Model Predictive Control

Chapter 10: Robust MPC - Extensions

Prof. Melanie Zeilinger

ETH Zurich

Spring 2022

Coauthors: Prof. Colin Jones, EPFL
Prof. Manfred Morari, University of Pennsylvania

Outline

1. Robust “Constraint Tightening” MPC

2. Robustness of Nominal MPC

Robust Open-loop vs. Tube-based MPC

Robust open-loop MPC

$$\min_{X,U} \sum_{i=0}^{N-1} l(x_i, u_i) + l_f(x_N)$$

$$\begin{aligned} \text{subj. to } & x_{i+1} = Ax_i + Bu_i \\ & x_i \in \mathcal{X} \ominus (\mathcal{W} \oplus A\mathcal{W} \oplus \dots A^{i-1}\mathcal{W}) \\ & u_i \in \mathcal{U} \\ & x_N \in \mathcal{X}_f \ominus (\mathcal{W} \oplus A\mathcal{W} \oplus \dots A^{N-1}\mathcal{W}) \\ & x_0 = x(k) \end{aligned}$$

$$\mu_{\text{ol}}(x) := u_0^\star(x)$$

Robust tube-based MPC

$$\min_{Z,V} \sum_{i=0}^{N-1} l(z_i, v_i) + l_f(z_N)$$

$$\begin{aligned} \text{subj. to } & z_{i+1} = Az_i + Bv_i \\ & z_i \in \mathcal{X} \ominus \mathcal{E} \\ & u_i \in \mathcal{U} \ominus K\mathcal{E} \\ & z_N \in \mathcal{X}_f \\ & x(k) \in z_0 \oplus \mathcal{E} \end{aligned}$$

$$\mu_{\text{tube}}(x) := K(x - z_0^\star(x)) + v_0^\star(x)$$

Can we combine these ideas? $\rightarrow$ Robust constraint tightening MPC

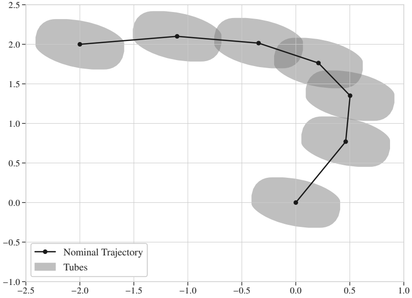
Tube MPC to "Constraint Tightening" MPC

Main idea: Use propagated error bound to tighten constraints

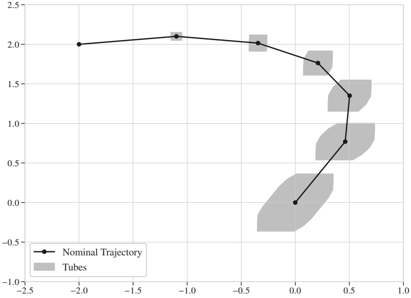
Recall: Error dynamics $e_{i+1} = (A + BK)e_i + w_i = A_K e_i + w_i$, $w_i \in \mathcal{W}$

If $e_0 = 0$, then $e_i = \sum_{j=0}^{i-1} A^j w_{i-1-j} \Rightarrow e_i \in \mathcal{W} \oplus A_K \mathcal{W} \oplus \ldots A_K^{i-1} \mathcal{W}$

Robust positively invariant (RPI) tubes:



Disturbance reachable sets (DRS):



Robust "Constraint-Tightening" MPC

$$
\begin{aligned}
\min_{Z,V} \; & \sum_{i=0}^{N-1} l(z_i, v_i) + l_f(z_N) \\
\text{subj. to } & z_{i+1} = Az_i + Bv_i \\
& z_i \in \mathcal{X} \ominus (\mathcal{W} \oplus A_K\mathcal{W} \oplus \dots A_K^{i-1}\mathcal{W}) \\
& u_i \in \mathcal{U} \ominus K(\mathcal{W} \oplus A_K\mathcal{W} \oplus \dots A_K^{i-1}\mathcal{W}) \\
& z_N \in \mathcal{X}_f \ominus (\mathcal{W} \oplus A_K\mathcal{W} \oplus \dots A_K^{N-1}\mathcal{W}) \\
& z_0 = x(k)
\end{aligned}
$$

- Applied control: $u(k) = v_0^* + K(x(k) - z_0) = v_0^*$
- Motivation: Can robustly ensure satisfaction of constraints at each time step
- Need a terminal set $\mathcal{X}_f$ that is robust invariant under the tube controller K

Outline

Nominal MPC with Noise

We want to control the noisy system:

$$x(k+1) = Ax(k) + Bu(k) + w(k)$$

What happens if we just ignore the noise and hope for the best?

Setup and solve a standard MPC problem:

$$\begin{aligned} J^\star(x_0) = \min_U \ & \sum_{i=0}^{N-1} l(x_i, u_i) + l_f(x_N) \\ \text{subj. to } \ & x_{i+1} = Ax_i + Bu_i \\ & x_i, u_i \in \mathcal{X} \times \mathcal{U} \\ & x_N \in \mathcal{X}_f \end{aligned}$$

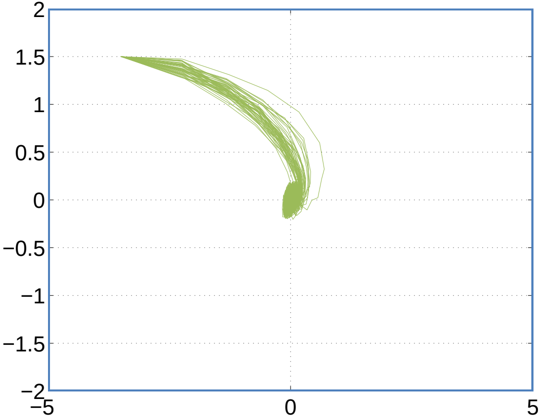
Our closed-loop system is now:

$$x(k+1) = Ax(k) + Bu_0^\star(x(k)) + w(k)$$

$\Rightarrow$ Can prove convergence to a neighborhood of the origin (for linear systems!)

Example

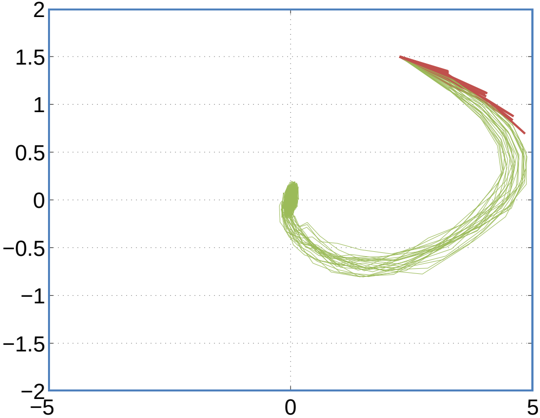
Consider system with noise, but we pretend it's not there in the controller.



- 100 trajectories with different noise realizations
- Seems to work fine?!

Example

Consider system with noise, but we pretend it's not there in the controller.



- 100 trajectories with different noise realizations
- Seems to work fine?!
- Can no longer be certain it will work!
- For some states it will work sometimes

How do we formalize this idea?

What Happens to Our Lyapunov Function?

Recall: The optimal cost $J^\star(x)$ is a Lyapunov function for the nominal system

$$J^\star(Ax + Bu^\star(x)) - J^\star(x) \leq -l(x, u^\star(x))$$

However, our state at the next point in time is now

$$x(k+1) = Ax(k) + Bu^\star(x(k)) + w(k)$$

Do we still have a Lyapunov decrease?

What Happens to Our Lyapunov Function?

Assume: Optimal cost $J^\star$ is continuous[1]

$$|J^\star(Ax + Bu^\star(x) + w) - J^\star(Ax + Bu^\star(x))| \\ \leq \gamma \|Ax + Bu^\star(x) + w - (Ax + Bu^\star(x))\| = \gamma \|w\|$$

Our Lyapunov decrease can be bounded as:

$$\begin{aligned} & J^\star(Ax + Bu^\star(x) + w) - J^\star(x) \\ &= J^\star(Ax + Bu^\star(x) + w) - J^\star(x) - J^\star(Ax + Bu^\star(x)) + J^\star(Ax + Bu^\star(x)) \\ &\leq J^\star(Ax + Bu^\star(x)) - J^\star(x) + \gamma \|w\| \\ &\leq -l(x, u^\star(x)) + \gamma \|w\| \end{aligned}$$

- Amount of decrease grows with $\|x\|$
- Amount of increase is upper bounded by $\max \{\|w\| \mid w \in \mathcal{W}\}$

Therefore we will move towards the origin until there is a balance between the size of x and the size of w

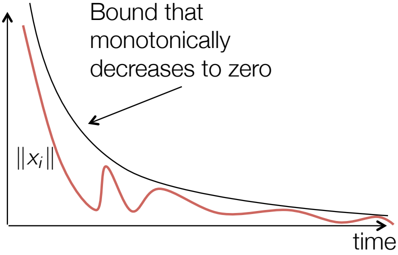
[1] True for linear systems, convex constraints and continuous stage costs.

Input-to-State Stability

What we have shown is that our system is **Input-to-State Stable**.

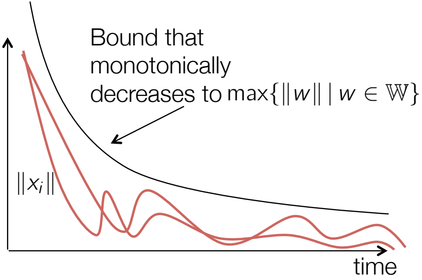
Much more general theory than what is given here[2]

Asymptotic stability



System converges to zero

ISS stability



Converges to set around zero, who's size is determined by size of the noise

[2] Limon, D., Alamo, T., Raimondo, D. M., Muñoz de la Peña, D., Bravo, J. M., Ferramosca, A., and Camacho, E. F. (2009). Input-to-State Stability: A Unifying Framework for Robust Model Predictive Control. In L. Magni, D. M. Raimondo, & F. Allgöwer (Eds.), Nonlinear Model Predictive Control (Vol. 384, pp. 1-26). Berlin, Heidelberg: Springer Berlin Heidelberg. doi:10.1007/978-3-642-01094-1

Nominal MPC for Uncertain Systems - Summary

Idea

- Ignore the noise and hope it works

Benefits

- Simple
- No knowledge of the noise set $\mathcal{W}$ required - 'just works'
- Often very effective in practice (this is what most practitioners do anyway)
- Feasible set is large (we can find a solution, but it may not work)
- Region of attraction may be larger than other approaches

Cons

- Very difficult to determine region of attraction (set of states in which the controller works)
- Hard to tune - no obvious way to tradeoff robustness against performance
- Works for linear systems, for nonlinear systems only under continuity assumptions