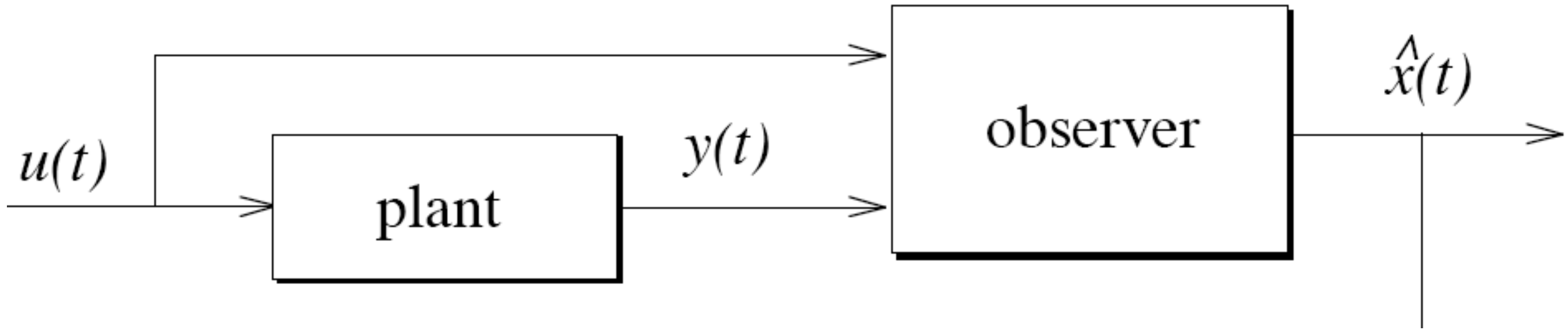
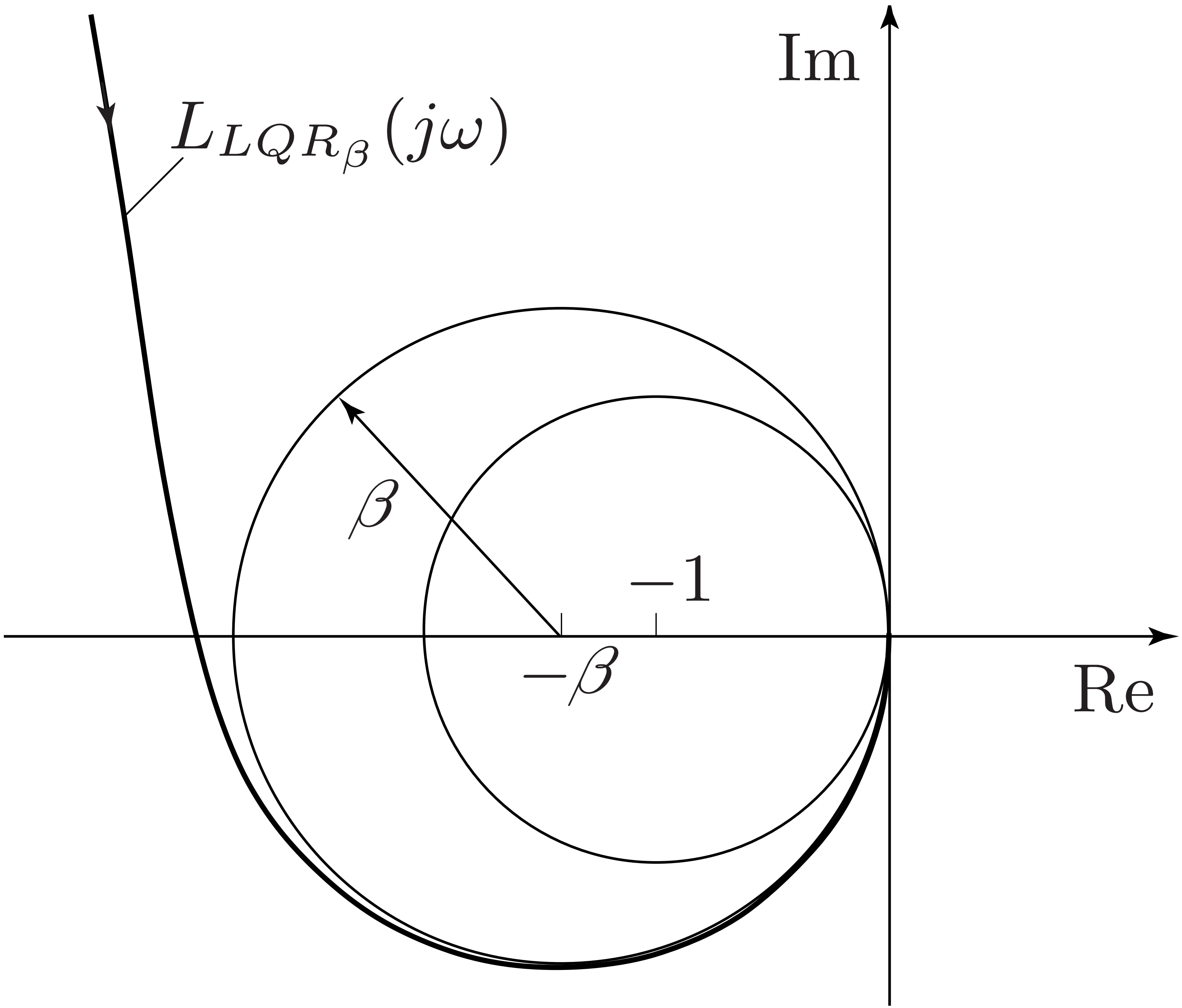
Plan:

1. $u(t) = f(x(t), t)$

2. 

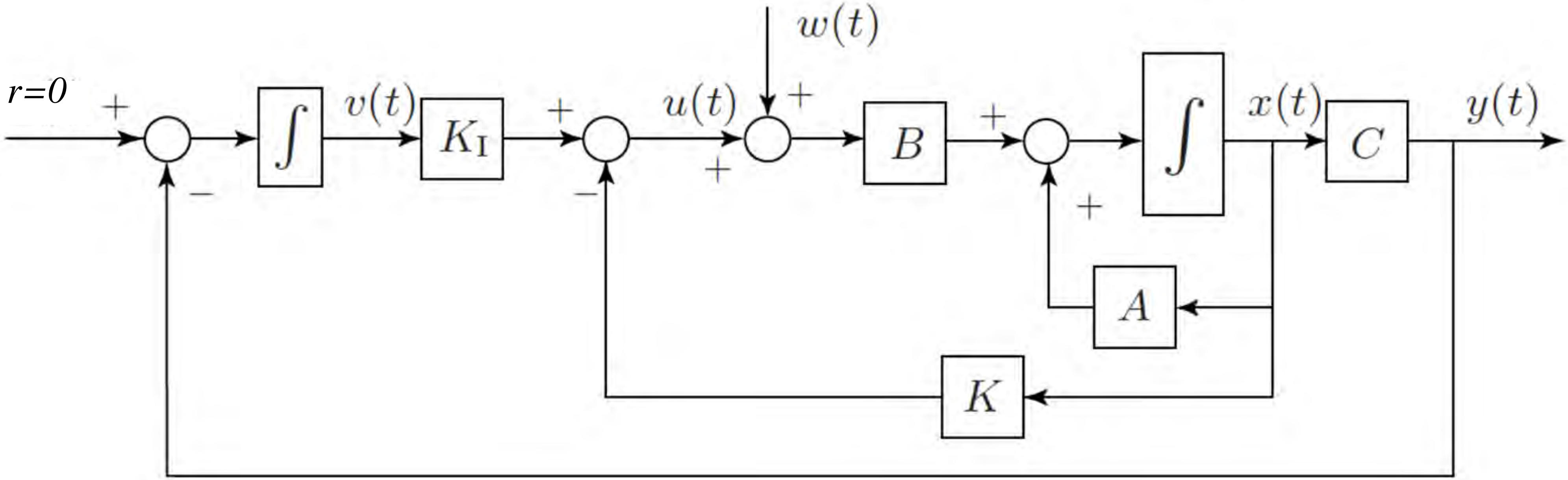
3.

4. Many improvements



Im
$L_{LQR_\beta}(j\omega)$
β
-1
$-\beta$
Re

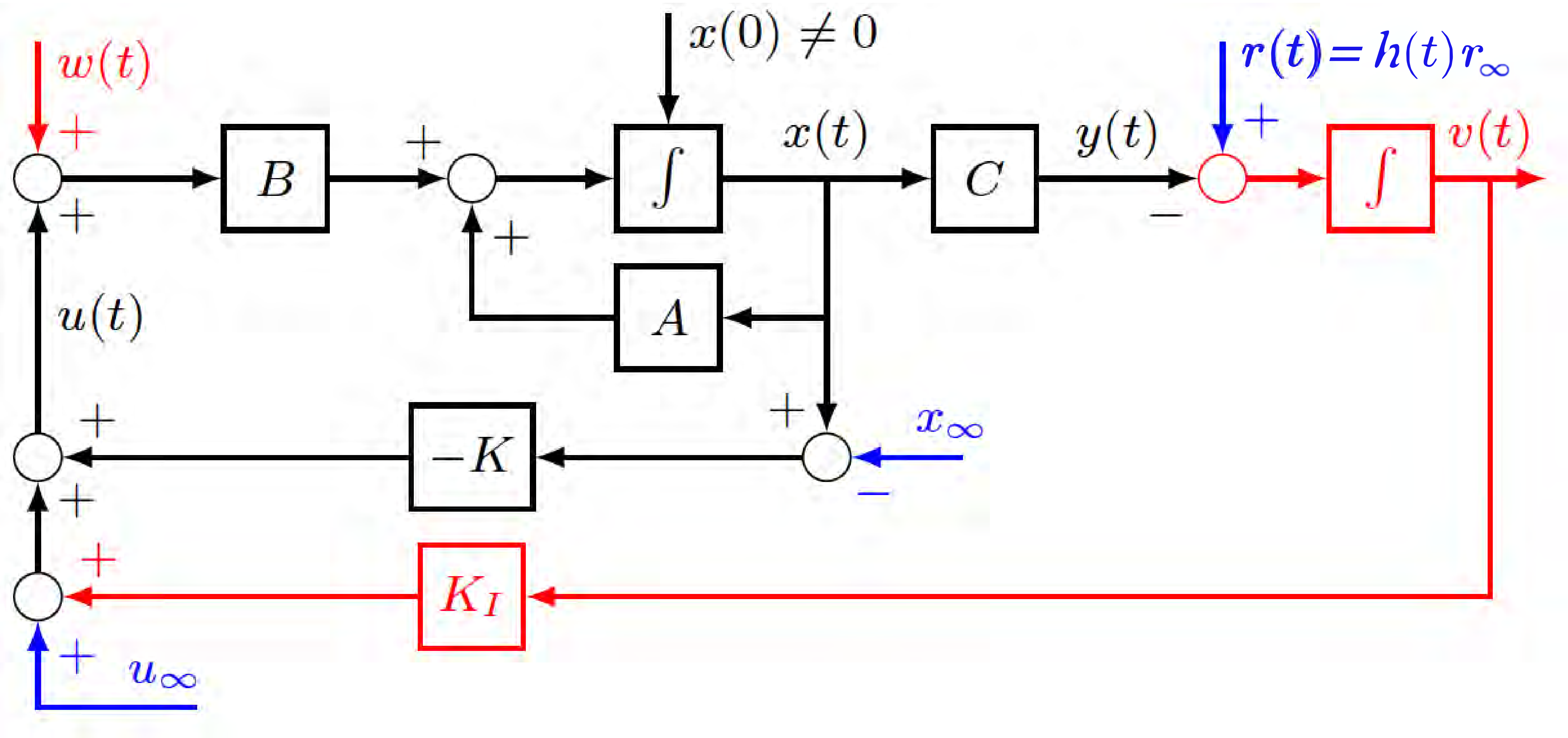
LQR-I Controller



Other structures possible

Re-arrange as standard LQR problem

Reference Tracking LQR-I



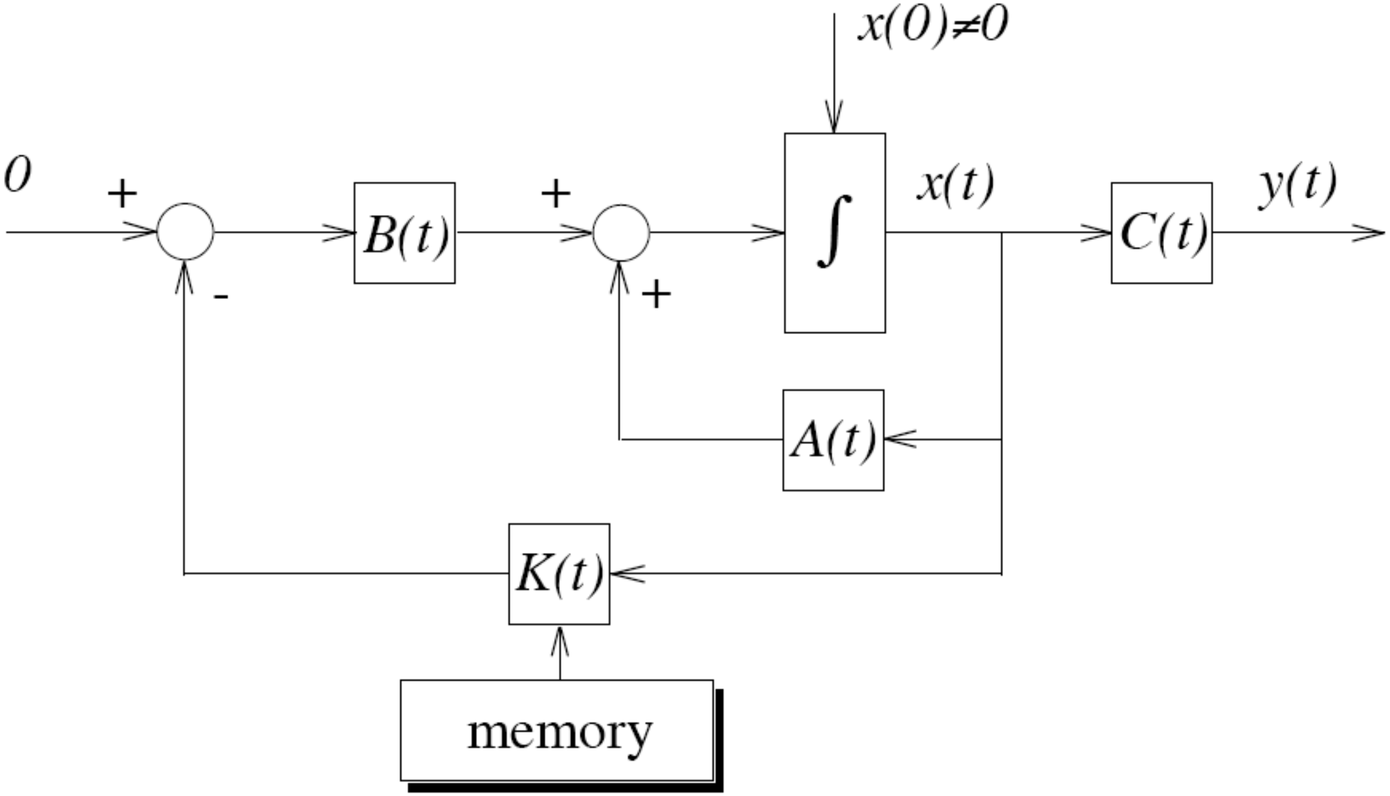
Full solution: See Section 4.4.3

Time-Varying Systems and Finite Horizons

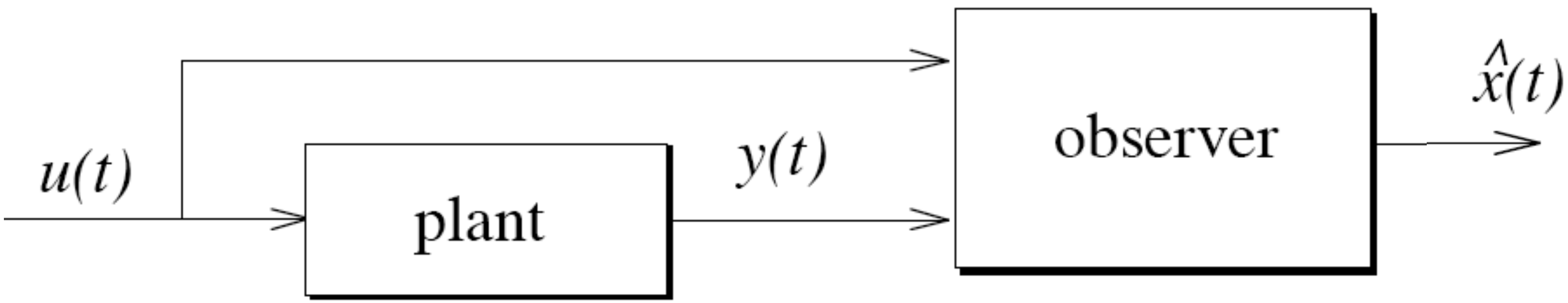
$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = A(t)\cdot x(t) + B(t)\cdot u(t), \quad x(t)\in\mathbb{R}^n, \; u(t)\in\mathbb{R}^m, \quad x(t_a) = x_a$$

$$J(u) = x^{\mathrm{T}}(t_b)\cdot P\cdot x(t_b) + \int_{t_a}^{t_b} \left(x^{\mathrm{T}}(u(t))\cdot Q(t)\cdot x(u(t)) + u^{\mathrm{T}}(t)\cdot R(t)\cdot u(t)\right) dt$$

$$\frac{d}{dt}\Phi(t) = \Phi(t)\cdot B(t)\cdot R^{-1}(t)\cdot B^{T}(t)\cdot\Phi(t) - \Phi(t)\cdot A(t) - A^{T}(t)\cdot\Phi(t) - Q(t)\,, \quad \Phi(t_b) = P$$

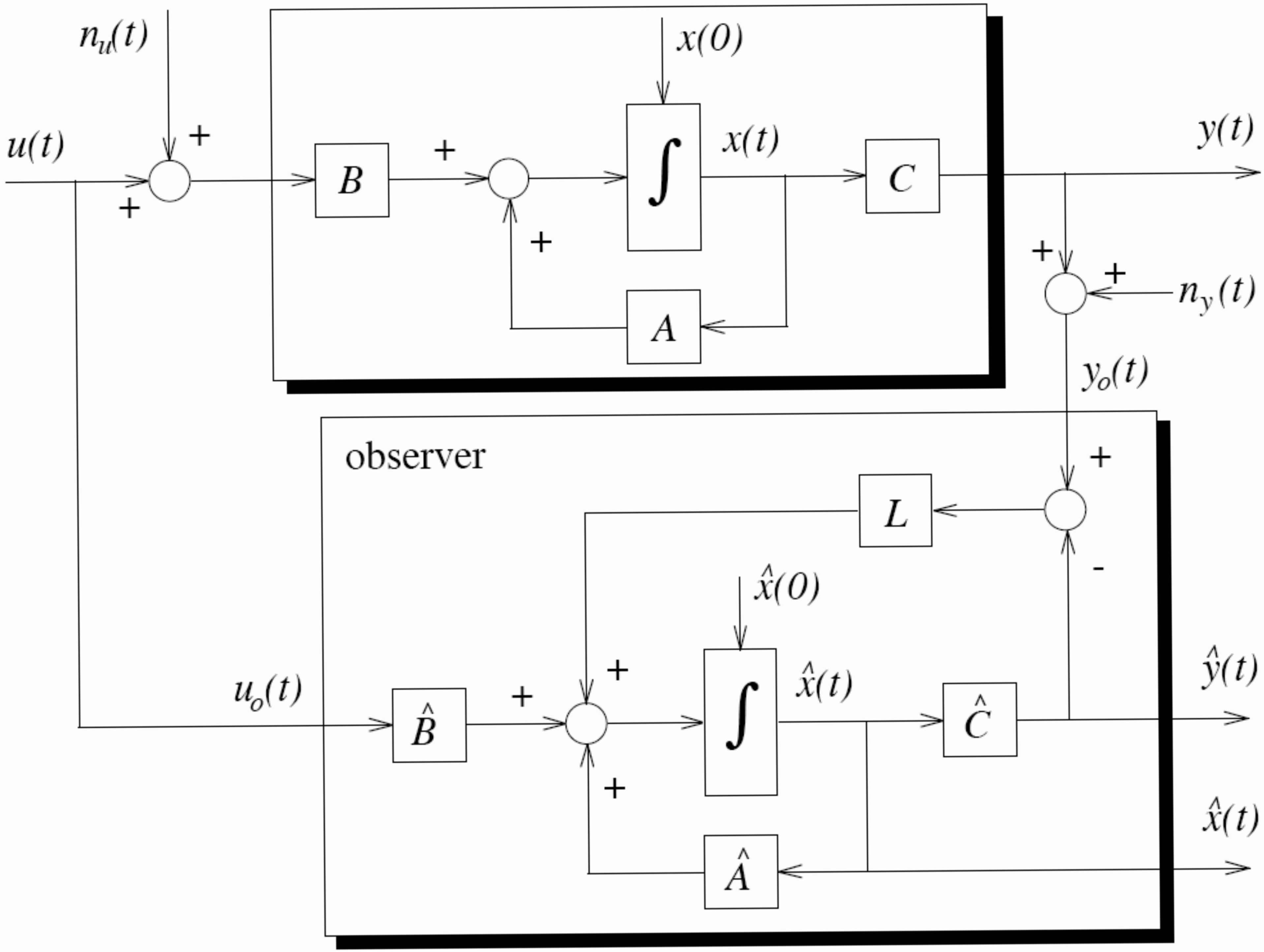


State Observers



Known: signals: $u(t)$ and $y(t)$
plant model $\{A,B,C,D\}$

Find: Estimation $\hat{x}(t)$ of $x(t)$



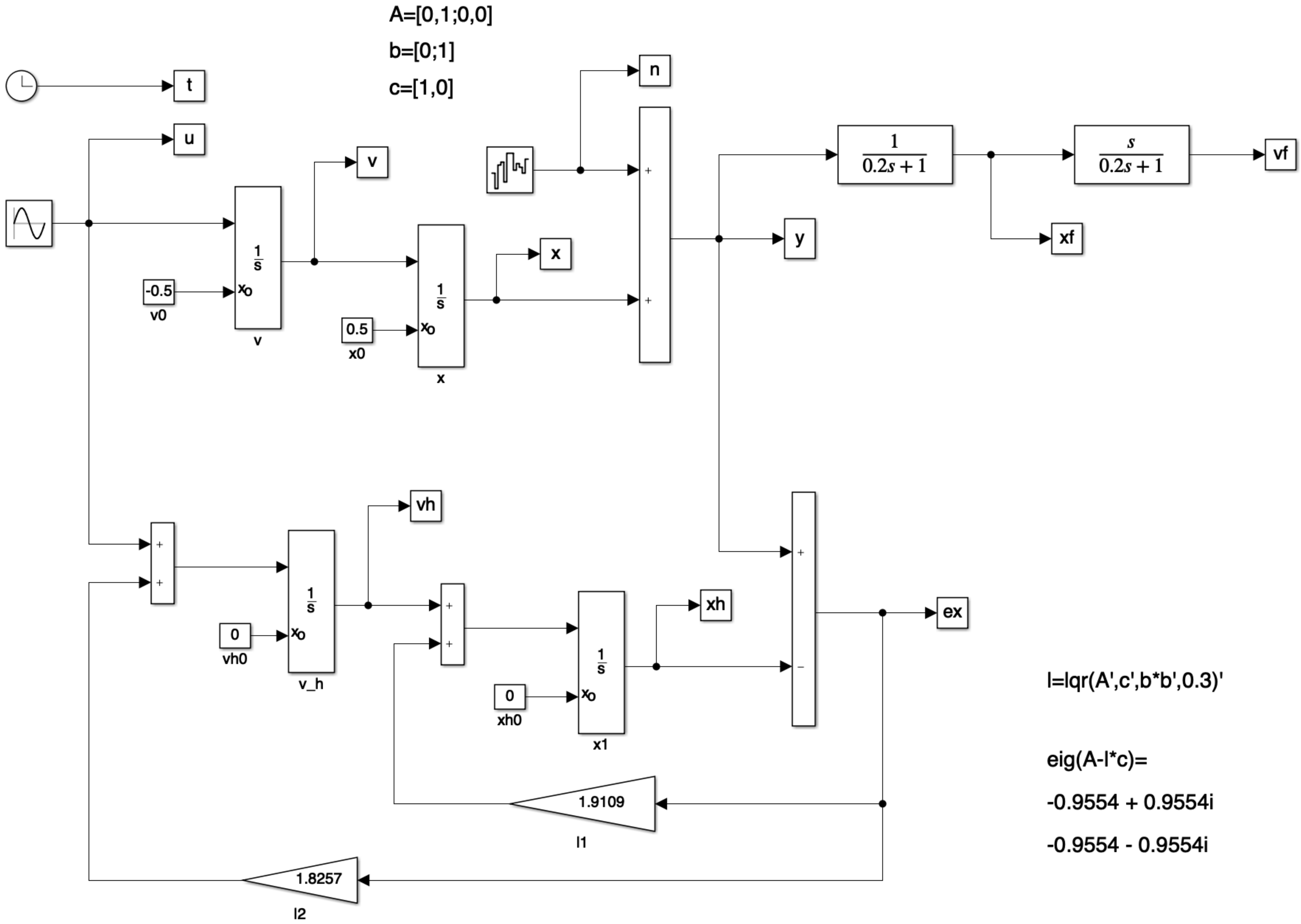
n_u(t)
+
u(t)
+
x(0)
B
+
x(t)
C
y(t)
+
+
A
+
+
n_y(t)
y_o(t)
observer
+
L
-
x̂(0)
u_o(t)
B̂
+
+
x̂(t)
Ĉ
ŷ(t)
+
Â
x̂(t)

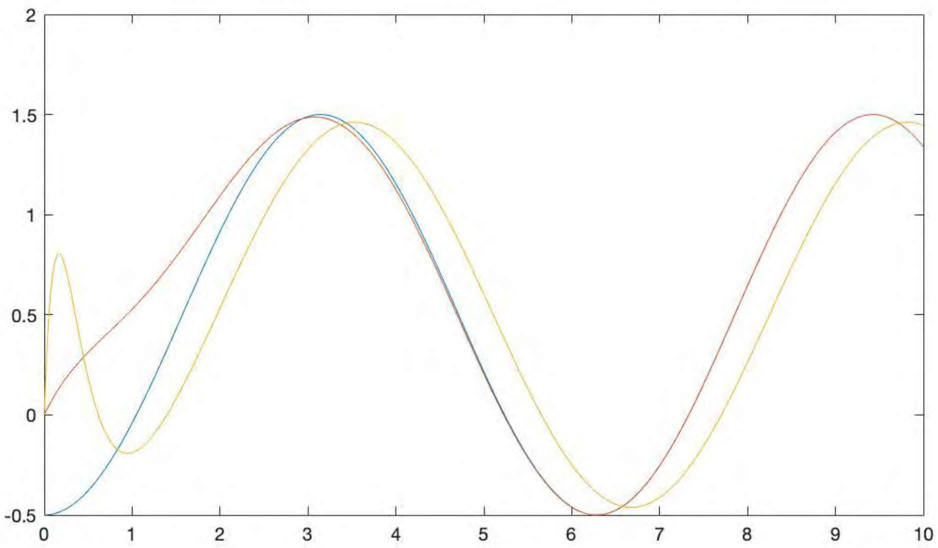
$$\bar{x}(t) = x(t) - \widehat{x}(t) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\bar{x}(t) = (A - L \cdot C) \cdot \bar{x}(t)$$

How to compute L?

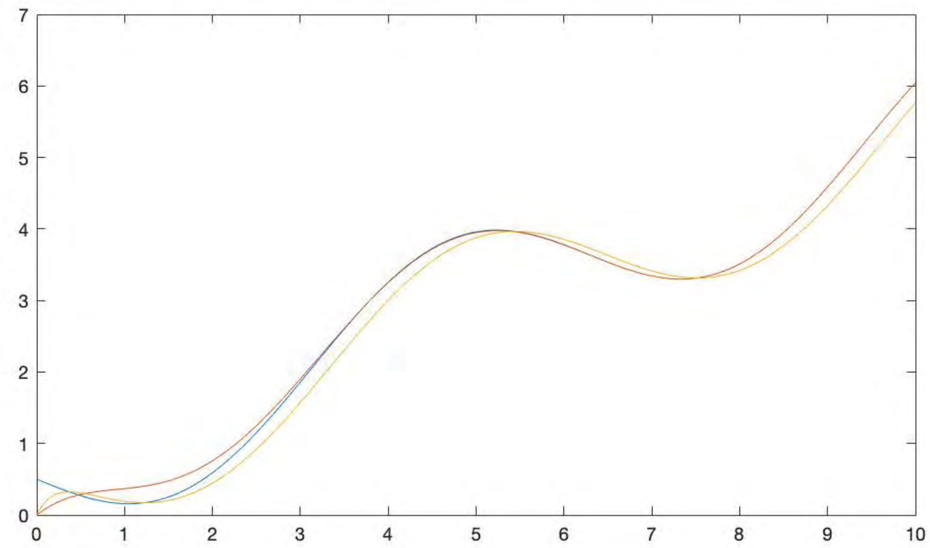
Dual LQR problem

$$\begin{array}{rcl} A & \rightarrow & A^T \\ B & \rightarrow & C^T \\ Q = \bar{C}^T \cdot \bar{C} & \rightarrow & \bar{B} \cdot \bar{B}^T \\ R = r \cdot I & \rightarrow & q \cdot I \end{array}$$

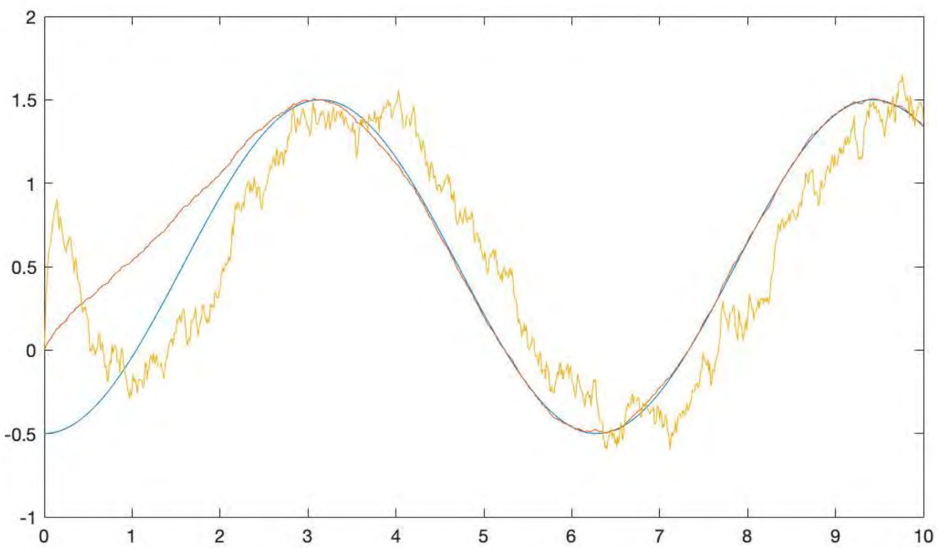

A=[0,1;0,0]
b=[0;1]
c=[1,0]
t
u
v
n
vf
xf
y
x
-0.5
v0
0.5
x0
vh
xh
ex
0
vh0
v_h
0
xh0
x1
1.9109
l1
1.8257
l2
l=lqr(A',c',b*b',0.3)'
eig(A-l*c)=
-0.9554 + 0.9554i
-0.9554 - 0.9554i



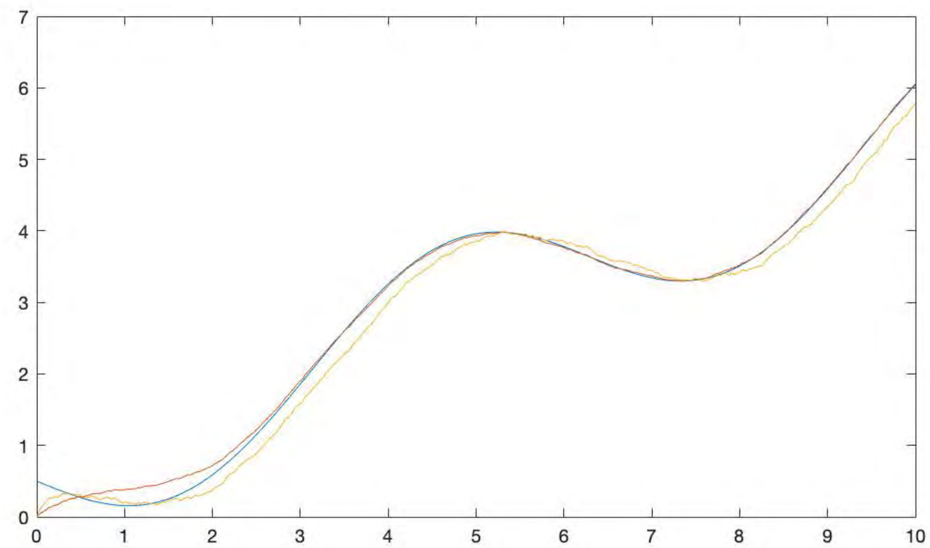
2
1.5
1
0.5
0
-0.5
0 1 2 3 4 5 6 7 8 9 10



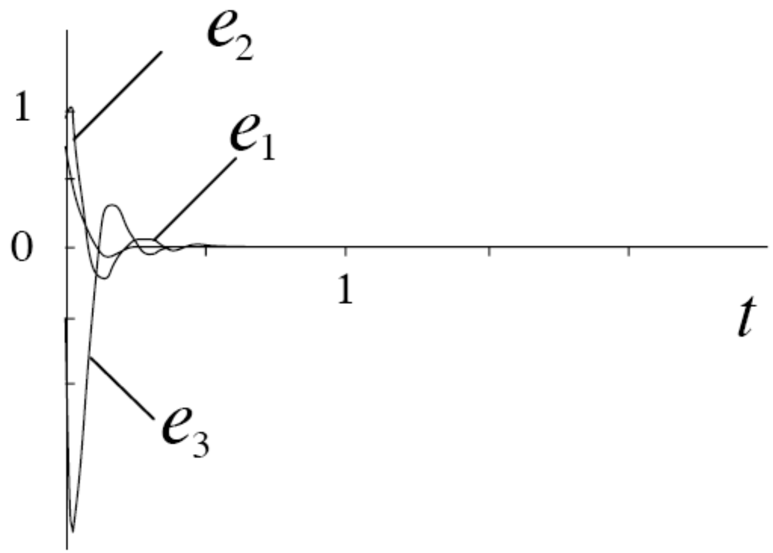
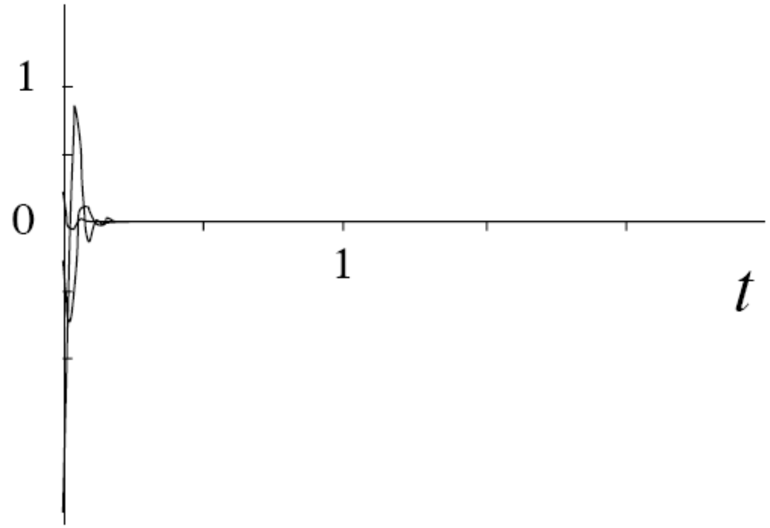
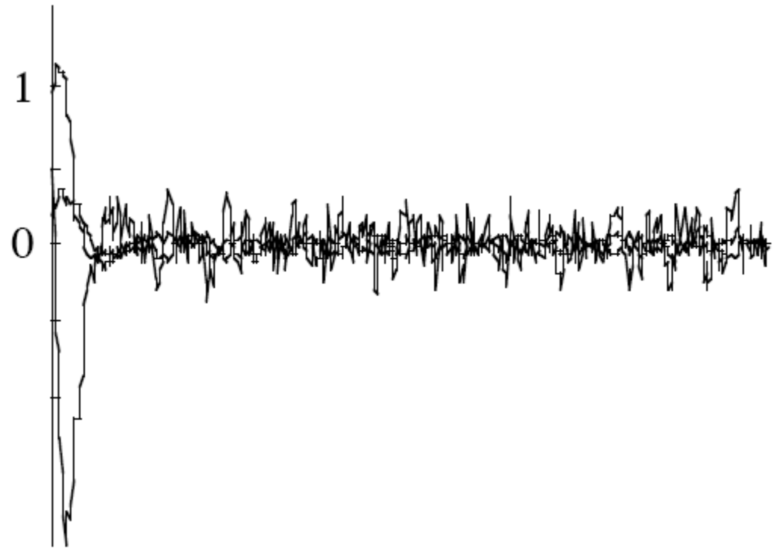
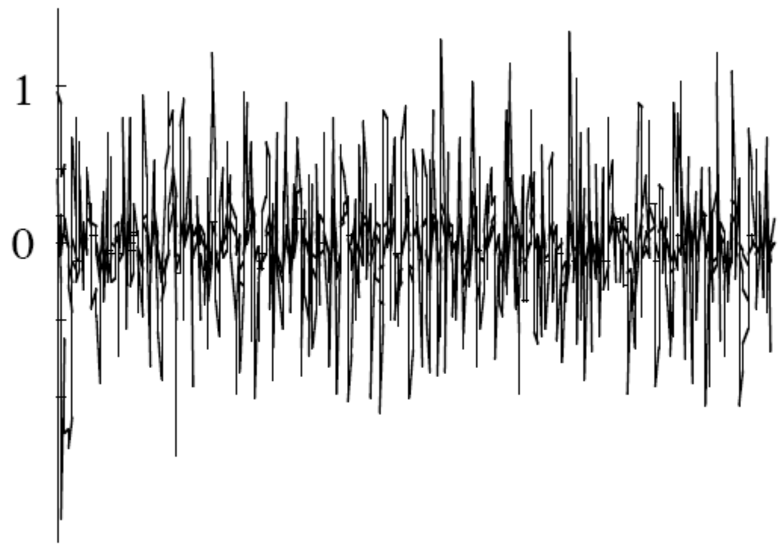
7
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8 9 10



2
1.5
1
0.5
0
-0.5
-1
0 1 2 3 4 5 6 7 8 9 10



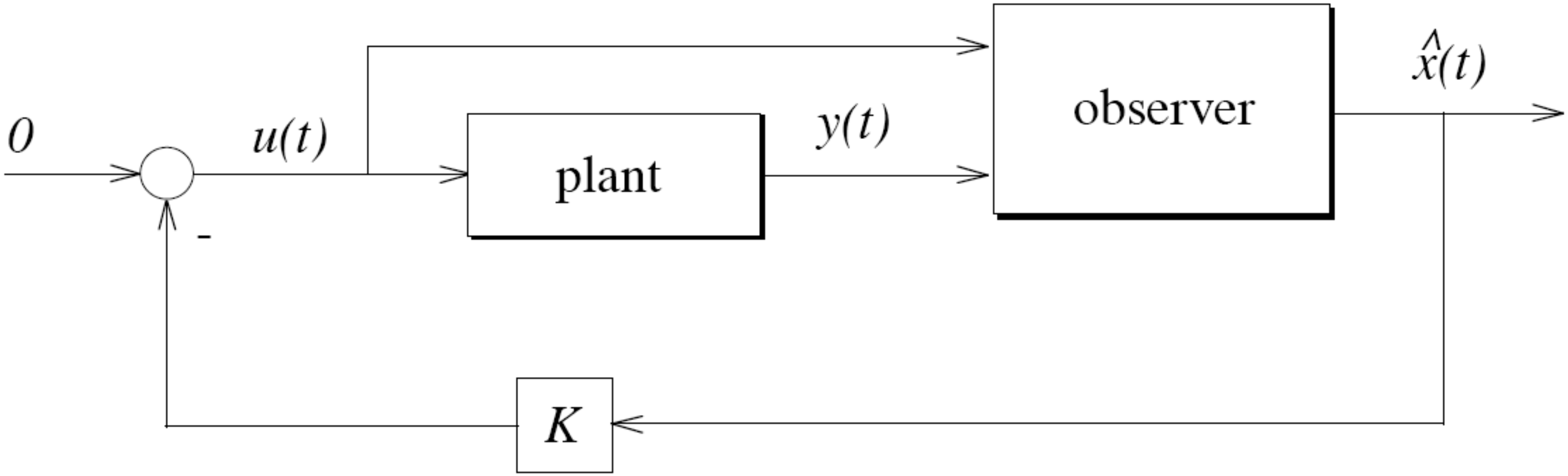
7
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8 9 10

<table>
<tr><th></th><th>“slow” observer</th><th>“fast” observer</th></tr>
<tr><td>no noise</td><td>
</td><td>
</td></tr>
<tr><td>some noise</td><td>
</td><td>
</td></tr>
</table>

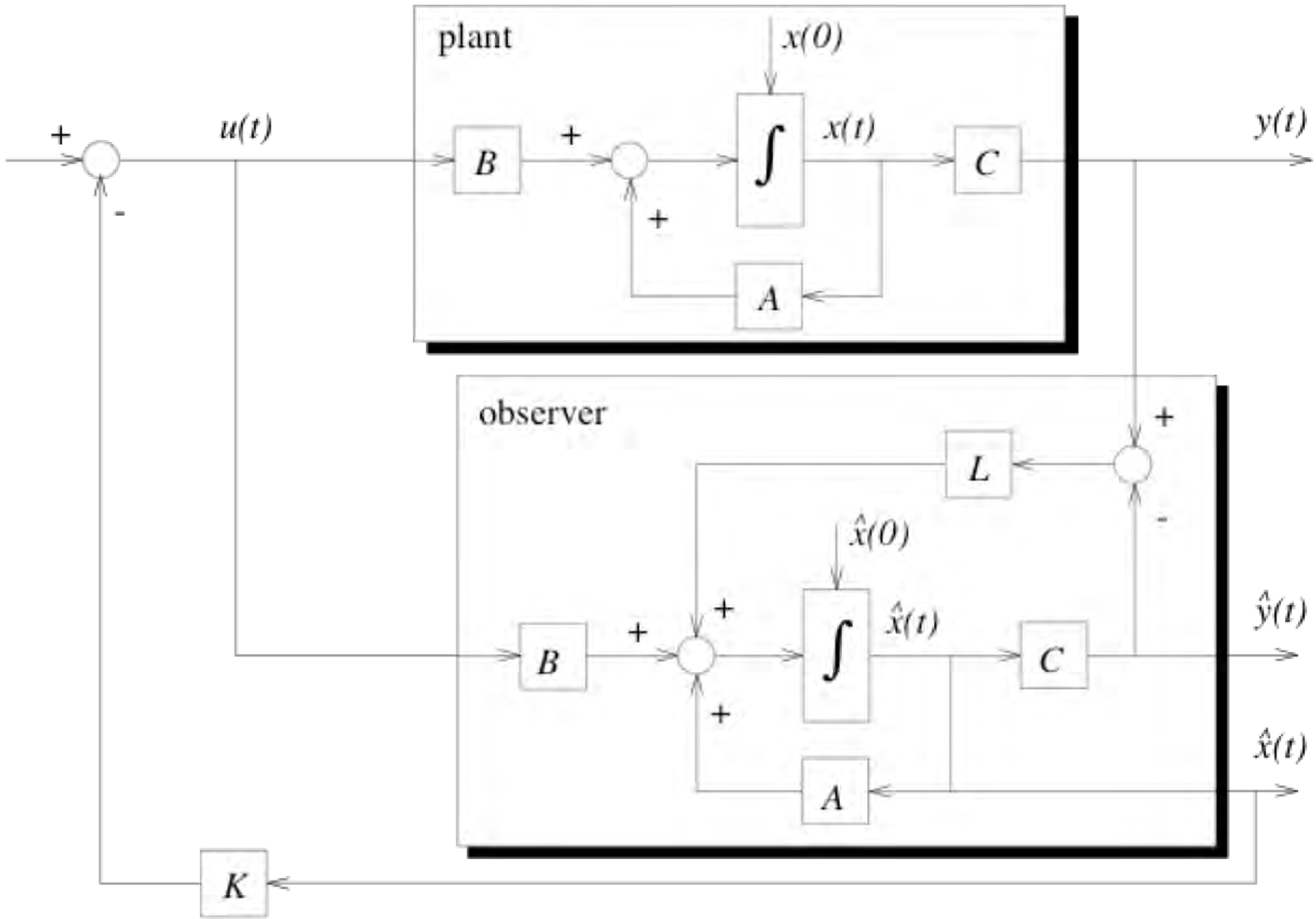
Lecture X – Observer-Based Output Feedback Control

Output Feedback Controllers

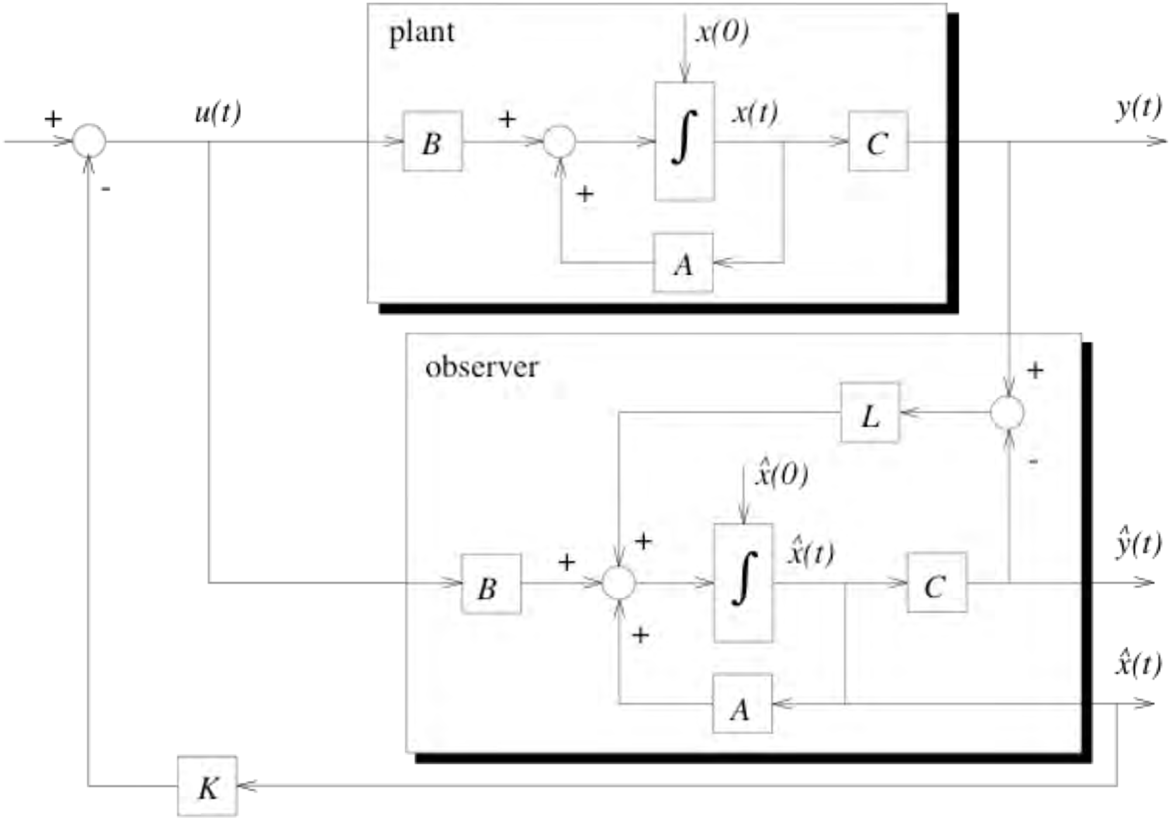
$$u(t) = -K \cdot \hat{x}(t)$$

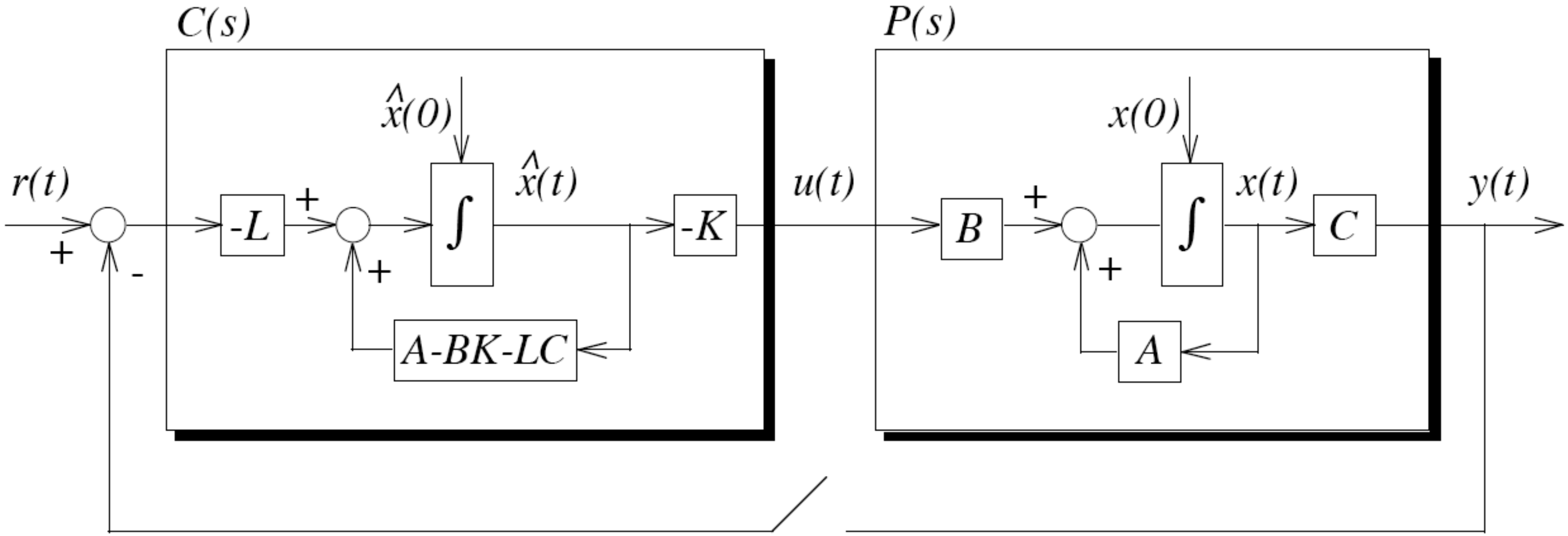


LQG(aussian): observer gain computed using dual LQR approach



plant
x(0)
+
u(t)
B
+
∫
x(t)
C
y(t)
-
+
A
observer
+
L
-
x̂(0)
+
+
B
∫
x̂(t)
C
ŷ(t)
+
x̂(t)
A
K


plant
x(0)
u(t)
x(t)
y(t)
B
C
A
observer
L
x̂(0)
x̂(t)
ŷ(t)
B
C
A
x̂(t)
K


C(s)
P(s)
r(t)
x̂(0)
x̂(t)
-L
-K
A-BK-LC
u(t)
x(0)
x(t)
y(t)
B
C
A

LQG = Output-Feedback, i.e., realizable!

State vector

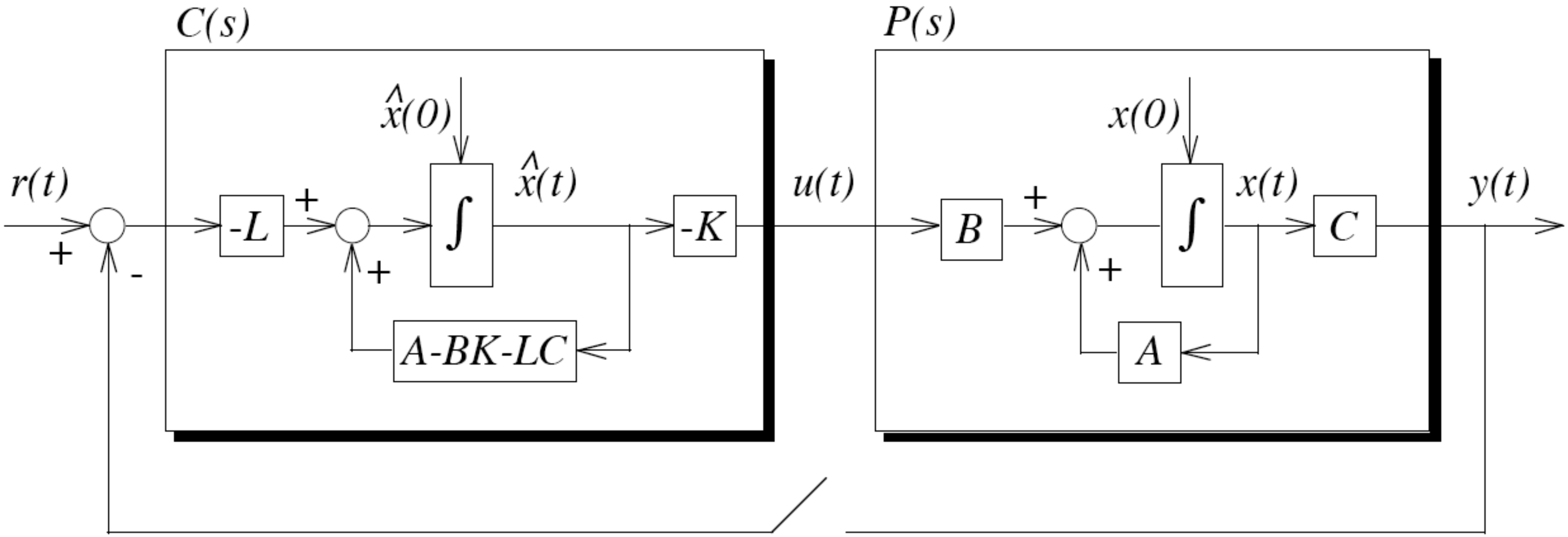
$$\widetilde{x}(t) = \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

System description:

a) open loop $\Rightarrow$ robustness

b) closed loop $\Rightarrow$ stability

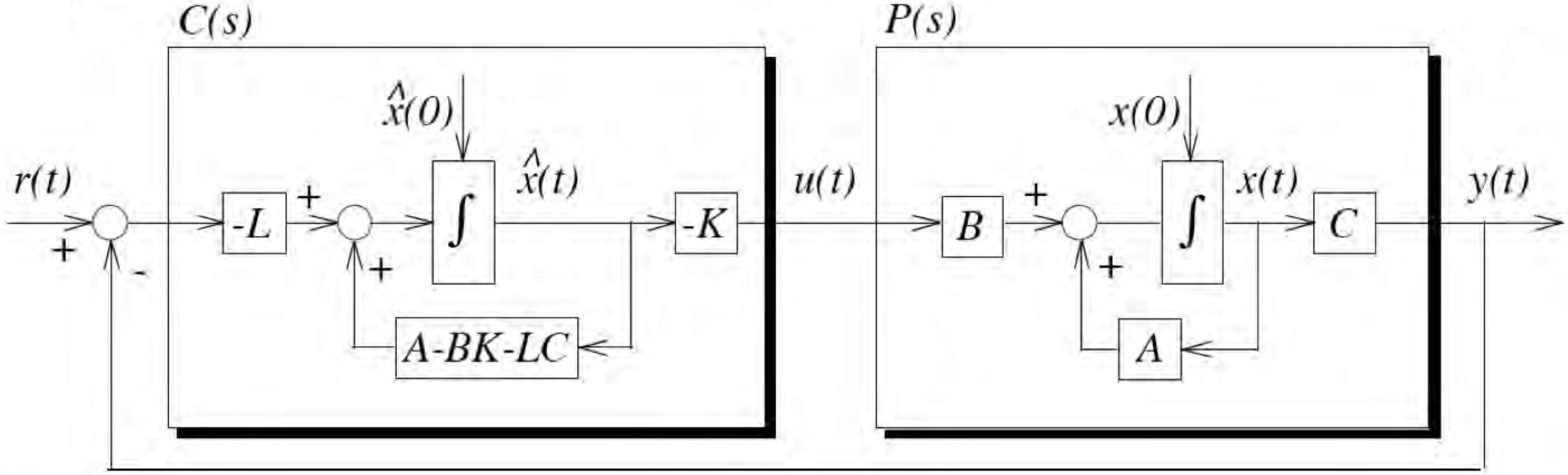
$$\frac{d}{dt}\widetilde{x}(t) = \widetilde{A}_{ol} \cdot \widetilde{x}(t) + \widetilde{B} \cdot r(t), \quad y(t) = \widetilde{C} \cdot \widetilde{x}(t)$$



$$\widetilde{A}_{ol} = \begin{bmatrix} A & -B \cdot K \\ 0 & A - B \cdot K - L \cdot C \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} 0 \\ -L \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} C & 0 \end{bmatrix}$$

$$L_{LQG}(s) = C \cdot [s\,I - A]^{-1} \cdot B \cdot K \cdot [s\,I - (A - B \cdot K - L \cdot C)]^{-1} \cdot L$$

$$\frac{d}{dt}\widetilde{x}(t) = \widetilde{A}_{cl} \cdot \widetilde{x}(t) + \widetilde{B} \cdot r(t), \quad y(t) = \widetilde{C} \cdot \widetilde{x}(t)$$



$$\widetilde{A}_{cl} = \begin{bmatrix} A & -B \cdot K \\ L \cdot C & A - B \cdot K - L \cdot C \end{bmatrix}$$

In particular, two points must be verified:

1. the stability of the closed-loop system (3.58); and
2. the robustness of the open-loop system (3.56).

Ad. 1.: Analyze $\widetilde{A}_{\text{cl}}$, i.e., is this matrix Hurwitz
(all of its Eigenvalues negative real part)?
Result know as "Separation Principle", today.

Ad. 2: Analyze return difference $I+L_{LQG}(j\omega)$

The Separation Principle

Assume $A - B \cdot K$ and $A - L \cdot C$ are both Hurwitz matrices. Then the output-feedback closed-loop system is asymptotically stable and n of its eigenvalues coincide with the eigenvalues of $A - B \cdot K$ and its other n eigenvalues coincide with the eigenvalues of $A - L \cdot C$.

Proof: The closed-loop system can be compactly described using the $2\,n$-dimensional vector

$$x_{cl}(t) = \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

using the ODE

$$\frac{d}{dt} x_{cl}(t) = A_{cl} \cdot x_c(t) + B_{cl} \cdot r(t), \quad y(t) = C_{cl} \cdot x_{cl}(t)$$

where the matrix A_{cl}, which fully determines the stability properties, is

$$\widetilde{A}_{\mathrm{cl}} = \begin{bmatrix} A & -B \cdot K \\ L \cdot C & A - B \cdot K - L \cdot C \end{bmatrix}$$

Using the coordinate transformation $x_{cl} = T \cdot z_{cl}$ with

$$T = \begin{bmatrix} I_{n\times n} & 0_{n\times n} \\ I_{n\times n} & -I_{n\times n} \end{bmatrix} = T^{-1}$$

yields the following transformed system equation

$$\frac{d}{dt} z_{cl}(t) = T^{-1} \cdot A_{cl} \cdot T \cdot z_{cl}(t) + T^{-1} \cdot B_{cl} \cdot r(t), \quad y(t) = C_{cl} \cdot T \cdot z_{cl}(t) \quad (139)$$

with

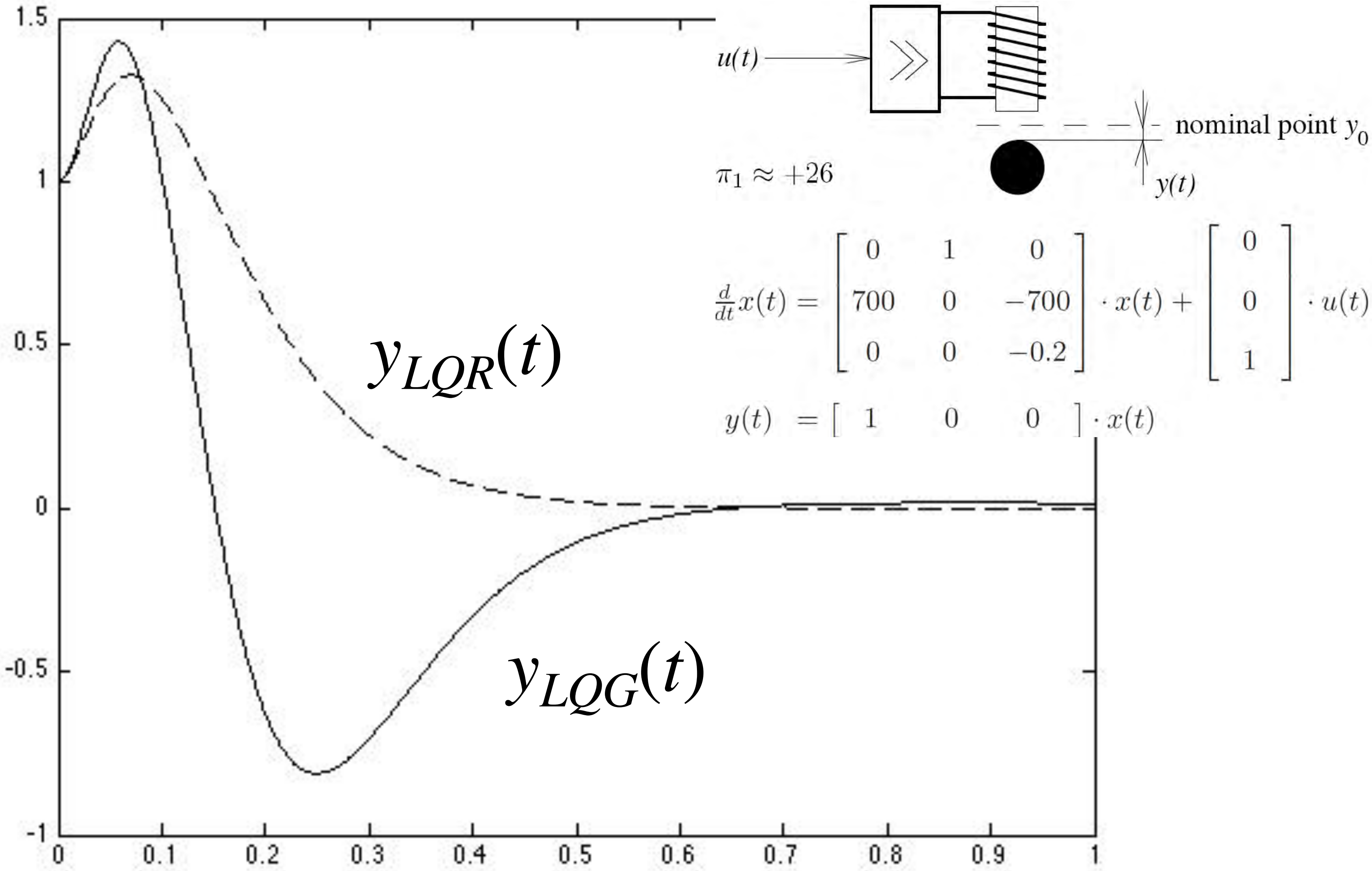
$$T^{-1} \cdot \widetilde{A}_{\mathrm{cl}} \cdot T = \begin{bmatrix} A - B \cdot K & B \cdot K \\ 0_{n\times n} & A - L \cdot C \end{bmatrix} \quad (140)$$

The Separation Principle immediately follows from the fact that the eigenvalues of block-triangular matrices are equal to the eigenvalues of the triangular blocks and from the fact that the eigenvalues are invariant with respect to coordinate ("similarity") transformations. $\mathbf{\Omega}$

Levitating Sphere – TD Behavior

$$x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



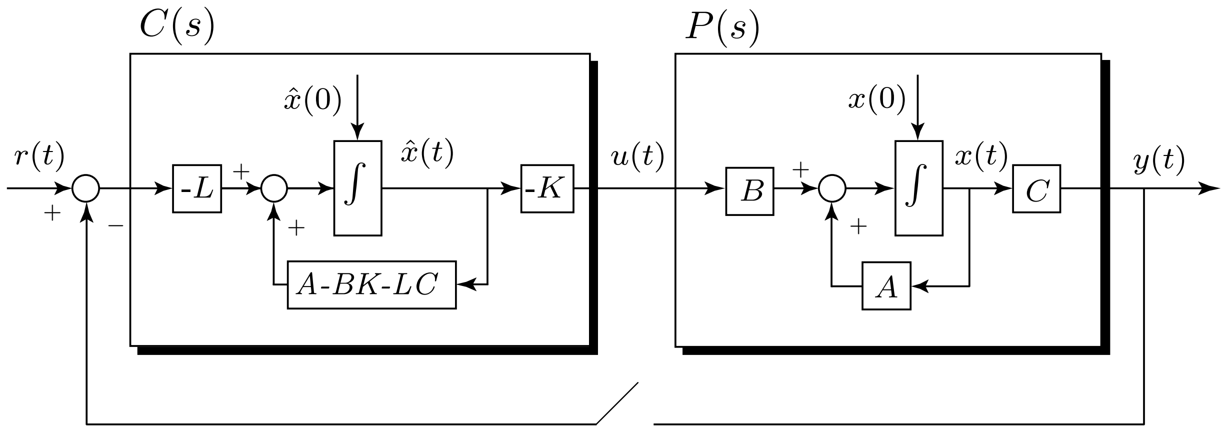
$\pi_1 \approx +26$

$$\frac{d}{dt}x(t) = \begin{bmatrix} 0 & 1 & 0 \\ 700 & 0 & -700 \\ 0 & 0 & -0.2 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot u(t)$$

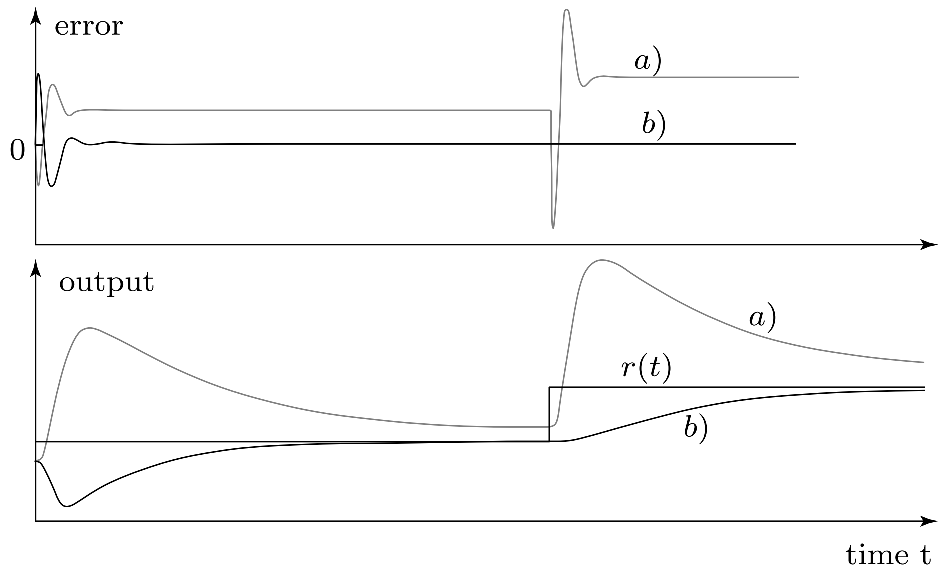
$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \cdot x(t)$$

Observer-Based Output Feedback for Reference Tracking

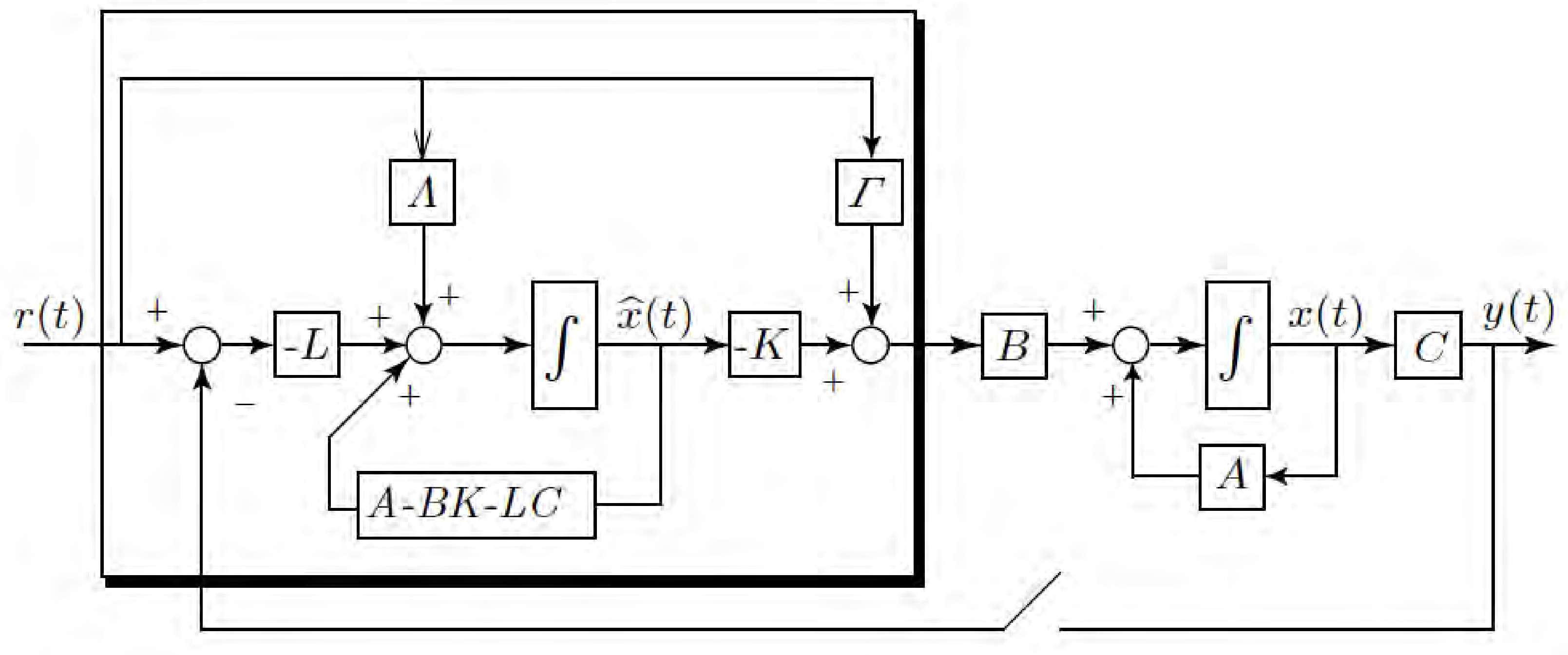
Applying a reference step to



yields curve a)



Unnecessary excitation of error by reference signal may be avoided (curve b) by an appropriate feedforward action ("2-dof controller")



Choose gains $\Lambda \in \Re^{m \times m}$ and $\Gamma \in \Re^{m \times m}$ such that:

- the error dynamics do not depend on $r(t)$,
- the static gain $r \rightarrow y$ is equal to I.

Of course more general feedforward systems can be formulated (specifications for all frequencies, not just for $\omega = 0$)!

Solution

$$\Lambda = L + B \cdot \Gamma, \qquad \Gamma = -\left(\widetilde{C} \cdot \widetilde{A}_{\mathrm{cl}}^{-1} \cdot \widetilde{B}_r\right)^{-1}$$

where the new matrices define the closed-loop system shown in the last figure, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}(t) = \widetilde{A}_{\mathrm{cl}} \cdot \widetilde{x}(t) + \widetilde{B}_r \cdot \Gamma \cdot r(t), \quad y(t) = \widetilde{C} \cdot \widetilde{x}(t)$$

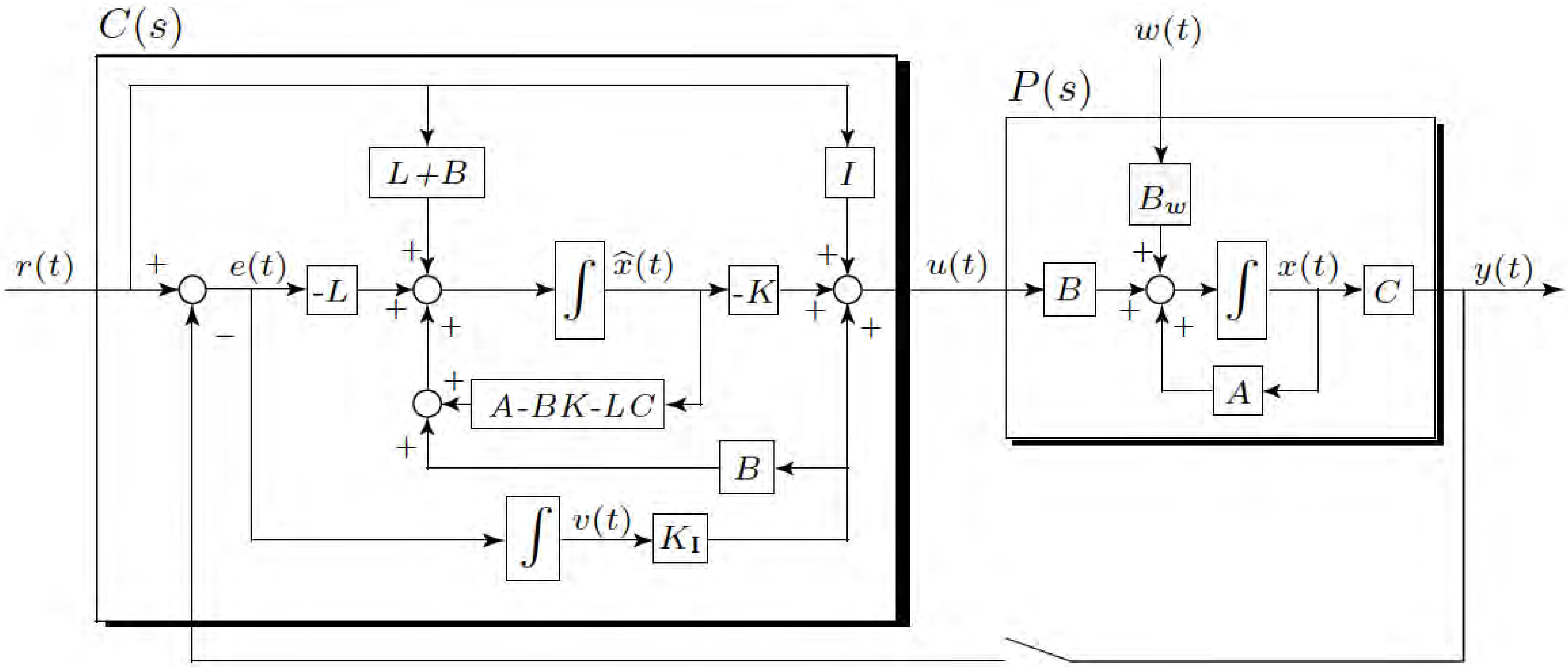
and

$$\widetilde{A}_{\mathrm{cl}} = \begin{bmatrix} A & -B\,K \\ L\,C & A - B\,K - L\,C \end{bmatrix}, \quad \widetilde{B}_r = \begin{bmatrix} B \\ B \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} C & 0 \end{bmatrix}$$

Notice: Without loss of generality one may assume that $\det \widetilde{A} \neq 0$. If the

ranks of B and C are full (which is always the case in well-posed problems) then it is also true that $\det\{\widetilde{C}\,\widetilde{A}^{-1}\widetilde{B}\} \neq 0$.

Output Feedback with Integral Action and Feedforward Part



State-space description open-loop gain $e \ \rightarrow \ y$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x(t) \\ \widehat{x}(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} A & -B\,K & B\,K_{\mathrm{I}} \\ 0 & A - B\,K - L\,C & B\,K_{\mathrm{I}} \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ \widehat{x}(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -L \\ I \end{bmatrix} \cdot e(t)$$

$$y(t) = \begin{bmatrix} C & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ \widehat{x}(t) \\ v(t) \end{bmatrix}$$

State-space description closed-loop $[r, d]^T \rightarrow y$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x(t) \\ \widehat{x}(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} A & -B\,K & B\,K_{\mathrm{I}} \\ L\,C & A - B\,K - L\,C & B\,K_{\mathrm{I}} \\ -C & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ \widehat{x}(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} B & B_w \\ B & 0 \\ I & 0 \end{bmatrix} \cdot \begin{bmatrix} r(t) \\ w(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} C & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ \widehat{x}(t) \\ v(t) \end{bmatrix}$$

Design procedure:

1. Form the extended system $\{\widetilde{A},\, \widetilde{B},\, \widetilde{C}\}$ following the approach outlined in the LQRI section.

2. Design a suitable state-feedback control gain $\widetilde{K}$ as a solution of the LQR problem defined by

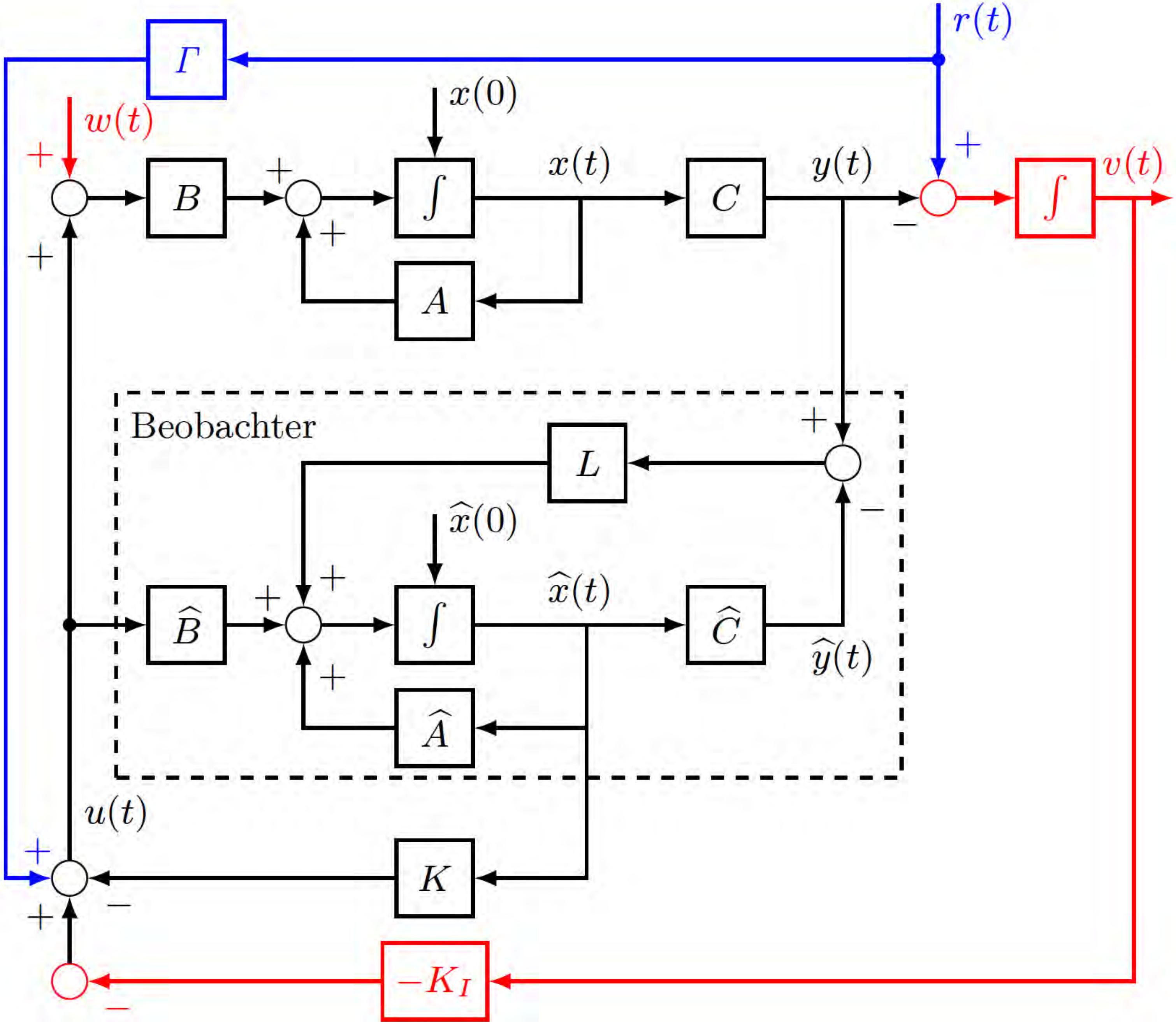
$$\widetilde{K} = [K, -K_I] = \mathrm{lqr}(\widetilde{A}, \widetilde{B}, \widetilde{C}^T \widetilde{C}, \rho\, I)$$

3. Design a observer gain L for the standard system $\{A,\, B,\, C\}$ using the duality approach

$$L = \mathrm{lqr}(A^T,\, C^T,\, B\, B^T, \mu\, I)^T$$

   Note: the state v is known and needs not to be included into the observer.

4. The feedforward part is straightforward because the closed-loop feedback system is known to have a DC gain of 1.


r(t)
Γ
w(t)
x(0)
x(t)
y(t)
v(t)
B
∫
C
A
Beobachter
L
x̂(0)
x̂(t)
B̂
Ĉ
ŷ(t)
Â
u(t)
K
−K_I