{n=0}^{\infty}\frac{(A^ct)^n}{n!}$

$$x(t)=e^{A^c(t-t_0)}x_0+\int_{t_0}^{t}e^{A^c(t-\tau)}B^cu(\tau)d\tau$$

1.1.2 Discrete Time

Euler Discretization (T_s = sampling time) (stability not guaranteed)

$\dot x^c\approx\frac{x^c(t+T_s)-x^c(t)}{T_s}$, $x(k):=x^c(t_0+kT_s)$, $u(k):=u^c(t_0+kT_s)$

Nonlinear System:

$$x(k+1)=x(k)+T_s(g^c(x(k),u(k)))=g(x(k),u(k))$$

$$y(k)=h^c(x(k),u(k))=h(x(k),u(k))$$

Linear System:

$$x(k+1)=A^dx(k)+B^du(k),\quad A^d=\mathbb{I}+T_sA^c,\ B^d=T_sB^c$$

$$y(k)=C^dx(k)+D^du(k),\quad C^d=C^c,\ D^d=D^c$$

Exact Discretization (only for linear systems), (stability guaranteed)

Exact solution (u assumed constant over T_s):

$$x(t_{k+1})=\underbrace{e^{A^cT_s}}_{=A}x(t_k)+\underbrace{\int_0^{T_s}e^{A^c(T_s-\tau)}B^cd\tau}_{B=(A^c)^{-1}(A-\mathbb{I})B^c}u(t_k)$$

We see the solution over k is then given by:

$$x(k+N)=A^Nx(k)+\sum_{i=0}^{N-1}A^iBu(k+N-1-i)$$

1.2 Linear System Analysis

DT Stability (Lyapunov indirect method)

$x(k+1)=Ax(k)$ stable iff $|\lambda_j|<1,\forall j\to$ NL system stable

if $|\lambda_i|=1$ NL system no info, if $|\lambda_i|>1$ NL system unstable

LTI DT Controllability can reach x^* from $x(0)$ in n steps

$$\mathcal{C}=[B\quad\cdots\quad A^{n-1}B]\Rightarrow\text{rank}(\mathcal{C})\overset{!}{=}n$$

DT Observability uniquely distinguish IC from output

$$\mathcal{O}=[C^\top\quad\cdots\quad(CA^{n-1})^\top]^\top\Rightarrow\text{rank}(\mathcal{O})\overset{!}{=}n$$

Stabilizability iff all uncontrollable modes stable $\Lambda_A^+=\{\lambda\mid1\le|\lambda|\}$

if $\text{rank}([\lambda_j\mathbb{I}-A\mid B])=n\ \forall\lambda_j\in\Lambda_A^+\Rightarrow(A,B)$ stabilizable

Detectablity iff all unobservable modes stable $\Lambda_A^+=\{\lambda\mid1\le|\lambda|\}$

if $\text{rank}([A^\top-\lambda_j\mathbb{I}\mid C^\top])=n\ \forall\lambda_j\in\Lambda_A^+\Rightarrow(A,C)$ detect.

1.3 Nonlinear System Analysis

Lyapunov Stability (w.r.t eq. point $\bar x$ of a system)

Lyapunov Stable if for every $\epsilon>0$ exists $\delta(\epsilon)$ s.t.

$$||x(0)-\bar x||<\delta(\epsilon)\to||x(k)-\bar x||<\epsilon$$

Globally Asympt. Stable if Lyap. stable & Attractive

$$\lim_{k\to\infty}||x(k)-\bar x||=0\ \forall x(0)$$

Lyapunov Function

Consider eq point $\bar x=0$. $V:\mathbb{R}^n\to\mathbb{R}$, continuous at origin, finite $\forall x$,

(1) $||x||\to\infty\Rightarrow V(x)\to\infty$

(2) $V(0)=0,\quad V(x)>0\quad\forall x\in\mathbb{R}^n\setminus\{0\}$

(3) $V(g(x))-V(x)\le-\alpha(x)\quad\forall x\in\mathbb{R}^n$

where $\alpha:\mathbb{R}^n\to\mathbb{R}$ continuous pos. def.

Lyapunov Stability

If sys admits a $V(x)\Rightarrow x=0$ is **Globally Asympt. Stable**

Caution if α pos. **semi**def $\Rightarrow x=0$ is **Globally Lyapunov Stable**

Asympt. Stable in pos invar set $\Omega\subseteq\mathbb{R}^n$ if Lyap. stable and attractive

$$\lim_{k\to\infty}||x(k)-\bar x||=0\quad\forall x(0)\in\Omega$$

Globally Asympt. Stable if asympt. stable & $\Omega=\mathbb{R}^n$

2 Linear Quadratic Optimal Control

2.1 Linear Quadratic Optimal Control

Problem Definition

$$J(x(0),U):=x_N^\top Px_N+\sum_{i=0}^{N-1}(x_i^\top Qx_i+u_i^\top Ru_i)$$

subj. to $x_{i+1}=Ax_i+Bu_i,\quad x_0=x(0)$

- N : horizon length • $Q\succeq0,\quad Q=Q^\top$ • $x(0)$: **current** state
- $P\succeq0,\quad P=P^\top$ • $R\succ0,\quad R=R^\top$ • x_i,u_i: **opt.** variable

2.1.1 Batch Approach

Idea explicitly represent $x_i\in\mathbb{R}^n$ through x_0 & $u_i\in\mathbb{R}^m$

$$\begin{bmatrix}x_0\\ \vdots\\ \vdots\\ x_N\end{bmatrix}=\begin{bmatrix}\mathbb{I}\\ A\\ \vdots\\ A^N\end{bmatrix}x(0)+\begin{bmatrix}0&\cdots&0\\ B&\cdots&0\\ \vdots&\ddots&0\\ A^{N-1}B&\cdots&B\end{bmatrix}\begin{bmatrix}u_0\\ u_1\\ \vdots\\ u_{N-1}\end{bmatrix}$$

Equivalent to $\mathcal{S}^x\in\mathbb{R}^{(N+1)n\times n},\quad\mathcal{S}^u\in\mathbb{R}^{(N+1)n\times Nm}$

$$X=\mathcal{S}^xx(0)+\mathcal{S}^uU\to J(x(0),U)=X^\top\bar QX+U^\top\bar RU$$

Cost:

$\bar Q:=\text{blockdiag}(Q,\dots,Q,P)$ & $\bar R:=\text{blockdiag}(R,\dots,R)$

Solve by setting gradient to zero: $2HU^\star+F^\top x(0)=0$

Optimal Input: $H=(\mathcal{S}^u)^\top\bar Q\mathcal{S}^u+\bar R$, $F=(\mathcal{S}^x)^\top\bar Q\mathcal{S}^u$

$$U^\star(x(0))=-((\mathcal{S}^u)^\top\bar Q\mathcal{S}^u+\bar R)^{-1}(\mathcal{S}^u)^\top\bar Q\mathcal{S}^xx(0)$$

Optimal Cost

$$J^\star=x(0)^\top[\mathcal{S}_x^\top\bar Q\mathcal{S}_x-\mathcal{S}_x^\top\bar Q\mathcal{S}_u\left(\mathcal{S}_u^\top\bar Q\mathcal{S}_u+\bar R\right)^{-1}\mathcal{S}_u^\top\bar Q\mathcal{S}_x]x(0)$$

2.1.2 Recursive Approach

Idea: Recursively compute optimal input $u_j^\star$ and optimal cost $J_j^\star$

$$J_j^\star(x(j)):=\min_{U_{j\to N}}x_N^\top Px_N+\sum_{i=j}^{N-1}(x_i^\top Qx_i+u_i^\top Ru_i)$$

$P=P_N$, $F\leftarrow f(P)$, Control input, $P\leftarrow f(F)$, Cost calculation, repeat

Optimal Control Policy

$$u_i^\star=-(B^\top P_{i+1}B+R)^{-1}B^\top P_{i+1}A\cdot x(i):=F_ix_i$$

Optimal Cost-To-Go $J_i^\star(x_i)=x_i^\top P_ix_i$

RDE – Riccati Difference Equation ($\mathbf{P_N=P}$)

$$P_i=A^\top P_{i+1}A+Q-A^\top P_{i+1}B(B^\top P_{i+1}B+R)^{-1}B^\top P_{i+1}A$$

Numerically Safer Alternative

$$P_i=Q+F_i^\top RF_i+(A+BF_i)^\top P(A+BF_i)$$

2.1.3 Comparison - Batch vs. Recursive

- **Batch** – sequence of numeric values $U^\star$
- **Recursive** – feedback policies $u_i^\star$
- Control actions identical if perfect model
- **Disturbances** – Recursive more robust to disturbances
- **Computational efficiency**
  - Recursive more efficient for large N
  - Matrix inversion in Batch approach expensive
- **Constraints** – **Neither** works with constraints on x_i or u_i
- **Batch Approach easier to adapt when contraints are present**
  constrained minimization (solving for J_{i+1} with constraints) hard

2.2 Receding Horizon Control

Compute optimal control policy for N steps, apply only first step, then re-compute

$$U^\star:=\text{argmin}\ x_N^\top Px_N+\sum_{i=0}^{N-1}x_i^\top Qx_i+u_i^\top Ru_i$$

subj. to $x_{i+1}=Ax_i+Bu_i\quad\Rightarrow U^\star$

- Extract first input in sequence: $U^\star=\{u_0^\star,\dots,u_{N-1}^\star\}\Rightarrow u_0^\star$
- Introduce feedback to sys: $x(k+1)=Ax(k)+Bu(k)\Rightarrow x$

Why Reoptimize Provides robustness to noise / modeling errors,
Sol'n at k subopt. (finite horizon) ⇝ reopt. potentially better performance

2.3 Infinite Horizon Control LQR

Solve LQOC for $N\to\infty$

$$J_\infty(x(0))=\min_u\sum_{i=0}^{\infty}x_i^\top Qx_i+u_i^\top Ru_i$$

subj. to $x_{i+1}=Ax_i+Bu_i,\quad x_0=x(0)$

As with recursive approach it must hold:

$$u^\star(k)=-(B^\top P_\infty B+R)^{-1}B^\top P_\infty A\cdot x(k):=F_\infty x(k)$$

with infinite cost to go: $J_\infty(x(k))=x(k)^\top P_\infty x(k)$

Algebraic Riccati Equation (ARE) to find P_∞:

$$P_\infty=A^\top P_\infty A+Q-A^\top P_\infty B(B^\top P_\infty B+R)^{-1}B^\top P_\infty A$$

LQR Lyapunov Function

If (A,B) stabilizable, $(Q^{1/2},A)$ detectable ⇝ $J^\star(x)=x^\top P_\infty x$ is Lyap. function for sys $x^+=(A+BF_\infty)x$

Choice of P in Finite Horizon Control

- Can choose to match ∞-Horizon sol'n ⇝ Make $P\approx J_{N\to\infty}$ with ARE
- Can Choose P assuming no control action after end of horizon
  This P determined from solving Lyap eqn $A^\top PA+Q=P$
  Only makes sense if system asympt. stable
- Assume we want state and input both to be 0 at end of horizon ⇝ no P but extra constraint $x_{i+N}=0$

3 Convex Optimization

3.1 Problem Formulation

$$\min_{x\in\text{dom}(f)}f(x)\quad\text{subj. to } g_i(x)\le0\quad i=1,\dots,m$$
$$h_i(x)=0\quad i=1,\dots,p$$

- $\mathcal{X}:\{x\in\text{dom}(f)\mid g_i\le0,h_i=0\}$ **feasible set**
- g_i : **ineq constraints**, h_i : **eq contraints**

Feasibility Point x satisfies $g_i\le0,h_i=0$ & eq contraints

Optimal Value lowest cost $p^*=f(x^*)=\min_{x\in\mathcal{X}}f(x)$

Strictly Feasible Point x satisfies $g_i<0$

Optimizer smallest $p^*,x\in\mathcal{X}$: $\text{argmin}_{x\in\mathcal{X}}f(x):=\{x\in\mathcal{X}|f(x)=p^*\}$

Caution NOT always unique

Active Contraints: when ineq const. are eq ⇝ "active"

Locally Optimal: $y\in\mathcal{X},||y-x||\le R\Rightarrow f(y)\ge f(x)$

Unbounded Below $p^*=-\infty$, **Unconstrained** $\mathcal{X}=\mathbb{R}^n$

Redundant Contraints do not change feasible set

Globally Optimal: $y\in\mathcal{X}\Rightarrow f(y)\ge f(x)$

Infeasible $p^*=\infty\Leftrightarrow\mathcal{X}=\{\}$

3.2 Convex Sets

Definition Set $\mathcal{X}$ is **convex** *iff* for any pair of points x and y in $\mathcal{X}$:

$$\lambda x+(1-\lambda)y\in\mathcal{X}\quad\forall\lambda\in[0,1],\quad\forall x,y\in\mathcal{X}$$

Interpretation: *All lines starting in $\mathcal{X}$ stay within $\mathcal{X}$*

Convex Combination:

$$x=\theta_1x_1+\theta_2x_2+\ldots+\theta_kx_k,\quad\text{with }\sum_i\theta_i=1,\theta_i\ge0$$

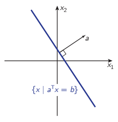
Hyperplane

$\{x\in\mathbb{R}^n\mid a^\top x=b\}$

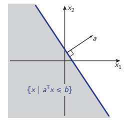
Halfspace

$\{x\in\mathbb{R}^n\mid a^\top x\le b\}$

open: $<$, closed: $\le$



A hyperplane



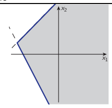
A closed halfspace

Polyhedron

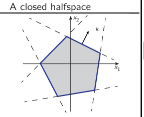
$P:=\{x\mid a_i^\top x\le b_i,i=\ldots\}$

$:=\{x\mid Ax\le b\}$

Polytope: *bounded* Polyhedron



An (unbounded) polyhedron



A polytope

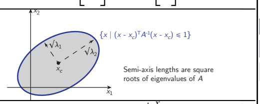
Intersection of Polytopes in inequality form:

$$\{x\mid Ax\le b\}\cap\{x\mid Cx\le d\}=\{x\mid\begin{bmatrix}A\\C\end{bmatrix}x\le\begin{bmatrix}b\\d\end{bmatrix}\}$$

Ellipsoid

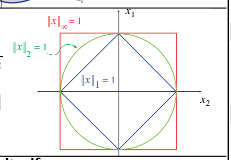
$\{x|(x-x_c)^\top A^{-1}(x-x_c)\le1\}$

x_c : center of ellipsoid



Norm Ball $\{x\mid||x-x_c||\le r\}$

- $p=2$ Euclidean Norm $||x||_2=\sqrt{\sum_ix_i^2}$
- $p=1$ Sum of Absolute $||x||_1=\sum_i|x_i|$
- $p=\infty$ Largest Absolute



Intersection $\bigcap$ of two convex sets is convex itself

Union $\bigcup$ of two convex sets is NOT convex in general

3.3 Convex Functions

Definition A function $f:\mathcal{D}\to\mathbb{R}$ is **convex** *if and only if* its domain $\mathcal{D}=\text{dom}(f)$ is a convex set and

$$f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y),\ \forall x,y\in\mathcal{D},\lambda\in[0,1]$$

$f:\mathcal{D}\to\mathbb{R}$ is **strictly** convex if the inequality is strict.

f is concave if $-f$ is convex.

First order condition

A differentiable function $f:\mathcal{D}\to\mathbb{R}$ with a convex domain $\mathcal{D}$ is convex if and only if

$$f(y)\ge f(x)+\nabla f(x)^\top(y-x),\quad\forall x,y\in\mathcal{D}$$

Gradient is given by: $\nabla f(x)=\left[\frac{\partial f(x)}{\partial x_1},\ldots,\frac{\partial f(x)}{\partial x_n}\right]^\top$

Second order condition

A twice-differentiable function $f:\mathcal{D}\to\mathbb{R}$ with a convex domain $\mathcal{D}$ is convex if and only if

$$\nabla^2f(x)\succeq0,\quad\forall x\in\mathcal{D},\quad\nabla^2f(x)_{ij}=\frac{\partial^2f(x)}{\partial x_i\partial x_j}$$

Strictly convex if $\nabla^2f(x)\succ0$.

Examples

Convex
- **Affine** $ax+b$ for any $a,b\in\mathbb{R}$
- **Exp.** e^{ax} for any $A\in\mathbb{R}$
- **Powers** x^α, $x\in\mathbb{R}_{++}$, $\alpha\ge1$, $\alpha\le0$
- **Vector norms** on $\mathbb{R}^n$: $||x||_p=(\sum_{i=1}^n|x|^p)^{1/p},\forall p\ge1$

Concave
- **Affine** $ax+b$ for any $a,b\in\mathbb{R}$
- **Powers** x^α, $x\in\mathbb{R}_{++}$, for $0\le\alpha\le1$
- **Log** $\log x$ on domain $\mathbb{R}_{++}$
- **Entropy** $-x\log x$ on domain $\mathbb{R}_{++}$

Convextiy Preserving Operations

- Nonnegative weighted sum: $f(x)=\sum_{i=1}^n\theta_if_i(x),\ \theta_i\ge0$
- Composition with an affine mapping: $f(x)=g(Ax+b)$
- Pointwise maximum: $f(x)=\max\{f_1(x),\ldots,f_m(x)\}$
- Partial minimization: $f(x,y)=\min_zg(x,y,z)$

3.3.1 Level and sublevel sets

Levelset

Definition The level set L_α of a function $f:\mathcal{D}\to\mathbb{R}$ is the set of points in the domain $\mathcal{D}$ which $f(x)=\alpha$.

$$L_\alpha=\{x\in\mathcal{D}\mid f(x)=\alpha\}$$

For $f:\mathbb{R}^2\to\mathbb{R}$, these are contour lines of constant height.

Sublevelset

Definition The sublevel set C_α of a function $f:\mathcal{D}\to\mathbb{R}$ is the set of points in the domain $\mathcal{D}$ which $f(x)\le\alpha$.

$$C_\alpha=\{x\in\mathcal{D}\mid f(x)\le\alpha\}$$

If f is convex, then C_α is convex for all α.

3.4 Convex Optimization Problem

A **convex optimization problem** in standard form:

$$\min_{x\in\text{dom}(f)}f(x),\quad\text{subj. to } g_i(x)\le0\quad i=1,\ldots,m$$
$$h_i(x)=a_i^\top x=b_i\quad i=1,\ldots,p$$

f, g_i are convex, h_i are affine.

Affine constraints are typically written in matrix form as $Ax=b$.

Important Property: Feasible set of a convex optimization problem is convex.

Local and Global Optimality: For a convex optimization problem, any local optimal solution is also a global optimal solution.

Equivalent Optimization Problems

Two problems are called equivalent if the solution from one can be inferred easily from the solution of the other.

Example: $\min_xf(A_ox+b)$ subj. to $g_i(A_ix+b_i)\le0,\ i=1,\ldots,m$ is equal to

$$\min_{x,y_i}f(y_0)\quad\text{subj. to } g_i(y_i)\le0,\quad A_ix+b_i=y_i,\quad i=0,\ldots,m$$

3.4.1 Linear Program

Problem

$$\min_{x\in\mathbb{R}^n}c^\top x$$

subj. to $Gx\le h,\quad Ax=b$

Solutions

Case 1: LP unbounded: $p^\star=-\infty$

Case 2: Bounded and unique

Case 3: LP bounded but not unique

3.4.2 Quadratic Program

Problem → solution is unique

$$\min_{x\in\mathbb{R}^n}\frac12x^\top Hx+q^\top x+r$$

subj. to $Gx\le h,\quad Ax=b$

r not needed (does not change optimal x), Convex if $H\succ0$

Case 1: optimizer lies strictly inside the feasible polyhedron

Case 2: optimizer lies on the boundary of the feasible polyhedron

3.5 Optimality Conditions

3.5.1 Lagrang Dual Problem

Lagrangian Function

$$L(x,\lambda,\nu)=f(x)+\sum_{i=1}^m\lambda_ig_i(x)+\sum_{i=1}^p\nu_ih_i(x)$$

Lagrange Dual Function (concave)

$$d(\lambda,\nu)=\inf_{x\in\text{dom}(f)}L(x,\lambda,\nu)\le p^\star$$

inf = infimum, the greatest lower bound of a set.

Primal and Dual Problem

$$(P):\ \begin{matrix}\min_x f(x)\\ \text{subj. to } g_i(x)\le0\\ h_i(x)=0\end{matrix}\qquad\Big|\qquad(D):\ \begin{matrix}\max_{\nu,\lambda}d(\nu,\lambda)\\ \text{subj. to }\lambda\ge0\end{matrix}$$

- $d(\lambda,\nu)$ always **concave**
- $d^\star\le p^\star$ ⇝ $d(\lambda,\nu)$ gens lower bound for p
- (D) **convex** even if (P) not
- Point (λ,ν) **dual feas.** if $\lambda\ge0$, $(\lambda,\nu)\in\text{dom}(d)$

LP – Dual

$$\text{(P)}:\min_{x\in\mathbb{R}^n}c^\top x\quad\text{subj. to } Ax=b,\quad Cx\le e$$

$$\text{(D)}:\max_{\lambda,\nu}-b^\top\nu-e^\top\lambda,\quad\text{s.t } A^\top\nu+C^\top\lambda+c=0,\ \lambda\ge0$$

QP – Dual with $Q\succ0$

$$\text{(P)}:\min_{x\in\mathbb{R}^n}\tfrac12x^\top Qx+c^\top x,\quad\text{subj. to } Cx\le e$$

$$\text{(D)}:\max_{\lambda,\nu}\tfrac12\lambda^\top CQ^{-1}C^\top\lambda+(CQ^{-1}c+e)^\top\lambda+\tfrac12c^\top Q^{-1}c$$

subj. to $\quad\lambda\ge0$

QP – Lagrangian

$$\min_x\tfrac12x^\top Hx+q^\top x+r\qquad L=\tfrac12x^\top Hx+q^\top x+r$$

$$\text{s.t } Gx\le h\qquad+\lambda^\top(Gx-h)+\nu^\top(Ax-b)$$

$$Ax=b\qquad\nabla_xL=Hx+q+G^\top\lambda+A^\top\nu$$

Weak & Strong Duality

Weak Duality – it is **always** true that $d^\star\le p^\star$

Stront Duality – it is **sometimes** true that $d^\star=p^\star$

- Strong duality usually does not hold for non-convex problems
- Can impose conditions on convex prob. to guarantee $d^\star=p^\star$
- Sometimes the dual much easier to solve than the primal
- **LP always has strong duality**

3.5.2 Slater Condition

If ∃ at least one **strictly feasible point** *i.e*

$$\{x \mid Ax = b, g_i(x) < 0 \ \forall i\} \neq \emptyset \ \Rightarrow \ p^\star = d^\star$$

3.5.3 KKT Conditions

(1) Primal Feasibility

$g_i(x^\star) \leq 0, \ i = 1 \dots m \qquad h_i(x^\star) = 0, \ i = 1 \dots p$

(2) Dual feasibility $\lambda^\star \geq 0$

(3) Complementary Slackness

$\lambda_i^\star g_i(x^\star) = 0 \qquad i = 1 \dots m$

(4) Stationarity

$$\nabla L = \nabla f(x^\star) + \sum_{i=1}^{m} \lambda_i^\star \nabla g_i(x^\star) + \sum_{i=1}^{p} \nu_i^\star \nabla h_i(x^\star) = 0$$

General Optimization Necessary condition

$x^\star, \lambda^\star, \nu^\star$ sol'n to (P), (D) with 0 duality gap $\Rightarrow$ $x^\star, \lambda^\star, \nu^\star$ satisfy KKT

Convex Optimization Sufficient condition

$x^\star, \lambda^\star, \nu^\star$ satisfy KKT $\Rightarrow$ $x^\star, \lambda^\star, \nu^\star$ sol'n to (P), (D) with 0 duality gap

Convex Opt. + Slater, Necessary & Sufficient condition

If Slater's cond. holds, $x^\star, \lambda^\star, \nu^\star$ are sol'n to (P), (D) **IFF** KKT satisfied

Remark for convex opt. problem, KKT conditions sufficent ⇝ if $x^\star, \lambda^\star, \nu^\star$ satisfy KKT then $p^\star = d^\star$

4 CFTOC

Constrained Finite Time Optimal Control

$$J^\star(x(k)) = \min_U l_f(x_N) + \textstyle\sum_{i=0}^{N-1} l(x_i, u_i)$$

$$\text{subj. to } x_{i+1} = Ax_i + Bu_i$$

$$x_i \in \mathcal{X}, \quad u_i \in \mathcal{U}$$

$$x_N \in \mathcal{X}_f, \quad x_0 = x(k)$$

- **Quad. Cost / Squared Euclidian Norm:**

$$J(x(k)) = x_N^\top P x_N + \textstyle\sum_{i=0}^{N-1} x_i^\top Q x_i + u_i R u_i$$

- **p-Norm:** $J(x(k)) = ||Px_N||_p + \sum_{i=0}^{N-1} ||Qx_i||_p + ||Ru_i||_p$

4.1 Transform CFTOC to QP

QP Problem

Goal – Rewrite Quad. Cost CFTOC as QP

⇝ easier to solve

$$\min_{z \in \mathbb{R}^n} \tfrac{1}{2} z^\top H z + q^\top z + r$$

$$\text{subj. to } Gz \leq h, \quad Az = b$$

4.1.1 Construction with Substitution, dense (good for large n)

Idea – Sub. state eqns $x_{i+1} = Ax_i + Bu_i$, $x_0 = x(k)$

Cost – Rewrite as (see Batch Approach for H and F)

$$J^\star(x(k)) = \min_U \begin{bmatrix} U^\top & x(k)^\top \end{bmatrix} \begin{bmatrix} H & F^\top \\ F & Y \end{bmatrix} \begin{bmatrix} U^\top & x(k)^\top \end{bmatrix}^\top$$

$$\text{subj. to } GU \leq w + Ex(k)$$

Constraints – Rewrite as $GU \leq w + Ex(k)$

$$\mathcal{X} = \{x | A_x x \leq b_x\} \quad \mathcal{U} = \{u | A_u u \leq b_u\} \quad \mathcal{X}_f = \{x | A_f x \leq b_f\}$$

$$G = \begin{bmatrix} A_u & \dots & \\ & & A_u \\ \hline 0 & \dots & 0 \\ A_x B & & \\ A_x AB & A_x B & 0 \\ \hline A_f A^{N-1} B & \dots & A_f B \end{bmatrix}, E = \begin{bmatrix} 0 \\ \dots \\ 0 \\ \hline -A_x \\ -A_x A \\ \dots \\ \hline -A_f A^N \end{bmatrix}, w = \begin{bmatrix} b_u \\ \dots \\ b_u \\ \hline b_x \\ b_x \\ \dots \\ \hline b_f \end{bmatrix}$$

Solution For a given $x(k)$, $U^\star$ can be found via QP solver

4.1.2 Construction without Substitution, sparse (good for large N)

Idea – Keep state eqns as eq. constraints

Cost with $z = \begin{bmatrix} x_1^\top & \dots x_N^\top & u_0^\top & \dots u_{N-1}^\top \end{bmatrix}^\top$

$$J^\star(x(k)) = \min_z \begin{bmatrix} z^\top & x(k)^\top \end{bmatrix} \begin{bmatrix} \bar{H} & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} z^\top & x(k)^\top \end{bmatrix}^\top$$

$$\text{subj. to } G_{in} z \leq w_{in} + E_{in} x(k)$$

$$G_{eq} z = E_{eq} x(k)$$

$$\bar{H} = \text{diag}(Q, \dots, Q, P, R, \dots, R)$$

Equality Constraints from System Dyn. $x_{i+1} = Ax_i + Bu_i$

$$G_{eq} = \left[\begin{array}{ccc|ccc} \mathbb{I} & & & -B & & \\ -A & \mathbb{I} & & & -B & \\ & -A & \mathbb{I} & & & -B \\ & \dots & \dots & & & \dots \end{array}\right], E_{eq} = \begin{bmatrix} A \\ 0 \\ \dots \\ 0 \end{bmatrix}$$

Inequality Constraints

$$\mathcal{X} = \{x | A_x x \leq b_x\} \quad \mathcal{U} = \{u | A_u u \leq b_u\} \quad \mathcal{X}_f = \{x | A_f x \leq b_f\}$$

$$G_{in} = \left[\begin{array}{cccc|cccc} 0 & & & & 0 & & & \\ \hline A_x & & & & 0 & & & \\ & A_x & & & & 0 & & \\ & & \dots & & & & \dots & \\ & & & A_f & & & & 0 \\ \hline 0 & & & & A_u & & & \\ & 0 & & & & A_u & & \\ & & \dots & & & & \dots & \\ & & & 0 & & & & A_u \end{array}\right], w_{in} = \begin{bmatrix} b_x \\ \hline b_x \\ b_x \\ \dots \\ b_f \\ \hline b_u \\ b_u \\ \dots \\ b_u \end{bmatrix},$$

$$E_{in} = \begin{bmatrix} -A_x^T, 0, \cdots, 0 \end{bmatrix}^\top$$

4.1.3 QP Feedback Solution

CFTOC problem as multiparametric QP

$$J^\star(x(k)) = \min_U \begin{bmatrix} U^\top x(k)^\top \end{bmatrix} \begin{bmatrix} H & F^\top \\ F & Y \end{bmatrix} \begin{bmatrix} U^\top x(k)^\top \end{bmatrix}^\top$$

$$\text{subj. to } GU \leq w + Ex(k)$$

Solution Properties

- **First component of optimal solution:**

$$u_0^\star = \kappa(x(k)), \quad \forall x(k) \in \mathcal{X}_0$$

$\kappa : \mathbb{R}^n \to \mathbb{R}^m$ is cont. and pw. affine on Polyhedra

$$\kappa(x) = F^j x + g^j \quad \text{if} \quad x \in CR^j, \quad j = 1, \dots, N^r$$

- Polyhedral sets $CR^j = \left\{x \in \mathbb{R}^n \mid H^j x \leq K^j\right\}, j = 1, \dots, N^r$ are partition of the feasible polyhedron $\mathcal{X}_0$.
- Value func. $J^\star(x(k))$ is convex and pw quad. on polyhedra.

4.1.4 Transform p-norm CFTOC to LP

ℓ_∞-Minimization

$$\begin{array}{c} \min_{x \in \mathbb{R}^m} ||x||_\infty \\ \text{subj. to } Fx \leq g \end{array} \iff \begin{array}{c} \min_{x, t} t \\ \text{subj. to } -\mathbf{1}t \leq x \leq \mathbf{1}t, \ Fx \leq g \end{array}$$

$-\mathbf{1}t \leq x \leq \mathbf{1}t$ bounds abs value of every elem. with scalar t

ℓ_1-Minimization

$$\begin{array}{c} \min_{x \in \mathbb{R}^m} ||x||_1 \\ \text{subj. to } Fx \leq g \end{array} \iff \begin{array}{c} \min_{x \in \mathbb{R}^m, t \in \mathbb{R}^m} \mathbf{1}^\top t \\ \text{subj. to } -t \leq x \leq t, \ Fx \leq g \end{array}$$

$||x||_1 = \sum_{i=1}^n |x_i| \leq \sum_{i=1}^n t_i = 1_n^\top t \ \rightsquigarrow \ -t \leq x \leq t$

bounds abs value of each component of x with a component of t

4.1.5 Consturction of ∞-norm

Cost (with substitution)

$$\min_z \epsilon_N^x + \textstyle\sum_{i-0}^{N-1} \epsilon_i^x + \epsilon_i^u$$

$$\text{subj. to } -\mathbf{1}_n \epsilon_i^x \leq \pm Q \left[A^i x_0 + \textstyle\sum_{j=0}^{i-1} A^j B u_{i-1-j}\right]$$

$$-\mathbf{1}_r \epsilon_N^x \leq \pm P \left[A^N x_0 + \textstyle\sum_{j=0}^{N-1} A^j B u_{N-1-j}\right]$$

$$-\mathbf{1}_m \epsilon_i^u \leq R u_i$$

$$x_i \in \mathcal{X}, \ u_i \in \mathcal{U}, \ x_f \in \mathcal{X}_f, \ x_0 = x(k)$$

Substitution: $z := \{\epsilon_0^x \dots \epsilon_N^x, \epsilon_0^u \dots \epsilon_{N-1}^u, u_0^\top \dots u_{N-1}^\top\} \in \mathbb{R}^s$

$s := (m+1)N + N + 1$ results in:

$$\min_z c^\top z \qquad \text{subj. to } \bar{G}z \leq \bar{w} + \bar{S}x(k)$$

$$\bar{G} = \begin{bmatrix} G_\epsilon & G_u \\ 0 & G \end{bmatrix}, \quad \bar{S} = \begin{bmatrix} S_\epsilon \\ S \end{bmatrix}, \quad \bar{w} = \begin{bmatrix} w_\epsilon \\ w \end{bmatrix}$$

Solution for given $x(k)$, $U^\star$ can be optained via LP solver

4.1.6 LP State Feedback Solution

Multiparam-LP $\quad \min_z c^\top z \quad \text{subj. to } \bar{G}z \leq \bar{w} + \bar{S}x(k)$

Properties

-First component of sol'n has form: $u_0^\star = \kappa(x(0))$, $\forall x(k) \in \mathcal{X}_0$

$\kappa : \mathbb{R}^n \to \mathbb{R}^m$ is cont. & pw affine on Polyhedra

$\kappa(x) = F^j x + g^j$ if $x \in CR^j$, $\quad j = 1, \dots, N^r$

-Polyhedral sets $CR^j = \{x \in \mathbb{R}^n \mid H^j x \leq K^j\}$ are partition of $\mathcal{X}_0$

-In case of multiple optimizers, a pw affine control law exists

-$J^\star(x(0))$ is convex, pw linear on polyhedra

Quad vs 1/∞-norm cost

Solution is either ($n = \#$ opt. var., FS = feas. set.)

Quadratic Cost	**Linear Cost**
-unique & in interior of FS (no constraints active)	-Unbounded
-unique & on boundary of FS (at least 1 const. active)	-unique at vertex of FS (at least n active constraints)
	-multiple optima (min. 1 active const.)

5 MPC vs Classical Controll

5.1 Difference to Classical Control

Classical Control main issues:	MPC main issues:
Distrubance rejections	Control constraints (input limits)
Noise insensitivity	Process/state constraints (saftey and physical constraints)
Model uncertainty	
Usually in frequency domain	Usually in time domain

MPC can better handle constraints as they are implemented into the control scheme. Classical controllers usually us ad hoc constraint management or suboptimal operation.

5.2 Advantages & Challenges

Advantages:

- Systematic and propper handling of constraints
- High performance controller

Challenges:

- Implementation: → real-time solving is challenging
- Feasibility: Oprimization problem may become infeasible in the future
- Stability: Closed-loop stability is not automatically guaranteed
- Robustness: Closed-loop system is not nessecarily robust against uncertainties or disturbances

6 Invariance

System

Autonomous $x(k+1) = g(x(k))$

Closed-Loop $x(k+1) = g(x(k), \kappa(x(k)))$ for given κ

Positively Invariant Set (Minkowski sum of invariant sets is also invariant)

Set $\mathcal{O}$ positively invariant for autonomous system if

$$x(k) \in \mathcal{O} \Rightarrow x(k+1) \in \mathcal{O}, \quad \forall k \in \{0, 1, \dots\}$$

Maximal Positively Invariant Set

$\mathcal{O}_\infty \subset \mathcal{X}$ positively invariant and contains all other $\mathcal{O}$

Initialize: $\Omega_0 \leftarrow \mathcal{X}$

Do: $\Omega_{i+1} \leftarrow \text{pre}(\Omega_i) \cap \Omega_i$

Until: $\Omega_{i+1} = \Omega_i \to \mathcal{O}_\infty = \Omega_i$

Pre-Set

Given set S, the pre-set of S is the set of states that evolve into S in 1 k:

$x(k+1) = g(x(k))$	$x(k+1) = Ax(k)$
$\Rightarrow \text{pre}(S) := \{x \mid g(x) \in S\}$	$\Rightarrow \text{pre}(S) := \{x \mid Ax \in S\}$

Pre-Set Computation Linear System

Set $S := \{x \mid Fx \leq f\}$, $x(k+1) = Ax(k)$ then

$$\text{pre}(S) := \{x \mid Ax \in S\} = \{x \mid FAx \leq f\}$$

- For $\{x \mid Fx \leq f\}$, if $F \downarrow$ or $f \uparrow$ ⇝ **Less Restrictive**
- $S \cap F$ ⇝ constraints from both sets active

Invariant Set Conditions

Given set S, the pre-set of S is the set of states that evolve into S in one time step. Set $\mathcal{O}$ is positively invariant set iff

$$\mathcal{O} \subseteq \text{pre}(\mathcal{O}) \iff \text{pre}(\mathcal{O}) \cap \mathcal{O} = \mathcal{O}$$

Necessary if $\mathcal{O} \nsubseteq \text{pre}(\mathcal{O})$, then $\exists \bar{x} \in \mathcal{O}$ s.t $\bar{x} \notin \text{pre}(\mathcal{O})$ ⇝ $\bar{x} \in \mathcal{O}, g(\bar{x}) \notin (\mathcal{O})$, thus $\mathcal{O}$ not positively invariant

Sufficient if $\mathcal{O}$ not pos invar set, then $\exists \bar{x} \in \mathcal{O}$ s.t $g(\bar{x}) \notin \mathcal{O}$ ⇝ $\bar{x} \in \mathcal{O}, \bar{x} \notin \text{pre}(\mathcal{O})$ thus $\mathcal{O} \notin \text{pre}(\mathcal{O})$

6.1 Control Invariance

Control Invariant Set

Set $\mathcal{C} \subseteq \mathcal{X}$ control invariant if

$$x(k) \in \mathcal{C} \quad \Rightarrow \ \exists u(k) \in \mathcal{U} \text{ s.t } g(x(k), u(k)) \in \mathcal{C} \ \forall k \in \mathbb{N}^+$$

Maximal Control Invariant Set

Set $\mathcal{C}_\infty$ maximal control invariant if it is control invariant and contains all control invariant sets contained in $\mathcal{X}$

For all states in $\mathcal{C}_\infty$, there exists control law s.t system constraints never violated ⇝ **The best any controller could ever do**

Pre-Set $\text{pre}(S) := \{x \mid \exists u \in \mathcal{U} \text{ s.t } g(x, u) \in S\}$

Control Invariant Set ⇒ Control Law

Let $\mathcal{C}$ be the control invariant set for $x(k+1) = g(x(k), u(k))$

The control law $\kappa(x(k))$ will guarantee that the system satisfies constraints $\forall t$ if $g(x, \kappa(x)) \in \mathcal{C} \ \forall x \in \mathcal{C}$ ⇝ With f as any function, synthesize control law κ: $\kappa(x) := \text{argmin}\{f(x, u) \mid g(x, u) \in \mathcal{C}\}$

- Does not ensure sys. will converge, but will satisfy constraints
- Don't often do because calculating control invariant sets is very hard
- **MPC implicitly** describes cont. invar. set s.t easy to represent/compute

6.2 Practical Invariant Set Computation

Minkowski-Weyl Theorem

For $P \subseteq \mathbb{R}^d$ following statements equivalent:

- P polytope, $\exists A, b$ s.t $P = \{x \mid Ax \leq b\}$
- P finitely generated, $\exists$ finite set of $\{v_i\}$ s.t $P = \text{co}(\{v_1 \dots v_s\})$

6.2.1 Invariant Sets from Lyapunov Functions

Lemma If $V : \mathbb{R}^n \to \mathbb{R}$ a Lyap. func. for sys. $x(k+1) = g(x(k))$, then $Y := \{x \mid V(x) \leq \alpha\}$ is an invariant set for all $\alpha \geq 0$

Proof: $V(g(x)) - V(x) < 0$ ⇝ once $V(x(k)) \leq \alpha$, stays there $\forall j \geq k$

6.2.2 Maximum Ellipsoidal Invariant Sets

For $x(k+1) = Ax(k)$ with $P \succ 0$ with $A^\top PA - P \prec 0$ then $V(x(k)) = x(k)^\top Px(k)$ is Lyap. function. Find largest α s.t set $Y_\alpha \in \mathcal{X}$

$Y_\alpha := \{x \mid x^\top Px \leq \alpha\} \subset \mathcal{X} := \{x \mid Fx \leq f\}$

Equivalent to $\quad \max_\alpha \alpha \quad \text{subj. to } h_{Y_\alpha}(F_i) \leq f_i \ \forall i \in \{1 \dots n\}$

Support of an ellipse: $h_{Y_\alpha}(F_i) = \max_x F_i x \quad \text{subj. to } x^\top Px \leq \alpha$

F_i and f_i are the rows of the polytopic description of $\mathcal{X}$ and $\mathcal{U}$

Change of Variables: $y := P^{1/2} x$

⇝ $h_{Y_\alpha}(F_i) = \max_x F_i P^{-1/2} y \quad \text{s.t } y^\top y \leq \sqrt{\alpha}^2$

Maximizer found by inspection:

$$h_{Y_\alpha}(F_i) = F_i P^{-1/2} \frac{P^{-1/2} F_i^\top}{||P^{-1/2} F_i^\top||} \sqrt{\alpha} = ||P^{-1/2} F_i^\top|| \sqrt{\alpha}$$

Largest ellipse now 1-dim optimization problem:

$$\alpha^\star = \max_\alpha \alpha \quad \text{s.t. } ||P^{-1/2} F_i^\top||^2 \alpha \leq f_i^2 \ \ \forall i \in \{1 \dots n\}$$

$$= \min_{i \in \{1 \dots n\}} \frac{f_i^2}{F_i P^{-1} F_i^\top}$$

7 MPC Formulation

System: $x(k+1) = Ax(k) + Bu(k)$, $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$

Control Law is defined by: $u = u^\star(0)$

$$J^\star(x(k)) = \min_U l_f(x_N) + \textstyle\sum_{i=1}^{N-1} l(x_i, u_i)$$

$$\text{subj. to } x_{i+1} = Ax_i + Bu_i$$

$$x_i \in \mathcal{X}, \quad u_i \in \mathcal{U}$$

$$x_N \in \mathcal{X}_f, \quad x_0 = x(k)$$

Assumptions that need to be met:

1. Stage cost pos def, strictly positive, only 0 at origin
2. (a) Terminal set invariant under local control law $\kappa_f(X)$: $x_{i+1} = Ax_i + B\kappa_f(x_i)$
   (b) All state and input constraints satisfied in $\mathcal{X}_f$
3. Terminal cost is cont. Lyap. func. in terminal set $\mathcal{X}_f$ and satisfies
   $$l_f(x_{i+1}) - l_f(x_i) \leq -l(x_i, \kappa_f(x_i)) \quad \forall x_i \in \mathcal{X}_f$$

If 1-3 are met: CL system under MPC control law $u_0^\star(x)$ asympt. stable and set $\mathcal{X}_N$ is positive invariant for system $x(k+1) = Ax(k) + Bu_0^\star(x(k))$

Often Quadratic Cost:

$$J^\star(x(k)) = \min x_N^T P x_N \textstyle\sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i$$

$$\text{subj. to } x_{i+1} = Ax_i + Bu_i$$

$$x_i \in \mathcal{X}, \quad u_i \in \mathcal{U}, \quad x_N \in \mathcal{X}_f, \quad x_0 = x(k)$$

$$Q = Q^\top \succeq 0, R = R^\top \succ 0$$

For 3. this implies: $A_{cl} = A + BK$

$$A_{cl}^T P A_{cl} - P \preceq -Q(-K^T RK), \quad A_{cl}^T P B_{cl} - P \preceq 0$$

7.1 Loss Of Feasibility & Stability

Infinite-Horizon Solve RHC for $N = \infty$, OL traj. are same as CL traj.

- If problem feasible, CL trajectories always feasible
- If cost finite, states and inputs will converge asympt. to origin

Finite-Horizon RHC "short-sighted" approximating ∞-horizon controller

- **Feasibility** – after some steps finite horizon optimal control problem may become infeasible (disturbances, model mismatch)
- **Stability** – generated inputs may not lead to traj. that converge to orgin

Solution Introduce terminal cost & constraints to ensure feas. & stab.

7.2 Feasibility & Stability Guarantees

Proof Strategy

Recursive Feasibility show existence of feasible control sequence for all time when starting from feasible initial point

- Assume feas. of $x(k)$, $\{u_0^\star, \dots, u_{N-1}^\star\}$, $\{x_0^\star, \dots, x_N^\star\}$
- At $x(k+1) \Rightarrow \{u_1^\star, \dots, \kappa_f(x_N^\star)\}$ should be feas.

Stability show that optimal cost is Lyapunov function

- l_f necessary to provide cost decrease for asympt. stability

Terminal Constraint At Zero $x_N \in \mathcal{X}_f = 0$

If at 0 and no input is given system stays there ⇝ stable and feasibly point. need large N to approximate maximum control invariant set

General Terminal Set $\mathcal{X}_f$

Need assumptions 1-3 for stability guarantees. Cost decrease proof:

$$J^*(x(k+1)) \leq \underbrace{\textstyle\sum_{i=1}^{N-1} l(x_i^*, u_i^*) + l\left(x_N^*, u_N^* = \kappa_f(x_N^*)\right)}_{\text{stage cost at } k+1} + \underbrace{l_f(Ax_N^* + B\kappa_f(x_N^*))}_{\text{cost of propagated state}}$$

$$= \underbrace{\textstyle\sum_{i=0}^{N-1} l(x_i^*, u_i^*)}_{J^*(x(k)) - l_f(x_N^*)} - l(x_0^*, u_0^*) + l(x_N^*, \kappa_f(x_N^*)) + l_f\left(Ax_N^* + B\kappa_f(x_N^*)\right)$$

$$= J^*(x(k)) - \underbrace{l(x(k), u_0^*)}_{\substack{\text{subtract cost} \\ \text{at stage } k>0}} + \underbrace{l(x_N^*, \kappa_f(x_N^*)) + l_f\left(Ax_N^* + B\kappa_f(x_N^*)\right) - l_f(x_N^*)}_{\leq 0 \text{ by assumption of Lyap func.in terminal set } \mathcal{X}_f}$$

⇒ $J^*(x)$ is a Lyap. function & the CL system under the MPC control law is AS.

Terminal Set & Cost – LQR

- Choose $P = P_\infty$ from (D)ARE
- Choose $\mathcal{X}_f$ to be max. invar. set for CL system $(A + BF_\infty)x_k$ ⇝ ellipsoidal inv. set with Lyap.
- All x, u constraints satisfied in $\mathcal{X}_f$

All assumptions of Feasibility & Stability Theorem Satisfied

Useful Properties

-X_1, X_2 convex invar. for $Ax(k)$ ⇝ $\alpha X_1 \oplus (1-\alpha) X_2$ invar $\forall \alpha \in [0, 1]$

-$X_1 \subseteq \mathcal{X}$, $X_2 \subseteq \mathcal{X}$, X_i, $\mathcal{X}$ convex $\to \alpha X_1 \oplus (1-\alpha) X_2 \subseteq \mathcal{X} \ \forall \alpha \in [0, 1]$

-$V_i(x(k)) = x^\top(k) P_i x(k)$ lyap. func. for $x(k+1) = Ax(k)$, rate of decrease $x^\top(k)\Gamma x(k)$ ⇝ $V(x(k)) = \alpha V_1(x(k)) + (1-\alpha) V_2(x(k))$ also lyap. func. with rate of decrease $x^\top(k)\Gamma x(k)$ for all $\alpha \in [0, 1]$

7.2.1 Feasibility & Stability Remarks

- Terminal constraint provides a **Sufficient Condition** for feas. & stab.
- Region of attraction w/o term. const. may be larger than with term. const.
- In practice: enlarge horizon and check stability by sampling. As $N \uparrow$, region of attraction appraoches max. control invariant set
- CL traj. may not follow assumptions made for OL predictions
- ∞-Horizon LQR controller locally optimal ⇝ best choice for quad. cost
- **∞-Horizon provides stab. and invariance. Finite-Horizon MPC may not be stable & may not satisfy constraints ∀ time**

Extension to Nonlinearity

- Assumptions on terminal set/cost did not rely on linearity
- Lyapunov stability is general framework (works for NL sys)
- **Results can be directly extended to NL systems**
- **However, computing sets $\mathcal{X}_f$ and function l_f can be very difficult**

8 Practical MPC

8.1 MPC Reference Tracking

8.1.1 Steady-State Target Tracking

Target Condition

$$x_s = Ax_s + Bu_s \quad z_s = Hx_s = r \iff \underbrace{\begin{bmatrix} \mathbb{I}-A & -B \\ H & 0 \end{bmatrix}}_{(n_x+n_r)\times(n_x+n_u)} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

- In presence of constraints, (x_s, u_s) must satisfy them
- In case of multiple feas. u_s, compute 'cheapest'

$$\min u_s^\top R_s u_s, \text{subj. to [Target Condition]}, x_s \in \mathcal{X}, u_s \in \mathcal{U}$$

- In general, asssume target problem is feasible
- If no sol'n ∃: compute reachable point 'closest' to r

$$\min (Hx_s - r)^\top Q_s (Hx_s - r), \quad \text{subj. to } x_s = Ax_s + Bu_s$$

8.1.2 Reference Tracking

MPC Design

$$\min_U ||z_N - Hx_s||^2_{P_z} + \sum_{i=1}^{N-1} ||z_i - Hx_s||^2_{Q_z} + ||u_i - u_s||^2_R$$

$$\text{subj. to [model, constraints]}, x_0 = x(k)$$

Delta Formulation

Set pt. tracking $\xrightarrow{\text{Coord.Trans.}}$ Regulation Problem

$$\Delta x := x - x_s \quad G_x \Delta x \leq h_x - G_x x_s$$
$$\Delta u := u - u_s \quad G_u \Delta u \leq h_u - G_u u_s$$

- Obtain target steady-state corresponding to reference r
- Initial state $\Delta x(k) = x(k) - x_s$
- Apply reg problem to new system in Δ-Formulation

$$\min \left[V_f(\Delta x_N) + \sum_{i=1}^{N=1} \Delta x_i^\top Q \Delta x_i + \Delta u_i^\top R \Delta u_i \right]$$

$$\text{subj. to } \Delta x_{i+1} = A\Delta x_i + B\Delta u_i, \quad G_x \Delta x_i \leq h_x - G_x x_s$$
$$G_u \Delta u_i \leq h_u - G_u u_s, \quad \Delta x_N \in \mathcal{X}_f, \quad \Delta x_0 = \Delta x(k)$$

- Find optimal sequence of $\Delta U^\star$
- Input applied to system $u_0^\star = \Delta u_0^\star + u_s$

Convergence

Assume target feasible with $x_s \in \mathcal{X}, u_s \in \mathcal{U}$, choose terminal weight $V_f(x)$ and constraint $\mathcal{X}_f$ as in regulation case satisfying
- $\mathcal{X}_f \subseteq \mathcal{X}, Kx \in \mathcal{U} \quad \forall x \in \mathcal{X}_f$
- $V_f(x(k+1)) - V_f(x(k)) \leq -l(x(k), Kx(k)) \quad \forall x \in \mathcal{X}_f$

If in addition the target reference x_s, u_s is such that
- $x_s \oplus \mathcal{X}_f \subseteq \mathcal{X}, \quad K\Delta x + u_s \in \mathcal{U}, \quad \forall \Delta x \in \mathcal{X}_f$

then CL system converges to target reference

$$x(k) \to x_s, z(k) = Hx(k) \xrightarrow{k\to\infty} r$$

Proof
- Invariance under local ctrol law inherited from regulation case
- Constraint satisfaction provided by extra conditions
  - $x_s \oplus \mathcal{X}_f \subseteq \mathcal{X} \to x \in \mathcal{X} \forall \Delta \in \mathcal{X}_f$
  - $K\Delta x + u_s \in \mathcal{U} \forall \Delta x \in \mathcal{X}_f \to u \in \mathcal{U}$
  - Fron asympt stability of the regulation problem: $\Delta x(k) \xrightarrow{k\to\infty} 0$

Terminal Set
- Set of feasible targets may be significantly reduced.
- Enlarge set of feasible targets by scaling terminal set for regulation $\mathcal{X}_f^{\text{scaled}} = \alpha \mathcal{X}_f$
- Invariance maintained if $\mathcal{X}_f$ invariant ⇝ so is $\alpha \mathcal{X}_f$
- Choose α s.t. x, u constraints still satisfied ⇝ scaling target dependent
- Targets at the boundary of the constraints: $x_N = x_s$, correspons to 0-terminal set in regulation case

8.2 Disturbance Rejection

Augmented Model

$$x_{k+1} = Ax_k + Bu_k + B_d d_k$$
$$d_{k+1} = d_k, \quad y_k = Cx_k + C_d d_k$$

Observability of aug. system: rank $\left(\begin{bmatrix} A-\mathbb{I} & B_d \\ C & C_d \end{bmatrix} \right) \overset{!}{=} n_x + n_d$

Inuition At steady-state $\begin{bmatrix} A-\mathbb{I} & B_d \\ C & C_d \end{bmatrix} \begin{bmatrix} x_s \\ d_s \end{bmatrix} = \begin{bmatrix} 0 \\ y_s \end{bmatrix}$ and given y_s, d_s must be uniquely defined

Linear State Estimation

Observer For Augmented Model:

$$\begin{bmatrix} \hat{x}(k+1) \\ \hat{d}(k+1) \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & \mathbb{I} \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \hat{d}(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k)$$
$$+ \begin{bmatrix} L_x \\ L_d \end{bmatrix} (-y(k) + C\hat{x}(k) + C_d \hat{d}(k))$$

Error Dynamics ⇒ choose L s.t error dynamics asympt. stable

$$\begin{bmatrix} x(k+1) - \hat{x}(k+1) \\ d(k+1) - \hat{d}(k+1) \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & \mathbb{I} \end{bmatrix} \begin{bmatrix} x(k) - \hat{x}(k) \\ d(k) - \hat{d}(k) \end{bmatrix}$$
$$- \begin{bmatrix} L_x \\ L_d \end{bmatrix} (C\hat{x}(k) + C_d \hat{d}(k) - Cx(k) - C_d d(k))$$
$$= \left(\begin{bmatrix} A & B_d \\ 0 & \mathbb{I} \end{bmatrix} + \begin{bmatrix} L_x \\ L_d \end{bmatrix} \begin{bmatrix} C & C_d \end{bmatrix} \right) \begin{bmatrix} x(k) - \hat{x}(k) \\ d(k) - \hat{d}(k) \end{bmatrix}$$

Observer Steady-State

Suppose observer asympt. stable and $n_y = n_d$

$$\begin{bmatrix} A-\mathbb{I} & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_\infty \\ u_\infty \end{bmatrix} = \begin{bmatrix} -B_d \hat{d}_\infty \\ y_\infty - C_d \hat{d}_\infty \end{bmatrix}$$

⇝ Observer output $C\hat{x}_\infty + C_d \hat{d}_\infty$ tracks y_∞ without offset

Offset-Free Tracking

Goal Track constant r: $z(k) = Hy(k) \to r$ as $k \to \infty$

Steady-State Condition

$$x_s = Ax_s + Bu_s + B_d \hat{d}_\infty, \quad z_s = H(Cx_s + C_d \hat{d}_\infty) = r$$

- Best forecast for d_∞ is current estimate $\hat{d}_\infty = \hat{d}$
- Same Procedure for regulation case with $r = 0$

Offset-Free Tracking Condition: $\begin{bmatrix} A-\mathbb{I} & B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} -B_d \hat{d} \\ r - HC_d \hat{d} \end{bmatrix}$

Offset-Free Tracking Procedure

1. Estimate $\hat{x}$ & $\hat{d}$
2. Obtain (x_s, u_s) from steady-state tgt problem using $\hat{d}$
3. Solve MPC problem for tracking using $\hat{d}$, $\tilde{x}_i := x_i - x_s$, $\tilde{u}_i = u_i - u_s$

$$\min_U V_f(\tilde{x}_N) + \sum_{i=0}^{N-1} (\tilde{x}_i)^\top Q(\tilde{x}_i) + (\tilde{u}_i)^\top R(\tilde{u}_i)$$
$$\text{subj. to } x_{i+1} = Ax_i + Bu_i + B_d d_i, \quad d_{i+1} = d_i$$
$$x_i \in \mathcal{X}, \quad u_i \in \mathcal{U}, \quad x_0 = \hat{x}(k), \quad d_0 = \hat{d}(k), \quad x_n - x_s \in \mathcal{X}_f$$

Offset-Free Tracking: Main Result

With $u_0^\star = \kappa(\hat{x}(k), \hat{d}(k), r)$. Assuming $n_d = n_y$, RHC recursively feasable and unconstrained for $k \geq j$, $j \in \mathbb{N}^+$ and the CL system:

$$x(k+1) = Ax(k) + B\kappa(\hat{x}(k), \hat{d}(k), r) + B_d d$$
$$\hat{x}(k+1) = (A + L_x C)\hat{x}(k) + (B_d + L_x C_d)\hat{d}(k)$$
$$+ B\kappa(\hat{x}(k), \hat{d}(k), r) - L_x y(k)$$
$$\hat{d}(k+1) = L_d C\hat{x}(k) + (\mathbb{I} + L_d C_d)\hat{d}(k) - L_d y(k)$$

converging, i.e. $((\hat{x}, \hat{d}) \xrightarrow{k\to\infty} (x_\infty, d_\infty))$

Then $z(k) = Hy(k) \xrightarrow{k\to\infty} r$

8.3 Enlarging Feasible Set

8.3.1 No Terminal Set

Motivation Terminal constraints reduce feasible set, Stability guarantees can add large number of constraints and adds state constraints to problems with only input constraints.

Goal MPC without terminal constraints with guaranteed stability

Note Feasible set without terminal constraint not invariant

MPC Without Terminal Set

Can remove terminal constraint while maintaining stability **if**
- Initial state lies in sufficiently small subset of feasible set
- N sufficiently large

s.t term. state satisfies term. const. without envorcing it in the optimization. ⇝ Sol'n of finite-horizon MPC problem corresponds to ∞-horizon sol'n

Advantage – Controller defined in larger feasible set

Disadvantage – Characterization of region of attaction of specification of required horizon length extremely difficult

- Term constraint provides sufficient condition for stab: Region of attraction without term constraint may be larger than with
- In practice: Enlarge horizon and check stability by sampling
- $N \uparrow$ ⇝ RoA approaches max control invar. set

8.3.2 Soft constraints

Motivation Input constraints usually 'hard' due to physical limits, state constraints rarely 'hard' (more safety and comfort reasons)

Goal Min size & duration of violation (**usually conflict!**)

MPC Problem Setup

$$\min_u x_N^\top P x_N + \boldsymbol{l_\epsilon(\epsilon_N)} + \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i + \boldsymbol{l_\epsilon(\epsilon_i)}$$

$$\text{s.t. } x_i = Ax_i + Bu_i, \; H_x x_i \leq k_x \boldsymbol{+\epsilon_i}, \; H_u u_i \leq k_u, \quad \boldsymbol{\epsilon_i \geq 0}$$

Requirement on l_ϵ

Original Problem	"Softened" Problem
$\min_z f(z) \quad \text{s.t } g(z) \leq 0$	$\min_z f(z) + l_\epsilon(\epsilon) \quad \text{s.t } g(z) \leq \epsilon, \; \epsilon \geq 0$

If original problem has feasible solution $z^\star$, Softened problem should have same solution $z^\star$, and $\epsilon = 0$.

Note $l_\epsilon(\epsilon_i) = s\epsilon_i^2$ does **not** fulfill requirement

Choice of Penalty
- **Quad. Penalty** $l_\epsilon(\epsilon_i) = \epsilon_i^\top S \epsilon_i$ (e.g $S = Q$)
- **Quad. + Linear Penalty** $l_\epsilon(\epsilon_i) = \epsilon_i^\top S \epsilon_i + v||\epsilon_i||_{1/\infty}$

Exact Penalty Function

$l_\epsilon(\epsilon) = v \cdot \epsilon$ satisfies requirement for any $v > \lambda^\star \geq 0$, where $\lambda^\star$ is optimal Lagrange multiplier for original problem.

In practice combined cost → exact penalty and tuning capabilities

$$l_\epsilon(\epsilon) = v \cdot \epsilon + \epsilon^\top S \epsilon$$

with $v > \lambda^\star$ and $S \succ 0$.

Tuning
- Increasing S leads to hardeing of constraints → reduced violation size but longer duration
- Increasing v leads to constraint satisfaction if possible → larger but shorter violation

Objective Seperation

1. Minimize violation over horizon:

$$\epsilon^{\min} = \text{argmin}_{u,\epsilon} \sum_{i=0}^{N-1} \epsilon_i^\top S \epsilon_i + v^\top \epsilon_i$$
$$\text{s.t } x_{i+1} = Ax_i + Bu_i H_x x_i \leq k_x + \epsilon_i$$
$$H_u u_i \leq k_u, \quad \epsilon_i \geq 0$$

2. Optimize Controller performance

$$\min_u x_N^\top P x_N + \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i$$
$$\text{s.t } x_{i+1} = Ax_i + Bu_i, \quad H_x x_i \leq k_x + \epsilon_i^{\min}, \quad H_u u_i \leq k_u$$

Simplifies tuning and constraint satisfied if possible, but two optimization problems have to be solved.

Note SC MPC does not provide stability guarantee for OL unstable sys.

9 Robust MPC for Linear Systems

9.1 Robust Open-Loop MPC

9.1.1 Uncertainty Models

Motivation: Random noise w influences system evolution, Model structure is unknown, Unknown parameters θ impact dynamics.

Uncertain Constrained System

$$x(k+1) = g(x(k), u(k), w(k); \theta),$$
$$x, u, w, \theta \in \mathcal{X}, \mathcal{U}, \mathcal{W}, \Theta$$

9.1.2 Robust Invariance

Robust Positive Invariant Set

Set $\mathcal{O}^\mathcal{W}$ said to be robust positive invariant for the autonomous system $x(k+1) = g(x(k), w(k))$ if

$$x \in \mathcal{O}^\mathcal{W} \Rightarrow g(x, w) \in \mathcal{O}^\mathcal{W}, \; \forall w \in \mathcal{W}, \; \forall k$$

Robust Pre Set

Given set Ω and dynamic system $x(k+1) = g(x(k), w(k))$,

$$\text{pre}^\mathcal{W}(\Omega) := \{x \mid g(x, w) \in \Omega \;\; \forall w \in \mathcal{W}\}$$

Maximal Robust Positively Invariant Set

$\mathcal{O}^\mathcal{W}_\infty \subset \mathcal{X}$ positively invariant and contains all other $\mathcal{O}^\mathcal{W}$:

Calculation using the algorithm for the nominal case.

Computing Robust Pre-Sets for Linear Systems

System $Ax(k) + w(k)$, set $\Omega := \{x \mid Fx \leq f\}$

$$\text{pre}^\mathcal{W}(\Omega) = \{x \mid FAx \leq f - \max_{w \in \mathcal{W}} Fw\}$$
$$= \{x \mid FAx \leq f - h_{\mathcal{W}^i}(F)\}$$

$h_\mathcal{W}$ is the support function

Robust Invariant Set Conditions

Set $\mathcal{O}^\mathcal{W}$ is a robust positive invariant set **iff**

$$\mathcal{O}^\mathcal{W} \subseteq \text{pre}^\mathcal{W}(\mathcal{O}^\mathcal{W}) \;\Leftrightarrow\; \text{pre}^\mathcal{W}(\mathcal{O}^\mathcal{W}) \cap \mathcal{O}^\mathcal{W} = \mathcal{O}^\mathcal{W}$$

9.1.3 Impact of Additive Bounded Noise

Additive Bounded Noise System:

$$x(k+1) = Ax(k) + Bu(k) + w(k),$$
$$x, u, w \in \mathcal{X}, \mathcal{U}, \mathcal{W}$$

Uncertain State Evolution:

$$\phi_i = \underbrace{A^i x_0 + \sum_{j=0}^{i-1} A^j B u_{i-1-j}}_{x_i \,\equiv\, \text{Nominal System}} + \underbrace{\sum_{j=0}^{i-1} A^j w_{i-1-j}}_{\text{Disturbance Offset}}$$

Robust Open-Loop MPC

Robust Open-Loop MPC

$$\min_U l_f(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i)$$
$$\text{subj. to } x_{i+1} = Ax_i + Bu_i$$
$$x_i \in \mathcal{X} \ominus (\textstyle\bigoplus_{j=0}^{i-1} A^j \mathcal{W}), \quad u_i \in \mathcal{U}$$
$$x_0 = x(k), \quad x_N \in \mathcal{X}_f \ominus (\textstyle\bigoplus_{j=0}^{N-1} A^j \mathcal{W})$$

where $\mathcal{X}_f \subseteq \mathcal{X}$ robust positive invariant set for system $(A + BK)x(k) + w(k)$ with $w \in \mathcal{W} \;\forall k$ for some stabilizing K, and $Kx \in \mathcal{U} \;\forall x \in \mathcal{X}_f$

Intuition Nominal MPC, but with tigher state constraints

Open-Loop: Not accounting for FB during solving, just plan ahead for w

Caution:

Unstable systems $A^{i-1}\mathcal{W}$ grows → use 'pre-stabilization' $u_i = Kx_i + u_i$

Potentially **very small region of attraction**, particularly for unstable sys

9.1.4 Robust Constrained Control

Goals: Design $u(k) = \kappa(x(k))$ such that the system
- **(a) Satisfies constraints**: $\{x(k)\} \subset \mathcal{X}, \{u(k)\} \subset \mathcal{U}$ for all disturbances
- **(b) Is Stable**: converges to a neighborhood of the origin
- **(c) Optimizes (expected/worst-case) 'Performance'**
- **(d) Maximizes Set** $\{x(0) \mid \text{Condition 1-3 met}\}$

(a) Robust Constraint Satisfaction

Ensure all states $\phi_i(x_0, U, W)$ satisfy system constraints $\mathcal{X}$:
- **State & Input Constraints** for $i = 0, \ldots, N-1$,
  Enforce constraints explicitly by imposing:
  $\phi_i \in \mathcal{X}, \; u_i \in \mathcal{U}, \; \forall W \in \mathcal{W}^N$
- **Terminal Constraints** for $i = N, \ldots$
  Enforce constraints implicitly by:
  Constraining $\phi_N \in$ robust invariant set $\mathcal{X}_f$ and $K\mathcal{X}_f \in \mathcal{U}$ for $\phi_{i+1} = (A + BK)\phi_i + w_i$

We want for all $i = 0, \ldots, N$:

$$\phi_i(x_0, U, W) = \left\{ x_i + \sum_{j=0}^{i-1} A^j w_{i-1-j} \mid W \in \mathcal{W}^i \right\} \subseteq \mathcal{X}$$

Assume $\mathcal{X} = \{x \mid Fx \leq f\}$ (polyhedron)

$$Fx_i \leq f - h_{\mathcal{W}^i}\left(F \sum_{j=0}^{i-1} A^j\right)$$

→ tightening constrains on the nominal system.

Support function $h_{\mathcal{W}^i}$ can be pre-computed offline.

Same goes for $i = N, \ldots, \infty$, i.e. $\phi_N(x_0, U, W) \subseteq \mathcal{X}_f$.

Requirement can be rewritten as:

$$\phi_i \in x_i \oplus (\mathcal{W} \oplus A\mathcal{W} \ldots A^{i-1}\mathcal{W}) \subseteq \mathcal{X}$$

or

$$x_i \in \mathcal{X} \ominus \left(\textstyle\bigoplus_{j=0}^{i-1} A^j \mathcal{W} \right)$$

$\mathcal{F}_i = \bigoplus_{j=0}^{i-1} A^j \mathcal{W}$ is called disturbance reachable set.

Note: $\mathcal{F}_{i+1} = A\mathcal{F}_i \oplus \mathcal{W}$

Caution: *Must ensure term state contained in robust invariant set*

Intuition: Tightening constraints on the nominal system

(b) Is stable: To show stability more general stability theory is needed.

(c) – Optimizes Performance

Cost to Minimize:

$$J(x_0, U, W) := l_f(\phi_N(x_0, U, W)) + \sum_{i=0}^{N-1} l(\phi_i(x_0, U, W), u_i)$$

Several options to eliminate dependence on W:
- Minimize expected value: $J_N(x_0, U) = \mathbb{E}\{J(x_0, U, W)\}$
- Take the worst case: $J_N(x_0, U) := \max_{W \in \mathcal{W}^{N-1}} J(x_0, U, W)$
- **Take the Nominal Case** $J_N(x_0, U) := J(x_0, U, 0)$

(d) Maximizes Set: potentially very small region of attraction

9.2 Robust Closed Loop MPC

Increase the feasibly set using closed-loop feedback.

9.2.1 Closed-Loop Predictions

Goal optimize over seq. of funcs $\{u_0, \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$ where $\mu_i(x_i) : \mathbb{R}^n \to \mathbb{R}^m$ is called control policy

Problem *Can't optimize over arbitrary functions!*

Solution assume some structure on functions μ_i

Pre-Stabilization $\mu_i(x_i) = Kx_i + v_i$
  Fixed K, s.t $A + BK$ stable → Simple, often conservative

Linear Feedback $\mu_i(x_i) = K_i x_i + v_i$
  Optimize over K_i, v_i, *Non-Convex* – Extremely difficult to solve

Disturbance Feedback $\mu_i(x_i) = \sum_{j=0}^{i-1} M_{ij} w_j + v_i$
  Optimize over M_{ij}, v_i → Equiv to linear feedback but *Convex* → Effective, but computationally intense

Tube-MPC $\mu_i(x_i) = v_i + K(x_i - \bar{x}_i)$
  Fixed K, s.t $A + BK$ stable → Optimize over $\bar{x}_i, v_i$ → Simple, can be effective

9.2.2 Tube-MPC

System: $x(k+1) = Ax(k) + Bu(k) + w(k) \quad x, u \in \mathcal{X}, \mathcal{U} \quad w \in \mathcal{W}$

Idea Seperate available control authority into 2 parts

(1) Portion that steers nominal system to origin:
  $z(k+1) = Az(k) + Bv(k)$

(2) Portion that compensates for deviations from this system $u_i = K(x_i - z_i) + v_i$ (keeps real traj close to nominal), for some linear K, which stabilizes nominal system

→ *Fix linear feedback K offline* and optimize over nominal trajectory $\{v_0, \ldots, v_{N-1}\}$ → convex problem

Error Dynamics

Define $e_i := x_i - z_i$ ⇝ $e_{i+1} = (A + BK)e_i + w_i$

Bound maximum error, how far 'real' traj from nominal

$$e_{i+1} = (A + BK)e_i + w_i \qquad w_i \in \mathcal{W}$$

Dynamics $A + BK$ are stable, set $\mathcal{W}$ bounded ⇝ Set $\mathcal{E}$ s.t e stays inside $\forall k$ → 'minimal robust invariant set'

Tube-MPC Procedure

(a) Compute set $\mathcal{E}$ that error remains inside
(b) Modify constraints on nominal traj $\{z_i\}$
(c) Formulate as convex optimization problem

(a) Minimum Robust Invariant Set (mRPI)

Minimum Robust Invariant Set

$$F_\infty = \bigoplus_{j=0}^{\infty} A^j \mathcal{W}$$
$$F_0 := \{0\}$$

If $F_n = F_{n+1} \Rightarrow F_n = F_\infty$

Algorithm to Compute F_∞

```
Ω_0 ← {0}
loop
    Ω_{i+1} ← Ω_i ⊕ A^i W
    if Ω_{i+1} = Ω_i then
        return F_∞ = Ω_i
    end if
end loop
```

-Finite n does not always exist, 'large' n often good approximation.

-If n not finite, other methods for small invariant sets, bit larger than F_∞

(b) Modify Nominal Trajectory Constraints

Noisy System Trajectory:
Given nominal trajectory z_i noisy system trajectory $x_i = z_i + e_i$ will be somewhere in $\mathcal{E}$

$$x_i \in z_i \oplus \mathcal{E} = \{z_i + e \mid e \in \mathcal{E}\}$$

Goal $x_i, u_i \in \mathcal{X}, \mathcal{U}$ for all $\{w_i\} \in \mathcal{W}^j$
State Condition Necessary & Sufficent Condition

$$z_i \oplus \mathcal{E} \subseteq \mathcal{X} \quad \Leftrightarrow \quad z_i \in \mathcal{X} \ominus \mathcal{E}$$

Input Condition:

$$u_i \in K\mathcal{E} \oplus v_i \subset \mathcal{U} \quad \Leftrightarrow \quad v_i \in \mathcal{U} \ominus K\mathcal{E}$$

Set $\mathcal{E}$ known offline – *can compute constraints offline!*
Ideally $\mathcal{E}$ is the minimum RPI set $\mathcal{F}_\infty = \bigoplus_{j=0}^{\infty} A^j \mathcal{W}$

(c) Convex Optimization Problem

Problem Formulation:

$$\min_{Z,V} \; l_f(z_N) + \sum_{i=0}^{N-1} l(z_i, v_i)$$

$$\left.\begin{aligned} \text{s.t. } & z_{i+1} = Az_i + Bv_i \\ & z_i \in \mathcal{X} \ominus \mathcal{E}, \quad u_i \in \mathcal{U} \ominus K\mathcal{E} \\ & z_N \in \mathcal{X}_f, \quad x(k) \in z_0 \oplus \mathcal{E} \end{aligned}\right\} =: \text{Set } \mathcal{Z}$$

$$\text{Control Law}: \mu_{\text{tube}}(x) := K(x - z_0^\star(x)) + v_0^\star(x)$$

- Optimizing nominal system with tightened state, input constraints
- **First tube center** z_0 **is opt. var.** $\rightsquigarrow$ has to be within $\mathcal{E}$ of x_0
- Cost is w.r.t tube centers, terminal set is w.r.t tightened constraints

Caution: $K(x - z_0^\star(x)) + v_0^\star(x)$ **NOT LINEAR** in CL

Robust Invariance

Suppose the terminal ingredients $(l_f, \mathcal{X}_f^{ct}, \pi_f)$ are designed such that $\mathcal{X}_f^{ct} \subset \mathcal{X}$ and for all $z \in \mathcal{X}_f^{ct}$:

- $\pi_f(z) \in \mathcal{U}$
- $Az + B\pi_f(z) + w \in \mathcal{X}_f^{ct} \; \forall w \in \mathcal{W}$
- $l_f(Az + B\pi_f(z)) - l_f(z) \le -l(z, \pi_f(z))$

Let $\mathcal{X}_N$ be the feasible set and $V^\star(x(k))$ be the optimizer of the robust constraint-tightening MPC problem.
Then $Ax(k) + Bv_0^\star(x(k)) + w(k) \in \mathcal{X}_N \;\; \forall w(k) \in \mathcal{W}$
$\rightarrow$ problem is recursively feasible

Robust Constraint Satisfaction

Tube-MPC Assumptions: almost the same as for nominal MPC

(1) Stage cost pos def, i.e strictly pos and only 0 at origin
(2) Terminal set is invariant **for the nominal system** under local control law $\kappa_f(z)$: $\;Az + B\kappa_f(z) \in \mathcal{X}_f \quad \forall z \in \mathcal{X}_f$
All **tightened state and input constraints** satisfied in $\mathcal{X}_f$:
$\mathcal{X}_f \subseteq \mathcal{X} \ominus \mathcal{E}, \quad \kappa_f(z) \in \mathcal{U} \ominus K\mathcal{E} \quad \forall z \in \mathcal{X}_f$
(3) Terminal cost is cont. Lyapunov function in terminal set $\mathcal{X}_f$:
$l_f(Az + B\kappa_f(z)) - l_f(z) \le -l(z, \kappa_f(z)) \quad \forall z \in \mathcal{X}_f$

Theorem: Robust Invariance of Tube-MPC

Set $\mathcal{Z} := \{x \mid \mathcal{Z} \ne \emptyset\}$ is robust invariant set of system $x(k+1) = Ax(k) + B\mu_{\text{tube}}(x(k)) + w(k)$ subject to constraints $x, u \in \mathcal{X}, \mathcal{U}$

Proof let $(\{v_0^\star \dots v_{N-1}^\star\}, \{z_0^\star \dots z_N^\star\})$ be optimal sol'n for $x(k)$ At next point in time, state $x(k+1)$ may have many possible values due to disturbance
By construction, state $x(k+1)$ in in the set $z_1^\star \oplus \mathcal{E} \;\forall \mathcal{W}$
Therefore the following sequence is feasible for all $x(k+1)$

$$(\{v_1^\star \dots v_{N-1}^\star, \kappa_f(z_N^\star)\}, \underbrace{\{z_1^\star \dots z_N^\star}_{\text{feas. IC}}, \underbrace{Az_N^\star + B\kappa_f(z_N^\star)\}}_{\in \mathcal{X}_f \rightsquigarrow \text{feas.}})$$

Robust Stability

Robust Stability of Tube-MPC

State $x(k)$ of system $x(k+1) = Ax(k) + B\mu_{\text{tube}}(x(k)) + w(k)$ converges in the limit to the set $\mathcal{E}$

Proof As in standard MPC we have

$$J^\star(z_0^\star(x(k))) = l_f(z_N^\star) + \sum_{i=0}^{N-1} l(z_i^\star, v_i^\star)$$

$$J^\star(z_0^\star(x(k+1))) \le l_f(z_N^\star) + \sum_{i=1}^{N-1} l(z_i^\star, v_i^\star)$$

$$+ l(z_0^\star, v_0^\star) - l(z_0^\star, v_0^\star) + l_f(z_N^\star) - l_f(z_N^\star)$$

$$= J^\star(x(k)) - \underbrace{l(z_0^\star, v_0^\star)}_{\ge 0} \underbrace{- l_f(z_N^\star) + l_f(z_{N+1}) + l(z_N^\star, \kappa_f(z_N^\star))}_{\le 0 \; (l_f \text{ is lyap function in } \mathcal{X}_f)}$$

This shows $\lim_{k\to\infty} J(z_0^\star(x(k))) = 0$, therefore $\lim_{k\to\infty} z_0^\star(x(k)) = 0$

Caution:

- $x(k)$ does **not tend to 0!** It only stays within robust invar set centered at $z_0^\star(x(k)) : \lim_{k\to 0} \text{dist}(x(k), \mathcal{E}) = 0$
- $\mathcal{E}$ must be robust positive invariant for proof (so error remains bounded)

9.3 Tube-MPC Implementation

Offline Design

(1) Choose stabilizing controller K s.t $||A + BK|| < 1$
(2) Compute mRPI set $\mathcal{E} = F_\infty$ for system $x(k+1) = (A + BK)x(k) + w(k), w \in \mathcal{W}$
(3) Compute tightened constraints $\tilde{\mathcal{X}} := \mathcal{X} \ominus \mathcal{E}, \tilde{\mathcal{U}} := \mathcal{U} \ominus K\mathcal{E}$
(4) Choose terminal weight function l_f and constraint $\mathcal{X}_f$ satisfying assumptions on tube MPC

LQR Terminal Constraint (typical choice)

- Choose LQR terminal control law $\kappa_f(x) = Kx$, $(Q, R$ same as MPC)
- Find $\mathcal{X}_f$ invar under this controller s.t satisfies constraints

Online Design

(1) Measure / Estimate state x
(2) Solve optimization problem:
$(V^\star(x_0), Z^\star(x_0)) = \text{argmin}_{V,Z}\{J(Z, V) \mid (Z, V) \in \mathcal{Z}(x_0)\}$
(3) Set input to $u = K(x - z_0^\star(x)) + v_0^\star(x)$

Tube-MPC Summary

Benefits

- Less conservative than OL robust MPC (now actively compensating for noise in prediction)
- Works for unstable systems
- Optimization problem to solve is 'simple'

Cons

- Sub-optimal MPC (optimal extremely difficult)
- Reduced feasible set when compared to nominal MPC
- We need to know what $\mathcal{W}$ is (usually not realistic)

9.4 Robust MPC for Uncertain Systems - Summary

Idea compensate for noise in prediction to ensure constraint satisfaction

Benefits

- Feasible set invariant – know exactly when controller will work
- Easier to tune – knobs to tradeoff robustness against performance

Cons

- Complex (tubes easy to implement, complex to understand)
- Must know largest noise $\mathcal{W}$
- Often conservative
- Feas set may be small

9.5 Robust MPC - Extensions

9.5.1 Robust Constraint Tightening MPC

Idea *Combine best of Robust OL and Tube-Based MPC*
$\rightarrow$ Use propagated error bound to tighten constraints

Error Dynamics:

$$e_{i+1} = (A + BK)e_i + w_i = A_K e_i + w_i, \qquad w_i \in \mathcal{W}$$

If $e_0 = 0$ then $e_i = \sum_{j=0}^{i-1} A^j w_{i-1-j} \in \mathcal{W} \oplus A_K\mathcal{W} \oplus \dots A_K^{i-1}\mathcal{W}$

Problem Setup:

$$\min_{Z,V} \; l_f(z_N) + \sum_{i=0}^{N-1} l(z_i, v_i)$$

$$\text{subj. to } z_{i+1} = Az_i + Bv_i$$

$$z_i \in \mathcal{X} \ominus (\mathcal{W} \oplus A_K\mathcal{W} \oplus \dots A_K^{i-1}\mathcal{W})$$

$$u_i \in \mathcal{U} \ominus K(\mathcal{W} \oplus A_K\mathcal{W} \oplus \dots A_K^{i-1}\mathcal{W})$$

$$z_N \in \mathcal{X}_f \ominus (\mathcal{W} \oplus A_K\mathcal{W} \oplus \dots A_K^{N-1}\mathcal{W})$$

$$\boldsymbol{z_0 = x(k)}$$

$$\text{Control Law } u(k) = v_0^\star + K(x(k) - z_0) = v_0^\star$$

Motivation can robustly ensure constraint satisfaction at each time step
Note need terminal set $\mathcal{X}_f$ that is robust invariant under tube controller K

9.5.2 Nominal MPC with Noise

Standard MPC Problem for $x(k+1) = Ax(k) + Bu(k) + \boldsymbol{w(k)}$

$$J^\star(x_0) = \min_U l_f(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i)$$

$$\text{s.t } x_{i+1} = Ax_i + Bu_i, \;\; x_i, u_i, x_N \in \mathcal{X}, \mathcal{U}, \mathcal{X}_f$$

Effect on Lyapunov Function

Assume Optimal cost $J^\star$ Lipschitz continuous

$$|J^\star(Ax + Bu^\star(x) + w) - J^\star(Ax + Bu^\star(x))|$$

$$\le \gamma||Ax + Bu^\star(X) + w - (Ax + Bu^\star(x))|| = \gamma||w||$$

Lyapunov Decrease can be bounded as

$$J^\star(Ax + Bu^\star + w) - J^\star(x) - J^\star(Ax + Bu^\star + w) + J^\star(x)$$

$$\le J^\star(Ax + Bu^\star) - J^\star(x) + \gamma||w|| \le -l(x, u^\star) + \gamma||w||$$

- Amount of decrease grows with $||x||$
- Amount of increase upper bounded by $\max\{||w|| \mid w \in \mathcal{W}\}$

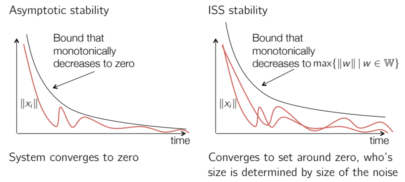
Benefits

- No special knowledge required – 'just works' (sometimes)
- Often very effective in practice
- Large feasible set
- Region of attraction may be relatively large

Cons

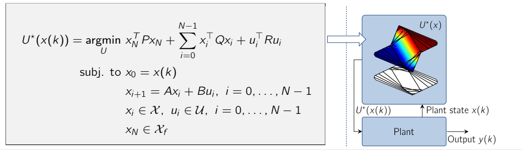
- Very difficult to determine region of attraction (set of states where controller works)
- Hard to tune
- Only works for NL systems under continuity assumptions

ISS – Input-To-State Stability



10 Implementation

10.1 Explicit MPC



Recall: Quadratic Cost State Feedback Solution

MP-QP – Multiparametric Quadratic Program

$$J^\star(x(k)) = \min_U \begin{bmatrix} U^\top & x(k)^\top \end{bmatrix} \begin{bmatrix} H & F^\top \\ F & Y \end{bmatrix} \begin{bmatrix} U^\top & x(k)^\top \end{bmatrix}^\top$$

$$\text{subj. to } GU \le w + Ex(k)$$

Solution Properties – $J^\star(x(k))$ **convex** and **PW Quad.** on polyhedra.

Active Set for $l = 1, \dots, m$

Define active set at x, $A(x)$, and it's complement $NA(x)$ as

$$A(x) := \{j \in l : G_j z^\star(x) - S_j x = w_j\} \quad \text{(satisfied with eq.)}$$

$$NA(x) := \{j \in l : G_j z^\star(x) - S_j x < w_j\} \quad \text{(strict inequality)}$$

Critical Region

CR_A is set of parameters x for which set $A \subseteq l$ of constraints i active at the optimum. For given $\bar{x} \in \mathcal{K}^\star$ let $(A, NA) := (A(\bar{x}), NA(\bar{X}))$. Then

$$CR_A := \{x \in \mathcal{K}^\star : A(x) = A\} \quad \text{(states share active set)}$$

Point Location

- **Sequential Search** – Computationally linear, very simple, works for all problems
- **Search Tree** – Potentially logarithmic, significant offline processing (reasonable for $< 1k$ regions)

Remarks on Explicit MPC

- Linear MPC + Quad / Linear-norm cost $\rightsquigarrow$ Controller PWA func.
- Can pre-compute this function offline
- Online evaluation of PWA function very fast (ns - μs)
- Can only do this for small systems (3-6 states, small horizon)

10.2 Iterative Optimization Methods

Generic Optimization Problem:
convex if $f : \mathbb{R}^n \to \mathbb{R}$ and set $\mathbb{Q}$ convex
Analytical sol'n cannot be obtained except simplest cases

$$\text{minimize } f(x)$$

$$\text{subj. to } x \in \mathbb{Q}$$

Iterative Optimization Methods Given initial guess $x^{(0)}$, produce sequence of iterates

$$x^{(i+1)} = \psi(x^{(i)}, f, \mathbb{Q}), \quad i = 0, \dots, m-1$$

such that $|f(x^{(m)}) - f(x^\star)| \le \epsilon$ and $\text{dist}(x^{(m)}, \mathbb{Q}) \le \delta$
where ϵ and δ are user defined tolerances

10.3 Unconstrained Minimization

Optimality Conditions

Assume $f(\cdot)$ diff'bar at $x^\star$. If f convex, then $x^\star$ global min iff $\nabla f(x^\star) = 0$

Descent Methods

$$x^{(i+1)} = x^{(i)} + h^{(i)}\Delta x^{(i)}$$

with $f(x^{(i+1)}) < f(x^{(i)})$

- Δx: step/search direction
- $h^{(i)}$: step size/length
- $f(x^{(i+1)}) < f(x^{(i)})$ i.e $\Delta x^{(i)}$ is descent function
- $\exists h^{(i)} > 0$ s.t $f(x^{(i+1)}) < f(x^{(i)})$ if $\nabla f(x^{(i)})^\top \Delta x^{(i)} < 0$

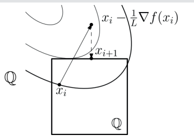
Input $x^{(0)} \in \text{dom}(f)$
repeat
  Compute descent dir. $\Delta x^{(i)}$
  Line Search: choose step size $h^{(i)} > 0$ s.t $f(x^{(i)} + h^{(i)}\Delta x^{(i)}) < f(x^{(i)})$
  Update $x^{(i+1)} := x^{(i)} + h\Delta x^{(i)}$
until termination condition
(e.g $f(x^{(m)}) - f(x^\star) \le \epsilon_1$)

Descent Direction

- **Gradient descent** $x^{(i+1)} = x^{(i)} - h^{(i)}\nabla f(x^{(i)})$
  - Assume ∇f Lipschitz-continuous $||\nabla f(x) - \nabla f(y)|| \le L||x - y||$
  - Choose constant step size $h^{(i)} = 1/L$
- **Newton Step** $x^{(i+1)} = x^{(i)} + h^{(i)}\Delta x_{nt}$
  - $\Delta x_{nt} = -(\nabla^2 f(x^{(i)}))^{-1}\nabla f(x^{(i)})$
  - Exact Line Search $h^{(i)\star} = \text{argmin}_{h>0} f(x^{(i)} + h^{(i)}\Delta x_{nt})$
    Optimization in 1 var $\rightsquigarrow$ solve by bisection, time consuming
  - Inexact Line search: find $h^{(i)}$ that decreases f by some amount

10.4 Constrained Minimization

Projected Gradient Methods



Incorporate Constraints in Gradient Step

$$x^{(i+1)} = \pi_{\mathbb{Q}}(x^{(i)} - h^{(i)}\nabla f(x^{(i)}))$$

Projection $\pi_{\mathbb{Q}} = \text{argmin}_x \frac{1}{2}||x - y||_2^2$ s.t $x \in \mathbb{Q}$

- Simple input constraints
- State constraints: hard $\rightsquigarrow$ solve for dual

Interior-Point Methods

System $\min f(x)$ s.t. $g_i(x) \le 0, i = 1, \dots, m$ **Assumptions** f, g_i convex, twice cont. diff'bar. $f(x^\star)$ is finite and attained, stict feasiblity $\exists g(\tilde{x}) < 0$, feasible set closed & compact **Idea** Reformulate as unconstrained problem

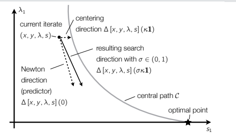
Primal-Dual Interior-Point Methods

Idea – Iteratively solve <u>relaxed</u> KKT system leave $\lambda_i^\star$ as variables, linearize and solve resulting sytem of linear eqns at each iteration

Search Direction $\Delta[x, \nu, \lambda, s](v)$

- $v = 0$ pure Newton direction "predictor"/"affine-scaling"
- $v = \kappa 1$ centering direction, approach central path

$\Rightarrow$ combine via **centering parameter** $\sigma \in (0, 1)$



11 Appendix

11.1 Set Operations

Minkowski Sum: $A \oplus B := \{x + y \mid x \in A, y \in B\}$
Pontryagin Difference: $A \ominus B := \{x \mid x + e \in A \;\forall e \in B\}$
Caution: $A \ominus B \oplus B \subseteq A$

11.2 Exercises

largest terminal invariant set determine largest terminal invariant set for the system $x_{i+1} = Ax_i + Bu_i$ with constraint $-c \le u_i \le c$ under a **stabilizing** linear terminal control law $u(k) = Kx(k)$

1) we need $(A + BK)$ *stable* $\Rightarrow K \in [\_,\_]$
2) $x \in \mathcal{X} = \{x | Kx \in \mathcal{U}\} = \left\{x \,\middle|\, \begin{bmatrix} K \\ -K \end{bmatrix} x \le \begin{bmatrix} c \\ c \end{bmatrix}\right\}$
3) max. control invariant set algorithm:

$$\Omega_0 = \mathcal{X} \to pre(\Omega_0) = \left\{x \,\middle|\, \begin{bmatrix} K \\ -K \end{bmatrix} x_{+1} \le \begin{bmatrix} c \\ c \end{bmatrix}\right\} = \left\{x \,\middle|\, \begin{bmatrix} K(A+BK) \\ -K(A+BK) \end{bmatrix} x \le \begin{bmatrix} c \\ c \end{bmatrix}\right\}$$

$$\Omega_1 = pre(\Omega_0) \cap \Omega_0 = \left\{x \,\middle|\, \begin{bmatrix} K \\ -K \\ K(A+BK) \\ -K(A+BK) \end{bmatrix} x \le \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix}\right\} = \Omega_0$$

$\Omega_1 = \Omega_0$ holds always for a stabilizing controller since $|A + BK| \le 1$ and thus Ω_0 is more restrictive than $pre(\Omega_0)$

Quadratic Program

- if the problem is convex & Slater's cond. satisfied $\Rightarrow$ KKT are necessary and sufficient for x^* being a global minimizer (convex = only 1 minimum)

$$\min_{x \in \mathcal{X}} \frac{1}{2}x^T Hx + f^T x$$
$$\text{s.t. } g(x) = Gx \le 0$$
$$h(x) = a^T x - b = 0$$

1) **Convexity:** ▪ check if all eigenvalues of the Hessian Matrix H are non-negative ▪ if furthermore $g(x), h(x)$ are linear / affine sets $\Rightarrow \mathcal{X}$ is convex
2) **Slater's Cond:** ▪ find a strictly feasible point s.t. Slater's Cond. is satisfied $\Rightarrow$ we have strong duality $p^* = d^*$
3) **KKT Cond.:** find suitable Lagrange Multipliers λ^*, ν^* s.t. KKT Cond. are satisfied $(\nabla_x L(x^*, \lambda^*, \nu^*) \overset{!}{=} 0 \Rightarrow \lambda^*, \nu^*)$ $\Rightarrow$ then x^* is a global minimizer

1D system: $x(k+1) = x(k) + u(k) + w(k)$
constraints: ▪ $u \in \mathcal{U} = \{u | -10 \le u \le 10\}$ ▪ $w \in \mathcal{W} = \{w | |w| < 1\}$

1) maximum robust invariant set $\mathcal{X}_f$ with control law $u(k) = -x(k)$
CL system: $x(k+1) = w(k)$ with constraint $-x(k) \in \mathcal{U} = \{u | -10 \le u \le 10\}$
We must be able to recover to the origin after the disturbance, therefore we compute the robust pre-set of $[-10,10]$:
$pre^{\mathcal{W}}([-10,10]) = \{x | x - x + w = [-1,1] \in [-10,10]\} = [-10,10] = \mathcal{X}_f$

2) terminal set for an OL robust MPC controller with horizon = 2
terminal set $\mathcal{X}_f \ominus (\bigoplus_{i=0}^{1} A^i \mathcal{W}) = \mathcal{X}_f \ominus 2\mathcal{W} = [-10,10] \ominus [-2,2] = [-8,8]$

3) state QP problem with $= 2, l(x_i, u_i) = 3x_i^T x_i + u_i^T u_i$ & terminal cost $5x_2^T x_2$ ▪ $u \in \mathcal{U}$ ▪ $x_{N=2} \in \mathcal{X}_f = [-8,8]$

$$\min [u_0 \; x_1 \; u_1 \; x_2] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} u_0 \\ x_1 \\ u_1 \\ x_2 \end{bmatrix}$$

$$s.t \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_0 \\ x_1 \\ u_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} x_0$$

$$s.t. \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \le \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 8 \\ 8 \end{bmatrix}$$

3) feasability of terminal set $\mathcal{X}_f$

show that $\mathcal{X}_f = [-1,1.5]$ will result in a recursively feasible controller for the terminal control law $u(k) = -0.3x(k) \Rightarrow$ e.g. the set $\mathcal{X}_f$ must be invariant for the nominal system subject to tightened constraints:

- Invariance: $(A + BK)\mathcal{X}_f = 0.8[-1,1.5] = [-0.8,1.2] \subseteq \mathcal{X}_f$ ✓
- Constraints: 1) $\mathcal{X}_f \subseteq \mathcal{X} \ominus \mathcal{E} = [-1,1.5]$ ✓
  2) $K\mathcal{X}_f = -0.3[-1,1.5] = [-0.45,0.3] \subseteq \mathcal{U} \ominus K\mathcal{E} = [-0.65,0.5]$ ✓