MODEL PREDICTIVE CONTROL

https://gitlab.ethz.ch/norrisg/mpc_summary

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1 SYSTEM THEORY

MODELS OF DYNAMIC SYSTEMS

NL TI CT SS Model

$\dot x = g(x,u)$, $y = h(x,u)$

$x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$, $y\in\mathbb{R}^p$

$g:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$, $h:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^p$

LINEARIZATION & DISCRETIZATION

Taylor Expansion around operating point $\bar x$ (first order) $f(x)\approx$

$$f(\bar x) + \frac{\partial f}{\partial x^\top}\Big|_{\bar x}(x-\bar x)$$

$$\dot x = \overbrace{\frac{\partial g}{\partial x^\top}\Big|_{\substack{x_s\\u_s}}}^{A^c\in\mathbb{R}^{n\times n}}\delta x + \overbrace{\frac{\partial g}{\partial u^\top}\Big|_{\substack{x_s\\u_s}}}^{B^c\in\mathbb{R}^{n\times m}}\delta u$$

$$y = \underbrace{\frac{\partial h}{\partial x^\top}\Big|_{\substack{x_s\\u_s}}}_{C\in\mathbb{R}^{p\times n}}\delta x + \underbrace{\frac{\partial h}{\partial u^\top}\Big|_{\substack{x_s\\u_s}}}_{D\in\mathbb{R}^{p\times m}}\delta u$$

Exact Solution $x(t) = e^{A^c(t-t_0)}x_0 + \int_{t_0}^t e^{A^c(t-\tau)}B^c u(\tau)d\tau$

Euler Discretiz. $\dot x^c \approx \frac{x^c(t+T_s)-x^c(t)}{T_s}$

$$x(k+1) = \overbrace{\mathbb{I}+T_sA^c}^{=A}x(k) + \overbrace{T_sB^c}^{=B}u(k)$$
$$y(k) = Cx(k)+Du(k)$$

Exact Discretization assume u constant over interval

$$x(t_{k+1}) = \underbrace{e^{A^cT_s}}_{=A}x(t_k) + \underbrace{\int_0^{T_s}e^{A^c(T_s-\tau)}B^c d\tau}_{B=(A^c)^{-1}(A-\mathbb{I})B^c}u(t_k)$$

DT LTI Solution

$$x(k+N) = A^Nx(k) + \sum_{i=0}^{N-1}A^iBu(k+N-1-i)$$

LINEAR SYSTEM ANALYSIS

DT Stability $x(k+1)=Ax(k)$ stable iff $|\lambda_j|<1, \forall j$

LTI DT Controllability can reach x^* from $x(0)$ in n steps

$$\mathcal{C} = [B \ \cdots \ A^{n-1}B] \Rightarrow \text{rank}(\mathcal{C}) \overset{!}{=} n$$

DT Observability uniquely distinguish IC from output

$$\mathcal{O} = [C^\top \ \cdots \ (CA^{n-1})^\top]^\top \Rightarrow \text{rank}(\mathcal{O}) \overset{!}{=} n$$

Stabilizability iff all uncontrollable modes stable

$$\text{if rank}([\lambda_j\mathbb{I}-A \mid B]) = n \ \forall\lambda_j\in\Lambda_A^+ \Rightarrow (A,B)\text{ stabilizable}$$

Detectablitiy iff all unobservable modes stable

$$\text{if rank}([A^\top-\lambda_j\mathbb{I} \mid C^\top]) = n \ \forall\lambda_j\in\Lambda_A^+ \Rightarrow (A,C)\text{ detect.}$$

NONLINEAR SYSTEM ANALYSIS

Lyapunov Stability (w.r.t **eq. point** $\bar x$ of a system)

Lyapunov Stable
for every $\epsilon>0$ exists $\delta(\epsilon)$ s.t.
$\|x(0)-\bar x\|<\delta(\epsilon)\to\|x(k)-\bar x\|<\epsilon$

Globally Asympt. Stable
Lyap. stable & Attractive
$\lim_{k\to\infty}\|x(k)-\bar x\| = 0 \ \forall x(0)$

Global Lyapunov Function (Candidate)

Consider eq point $\bar x = 0$. $V:\mathbb{R}^n\to\mathbb{R}$, continuous at origin, finite $\forall x$,
(1) $\|x\|\to\infty \Rightarrow V(x)\to\infty$
(2) $V(0)=0$, $V(x)>0 \quad \forall x\in\mathbb{R}^n\setminus\{0\}$
(3) $V(g(x))-V(x)\le-\alpha(x) \quad \forall x\in\mathbb{R}^n$
where $\alpha:\mathbb{R}^n\to\mathbb{R}$ continuous pos. def.

Global Lyapunov Stability

If sys admits a $V(x) \Rightarrow x=0$ is **Globally Asympt. Stable**

ACHTUNG if α pos. **semi**def $\Rightarrow x=0$ is **Globally Lyapunov Stable**

2 UNCONSTRAINED LQR CONTROL

LINEAR QUADRATIC OPTIMAL CONTROL

Dynamics $x_{i+1} = Ax(k)+Bu(k)$

Constraints **NONE** for state **OR** input

Goal ⇝ minimize **Quadratic Cost** subj. to dynamics

$$J^*(x(0)) := \min_U\left[x_N^\top Px_N + \sum_{i=0}^{N-1}(x_i^\top Qx_i + u_i^\top Ru_i)\right]$$

- N : horizon length
- $P\succeq0, \quad P=P^\top$
- $Q\succeq0, \quad Q=Q^\top$
- $R\succ0, \quad R=R^\top$
- $x(0)$: **current** state
- x_i, u_i: **opt.** variable

BATCH APPROACH

Idea explicitly represent $x_i\in\mathbb{R}^n$ through x_0 & $u_i\in\mathbb{R}^m$

$$\underbrace{\begin{bmatrix}x_0\\ \vdots\\ x_N\end{bmatrix}}_{X} = \underbrace{\begin{bmatrix}\mathbb{I}\\ A\\ \vdots\\ A^N\end{bmatrix}}_{\mathcal{S}^x\in\mathbb{R}^{(N+1)n\times n}}x(0) + \underbrace{\begin{bmatrix}0 & \cdots & 0\\ B & \cdots & 0\\ \vdots & \ddots & 0\\ A^{N-1}B & \cdots & B\end{bmatrix}}_{\mathcal{S}^u\in\mathbb{R}^{(N+1)n\times Nm}}\underbrace{\begin{bmatrix}u_0\\ u_1\\ \vdots\\ u_{N-1}\end{bmatrix}}_{U}$$

Cost $\overline{Q} :=$ blockdiag$(Q,\dots,Q,P)$ & $\overline{R} :=$ blockdiag$(R,\dots,R)$

Optimal Input

$$U^\star(x(0)) = -\underbrace{\left((\mathcal{S}^u)^\top\overline{Q}\mathcal{S}^u+\overline{R}\right)}_{H\text{ (Hessian)}}^{-1}\underbrace{(\mathcal{S}^u)^\top\overline{Q}\mathcal{S}^x}_{F^\top}x(0)$$

Optimal Cost

$$J^\star(x(0)) = x(0)^\top\left[\mathcal{S}_x^\top\overline{Q}\mathcal{S}_x - \mathcal{S}_x^\top\overline{Q}\mathcal{S}_u(\mathcal{S}_u^\top\overline{Q}\mathcal{S}_u+\overline{R})^{-1}\mathcal{S}_u^\top\overline{Q}\mathcal{S}_x\right]x(0)$$

RECURSIVE APPROACH

Idea apply DPOC ⇝ solve j-step optimal cost-to-go

$$J_j^\star(x(j)) := \min_{U_{j\to N}} x_N^\top Px_N + \sum_{i=j}^{N-1}(x_i^\top Qx_i + u_i^\top Ru_i)$$

$P \leftarrow P_N$
for $i = N : 1$ **do**
$F \leftarrow f(P)$
$P \leftarrow f(F)$
end for

Optimal Control Policy

$$u_i^\star = -(B^\top P_{i+1}B+R)^{-1}B^\top P_{i+1}A\cdot x(i) := F_ix_i$$

Optimal Cost-To-Go $J_i^\star(x_i) = x_i^\top P_ix_i$

RDE – Riccati Difference Equation ($\mathbf{P_N = P}$)

$$P_i = A^\top P_{i+1}A + Q - A^\top P_{i+1}B(B^\top P_{i+1}B+R)^{-1}B^\top P_{i+1}A$$

Numerically Safer Alternative $P_i = Q + F_i^\top RF_i + (A+BF_i)^\top P(A+BF_i)$

COMPARISON – BATCH VS RECURSIVE

- **Return**
  - **Batch** – sequence of numeric values $U^\star$
  - **Recursive** – feedback policies $u_i^\star$
- Control actions identical if perfect model
- **Disturbances** – Recursive more robust to disturbances
- **Computational efficiency**
  - Recursive more efficient for large N
  - Matrix inversion in Batch approach expensive
- **Constraints** – **Neither** works with constraints on x_i or u_i
- **Batch Approach easier to adapt when contraints are present** constrained minimization (solving for J_{i+1} with constraints) hard

RHC – RECEDING HORIZON CONTROL

Idea Compute opt. sequence over N-step horizon

$$U^\star := \text{argmin } x_N^\top Px_N + \sum_{i=0}^{N-1}x_i^\top Qx_i + u_i^\top Ru_i \quad \text{subj. to } x_{i+1} = Ax_i + Bu_i \quad \Rightarrow U^\star$$

- Extract first input in sequence: $U^\star = \{u_0^\star,\dots,u_{N-1}^\star\} \Rightarrow u_0^\star$
- Introduce feedback to sys: $x(k+1) = Ax(k)+Bu(k) \Rightarrow x$

Why Reoptimize Provides robustness to noise / modeling errors, Sol'n at k subopt. (finite horizon) ⇝ reopt. potentially better performance

INFINITE HORIZON LQR

- **Cost** Let $N\to\infty$ ⇝ $J_\infty(x(0)) = \min_u\sum_{i=0}^\infty x_i^\top Qx_i + u_i^\top Ru_i$
- RDE satisfied with $P_i = P_{i+1} = P_\infty$
- **Input** Feedback matrix F_∞ ⇝ $u^\star(k) := F_\infty x(k)$

LQR Lyapunov Function

If (A,B) stabilizable, $(Q^{1/2},A)$ detectable ⇝ $J^\star(x) = x^\top P_\infty x$ is Lyap. func. for system $x^+ = (A+BF_\infty)x$

Choice of P in Finite Horizon Control

- Can choose to match ∞-Horizon sol'n ⇝ Make $P\approx J_{N\to\infty}$ with ARE
- Can Choose P assuming no control action after end of horizon. This P determined from solving Lyap eqn $A^\top PA + Q = P$. **Only makes sense if system asympt. stable**
- Assume we want state and input both to be 0 at end of horizon ⇝ no P but extra constraint $x_{i+N} = 0$

3 CONVEX OPTIMIZATION

PROBLEM FORMULATION

$$\min_{x\in\text{dom}(f)} f(x) \quad \text{subj. to } g_i(x)\le0 \quad i=1,\dots,m; \quad h_i(x)=0 \quad i=1,\dots,p$$

- $\mathcal{X}$: $\{x\in\text{dom}(f) \mid g_i\le0, h_i=0\}$ **feasible set**
- g_i : **ineq constraints**
- h_i : **eq contraints**

Feasibility Point x satisfies $g_i\le0$, $h_i=0$ & eq contraints

Strictly Feasible Point x satisfies $g_i<0$

Optimal Value lowest cost $p^\star = f(x^\star) = \min_{x\in\mathcal{X}}f(x)$

Optimizer feas. $x^\star$ ⇝ smallest $p^\star$

$$\text{argmin}_{x\in\mathcal{X}}f(x) := \{x\in\mathcal{X} \mid f(x) = p^\star\}$$

ACHTUNG NOT always unique

Active Contraints when ineq const. are eq ⇝ "active"

Redundant Contraints do not change feasible set

Locally Optimal $y\in\mathcal{X}, \|y-x\|\le R \Rightarrow f(y)\ge f(x)$

Globally Optimal $y\in\mathcal{X} \Rightarrow f(y)\ge f(x)$

Unbounded Below $p^\star = -\infty$

Infeasible $p^\star = \infty \Leftrightarrow \mathcal{X} = \{\}$

Unconstrained $\mathcal{X} = \mathbb{R}^n$

CONVEX SETS

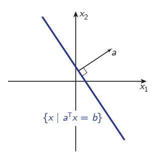
Definition Convex iff $\lambda x + (1-\lambda)y\in\mathcal{X}, \quad \forall\lambda\in[0,1], \ \forall x,y\in\mathcal{X}$

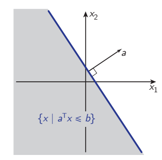
Interpretation ⇝ **All lines starting in $\mathcal{X}$ stay within $\mathcal{X}$**

Hyperplane $\{x\in\mathbb{R}^n \mid a^\top x = b\}$

Halfspace $\{x\in\mathbb{R}^n \mid a^\top x \le b\}$

open: $<$, closed: $\le$

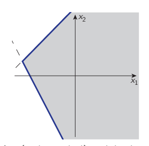

A hyperplane

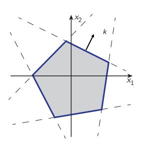

A closed halfspace

Polyhedron

$P := \{x \mid a_i^\top x \le b_i, i = \dots\}$
$:= \{x \mid Ax\le b\}$

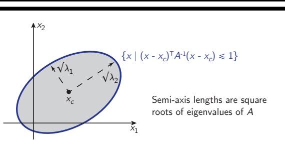
Polytope bounded polyhedron


An (unbounded) polyhedron


A polytope

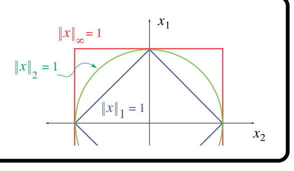
Ellipsoid $\{x \mid (x-x_c)^\top A^{-1}(x-x_c)\le1\}$

x_c : center of ellipsoid



Norm Ball $\{x \mid \|x-x_c\|\le r\}$

- $p=2$ Euclidean Norm $\|x\|_2 = \sqrt{\sum_i x_i^2}$
- $p=1$ Sum of Absolute $\|x\|_1 = \sum_i|x_i|$
- $p=\infty$ Largest Absolute



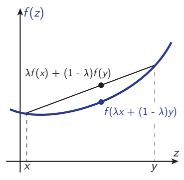
Intersections & Unions

Intersection – Intersection of two or more convex sets is itself convex
Union – Union of two sets is **NOT** convex in general

CONVEX FUNCTIONS

Definition convex iff dom(f) convex &

$$f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y) \quad \forall\lambda\in(0,1), \quad \forall x,y\in\text{dom}(f)$$



Strictly Convex if inequality is strict

1st-Order Condition $f(x)$ convex **iff** $f(y)\ge f(x)+\nabla f(x)^\top(y-x)$

2nd-Order Condition $f(x)$ convex **iff** $\nabla^2f(x)\succeq0, \quad \nabla^2f(x)_{ij} = \frac{\partial^2f(x)}{\partial x_i\partial x_j}$

Level Set

L_α of f, set for which
$L_\alpha := \{x \mid x\in\text{dom}(f), f(x)=\alpha\}$
Equiv to contour lines of const 'height'

Sublevel Set

C_α of f for value α is defined by
$C_\alpha := \{x \mid x\in\text{dom}(f), f(x)\le\alpha\}$
f convex $\Rightarrow$ sublevel sets convex $\forall\alpha$

Examples

Convex
- **Affine** $ax+b$ for any $a,b\in\mathbb{R}$
- **Exp.** e^{ax} for any $A\in\mathbb{R}$
- **Powers** x^α on domain $\mathbb{R}_{++}$, for $\alpha\ge1$ or $\alpha\le0$
- **Vector norms** on $\mathbb{R}^n$: $\|x\|_p = (\sum_{i=1}^n|x|^p)^{1/p}$, for $p\ge1$,

Concave
- **Affine** $ax+b$ for any $a,b\in\mathbb{R}$
- **Powers** x^α on domain $\mathbb{R}_{++}$, for $0\le\alpha\le1$
- **Log** $\log x$ on domain $\mathbb{R}_{++}$
- **Entropy** $-x\log x$ on domain $\mathbb{R}_{++}$

OPTIMALITY CONDITIONS

Lagrange Dual Function

$$d(\lambda,\nu) = \underbrace{\inf_{x\in\text{dom}(f)}}_{\approx\min_x}\underbrace{\left[f(x)+\sum_{i=1}^m\lambda_ig_i(x)+\sum_{i=1}^p\nu_ih_i(x)\right]}_{L(x,\lambda,\nu)\text{: Lagrange Function}}$$

P – Primal Problem

$$(P): \ \min_x f(x) \quad \text{subj. to } g_i(x)\le0, \quad h_i(x)=0$$

D – Dual Problem

$$(D): \ \max_{\nu,\lambda} d(\nu,\lambda) \quad \text{subj. to } \lambda\ge0$$

- $d(\lambda,\nu)$ always **concave**
- $d^\star\le p^\star$ ⇝ $d(\lambda,\nu)$ gens lower bound for p
- (D) **convex** even if (P) not
- Point (λ,ν) **dual feas.** if $\lambda\ge0$, $(\lambda,\nu)\in\text{dom}(d)$

Weak & Strong Duality

Weak Duality – it is **always** true that $d^\star\le p^\star$
Stront Duality – it is **sometimes** true that $d^\star = p^\star$
- Strong duality usually does not hold for non-convex problems
- Can impose conditions on convex problems to guarantee that $d^\star = p^\star$
- Sometimes the dual much easier to solve than the primal
  - **LP always has strong duality**

Slater Condition

If $\exists$ at least one **strictly feasible point** *i.e* $\{x \mid Ax=b, g_i(x)<0 \ \forall i\}$ $\Rightarrow p^\star = d^\star$

KKT – KARUSH-KUHN-TUCKER CONDITIONS

(1) Primal Feasibility $g_i(x^\star)\le0, \ i=1\dots m \quad h_i(x^\star)=0, \ i=1\dots p$
(2) Dual feasibility $\lambda^\star\ge0$
(3) Complementary Slackness $\lambda_i^\star g_i(x^\star)=0 \quad i=1\dots m$
(4) Stationarity $\nabla L = \nabla f(x^\star)+\sum_{i=1}^m\lambda_i^\star\nabla g_i(x^\star)+\sum_{i=1}^p\nu_i^\star\nabla h_i(x^\star)=0$

General Optimization Necessary condition

$x^\star,\lambda^\star,\nu^\star$ sol'n to $(P),(D)$ with 0 duality gap $\Rightarrow x^\star,\lambda^\star,\nu^\star$ satisfy KKT

Convex Optimization Sufficient condition

$x^\star,\lambda^\star,\nu^\star$ satisfy KKT $\Rightarrow x^\star,\lambda^\star,\nu^\star$ sol'n to $(P),(D)$ with 0 duality gap

Convex Opt. + Slater Necessary & Sufficient condition

If Slater's cond. holds, $x^\star,\lambda^\star,\nu^\star$ are sol'n to $(P),(D)$ **IFF** KKT satisfied

Remark for convex opt. problem, KKT conditions sufficent ⇝ if $x^\star,\lambda^\star,\nu^\star$ satisfy KKT then $p^\star = d^\star$

MATRIX CALCULUS

Basics

$xy^\top = [x_1y \ \dots \ x_ny]$
$\langle x,y\rangle = x^\top y = \sum x_iy_i$

Vector Derivatives

$\frac{\partial}{\partial x}x^\top A = \frac{\partial}{\partial x}A^\top x = A$
$\frac{\partial}{\partial x}x^\top Ax = (A+A^\top)x$

Del-Operator (Gradient) $\nabla_xf(x) = \left[\frac{\partial}{\partial x_1}f(x) \ \cdots \ \frac{\partial}{\partial x_n}f(x)\right]^\top$

Jacobian (Gradient of multivar func) $\frac{\partial f}{\partial x^\top} = \begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}\end{bmatrix}$

EXAMPLES

LP – Dual

$(P): \ \min_{x\in\mathbb{R}^n} \ c^\top x, \quad \text{subj. to } Ax=b, \quad Cx\le e$
$(D): \ \max_{\lambda,\nu} \ -b^\top\nu - e^\top\lambda, \quad \text{s.t } A^\top\nu + C^\top\lambda + c = 0, \quad \lambda\ge0$

QP – Dual with $Q\succ0$

$(P): \ \min_{x\in\mathbb{R}^n} \ \frac12x^\top Qx + c^\top x, \quad \text{subj. to } Cx\le e$
$(D): \ \max_{\lambda,\nu} \ \frac12\lambda^\top CQ^{-1}C^\top\lambda + (CQ^{-1}c+e)^\top\lambda + \frac12c^\top Q^{-1}c \quad \text{subj. to } \lambda\ge0$

QP – Lagrangian

$\min_x \frac12x^\top Hx + q^\top x + r$ s.t $Gx\le h$, $Ax=b$

$L = \frac12x^\top Hx + q^\top x + r + \lambda^\top(Gx-h) + \nu^\top(Ax-b)$

$\nabla_xL = Hx + q + G^\top\lambda + A^\top\nu$

4 CFTOC

CFTOC – CONSTRAINED FINITE-TIME OPT. CONTROL

Constrained Linear Optimal Control

$$\begin{aligned} J^\star(x(k)) = \min_U \; & l_f(x_N) + \textstyle\sum_{i=0}^{N-1} l(x_i, u_i) \\ \text{subj. to } & x_{i+1} = Ax_i + Bu_i \\ & x_i \in \mathcal{X}, \quad u_i \in \mathcal{U} \\ & x_N \in \mathcal{X}_f, \quad x_0 = x(k) \end{aligned}$$

- **Quad. Cost / Squared Euclidian Norm:**

$$J(x(k)) = x_N^\top P x_N + \textstyle\sum_{i=0}^{N-1} x_i^\top Q x_i + u_i R u_i$$

- **p-Norm:** $J(x(k)) = ||Px_N||_p + \sum_{i=0}^{N-1} ||Qx_i||_p + ||Ru_i||_p$

TRANSFORM QUAD CFTOC TO QP

QP Problem
Goal – Rewrite Quad. Cost CFTOC as QP
⇝ easier to solve

$$\begin{aligned} \min_{z \in \mathbb{R}^n} \; & \tfrac{1}{2} z^\top H z + q^\top z + r \\ \text{subj. to } & Gz \le h \\ & Az = b \end{aligned}$$

CONSTRUCTION WITH SUBSTITUTION

IDEA – Sub. state eqns $x_{i+1} = Ax_i + Bu_i, \quad x_0 = x(k)$
Cost – Rewrite as

$$\begin{aligned} J^\star(x(k)) = \min_U \; & \begin{bmatrix} U^\top & x(k)^\top \end{bmatrix} \begin{bmatrix} H & F^\top \\ F & Y \end{bmatrix} \begin{bmatrix} U^\top & x(k)^\top \end{bmatrix}^\top \\ \text{subj. to } & GU \le w + Ex(k) \end{aligned}$$

Constraints – Rewrite as $GU \le w + Ex(k)$

$$\mathcal{X} = \{x | A_x x \le b_x\} \quad \mathcal{U} = \{u | A_u u \le b_u\} \quad \mathcal{X}_f = \{x | A_f x \le b_f\}$$

$$G = \begin{bmatrix} A_u & & \\ & \ddots & \\ & & A_u \\ 0 & \dots & 0 \\ A_x B & & \\ A_x AB & A_x B & 0 \\ \vdots & & \\ A_f A^{N-1} B & \dots & A_f B \end{bmatrix}, \; E = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -A_x \\ -A_x A \\ \vdots \\ -A_f A^N \end{bmatrix}, \; w = \begin{bmatrix} b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ \vdots \\ b_f \end{bmatrix}$$

Solution For a given $x(k)$, $U^\star$ can be found via QP solver

CONSTRUCTION WITHOUT SUBSTITUTION

Idea – Keep state eqns as eq. constraints
Cost with $z = \begin{bmatrix} x_1^\top & \dots x_N^\top & u_0^\top & \dots u_{N-1}^\top \end{bmatrix}^\top$

$$\begin{aligned} J^\star(x(k)) = \min_z \; & \begin{bmatrix} z^\top & x(k)^\top \end{bmatrix} \begin{bmatrix} \bar{H} & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} z^\top & x(k)^\top \end{bmatrix}^\top \\ \text{subj. to } & G_{in} z \le w_{in} + E_{in} x(k) \\ & G_{eq} z = E_{eq} x(k) \end{aligned}$$

$$\bar{H} = \operatorname{diag}(Q, \dots, Q, P, R, \dots, R)$$

Equality Constraints from System Dyn. $x_{i+1} = Ax_i + Bu_i$

$$G_{eq} = \left[\begin{array}{ccc|ccc} \mathbb{I} & & & -B & & \\ -A & \mathbb{I} & & & -B & \\ & -A & \mathbb{I} & & & -B \\ & \dots & \dots & & & \dots \end{array} \right], \; E_{eq} = \begin{bmatrix} A \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Inequality Constraints

$$\mathcal{X} = \{x | A_x x \le b_x\} \quad \mathcal{U} = \{u | A_u u \le b_u\} \quad \mathcal{X}_f = \{x | A_f x \le b_f\}$$

$$G_{in} = \left[\begin{array}{cccc|cccc} 0 & & & & 0 & & & \\ A_x & & & & & 0 & & \\ & A_x & & & & & \dots & \\ & & \dots & & & & & 0 \\ & & & A_f & & & & \\ 0 & & & & A_u & & & \\ & 0 & & & & A_u & & \\ & & \dots & & & & \dots & \\ & & & 0 & & & & A_u \end{array} \right], \; w_{in} = \begin{bmatrix} b_x \\ b_x \\ b_x \\ \vdots \\ b_f \\ b_u \\ b_u \\ \vdots \\ b_u \end{bmatrix}, \; E_{in} = \begin{bmatrix} -A_x^\top \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

QP FEEDBACK SOLUTION

From CFTOC problem as multiparametric QP

$$\begin{aligned} J^\star(x(k)) = \min_U \; & \begin{bmatrix} U^\top x(k)^\top \end{bmatrix} \begin{bmatrix} H & F^\top \\ F & Y \end{bmatrix} \begin{bmatrix} U^\top x(k)^\top \end{bmatrix}^\top \\ \text{subj. to } & GU \le w + Ex(k) \end{aligned}$$

Solution Properties

- **First component of optimal solution:**

$$u_0^\star = \kappa(x(k)), \quad \forall x(k) \in \mathcal{X}_0$$

$\kappa : \mathbb{R}^n \to \mathbb{R}^m$ is cont. and pw. affine on Polyhedra

$$\kappa(x) = F^j x + g^j \quad \text{if} \quad x \in CR^j, \quad j = 1, \dots, N^r$$

- Polyhedral sets $CR^j = \{x \in \mathbb{R}^n \mid H^j x \le K^j\}, j = 1, \dots, N^r$ are partition of the feasible polyhedron $\mathcal{X}_0$.
- Value func. $J^\star(x(k))$ is convex and pw quad. on polyhedra.

TRANSFORM P-NORM CFTOC TO LP

ℓ_∞-Minimization

$$\begin{aligned} \min_{x \in \mathbb{R}^m} \; & ||x||_\infty \\ \text{subj. to } & Fx \le g \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \min_{x,t} \; & t \\ \text{subj. to } & -1_m t \le x \le 1_m t, \; Fx \le g \end{aligned}$$

Intuition $-1_m t \le x \le 1_m t$ bounds abs value of every elem. with scalar t

ℓ_1-Minimization

$$\begin{aligned} \min_{x \in \mathbb{R}^m} \; & ||x||_1 \\ \text{subj. to } & Fx \le g \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \min_{x \in \mathbb{R}^m, t \in \mathbb{R}^m} \; & 1_m^\top t \\ \text{subj. to } & -t \le x \le t, \; Fx \le g \end{aligned}$$

Intuition $\|x\|_1 = \sum_{i=1}^n |x_i| \le \sum_{i=1}^n t_i = 1_n^\top t \rightsquigarrow -t \le x \le t$ bounds abs value of each component of x with a component of vector t

CONSTRUCTION OF ∞-NORM WITH SUBSTITUTION

Cost

$$\begin{aligned} \min_z \; & \epsilon_N^x \textstyle\sum_{i-0}^{N-1} \epsilon_i^x + \epsilon_i^u \\ \text{subj. to } & -\mathbf{1}_n \epsilon_i^x \le \pm Q \left[A^i x_0 + \textstyle\sum_{j=0}^{i-1} A^j B u_{i-1-j} \right] \\ & -\mathbf{1}_r \epsilon_N^x \le \pm P \left[A^N x_0 + \textstyle\sum_{j=0}^{N-1} A^j B u_{N-1-j} \right] \\ & -\mathbf{1}_m \epsilon_i^u \le R u_i \\ & x_i \in \mathcal{X}, \; u_i \in \mathcal{U}, \; x_f \in \mathcal{X}_f, \; x_0 = x(k) \end{aligned}$$

Substitution with $z := \{\epsilon_0^x \dots \epsilon_N^x, \epsilon_0^u \dots \epsilon_{N-1}^u, u_0^\top \dots u_{N-1}^\top\} \in \mathbb{R}^s$, $s := (m+1)N + N + 1$

$$\min_z c^\top z \quad \text{subj. to } \bar{G} z \le \bar{w} + \bar{S} x(k)$$

$$\bar{G} = \begin{bmatrix} G_\epsilon & G_u \\ 0 & G \end{bmatrix}, \quad \bar{S} = \begin{bmatrix} S_\epsilon \\ S \end{bmatrix}, \quad \bar{w} = \begin{bmatrix} w_\epsilon \\ w \end{bmatrix}$$

Solution for given $x(k)$, $U^\star$ can be optained via LP solver

LP STATE FEEDBACK SOLUTION

MP-LP
multiparam. LP

$$\min_z c^\top z \; \text{subj. to } \bar{G} z \le \bar{w} + \bar{S} x(k)$$

Properties

- First component of mp sol'n has form $u_0^\star = \kappa(x(0)), \quad \forall x(k) \in \mathcal{X}_0$
  $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ cont. & pw affine on Polyhedra
  $\kappa(x) = F^j x + g^j$ if $x \in CR^j, \quad j = 1, \dots, N^r$
- Polyhedral sets $CR^j = \{x \in \mathbb{R}^n \mid H^j x \le K^j\}$ are partition of feasible Polyhedron $\mathcal{X}_0$
- In case of multiple optimizers, a pw affine control law exists
- $J^\star(x(0))$ is convex, pw linear on polyhedra

QUAD VS $1/\infty$-NORM COST

n = # opt. var., FS = feas. set. **Solution is either**

Quadratic Cost

- unique & in interior of FS (no constraints active)
- unique & on boundary of FS (at least 1 const. active)

Linear Cost

- Unbounded
- unique at vertex of FS (at least n active constraints)
- multiple optima (at least 1 active const.)

5 INVARIANCE

INVARIANCE

System
Autonomous $x(k+1) = g(x(k))$
Closed-Loop $x(k+1) = g(x(k), \kappa(x(k)))$ for given κ

Positively Invariant Set

Set $\mathcal{O}$ positively invariant for autonomous system if

$$x(k) \in \mathcal{O} \Rightarrow x(k+1) \in \mathcal{O}, \quad \forall k \in \{0, 1, \dots\}$$

Maximal Positively Invariant Set

$\mathcal{O}_\infty \subset \mathcal{X}$ positively invariant and contains all other $\mathcal{O}$

Pre-Set

Given set S, the pre-set of S is the set of states that evolve into S in one time step

$x(k+1) = g(x(k))$	$x(k+1) = Ax(k)$
$\Rightarrow \text{pre}(S) := \{x \mid g(x) \in S\}$	$\Rightarrow \text{pre}(S) := \{x \mid Ax \in S\}$

Invariant Set Conditions

Set $\mathcal{O}$ is positively invariant set iff

$$\mathcal{O} \subseteq \text{pre}(\mathcal{O}) \;\Leftrightarrow\; \text{pre}(\mathcal{O}) \cap \mathcal{O} = \mathcal{O}$$

Necessary if $\mathcal{O} \nsubseteq \text{pre}(\mathcal{O})$, then $\exists \bar{x} \in \mathcal{O}$ s.t $\bar{x} \notin \text{pre}(\mathcal{O}) \rightsquigarrow \bar{x} \in \mathcal{O}, \bar{x} \notin \text{pre}(\mathcal{O})$, thus $\mathcal{O}$ not positively invariant
Sufficient if $\mathcal{O}$ not pos invar set, then $\exists \bar{x} \in \mathcal{O}$ s.t $g(\bar{x}) \notin \mathcal{O} \rightsquigarrow \bar{x} \in \mathcal{O}, \bar{x} \notin \text{pre}(\mathcal{O})$ thus $\mathcal{O} \notin \text{pre}(\mathcal{O})$

Pre-Set Computation

Set $S := \{x \mid Fx \le f\}$, $x(k+1) = Ax(k)$ then

$$\begin{aligned} \text{pre}(S) :=& \{x \mid Ax \in S\} \\ =& \{x \mid FAx \le f\} \end{aligned}$$

```
Ω₀ ← 𝒳
loop
    Ω_{i+1} ← pre(Ω_i) ∩ Ω_i
    if Ω_{i+1} = Ω_i then
        return 𝒪_∞ = Ω_i
    end if
end loop
```

- For $\{x \mid Fx \le f\}$, if $F \downarrow$ or $f \uparrow \rightsquigarrow$ ***Less* Restrictive**
- $S \cap F \rightsquigarrow$ constraints from both sets active

CONTROL INVARIANCE

Control Invariant Set

Set $\mathcal{C} \subseteq \mathcal{X}$ control invariant if

$$x(k) \in \mathcal{C} \quad \Rightarrow \; \exists u(k) \in \mathcal{U} \text{ s.t } g(x(k), u(k)) \in \mathcal{C} \; \forall k$$

Maximal Control Invariant Set

Set $\mathcal{C}_\infty$ maximal control invariant if it is control invariant and contains all control invariant sets contained in $\mathcal{X}$

Intuition For all states in $\mathcal{C}_\infty$, there exists control law s.t system constraints never violated ⇝ **The best any controller could ever do**
Pre-Set $\text{pre}(S) := \{x \mid \exists u \in \mathcal{U} \text{ s.t } g(x, u) \in S\}$
Control Invariant $\mathcal{C}$ control invariant set iff $\mathcal{C} \subseteq \text{pre}(\mathcal{S})$
Algorithm Same, but **much harder to compute pre-set**
Control Invariant Set ⇒ Control Law

$\mathcal{C}$ control invariant set for $x(k+1) = g(x(k), u(k))$ Control law $\kappa(x(k))$ will guarantee that system satisfies constraints $\forall t$ if $g(x, \kappa(x)) \in \mathcal{C} \; \forall x \in \mathcal{C} \rightsquigarrow$ With f as any function Synthesize control law κ:

$$\kappa(x) := \operatorname{argmin}\{f(x, u) \mid g(x, u) \in \mathcal{C}\}$$

- Does not ensure sys. will converge, but will satisfy constraints
- Don't often do because calculating control invariant sets is very hard
- **MPC implicitly** describes cont. invar. set s.t easy to represent/compute

PRACTICAL INVARIANT SET COMPUTATION

Minkowski-Weyl Theorem

For $P \subseteq \mathbb{R}^d$ following statements equivalent:
- P polytope, $\exists A, b$ s.t $P = \{x \mid Ax \le b\}$
- P finitely generated, $\exists$ finite set of vectors $\{v_i\}$ s.t $P = \text{co}(\{v_1 \dots v_s\})$

Invariant Sets from Lyapunov Functions

Lemma If $V : \mathbb{R}^n \to \mathbb{R}$ a Lyap. func. for sys. $x(k+1) = g(x(k))$, then $Y := \{x \mid V(x) \le \alpha\}$ is an invariant set for all $\alpha \ge 0$
Proof
- $V(x) \ge 0 \; \forall x$
- $V(g(x)) - V(x) < 0 \rightsquigarrow$ once $V(x(k)) \le \alpha$, will remain there for all $j \ge k \rightsquigarrow$ Invariance

Example System for $x(k+1) = Ax(k)$ with $P \succ 0$ that satisfies $A^\top P A - P \prec 0 \rightsquigarrow$ then $V(x(k)) = x(k)^\top P x(k)$ is Lyap. function
Goal – find largest α s.t $Y_\alpha \in \mathcal{X}$
$Y_\alpha := \{x \mid x^\top P x \le \alpha\} \subset \mathcal{X} := \{x \mid Fx \le f\}$
Equivalent to $\max_\alpha \alpha$ subj. to $h_{Y_\alpha}(F_i) \le f_i \; \forall i \in \{1 \dots n\}$

Maximum Ellipsoidal Invariant Sets

Support of an ellipse: $h_{Y_\alpha}(F_i) = \max_x F_i x$ subj. to $x^\top P x \le \alpha$
Change of Variables: $y := P^{1/2} x$
$\rightsquigarrow h_{Y_\alpha}(F_i) = \max_x F_i P^{-1/2} y$ s.t $y^\top y \le \sqrt{\alpha}^2$
Maximizer found by inspection:

$$h_{Y_\alpha}(F_i) = F_i P^{-1/2} \frac{P^{-1/2} F_i^\top}{||P^{-1/2} F_i^\top||} \sqrt{\alpha} = ||P^{-1/2} F_i^\top|| \sqrt{\alpha}$$

Largest ellipse now 1-dim optimization problem:

$$\begin{aligned} \alpha^\star &= \max_\alpha \alpha \quad \text{s.t.} \; ||P^{-1/2} F_i^\top||^2 \alpha \le f_i^2 \; \forall i \in \{1 \dots n\} \\ &= \min_{i \in \{1 \dots n\}} \frac{f_i^2}{F_i P^{-1} F_i^\top} \end{aligned}$$

6 FEASIBILITY AND STABILITY

LQR MPC COMPARISON

LQR	MPC
$J_\infty^\star(x(k)) = \min \sum_{i=0}^{\infty} x_i^\top Q x_i + u_i^\top R u_i$	$J^\star(x(k)) = \min \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i$
subj. to $x_{i+1} = Ax_i + Bu_i$	subj. to $x_{i+1} = Ax_i + Bu_i$
$x_0 = x(k)$	$x_i \in \mathcal{X}, \quad u_i \in \mathcal{U}$
	$x_0 = x(k)$

• **Quad. Cost** • **Linear System Dynamics** • **Linear Constraints on** u, x
Assume: $Q = Q^\top \succeq 0, R = R^\top \succ 0$

LOSS OF FEASIBILITY & STABILITY

Infinite-Horizon Solve RHC for $N = \infty$, OL traj. are same as CL traj.
- If problem feasible, CL trajectories always feasible
- If cost finite, states and inputs will converge asympt. to origin

Finite-Horizon RHC "short-sighted" approximating ∞-horizon controller
- **Feasibility** – after some steps finite horizon optimal control problem may become infeasible (disturbances, model mismatch)
- **Stability** – generated inputs may not lead to traj. that converge to orgin

Solution

Introduce terminal cost & constraints to explicitly ensure feas. & stab.

$$\begin{aligned} J^\star(x(k)) = \min_U \; & l_f(x_N) + \textstyle\sum_{i=1}^{N-1} l(x_i, u_i) \\ \text{subj. to } & x_{i+1} = Ax_i + Bu_i \end{aligned} \quad \begin{aligned} & x_i \in \mathcal{X}, \quad u_i \in \mathcal{U} \\ & x_N \in \mathcal{X}_f, \quad x_0 = x(k) \end{aligned}$$

$l_f(\cdot)$, $\mathcal{X}_f$ chosen to mimic infinite horizon

LYAPUNOV STABILITY

System NL, TI, DT $x(k+1) = g(x(k))$
Asymptotic stability eq point $\bar{x} \in \Omega \; (g(\bar{x}) = \bar{x})$

Asympt. Stable in pos invar set $\Omega \subseteq \mathbb{R}^n$ if Lyap. stable and attactive

$$\lim_{k \to \infty} ||x(k) - \bar{x}|| = 0 \quad \forall x(0) \in \Omega$$

Globally Asympt. Stable if asympt. stable & $\Omega = \mathbb{R}^n$

FEASIBILITY & STABILITY GUARANTEES OF MPC

Proof Strategy

Recursive Feasibility show existence of feasible control sequence for all time when starting from feasible initial point
- Assume feas. of $x(k)$, $\{u_0^\star, \dots, u_{N-1}^\star\}$, $\{x_0^\star, \dots, x_N^\star\}$
- At $x(k+1) \Rightarrow \{u_1^\star, \dots, \kappa_f(x_N^\star)\}$ should be feas.

Stability show that optimal cost is lyap function
- l_f necessary to provide cost decrease for asympt. stability

General Terminal Set $\mathcal{X}_f$

Assumptions
1. Stage cost pos def, strictly positive, only 0 at origin
2. Terminal set invariant under local control law
   All state and input constraints satisfied in $\mathcal{X}_f$
3. Terminal cost is cont. Lyap. func. in terminal set $\mathcal{X}_f$ and satisfies

$$l_f(x_{i+1}) - l_f(x_i) \le -l(x_i, \kappa_f(x_i)) \quad \forall x_i \in \mathcal{X}_f$$

Theorem – CL system under MPC control law $u_0^\star(x)$ asympt. stable and set $\mathcal{X}_N$ is positive invariant for system $x(k+1) = Ax(k) + Bu_0^\star(x(k))$

Terminal Constraint At Zero $x_N \in \mathcal{X}_f = 0$
⇝ need large N to approx. max. cont. invar. set

Terminal Set & Cost – LQR

$$J^\star(x(k)) = \min_U x_N^\top P x_N + \textstyle\sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i$$

- Choose $P = P_\infty$ from (D)ARE
- Choose $\mathcal{X}_f$ to be max. invar. set for CL system $(A + BF_\infty)x_k$
  ⇝ ellipsoidal inv. set with Lyap.
- All x, u constraints satisfied in $\mathcal{X}_f$

All assumptions of Feasibility & Stability Theorem Satisfied

Useful Properties

- X_1, X_2 convex invar. for $Ax(k) \rightsquigarrow \alpha X_1 \oplus (1-\alpha) X_2$ invar $\forall \alpha \in [0, 1]$
- $X_1 \subseteq \mathcal{X}, X_2 \subseteq \mathcal{X}, X_i, \mathcal{X}$ convex $\rightsquigarrow \alpha X_1 \oplus (1-\alpha) X_2 \subseteq \mathcal{X} \; \forall \alpha \in [0, 1]$
- $V_i(x(k)) = x^\top(k) P_i x(k)$ lyap. func. for $x(k+1) = Ax(k)$, rate of decrease $x^\top(k) \Gamma x(k) \rightsquigarrow V(x(k)) = \alpha V_1(x(k)) + (1-\alpha) V_2(x(k))$ also lyap. func. with rate of decrease $x^\top(k) \Gamma x(k)$ for all $\alpha \in [0, 1]$

FEASIBILITY & STABILITY REMARKS

- Terminal constraint provides a **Sufficient Condition** for feas. & stab.
- Region of attraction w/o term. const. may be larger than with term. const.
- In practice: enlarge horizon and check stability by sampling. As $N\uparrow$, region of attraction appraoches max. control invariant set
- CL traj. may not follow assumptions made for OL predictions
- ∞-Horizon LQR controller locally optimal $\rightsquigarrow$ best choice for quad. cost
- **∞-Horizon provides stab. and invariance. Finite-Horizon MPC may not be stable & may not satisfy constraints $\forall$ time**

Extension to Nonlinearity

- Assumptions on terminal set/cost did not rely on linearity
- Lyapunov stability is general framework (works for NL sys)
- **Results can be directly extended to NL systems**
- **However, computing sets $\mathcal{X}_f$ and function l_f can be very difficult**

7 PRACTICAL MPC

REFERENCE TRACKING

System $x(k+1) = Ax(k) + Bu(k), \quad x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}$
Constraints $\mathcal{X} = \{x \mid G_x x \leq h_x\}, \mathcal{U} = \{u \mid G_u u \leq h_u\}$

STEADY-STATE TARGET PROBLEM

Target Condition

$$\begin{aligned} x_s &= Ax_s + Bu_s \\ z_s &= Hx_s = r \end{aligned} \iff \begin{bmatrix} \mathbb{I}-A & -B \\ H & 0 \end{bmatrix}\begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

- In presence of constraints, (x_s, u_s) must satisfy them
- In case of multiple feas. u_s, compute 'cheapest'

$$\min u_s^\top R_s u_s \quad \text{subj. to } [\text{Target Condition}], x_s \in \mathcal{X}, u_s \in \mathcal{U}$$

- In general, asssume target problem is feasible
- If no sol'n $\exists$: compute reachable point 'closest' to r

$$\min (Hx_s - r)^\top Q_s (Hx_s - r) \quad \text{subj. to } x_s = Ax_s + Bu_s$$

MPC FOR REFERENCE TRACKING

MPC Design

$$\min_U ||z_N - Hx_s||^2_{P_z} + \sum_{i=1}^{N-1} ||z_i - Hx_s||^2_{Q_z} + ||u_i - u_s||^2_R$$
$$\text{subj. to [model, constraints]}, x_0 = x(k)$$

Delta Form. Set pt. tracking $\xrightarrow{\text{Coord. Trans.}}$ Regulation Problem

$$\begin{aligned} \Delta x &:= x - x_s & G_x \Delta x &\leq h_x - G_x x_s \\ \Delta u &:= u - u_s & G_u \Delta u &\leq h_u - G_u u_s \end{aligned}$$

- Obtain target steady-state corresponding to reference r
- Initial state $\Delta x(k) = x(k) - x_s$
- Apply reg problem to new system in Δ-Formulation

$$\min \left[V_f(\Delta x_N) + \sum_{i=1}^{N=1} \Delta x_i^\top Q \Delta x_i + \Delta u_i^\top R \Delta u_i \right]$$
$$\begin{aligned} \text{subj. to } \Delta x_{i+1} &= A\Delta x_i + B\Delta u_i, \quad G_x \Delta x_i \leq h_x - G_x x_s \\ G_u \Delta u_i &\leq h_u - G_u u_s, \quad \Delta x_N \in \mathcal{X}_f, \quad \Delta x_0 = \Delta x(k) \end{aligned}$$

- Find optimal sequence of $\Delta U^\star$
- Input applied to system $u_0^\star = \Delta u_0^\star + u_s$

Convergence

Assume target feasible with $x_s \in \mathcal{X}, u_s \in \mathcal{U}$, choose terminal weight $V_f(x)$ and constraint $\mathcal{X}_f$ as in regulation case satisfying

- $\mathcal{X}_f \subseteq \mathcal{X}, Kx \in \mathcal{U} \quad \forall x \in \mathcal{X}_f$
- $V_f(x(k+1)) - V_f(x(k)) \leq -l(x(k), Kx(k)) \quad \forall x \in \mathcal{X}_f$

If in addition the target reference x_s, u_s is such that

- $x_s \oplus \mathcal{X}_f \subseteq \mathcal{X}, K\Delta x + u_s \in \mathcal{U}, \quad \forall \Delta x \in \mathcal{X}_f$

then CL system converges to target reference

$$x(k) \to x_s, z(k) = Hx(k) \xrightarrow{k\to\infty} r$$

Proof

- Invariance under local ctrol law inherited from regulation case
- Constraint satisfaction provided by extra conditions
  - $x_s \oplus \mathcal{X}_f \subseteq \mathcal{X} \to x \in \mathcal{X} \forall \Delta \in \mathcal{X}_f$
  - $K\Delta x + u_s \in \mathcal{U} \forall \Delta x \in \mathcal{X}_f \to u \in \mathcal{U}$
- Fron asympt stability of the regulation problem: $\Delta x(k) \xrightarrow{k\to\infty} 0$

Terminal Set

- Set of feasible targets may be significantly reduced. Enlarge set of feasible targets by scaling terminal set for regulation $\mathcal{X}_f^{\text{scaled}} = \alpha \mathcal{X}_f$
- Invariance maintained if $\mathcal{X}_f$ invariant $\rightsquigarrow$ so is $\alpha\mathcal{X}_f$
- Choose α s.t. x, u constraints still satisfied $\rightsquigarrow$ scaling target dependent
- Targets at the boundary of the constraints: $x_N = x_s$, corresponds to 0-terminal set in regulation case

MPC FOR REFERENCE TRACKING WITHOUT OFFSET

Augmented Model

$$x_{k+1} = Ax_k + Bu_k + B_d d_k, \quad d_{k+1} = d_k, \quad y_k = Cx_k + C_d d_k$$

Observability of aug. system: rank $\left(\begin{bmatrix} A-\mathbb{I} & B_d \\ C & C_d \end{bmatrix}\right) = n_x + n_d$

Inuition At steady-state $\begin{bmatrix} A-\mathbb{I} & B_d \\ C & C_d \end{bmatrix}\begin{bmatrix} x_s \\ d_s \end{bmatrix} = \begin{bmatrix} 0 \\ y_s \end{bmatrix}$, y_s, d_s unique

Linear State Estimation

Observer For Augmented Model

$$\begin{bmatrix} \hat{x}(k+1) \\ \hat{d}(k+1) \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & \mathbb{I} \end{bmatrix}\begin{bmatrix} \hat{x}(k) \\ \hat{d}(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} L_x \\ L_d \end{bmatrix}(-y(k) + C\hat{x}(k) + C_d \hat{d}(k))$$

Error Dynamics $\Rightarrow$ choose L s.t error dynamics asympt. stable

$$\begin{bmatrix} x(k+1) - \hat{x}(k+1) \\ d(k+1) - \hat{d}(k+1) \end{bmatrix} = \left(\begin{bmatrix} A & B_d \\ 0 & \mathbb{I} \end{bmatrix} + \begin{bmatrix} L_x \\ L_d \end{bmatrix}\begin{bmatrix} C & C_d \end{bmatrix}\right)\begin{bmatrix} x(k) - \hat{x}(k) \\ d(k) - \hat{d}(k) \end{bmatrix}$$

MPC FOR REFERENCE TRACKING WITHOUT OFFSET

Observer Steady-State

Suppose observer asympt. stable and $n_y = n_d$

$$\begin{bmatrix} A-\mathbb{I} & B \\ C & 0 \end{bmatrix}\begin{bmatrix} \hat{x}_\infty \\ u_\infty \end{bmatrix} = \begin{bmatrix} -B_d \hat{d}_\infty \\ y_\infty - C_d \hat{d}_\infty \end{bmatrix}$$

$\rightsquigarrow$ Observer output $C\hat{x}_\infty + C_d \hat{d}_\infty$ tracks y_∞ without offset

Offset-Free Tracking

Goal Track constant r: $z(k) = Hy(k) \to r$ as $k \to \infty$

Steady-State Condition

$$x_s = Ax_s + Bu_s + B_d \hat{d}_\infty, \quad z_s = H(Cx_s + C_d \hat{d}_\infty) = r$$

- Best forecast for d_∞ is current estimate $\hat{d}_\infty = \hat{d}$
- Same Procedure for regulation case with $r = 0$

Offset-Free Tracking Condition

$$\begin{bmatrix} A-\mathbb{I} & B \\ HC & 0 \end{bmatrix}\begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} -B_d \hat{d} \\ r - HC_d \hat{d} \end{bmatrix}$$

Offset-Free Tracking Procedure

1. Estimate $\hat{x}$ & $\hat{d}$
2. Obtain (x_s, u_s) from steady-state tgt problem using $\hat{d}$
3. Solve MPC problem for tracking using $\hat{d}$ $\tilde{x}_i := x_i - x_s$, $\tilde{u}_i = u_i - u_s$

$$\min_U V_f(\tilde{x}_N) + \sum_{i=0}^{N-1} (\tilde{x}_i)^\top Q(\tilde{x}_i) + (\tilde{u}_i)^\top R(\tilde{u}_i)$$
$$\begin{aligned} \text{subj. to } & x_{i+1} = Ax_i + Bu_i + B_d d_i, \quad d_{i+1} = d_i \\ & x_i \in \mathcal{X}, \quad u_i \in \mathcal{U}, \\ & x_0 = \hat{x}(k), \quad d_0 = \hat{d}(k), \quad x_n - x_s \in \mathcal{X}_f \end{aligned}$$

Offset-Free Tracking: Main Result

With $u_0^\star = \kappa(\hat{x}(k), \hat{d}(k), r) = \kappa(\cdot)$. Assuming $n_d = n_y$, RHC recursively feas., unconstrained for $k \geq j, j \in \mathbb{N}^+$, CL system:

$$\begin{aligned} x(k+1) &= Ax(k) + B\kappa(\cdot) + B_d d \\ \hat{x}(k+1) &= (A + L_x C)\hat{x}(k) + (B_d + L_x C_d)\hat{d}(k) + B\kappa(\cdot) - L_x y(k) \\ \hat{d}(k+1) &= L_d C\hat{x}(k) + (\mathbb{I} + L_d C_d)\hat{d}(k) - L_d y(k) \end{aligned}$$

converges $((\hat{x}, \hat{d}) \xrightarrow{k\to\infty} (x_\infty, d_\infty))$, then $z(k) = Hy(k) \xrightarrow{k\to\infty} r$

ENLARGING FEASIBLE SET – NO TERMINAL SET

Motivation Term. constraints reduces feasible set
Goal MPC without term. constraint with guaranteed stability
Note Feasible set without term. constraint not invariant

MPC Without Terminal Set

Can remove Term. constraint while maintaining stability **if**

- Initial state lies in sufficiently small subset of feasible set
- N sufficiently large

s.t term. state satisfies term. const. without envorcining it in the optimization. $\rightsquigarrow$ Sol'n of finite-horizon MPC problem corresponds to ∞-horizon sol'n

Advantage – Controller defined in larger feasible set
Disadvantage – Characterization of region of attaction of specification of required horizon length extremely difficult

- Term constr provides sufficient cond. for stab: RoA w/o term constr may be larger than w/
- In practice: Enlarge horizon and check stability by sampling
- $N\uparrow \rightsquigarrow$ RoA approachees max control invar. set

ENLARGING FEASIBLE SET – SOFT CONSTRAINTS

Motivation Input constraints usually 'hard', state constraints rarely 'hard' $\rightsquigarrow$ breakable
Goal Min size & duration of violation (**usually conflict!**)

MPC Problem Setup

$$\min_u \left[x_N^\top P x_N + \mathbf{l}_\epsilon(\epsilon_N) + \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i + \mathbf{l}_\epsilon(\epsilon_i) \right]$$
$$\text{s.t. } x_i = Ax_i + Bu_i, \quad H_x x_i \leq k_x + \boldsymbol{\epsilon}_i, \; H_u u_i \leq k_u, \quad \boldsymbol{\epsilon}_i \geq \mathbf{0}$$

Requirement on $\mathbf{l}_\epsilon$

Original Problem	"Softened" Problem
$\min_z f(z) \quad \text{s.t } g(z) \leq 0$	$\min_z f(z) + l_\epsilon(\epsilon) \quad \text{s.t } g(z) \leq \epsilon, \quad \epsilon \geq 0$

If original problem has feasible solution $z^\star$, Softened problem should have same solution $z^\star$, and $\epsilon = 0$.
Note $l_\epsilon(\epsilon_i) = s\epsilon_i^2$ does **not** fulfill requirement

Choice of Penalty

- **Quad. Penalty** $l_\epsilon(\epsilon_i) = \epsilon_i^\top S \epsilon_i$ (e.g $S = Q$)
- **Quad. + Linear Penalty** $l_\epsilon(\epsilon_i) = \epsilon_i^\top S \epsilon_i + v||\epsilon_i||_{1/\infty}$

Exact Penalty Function

$l_\epsilon(\epsilon) = v \cdot \epsilon$ satisfies requirement for any $v > \lambda^\star \geq 0$, where $\lambda^\star$ is optimal Lagrange multiplier for original problem

- In practice, combined cost used for exact penalty and tuning capabilities

Tuning

1. Minimize violation over horizon:

$$\epsilon^{\min} = \operatorname{argmin}_{u,\epsilon} \sum_{i=0}^{N-1} \epsilon_i^\top S \epsilon_i + v^\top \epsilon_i, \quad \text{s.t } x_{i+1} = Ax_i + Bu_i$$
$$H_x x_i \leq k_x + \epsilon_i, \quad H_u u_i \leq k_u, \quad \epsilon_i \geq 0$$

2. Optimize Controller performance

$$\min_u x_N^\top P x_N + \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i$$
$$\text{s.t } x_{i+1} = Ax_i + Bu_i, \quad H_x x_i \leq k_x + \epsilon_i^{\min}, \quad H_u u_i \leq k_u$$

Note Standard SC MPC does not provide stability guarantee for OL unstable sys.

8 ROBUST MPC I

UNCERTAINTY MODELS

Motivation Random Noise w changes sys. evolution, Model structure unknown, Unknown parameters θ impact dynamics

Uncertain Constrained System

$$x(k+1) = g(x(k), u(k), w(k); \theta), \quad x, u, w, \theta \in \mathcal{X}, \mathcal{U}, \mathcal{W}, \Theta$$

Additive Bounded Noise System

$$x(k+1) = Ax(k) + Bu(k) + w(k), \quad x, u, w \in \mathcal{X}, \mathcal{U}, \mathcal{W}$$

IMPACT OF BOUNDED ADDITIVE NOISE

Goals

Design $u(k) = \kappa(x(k))$ **s.t the system**

- **(a) Satisfies constraints**: $\{x(k)\} \subset \mathcal{X}, \{u(k)\} \subset \mathcal{U}$ for all disturbances
- **(b) Is Stable**: converges to neighbourhood of origin
- **(c) Optimizes (expected/worst-case) 'Performance'**
- **(d) Maximizes Set** $\{x(0) \mid \text{Condition 1-3 met}\}$

Uncertain State Evolution

$$\phi_i = \underbrace{A^i x_0 + \sum_{j=0}^{i-1} A^j B u_{1-1-j}}_{x_i \equiv \text{Nominal System}} + \underbrace{\sum_{j=0}^{i-1} A^j w_{i-1-j}}_{\text{Disturbance Offset}}$$

(c) – OPTIMIZES PERFORMANCE

Cost to Minimize

Cost now func of Disturbance $\rightsquigarrow$ Need to eliminate W

$$J(x_0, U, W) := l_f(\phi_N(x_0, U, W)) + \sum_{i=0}^{N-1} l(\phi_i(x_0, U, W), u_i)$$

Several Options

- Minimize expected value $J_N(x_0, U) = \mathbb{E}\{J(x_0, U, W)\}$
- Take worst case $J_N(x_0, U) := \max_{W \in \mathcal{W}^{N-1}} J(x_0, U, W)$
- **Take Nominal Case** $J_N(x_0, U) := J(x_0, U, 0)$

(a) – SATISFIES CONSTRAINTS

Robust Constraint Satisfaction

- **State & Input Constraints** for $i = 0, \dots, N-1$,
  Enforce constraints explicitly by imposing $\phi_i \in \mathcal{X}, \; u_i \in \mathcal{U}, \; \forall W \in \mathcal{W}^N$
- **Terminal Constraints** for $i = N, \dots$
  Enforce constraints implicitly $\phi_N \in$ robust invariant set $\mathcal{X}_f$, $K\mathcal{X}_f \in \mathcal{U}$ for $\phi_{i+1} = (A + BK)\phi_i + w_i$

Robust Positive Invariant Set

Set $\mathcal{O}^\mathcal{W}$ said to be robust pos. invar. for autonomous system $x(k+1) = g(x(k), w(k))$ if

$$x \in \mathcal{O}^\mathcal{W} \Rightarrow g(x, w) \in \mathcal{O}^\mathcal{W}, \; \forall w \in \mathcal{W}$$

Robust Pre-Set

Given set Ω and dynamic system $x(k+1) = g(x(k), w(k))$,

$$\text{pre}^\mathcal{W}(\Omega) := \{x \mid g(x, w)\} \in \Omega \; \forall w \in \mathcal{W}$$

Computing Robust Pre-Sets for Linear Systems

System $Ax(k) + w(k)$, set $\Omega := \{x \mid Fx \leq f\}$

$$\text{pre}^\mathcal{W}(\Omega) = \{x \mid FAx \leq f - \max_{w \in \mathcal{W}} Fw\} = \{x \mid FAx \leq f - h_{\mathcal{W}^i}(F)\}$$

Robust Invariant Set Conditions

Set $\mathcal{O}^\mathcal{W}$ is robust positive invariant set **iff**

$$\mathcal{O}^\mathcal{W} \subseteq \text{pre}^\mathcal{W}(\mathcal{O}^\mathcal{W}) \Leftrightarrow \text{pre}^\mathcal{W}(\mathcal{O}^\mathcal{W}) \cap \mathcal{O}^\mathcal{W} = \mathcal{O}^\mathcal{W}$$

Robust Constraint Satisfaction

Ensure constraints are satisfied for MPC sequence

$$\phi_i(x_0, U, W) = \left\{ x_i + \sum_{j=0}^{i-1} A^j w_{i-1-j} \mid W \in \mathcal{W}^i \right\} \subseteq \mathcal{X} \quad (1)$$

Assume $\mathcal{X} = \{x \mid Fx \leq f\}$ (polyhedron)

$$Fx_i \leq f - h_{\mathcal{W}^i}\left(F \sum_{j=0}^{i-1} A^j\right)$$

ACHTUNG Must ensure term state contained in robust invariant set
Intuition Tightening constraints on nominal system

SET OPERATORS

Minkowski Sum	**Pontryagin Difference**
$A \oplus B := \{x + y \mid x \in A, y \in B\}$	$A \ominus B := \{x \mid x + e \in A \; \forall e \in B\}$

ACHTUNG $A \ominus B \oplus B \subseteq A$

Robust Constraint Satisfaction

Eqn. (1) can be rewritten $\phi_i \in x_i \oplus (\mathcal{W} \oplus \dots A^{i-1}\mathcal{W}) \subseteq \mathcal{X}$

Enforcing this cond. requires Tightened Constraints

$$x_i \in \mathcal{X} \ominus \left(\bigoplus_{j=0}^{i-1} A^j \mathcal{W}\right)$$

ROBUST OPEN-LOOP MPC

Robust Open-Loop MPC

$$\min_U \left[l_f(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i) \right]$$
$$\begin{aligned} \text{subj. to } & x_{i+1} = Ax_i + Bu_i \\ & x_i \in \mathcal{X} \ominus (\textstyle\bigoplus_{j=0}^{i-1} A^j \mathcal{W}), \quad u_i \in \mathcal{U} \\ & x_0 = x(k), \quad x_N \in \mathcal{X}_f \ominus (\textstyle\bigoplus_{j=0}^{N-1} A^j \mathcal{W}) \end{aligned}$$

$\mathcal{X}_f \subseteq \mathcal{X}$ robust pos invar set for system $(A + BK)x(k) + w(k)$ with $w \in \mathcal{W} \; \forall k$ for some stabilizing K, and $Kx \in \mathcal{U} \; \forall x \in \mathcal{X}_f$

Intuition Nominal MPC, but with tigher state constraints
Open-Loop? Not accounting for FB during solving, just plan ahead for w
Achtung

- Unstable systems $A^{i-1}\mathcal{W}$ grows $\rightsquigarrow$ use 'pre-stabilization' $u_i = Kx_i + u_i$
- Potentially **very small region of attraction**, particularly for unstable sys

Robust Invariance

If $U^\star(x(k))$ is optimizer of robust OL MPC problem for $x(k) \in \mathcal{X}$, then system $Ax(k) + Bu_0^\star(x(k)) + w(k) \in \mathcal{X}$ for all $w \in \mathcal{W}$

9 ROBUST MPC II

CLOSED-LOOP PREDICTIONS

Goal optimize over seq. of funcs $\{u_0, \mu_1(\cdot), \dots, \mu_{N-1}(\cdot)\}$ with $(\mu_i(x_i) : \mathbb{R}^n \to \mathbb{R}^m$ control policy)

Problem Can't optimize over arbitrary functions!

Solution assume some structure on functions μ_i

Pre-Stabilization $\mu_i(x_i) = Kx_i + v_i$
Fixed K, s.t $A + BK$ stable $\rightsquigarrow$ Simple, often conservative

Linear Feedback $\mu_i(x_i) = K_i x_i + v_i$
Optimize over K_i, v_i, $\rightsquigarrow$ **Non-Convex** – Extremely difficult to solve

Disturbance Feedback $\mu_i(x_i) = \sum_{j=0}^{i-1} M_{ij} w_j + v_i$
Optimize over $M_{ij}, v_i \rightsquigarrow$ Equiv to linear feedback but **Convex** $\rightsquigarrow$ Effective, but computationally intense

Tube-MPC $\mu_i(x_i) = v_i + K(x_i - \bar{x}_i)$
Fixed K, s.t $A + BK$ stable $\rightsquigarrow$ Optimize over $\bar{x}_i, v_i \rightsquigarrow$ Simple, can be effective

TUBE-MPC

System

$$x(k+1) = Ax(k) + Bu(k) + w(k) \quad x, u \in \mathcal{X}, \mathcal{U} \quad w \in \mathcal{W}$$

Idea Seperate available control authority into 2 parts

(1) Portion that steers nominal sys to origin $z(k+1) = Az(k) + Bv(k)$
(2) Portion that compensates for deviations from this system $u_i = K(x_i - z_i) + v_i$ (keeps real traj close to nominal), for some linear K, which stabilizes nominal system

$\rightsquigarrow$ **Fix linear FB K offline** and optimize over nominal trajectory $\{v_0, \dots, v_{N-1}\} \rightsquigarrow$ convex problem

Error Dynamics

Define $e_i := x_i - z_i \rightsquigarrow e_{i+1} = (A + BK)e_i + w_i$
Bound maximum error, how far 'real' traj from nominal

$$e_{i+1} = (A + BK)e_i + w_i \qquad w_i \in \mathcal{W}$$

Dynamics $A + BK$ are stable, set $\mathcal{W}$ bounded $\rightsquigarrow$ Set $\mathcal{E}$ s.t e stays inside $\forall k \rightsquigarrow$ want 'minimal robust invariant set'

Tube-MPC Procedure

(a) Compute set $\mathcal{E}$ that error remains inside
(b) Modify constraints on nominal traj $\{z_i\}$
(c) Formulate as convex optimization problem

(a) – MINIMUM ROBUST INVARIANT SET

mRPI – Minimum Robust Invariant Set

$$F_\infty = \bigoplus_{j=0}^{\infty} A^j \mathcal{W}$$
$$F_0 := \{0\}$$
If $F_n = F_{n+1} \Rightarrow F_n = F_\infty$

$\Omega_0 \leftarrow \{0\}$
loop
  $\Omega_{i+1} \leftarrow \Omega_i \oplus A^i \mathcal{W}$
  if $\Omega_{i+1} = \Omega_i$ **then**
    return $F_\infty = \Omega_i$
  end if
end loop

- Finite n does not always exist, 'large' n often good approx.
- If n not finite, other methods of computing small invariant sets, slightly larger than F_∞

(b) – MODIFY NOMINAL TRAJECTORY CONSTRAINTS

Noisy System Trajectory

Given nominal traj z_i noisy sytem traj $x_i = z_i + e_i \rightsquigarrow$ will be smewhr in $\mathcal{E}$
$$x_i \in z_i \oplus \mathcal{E} = \{z_i + e \mid e \in \mathcal{E}\}$$

Goal $x_i, u_i \in \mathcal{X}, \mathcal{U}$ for all $\{w_i\} \in \mathcal{W}^j$

State Condition

Necessary & Sufficient Condition
$$z_i \oplus \mathcal{E} \subseteq \mathcal{X} \quad \Leftrightarrow \quad z_i \in \mathcal{X} \ominus \mathcal{E}$$
Set $\mathcal{E}$ known offline – **can compute constraints offline!**

Input Condition

$$u_i \in K\mathcal{E} \oplus v_i \subset \mathcal{U} \quad \Leftrightarrow \quad v_i \in \mathcal{U} \ominus K\mathcal{E}$$

(c) – CONVEX OPTIMIZATION PROBLEM

Problem Formulation

$$\min_{Z,V} \; l_f(z_N) + \sum_{i=1}^{N-1} l(z_i, v_i)$$
$$\left.\begin{aligned} \text{s.t. } & z_{i+1} = Az_i + Bv_i \\ & z_i \in \mathcal{X} \ominus \mathcal{E}, \quad u_i \in \mathcal{U} \ominus K\mathcal{E} \\ & z_N \in \mathcal{X}_f, \quad x(k) \in z_0 \oplus \mathcal{E} \end{aligned}\right\} =: \text{Set } \mathcal{Z}$$
$$\text{Control Law} : \mu_{\text{tube}}(x) := K(x - z_0^\star(x)) + v_0^\star(x)$$

Remarks

- Optimizing nominal system with tightened state, input constraints
- **First tube center** z_0 **is opt. var.** $\rightsquigarrow$ has to be within $\mathcal{E}$ of x_0
- Cost is w.r.t tube centers, terminal set is w.r.t tightened constraints

ACHTUNG $K(x - z_0^\star(x)) + v_0^\star(x)$ **NOT LINEAR** in CL

ROBUST CONSTRAINT SATISFACTION

Assumptions almost the same as for nominal MPC

(1) Stage cost pos def, i.e strictly pos and only 0 at origin
(2) Terminal set invar **for the nominal sys** under local control law $\kappa_f(z)$:
$$Az + B\kappa_f(z) \in \mathcal{X}_f \quad \forall z \in \mathcal{X}_f$$
All **tightened state and input constraints** satisfied in $\mathcal{X}_f$:
$$\mathcal{X}_f \subseteq \mathcal{X} \ominus \mathcal{E}, \kappa_f(z) \in \mathcal{U} \ominus K\mathcal{E} \quad \forall z \in \mathcal{X}_f$$
(3) Terminal cost is continuous Lyapunov function in terminal set $\mathcal{X}_f$:
$$l_f(Az + B\kappa_f(z)) - l_f(z) \le -l(z, \kappa_f(z)) \quad \forall z \in \mathcal{X}_f$$

Theorem: Robust Invariance of Tube-MPC

Set $\mathcal{Z} := \{x \mid \mathcal{Z} \ne \emptyset\}$ is robust invariant set of system $x(k+1) = Ax(k) + B\mu_{\text{tube}}(x(k)) + w(k)$ subject to constraints $x, u \in \mathcal{X}, \mathcal{U}$

Proof let $(\{v_0^\star \dots v_{N-1}^\star\}, \{z_0^\star \dots z_N^\star\})$ be optimal sol'n for $x(k)$ At next point in time, state $x(k+1)$ may have many possible values due to disturbance
By construction, state $x(k+1)$ in in the set $z_1^\star \oplus \mathcal{E}\ \forall \mathcal{W}$
Therefore the following sequence is feasible for all $x(k+1)$
$$(\underbrace{\{v_1^\star \dots v_{N-1}^\star, \kappa_f(z_N^\star)\}, \{z_1^\star \dots z_N^\star}_{\text{feas. IC}}, \underbrace{Az_N^\star + B\kappa_f(z_N^\star)}_{\in \mathcal{X}_f \rightsquigarrow \text{feas.}}\})$$

ROBUST STABILITY

Robust Stability of Tube-MPC

State $x(k)$ of system $x(k+1) = Ax(k) + B\mu_{\text{tube}}(x(k)) + w(k)$ converges in the limit to the set $\mathcal{E}$

Proof As in standard MPC we have
$$J^\star(z_0^\star(x(k))) = l_f(z_N^\star) + \textstyle\sum_{i=0}^{N-1} l(z_i^\star, v_i^\star)$$
$$J^\star(z_0^\star(x(k+1))) \le l_f(z_N^\star) + \textstyle\sum_{i=1}^{N-1} l(z_i^\star, v_i^\star)$$
$$+ l(z_0^\star, v_0^\star) - l(z_0^\star, v_0^\star) + l_f(z_N^\star) - l_f(z_N^\star)$$
$$= J^\star(x(k)) - \underbrace{l(z_0^\star, v_0^\star)}_{\ge 0} \underbrace{- l_f(z_N^\star) + l_f(z_{N+1}) + l(z_N^\star, \kappa_f(z_N^\star))}_{\le 0 \ (l_f \text{ is lyap function in } \mathcal{X}_f)}$$

This shows $\lim_{k\to\infty} J(z_0^\star(x(k))) = 0$, therefore $\lim_{k\to\infty} z_0^\star(x(k)) = 0$

ACHTUNG

- $x(k)$ does **not tend to 0!** It only stays within robust invar set centered at $z_0^\star(x(k)) : \lim_{k\to 0} \text{dist}(x(k), \mathcal{E}) = 0$
- Can remove constr. $z_0 \in \mathcal{X} \oplus \mathcal{E}$, doesn't affect recursive stability
- $\mathcal{E}$ must be robust positive invariant for proof (so error remains bounded)

TUBE-MPC IMPLEMENTATION

Offline Design

(1) Choose stabilizing controller K s.t $||A + BK|| < 1$
(2) Compute mRPI set $\mathcal{E} = F_\infty$ for system $x(k+1) = (A + BK)x(k) + w(k)$, $w \in \mathcal{W}$
(3) Compute tightened constaints $\tilde{\mathcal{X}} := \mathcal{X} \ominus \mathcal{E}, \tilde{\mathcal{U}} := \mathcal{U} \ominus K\mathcal{E}$
(4) Choose terminal weight function l_f and constraint $\mathcal{X}_f$ satisfying assumptions on tube MPC (see **Robust Constraint Satisfaction**)
- Assumption on Terminal set ensures **Recursive Feasibility**
- Assumption on terminal cost ensures **Asymptotic Stability**

LQR Terminal Constraint (typical choice)

- Choose LQR terminal control law $\kappa_f(x) = Kx$, $(Q, R$ same as MPC)
- Find $\mathcal{X}_f$ invar under this controller s.t satisfies constraints

Online Design

(1) Measure / Estimate state x
(2) Solve optimization problem
$(V^\star(x_0), Z^\star(x_0)) = \text{argmin}_{V,Z}\{J(Z, V) \mid (Z, V) \in \mathcal{Z}(x_0)\}$
(3) Set input to $u = K(x - z_0^\star(x)) + v_0^\star(x)$

Benefits
- Less conservative than OL robust MPC (now actively compensating for noise in prediction)
- Works for unstable systems
- Optimization problem to solve is 'simple'

Cons
- Sub-optimal MPC (optimal extremely difficult)
- Reduced feasible set when compared to nominal MPC
- We need to know what $\mathcal{W}$ is (usually not realistic)

ROBUST MPC FOR UNCERTAIN SYSTEMS – SUMMARY

Idea compensate for noise in prediction to ensure constraint satisfaction

Benefits
- Feasible set invariant – know exactly when controller will work
- Easier to tune – knobs to tradeoff robustness against performance

Cons
- Complex (tubes easy to implement, complex to understand)
- Must know largest noise $\mathcal{W}$
- Often conservative
- Feas set may be small

10 ROBUST MPC III – EXTENSIONS

ROBUST CONSTRAINT TIGHTENING MPC

Idea Combine best of Robust OL and Tube-Based MPC
$\rightsquigarrow$ Use propagated error bound to tighten constraints

Error Dynamics $e_{i+1} = (A + BK)e_i + w_i = A_K e_i + w_i, w_i \in \mathcal{W}$

If $e_0 = 0$ then $e_i = \sum_{j=0}^{i-1} A^j w_{i-1-j} \Rightarrow e_i \in \mathcal{W} \oplus A_K\mathcal{W} \oplus \dots A_K^{i-1}\mathcal{W}$

$$\min_{Z,V} \; l_f(z_N) + \sum_{i=0}^{N-1} l(z_i, v_i)$$
$$\text{subj. to } z_{i+1} = Az_i + Bv_i$$
$$z_i \in \mathcal{X} \ominus (\mathcal{W} \oplus A_K\mathcal{W} \oplus \dots A_K^{i-1}\mathcal{W})$$
$$u_i \in \mathcal{U} \ominus K(\mathcal{W} \oplus A_K\mathcal{W} \oplus \dots A_K^{i-1}\mathcal{W})$$
$$z_N \in \mathcal{X}_f \ominus (\mathcal{W} \oplus A_K\mathcal{W} \oplus \dots A_K^{N-1}\mathcal{W})$$
$$z_0 = x(k)$$
$$\text{Control Law } u(k) = v_0^\star + K(x(k) - z_0) = v_0^\star$$

Motivation can robustly ensure constraint satisfactkon at each time step

Note need terminal set $\mathcal{X}_f$ that is robust invariant under tube controller K

NOMINAL MPC WITH NOISE

Standard MPC Problem for $x(k+1) = Ax(k) + Bu(k) + \mathbf{w(k)}$

$$J^\star(x_0) = \min_U l_f(x_N) + \textstyle\sum_{i=0}^{N-1} l(x_i, u_i), \text{ s.t } x_{i+1} = Ax_i + Bu_i, \; x_i, u_i, x_N \in \mathcal{X}, \mathcal{U}, \mathcal{X}_f$$

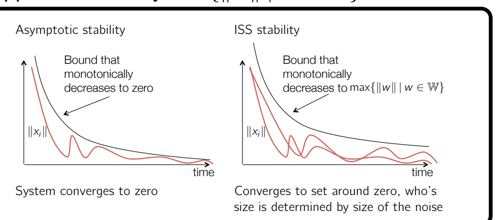
Effect on Lyapunov Function

Assume Optimal cost $J^\star$ Lipschitz continuous $(|f(y) - f(x)| \le \gamma||y - x||)$
$$|J(Ax + Bu + w) - J(Ax + Bu)| \le \gamma||Ax + Bu + w - (Ax + Bu)|| = \gamma||w||$$

Lyapunov Decrease can be bounded as
$$J^\star(Ax + Bu^\star + w) - J^\star(x) - J^\star(Ax + Bu^\star + w) + J^\star(x)$$
$$\le J^\star(Ax + Bu^\star) - J^\star(x) + \gamma||w|| \le -l(x, u^\star) + \gamma||w||$$

- Amount of decrease grows with $||x||$
- Amount of increase upper bounded by $\max\{||w|| \mid w \in \mathcal{W}\}$

ISS – Input-To-State Stability



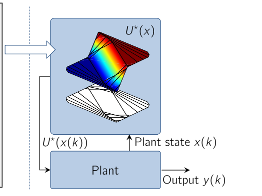
Benefits
- No special knowledge required – 'just works' (sometimes)
- Often very effective in practice
- Large feasible set
- Region of attraction may be relatively large

Cons
- Very difficult to determine region of attraction (set of states where controller works)
- Hard to tune
- Only works for NL systems under continuity assumptions

11 IMPLEMENTATION

EXPLICIT MPC

$$U^\star(x(k)) = \underset{U}{\text{argmin}} \; x_N^T P x_N + \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i$$
$$\text{subj. to } x_0 = x(k)$$
$$x_{i+1} = Ax_i + Bu_i, \; i = 0, \dots, N-1$$
$$x_i \in \mathcal{X}, \; u_i \in \mathcal{U}, \; i = 0, \dots, N-1$$
$$x_N \in \mathcal{X}_f$$



Recall: Quadratic Cost State Feedback Solution

MP-QP – Multiparametric Quadratic Program
$$J^\star(x(k)) = \min_U \left[U^\top x(k)^\top\right] \left[\begin{smallmatrix} H & F^\top \\ F & Y \end{smallmatrix}\right] \left[U^\top x(k)^\top\right]^\top$$
$$\text{subj. to } GU \le w + Ex(k)$$

Solution Properties – $J^\star(x(k))$ **convex** and **PW Quad.** on polyhedra.

Active Set for $l = 1, \dots, m$

Define active set at x, $A(x)$, and it's complement $NA(x)$ as
$$A(x) := \{j \in I : G_j z^\star(x) - S_j x = w_j\} \quad \text{(satisfied with eq.)}$$
$$NA(x) := \{j \in I : G_j z^\star(x) - S_j x < w_j\} \quad \text{(strict inequality)}$$

Critical Region

CR_A is set of parameters x for which set $A \subseteq I$ of constraints i active at the optimum. For given $\bar{x} \in \mathcal{K}^\star$ let $(A, NA) := (A(\bar{x}), NA(\bar{x}))$. Then
$$CR_A := \{x \in \mathcal{K}^\star : A(x) = A\} \quad \text{(states share active set)}$$

Point Location

- **Sequential Search** – Computationally linear, very simple, works for all problems
- **Search Tree** – Potentially logarithmic, significant offline processing (reasonable for <1k regions)

Remarks on Explicit MPC

- Linear MPC + Quad / Linear-norm cost $\rightsquigarrow$ Controller PWA func.
- Can pre-compute this function offline
- Online evaluation of PWA function very fast (ns - μs)
- Can only do this for small systems (3-6 states, small horizon)

ITERATIVE OPTIMIZATION METHODS

Generic Optimization Problem

convex if $f : \mathbb{R}^n \to \mathbb{R}$ and set $\mathbb{Q}$ convex
Analytical sol'n cannot be obtained except simplest cases

$$\text{minimize } f(x) \quad \text{subj. to } x \in \mathbb{Q}$$

Iterative Optimization Methods

Given initial guess $x^{(0)}$, produce sequence of iterates
$$x^{(i+1)} = \psi(x^{(i)}, f, \mathbb{Q}), \quad i = 0, \dots, m-1$$
such that $|f(x^{(m)}) - f(x^\star)| \le \epsilon$ and $\text{dist}(x^{(m)}, \mathbb{Q}) \le \delta$
where ϵ and δ are user defined tolerances

UNCONSTRAINED MINIMIZATION

Optimality Conditions

Assume $f(\cdot)$ diff'bar at $x^\star$. If f convex, then $x^\star$ global min iff $\nabla f(x^\star) = 0$

Descent Methods

$$x^{(i+1)} = x^{(i)} + h^{(i)}\Delta x^{(i)}$$
with $f(x^{(i+1)}) < f(x^{(i)})$

Input $x^{(0)} \in \text{dom}(f)$
repeat
  Compute descent dir. $\Delta x^{(i)}$
  Line Search: choose step size $h^{(i)} > 0$ s.t $f(x^{(i)} + h^{(i)}\Delta x^{(i)}) < f(x^{(i)})$
  Update $x^{(i+1)} := x^{(i)} + h\Delta x^{(i)}$
until termination condition
(e.g $f(x^{(m)}) - f(x^\star) \le \epsilon_1$)

- Δx: step/search direction
- $h^{(i)}$: step size/length
- $f(x^{(i+1)}) < f(x^{(i)})$ i.e $\Delta x^{(i)}$ is descent function
- $\exists h^{(i)} > 0$ s.t $f(x^{(i+1)}) < f(x^{(i)})$ if $\nabla f(x^{(i)})^\top \Delta x^{(i)} < 0$

Descent Direction

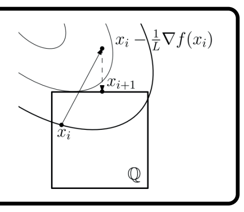
- **Gradient descent** $x^{(i+1)} = x^{(i)} - h^{(i)}\nabla f(x^{(i)})$
  - Assume ∇f Lipschitz-continuous $||\nabla f(x) - \nabla f(y)|| \le L||x - y||$
  - Choose constant step size $h^{(i)} = 1/L$
- **Newton Step** $x^{(i+1)} = x^{(i)} + h^{(i)}\Delta x_{nt}$
  - $\Delta x_{nt} = -(\nabla^2 f(x^{(i)}))^{-1}\nabla f(x^{(i)})$
  - Exact Line Search $h^{(i)\star} = \text{argmin}_{h>0} f(x^{(i)} + h^{(i)}\Delta x_{nt})$
    Optimization in 1 var $\rightsquigarrow$ solve by bisection, time consuming
  - Inexact Line search: find $h^{(i)}$ that decreases f by some amount

CONSTRAINED MINIMIZATION

Projected Gradient Methods

Incorporate Constraints in Gradient Step
$$\boxed{x^{(i+1)} = \pi_{\mathbb{Q}}(x^{(i)} - h^{(i)}\nabla f(x^{(i)}))}$$

Projection $\pi_{\mathbb{Q}} = \text{argmin}_x \frac{1}{2}||x - y||_2^2$ s.t $x \in \mathbb{Q}$
- Simple input constraints: ezpz
- State constraints: hard $\rightsquigarrow$ solve for dual



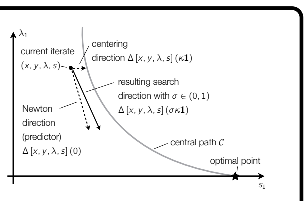
Interior-Point Methods

System $\min f(x)$ s.t. $g_i(x) \le 0, i = 1, \dots, m$
Assumptions f, g_i convex, twice cont. diff'bar. $f(x^\star)$ is finite and attained, stict feasiblity $\exists g(\tilde{x}) < 0$, feasible set closed & compact
Idea Reformulate as unconstrained problem

Primal-Dual Interior-Point Methods

Idea – Iteratively solve relaxed KKT system leave $\lambda_i^\star$ as variables, linearize and solve resulting sytem of linear eqns at each iteration

Search Direction $\Delta[x, \nu, \lambda, s](v)$
- $v = 0$ pure Newton direction "predictor"/"affine-scaling"
- $v = \kappa 1$ centering direction, approach central path



$\Rightarrow$ combine via **centering parameter** $\sigma \in (0, 1)$