Slides “Regelungstechnik II”

Part 2: Introduction to MIMO Methods

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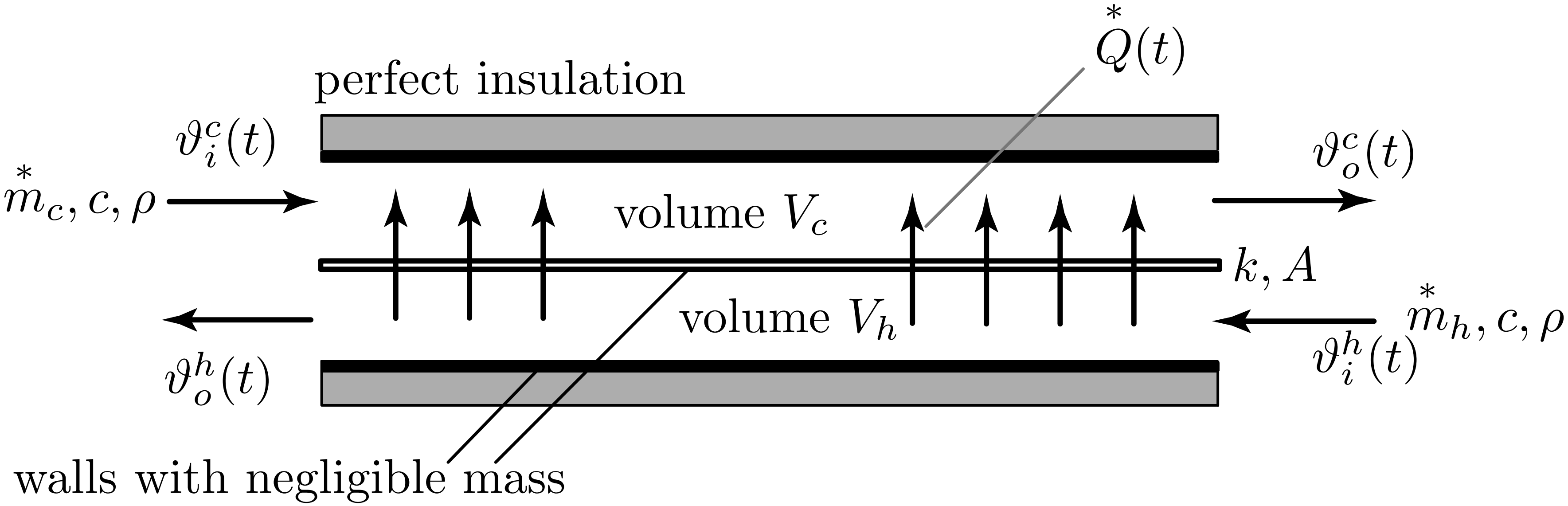
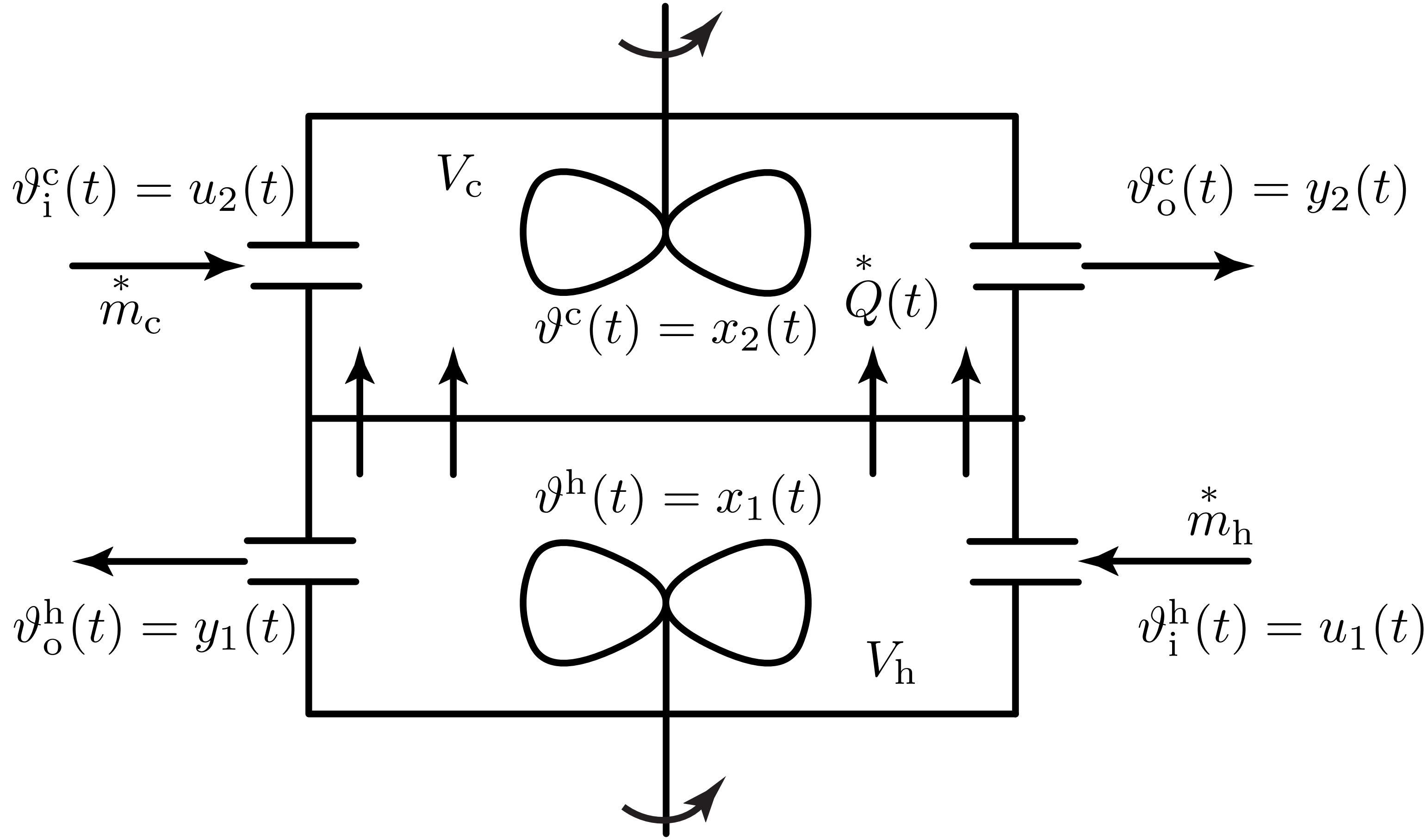
Lecture V – Differences between SISO and MIMO Systems

Introduction

Counter-flow heat exchanger (one strand)



Heat exchanger simplified (warning: not sufficiently accurate for most applications!)



Note: use several such elements in series connection to increase model prediction quality.

Causality:

1. state variables: $x_1 = \vartheta^h$ = temperature of hot fluid,
   $x_2 = \vartheta^c$ = temperature of cold fluid

2. inputs: $u_1 = \vartheta_i^h$ = temperature of hot fluid entering,
   $u_2 = \vartheta_i^c$ = temperature of cold fluid entering

3. outputs: $y_1 = \vartheta_o^h$ = temperature of hot fluid exiting,
   $y_2 = \vartheta_o^c$ = temperature of cold fluid exiting

Simplifications and assumptions:

1. no mass storage (incompressible fluid, say water)

2. perfect insulation

3. pipe walls are very thin and store no heat

4. perfect mixing inside the heat exchanger $\vartheta_o^{h/c} = \vartheta^{h/c}$

5. constant specific heat c, heat transfer coefficient k, and fluid mass flow $\overset{*}{m}$

Energy conservation:

$$\frac{d}{dt}U^h(t) = \overset{*}{H}{}_i^h(t) - \overset{*}{H}{}_o^h(t) - \overset{*}{Q}(t) \tag{1}$$

$$\frac{d}{dt}U^c(t) = \overset{*}{H}{}_i^c(t) - \overset{*}{H}{}_o^c(t) + \overset{*}{Q}(t) \tag{2}$$

Thermodynamics:

$$U^{h/c}(t) = \rho \cdot V \cdot c \cdot \vartheta_o^{h/c}(t) \tag{3}$$

$$\overset{*}{H}{}_{i/o}^{h/c}(t) = \overset{*}{m} \cdot c \cdot \vartheta_{i/o}^{h/c} \tag{4}$$

$$\overset{*}{Q}(t) = k \cdot A \cdot \left(\vartheta^h - \vartheta^c\right) \tag{5}$$

Control-oriented formulation

$$\rho \cdot V \cdot c \cdot \frac{d}{dt} x_1(t) = \overset{*}{m} \cdot c \cdot (u_1(t) - x_1(t)) - k \cdot A \cdot (x_1(t) - x_2(t)) \tag{6}$$

$$\rho \cdot V \cdot c \cdot \frac{d}{dt} x_2(t) = \overset{*}{m} \cdot c \cdot (u_2(t) - x_2(t)) + k \cdot A \cdot (x_1(t) - x_2(t)) \tag{7}$$

or

$$\tau \cdot \frac{d}{dt} x_1(t) = -x_1(t) + \sigma \cdot x_2(t) + \beta \cdot u_1(t) \tag{8}$$

$$\tau \cdot \frac{d}{dt} x_2(t) = -x_2(t) + \sigma \cdot x_1(t) + \beta \cdot u_2(t) \tag{9}$$

with

$$\tau = \frac{\rho \cdot V \cdot c}{\overset{*}{m} \cdot c + k \cdot A}, \quad \sigma = \frac{k \cdot A}{\overset{*}{m} \cdot c + k \cdot A}, \quad \beta = \frac{\overset{*}{m} \cdot c}{\overset{*}{m} \cdot c + k \cdot A} \tag{10}$$

Note: since all physical parameters are greater than 0, the control-oriented parameters satisfy the inequalities $\tau > 0$, $1 > \sigma > 0$, and $1 > \beta > 0$

State-space form

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t) \tag{11}$$

$$y(t) = C \cdot x(t) + D \cdot u(t) \tag{12}$$

where

$$A = \begin{bmatrix} -1/\tau & \sigma/\tau \\ \sigma/\tau & -1/\tau \end{bmatrix}, \quad B = \begin{bmatrix} \beta/\tau & 0 \\ 0 & \beta/\tau \end{bmatrix} \tag{13}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{14}$$

Number of state variables $n = 2$

"Square system," i.e., number of inputs $m = 2$ equal to number of measurements $p = 2$

Obviously, system completely controllable and observable ...

System stable?

Compute eigenvalues of A, i.e., roots of

$$\det(s \cdot I - A) = 0 \tag{15}$$

where

$$\det(s \cdot I - A) = (s + 1/\tau)^2 - (\sigma/\tau)^2 \tag{16}$$

Eigenvalues

$$\lambda_{1,2} = \frac{-2 \cdot \tau \pm \sqrt{4 \cdot \tau^2 - 4 \cdot \tau^2 \cdot (1 - \sigma^2)}}{2 \cdot \tau^2} = \frac{-1 \pm \sigma}{\tau} \tag{17}$$

Recall: $1 > \sigma > 0$ and $\tau > 0$.

Therefore, for all physically meaningful parameter values, the system is asymptotically stable and has two real ("non-oscillatory") eigenvalues.

Transfer function:

$$P(s) = C \cdot [s \cdot I - A]^{-1} \cdot B + D \tag{18}$$

Insert $\{A, B, C, D\}$

$$\Rightarrow P(s) = \begin{bmatrix} s + 1/\tau & -\sigma/\tau \\ \\ -\sigma/\tau & s + 1/\tau \end{bmatrix}^{-1} \cdot \frac{\beta}{\tau} \tag{19}$$

Using Cramer's rule

$$M^{-1} = \frac{1}{\det(M)} \cdot \mathrm{Adj}\,\{M\} \tag{20}$$

$$\Rightarrow P(s) = \frac{\tau^2}{\tau^2 \cdot s^2 + 2 \cdot \tau \cdot s + (1 - \sigma^2)} \cdot \begin{bmatrix} s + 1/\tau & \sigma/\tau \\ \\ \sigma/\tau & s + 1/\tau \end{bmatrix} \cdot \frac{\beta}{\tau} \tag{21}$$

Transfer function (contd.)

$$P(s) = \begin{bmatrix} \frac{\beta\cdot(\tau\cdot s+1)}{\tau^2\cdot s^2+2\cdot\tau\cdot s+(1-\sigma^2)} & \frac{\beta\cdot\sigma}{\tau^2\cdot s^2+2\cdot\tau\cdot s+(1-\sigma^2)} \\ \\ \frac{\beta\cdot\sigma}{\tau^2\cdot s^2+2\cdot\tau\cdot s+(1-\sigma^2)} & \frac{\beta\cdot(\tau\cdot s+1)}{\tau^2\cdot s^2+2\cdot\tau\cdot s+(1-\sigma^2)} \end{bmatrix} \tag{22}$$

Looking at the four SISO transfer functions it seems that

1. the system has 8 poles, but we know it has only two eigenvalues;
2. the system has 2 minimumphase zeros, but are these zeros really active?

Remark: using the definition of the zeros as the solution of the equation

$$\det \begin{bmatrix} (\zeta \cdot I - A) & -B \\ \\ C & D \end{bmatrix} = 0 \tag{23}$$

it is easy to see that there are no finite zeros (the determinant is constant and equal to β^2/τ^2).

Transmission zeros

$$\begin{aligned} (sI - A) \cdot x - B \cdot u &= 0 \\ C \cdot x + D \cdot u &= 0 \end{aligned} \tag{46}$$

has a nontrivial solution $[x_i, u_i]$ iff the matrix

$$\begin{bmatrix} (sI - A) & -B \\ C & D \end{bmatrix} \tag{47}$$

is singular. The values $s = \zeta_i$ for which this is true are exactly the transmission zeros of (34).

Numerical values (just reasonable examples)

$$\overset{*}{m} = 0.5\,kg/s,\;\; \rho = 1000\,kg/m^3,\;\; c = 4200\,J/(kg\,K) \tag{24}$$

$$A = 2\,m^2,\;\; V = 0.01\,m^3,\;\; k = 100\,W/(m^2\,K) \tag{25}$$

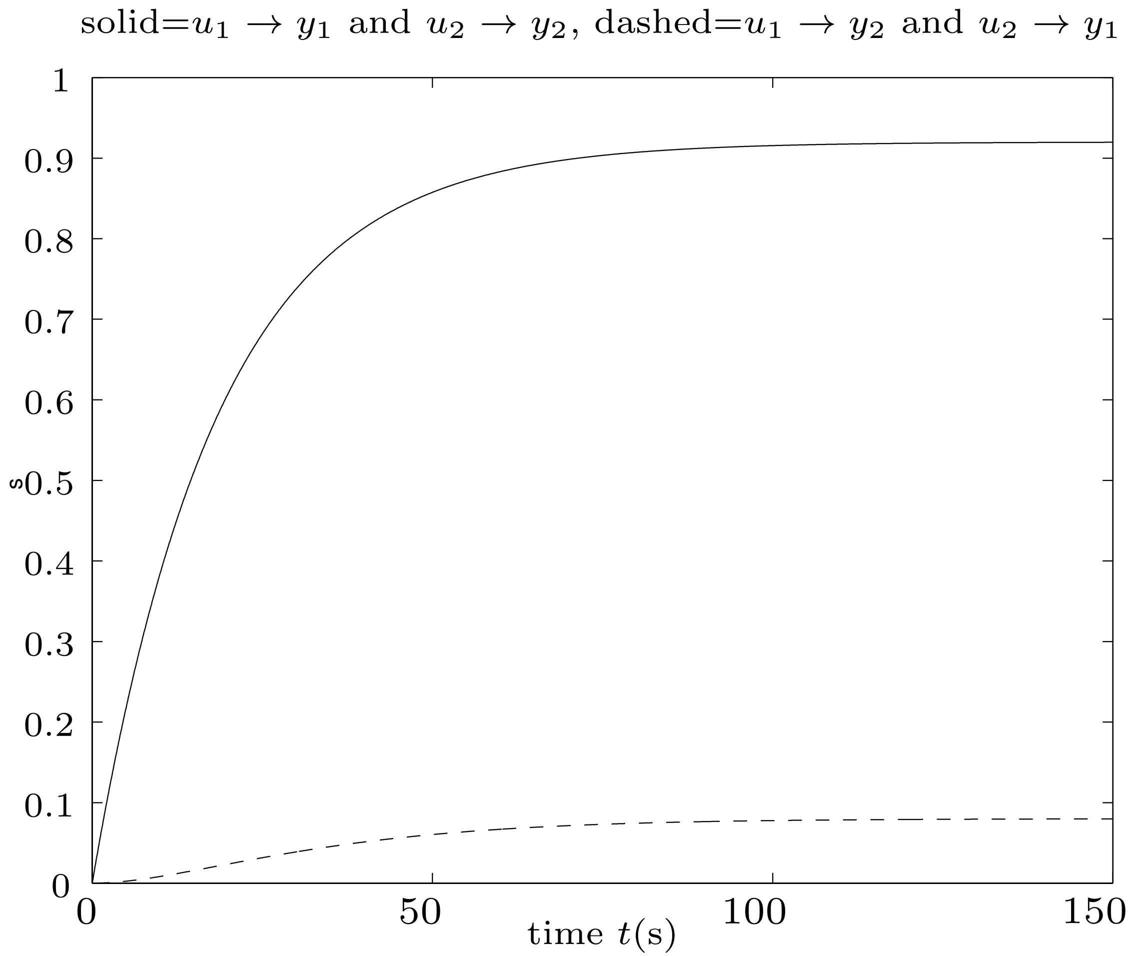
Therefore

$$\tau \approx 18.26\,s, \quad \sigma \approx 0.087, \quad \beta \approx 0.913 \tag{26}$$

and eigenvalues

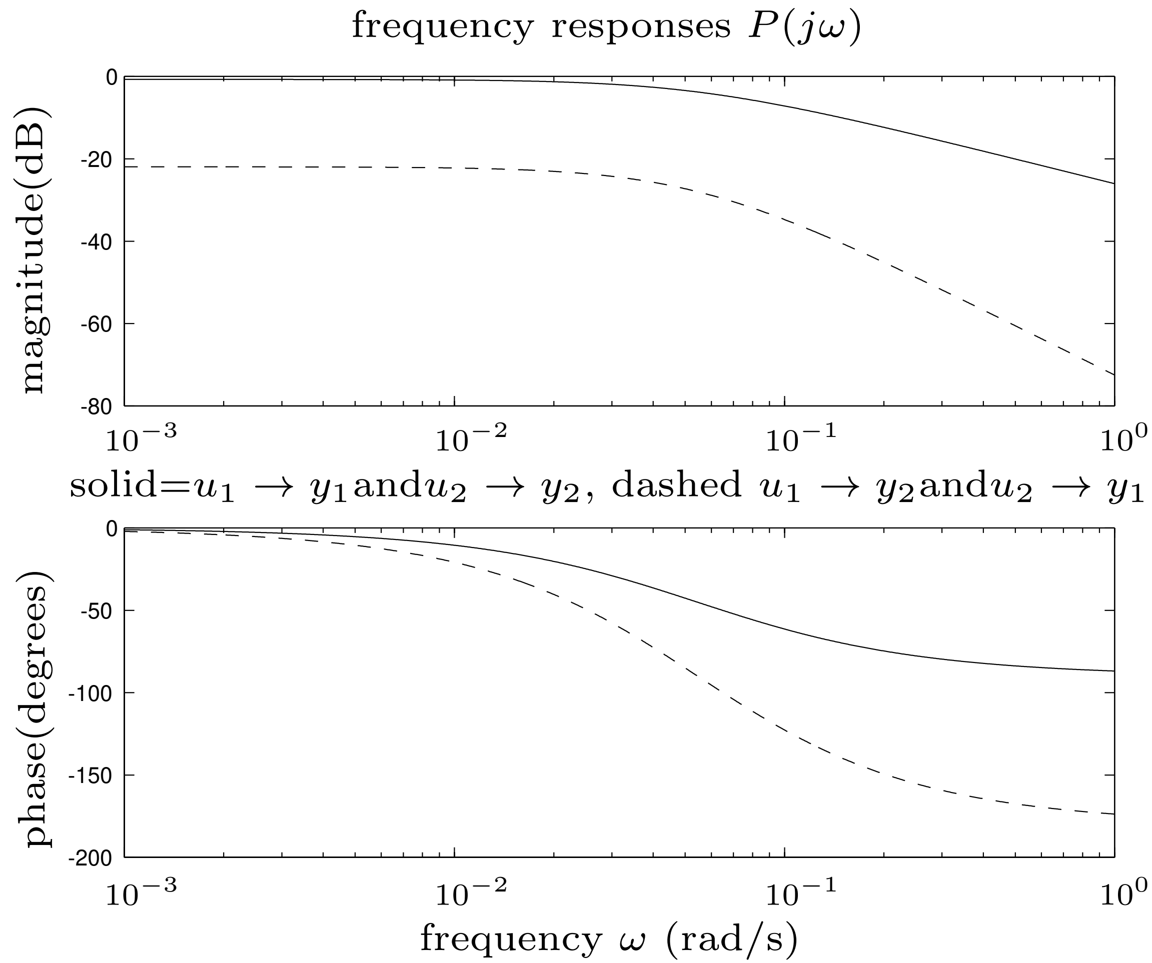
$$\lambda_1 = -0.05\,s^{-1}, \lambda_2 = -0.0595\,s^{-1} \tag{27}$$

Step responses plant



Physical interpretation?

Bode diagrams plant

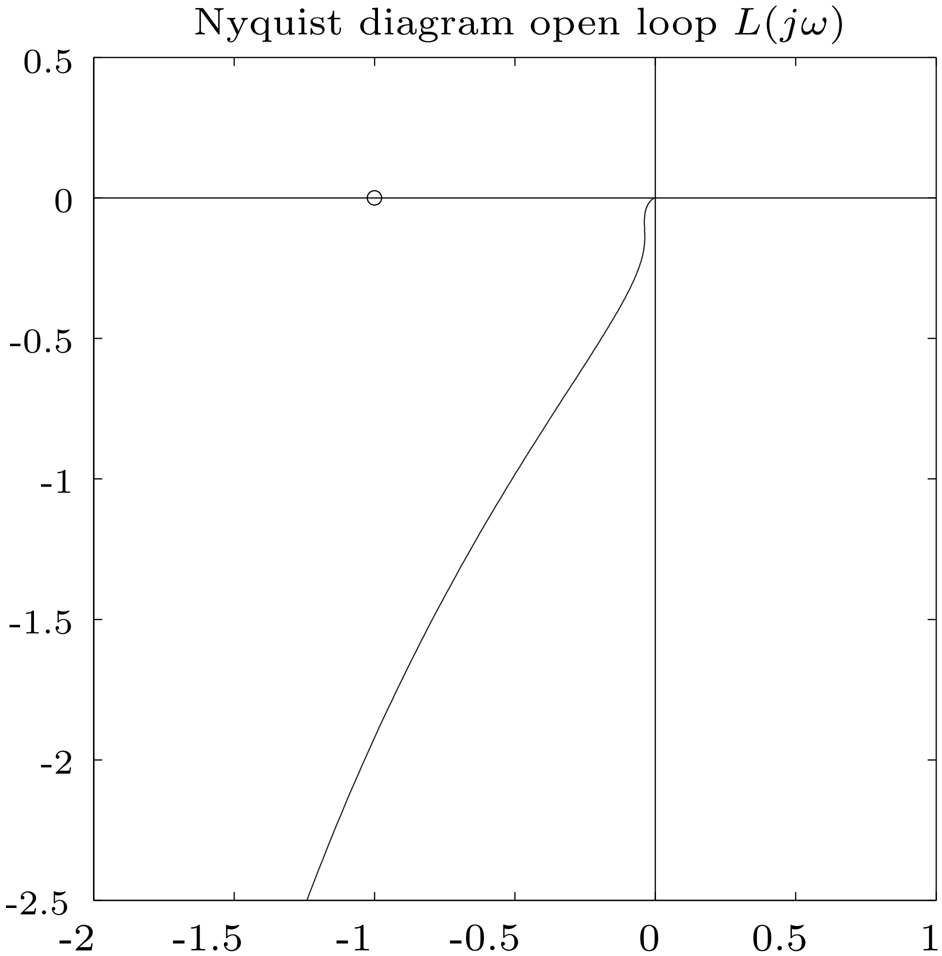


Physical interpretation? Coupling?

Controller design, chosen structure

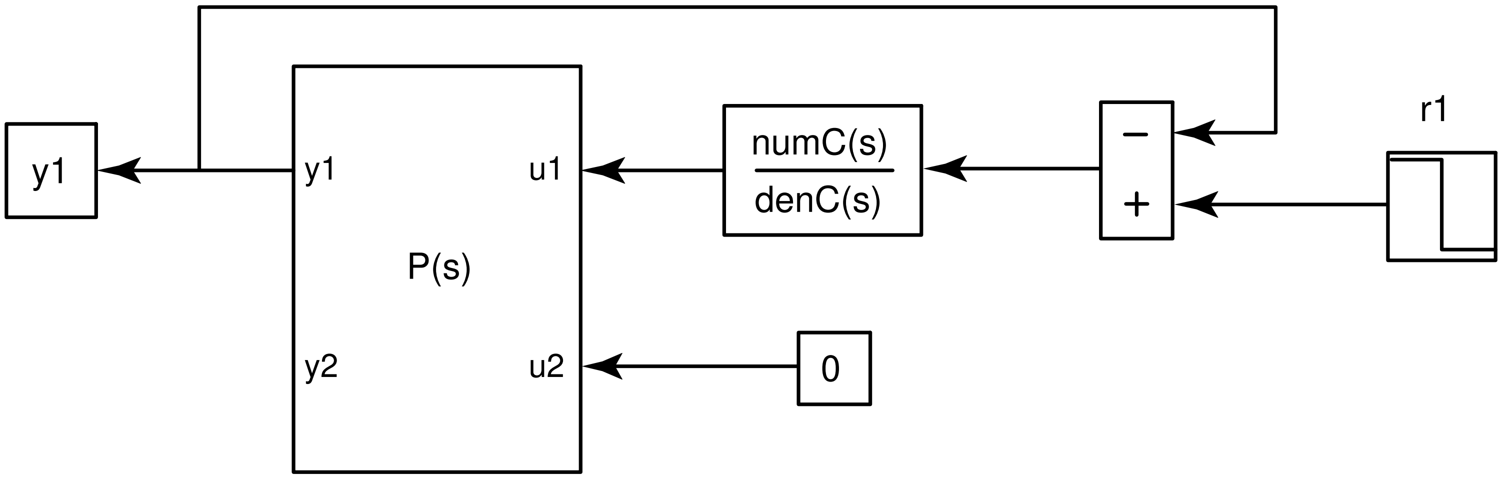
$$C(s) = k_p \cdot (1 + \frac{1}{T_i \cdot s}) \cdot \frac{1}{\tau_{ro} \cdot s + 1} \tag{28}$$

Parameters $k_p = 3$, $T_i = 7\,s$, and $\tau_{ro} = 0.7\,s$ yield following SISO loop gain

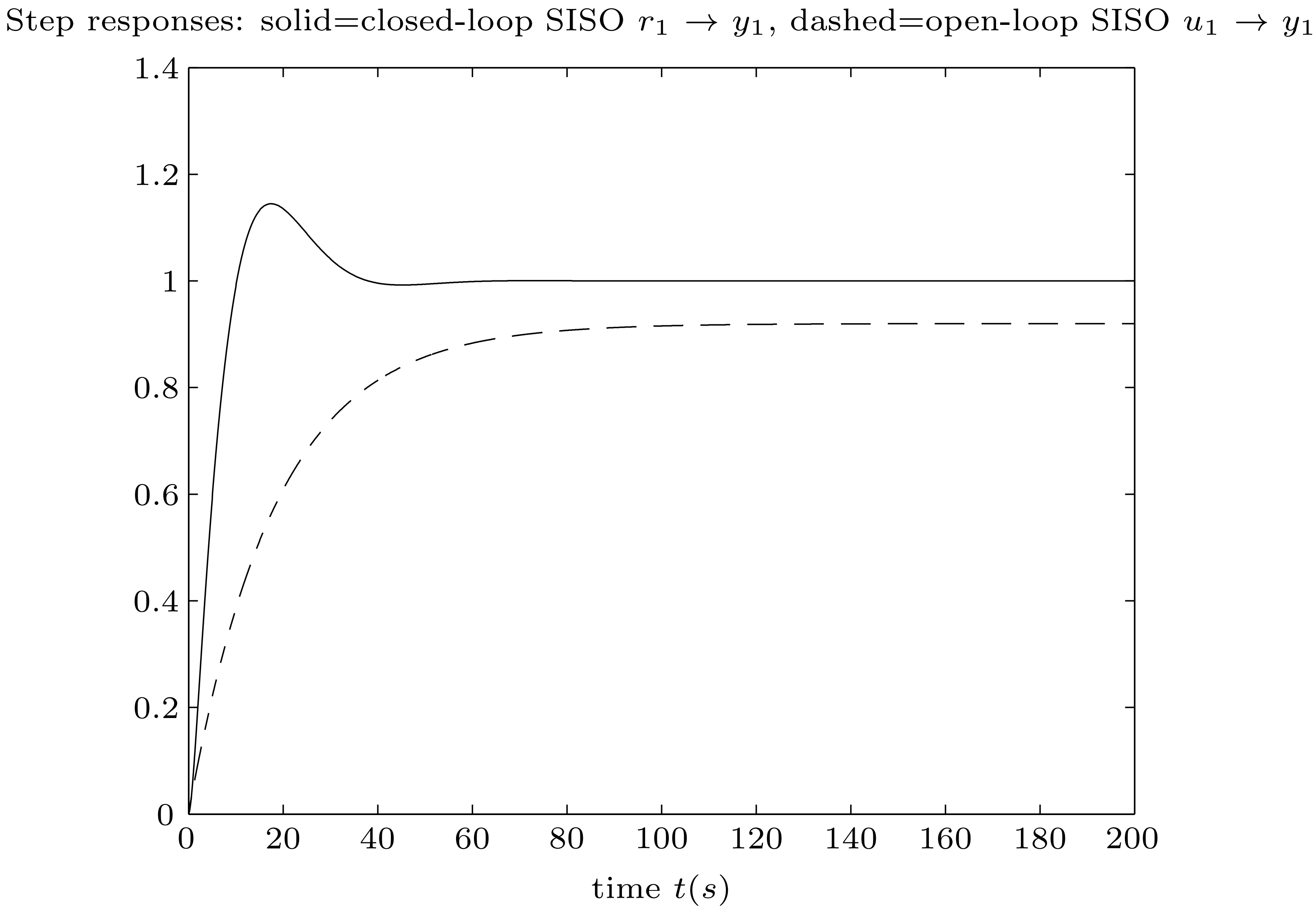


SISO *closed-loop* time-domain behavior (one channel only)

Setup:



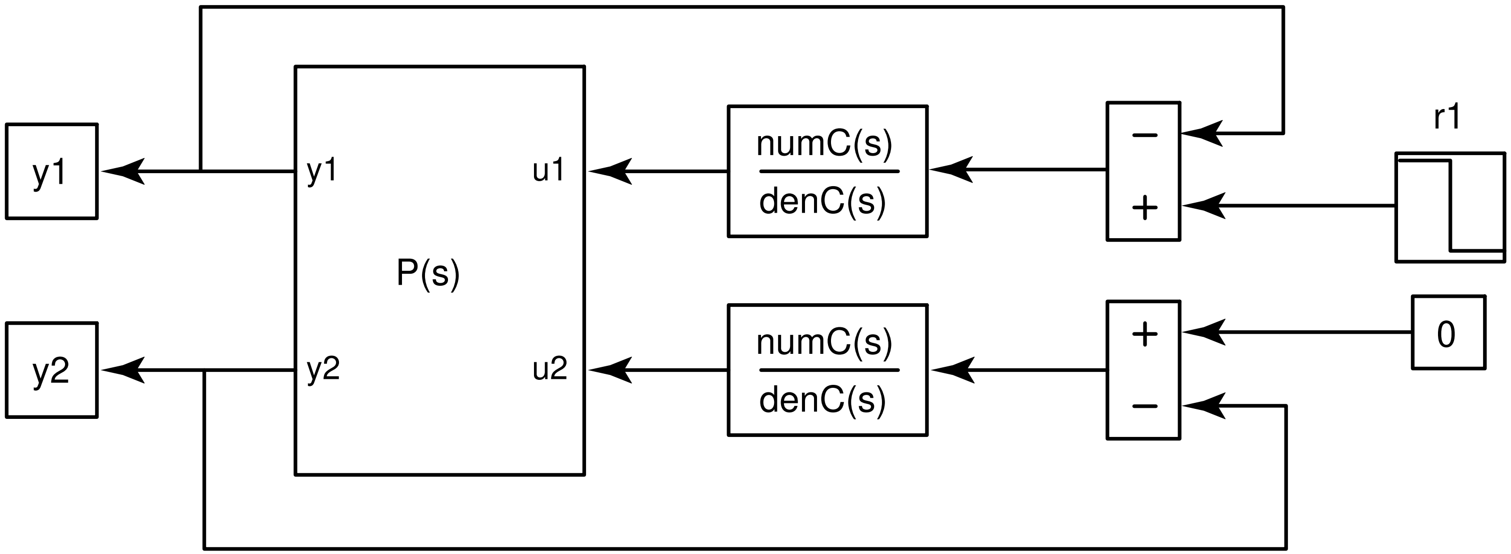
SISO *closed-loop* time-domain behavior (one channel only)



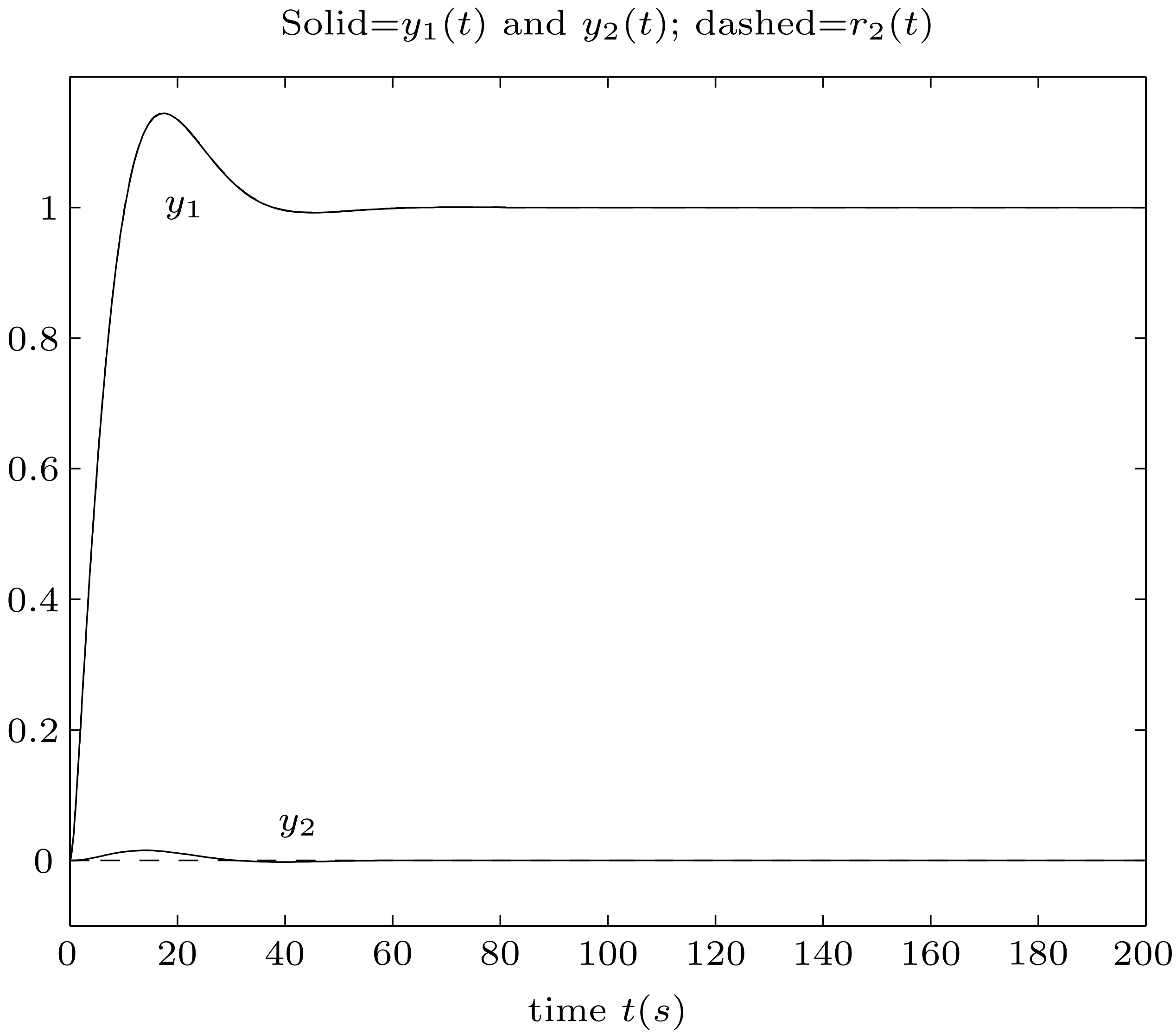
Looks OK, of course . . .

MIMO *closed-loop* time-domain behavior (both channels)

Setup:



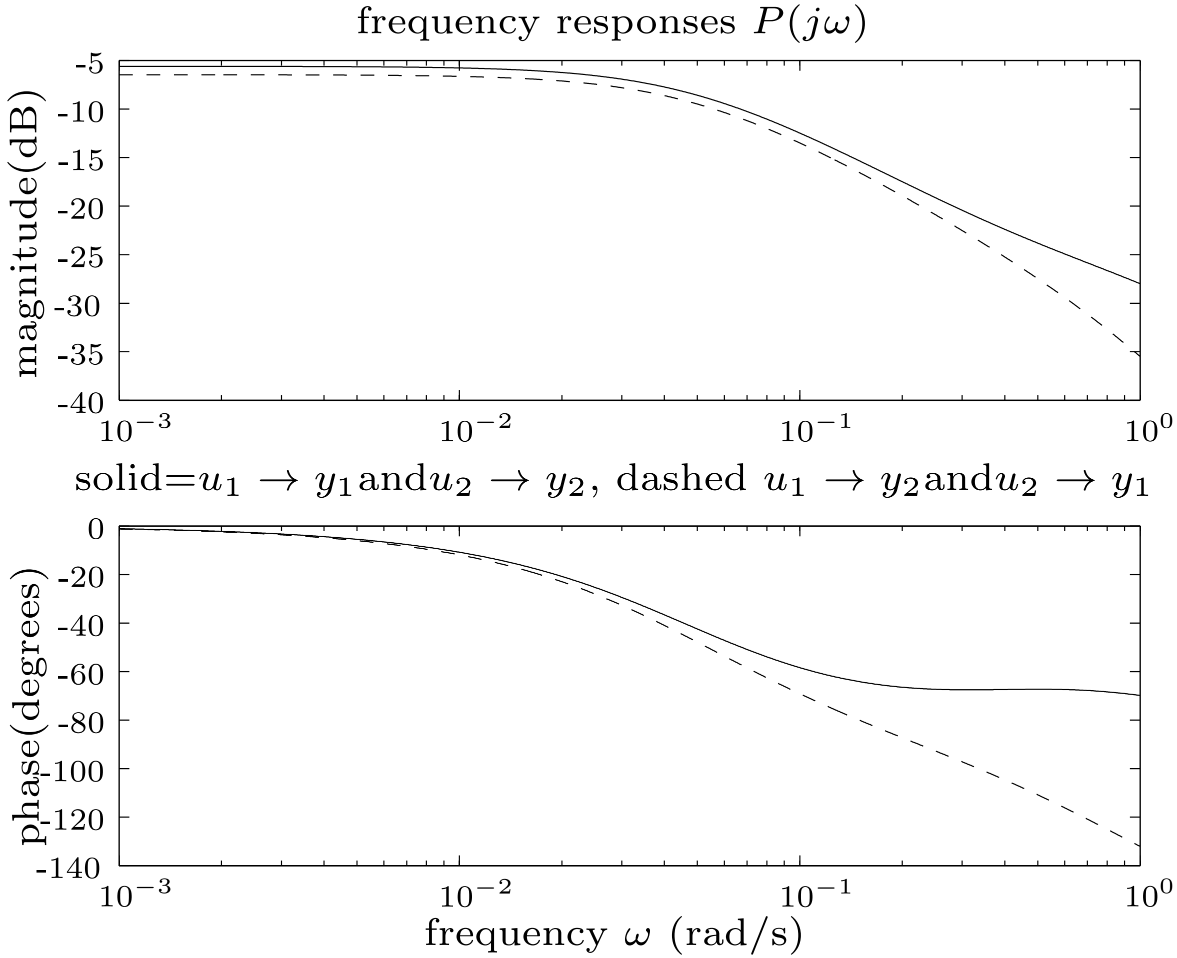
MIMO *closed-loop* time-domain behavior (both channels)



Looks still OK! MIMO and SISO close, why? Always the case?

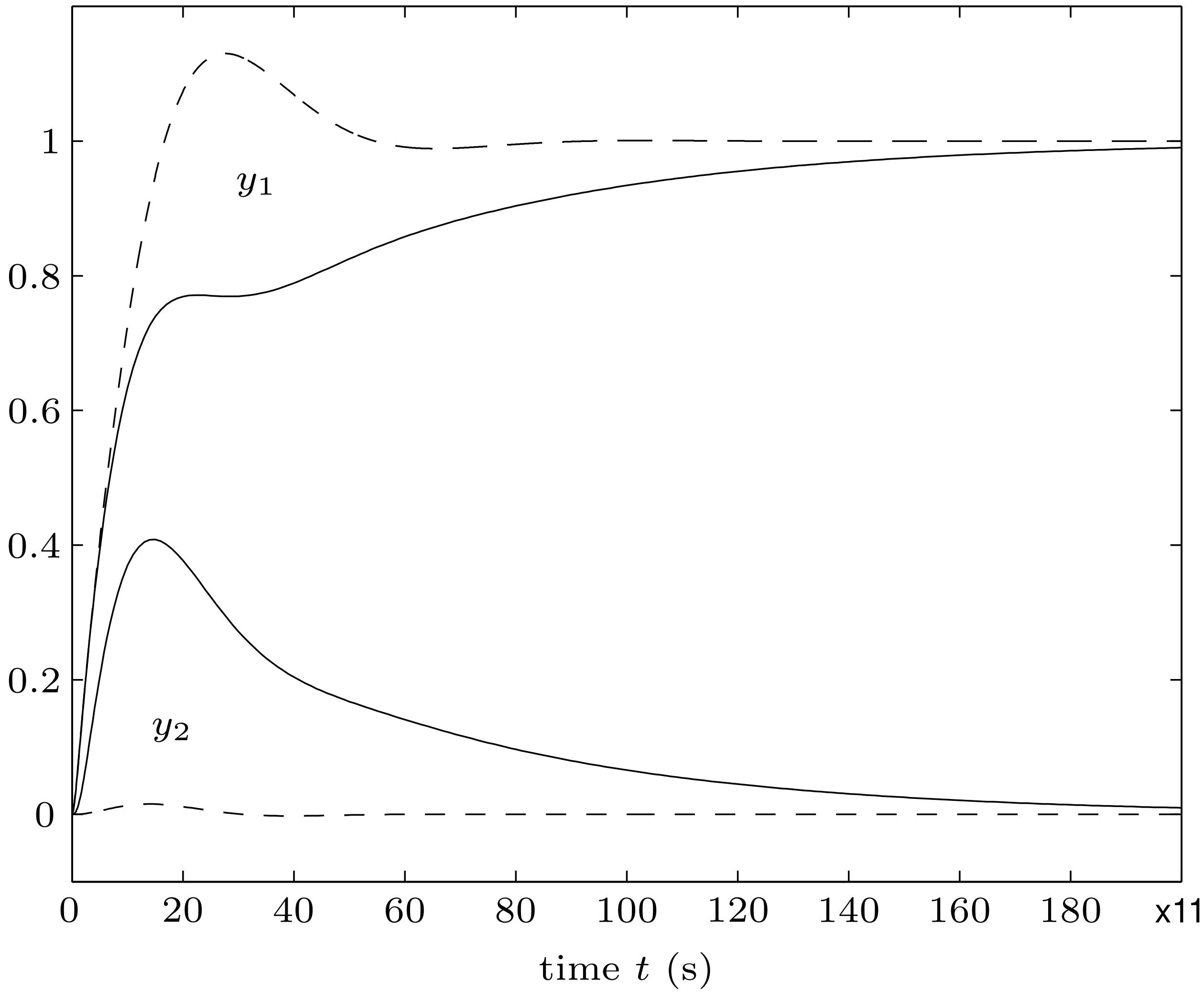
Assume much higher heat transfer coefficient

$$\widetilde{k} = 10000\,W/(m^2K) \tag{29}$$



Use same controller and check closed-loop MIMO behavior

solid: $k = 10000$ W/K m^2; dashed: $k = 100$ W/K m^2



Bad time-domain behavior, much slower convergence to desired value!

Explanation:

In the first case ($k = 100\, W/(m^2\, K)$) the cross-coupling transfer functions have a gain that is more than ten times smaller than the gain of the main channel in all frequencies.

Note that this yields an attenuation of $1/10^2$ when the loop is closed as a full MIMO system.

In the second case ($k = 10000\, W/(m^2\, K)$) the cross-coupling transfer functions have a gain that is close to the gain of the main channel in all relevant frequencies.

Therefore, in this case the system is “a hundred times more MIMO” than in the previous case.

Plants of the first type are called *diagonally dominant.* Such plants may be controlled using SISO controller design techniques by “breaking one loop at the time.” Plants of the second type require more powerful design techniques.

SISO

$$\dot{x}(t) = Ax(t) + bu(t)$$
$$y(t) = cx(t) + du(t)$$

easy $\updownarrow$

$$P(s) = \frac{b(s)}{a(s)}$$

MIMO

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

easy $\downarrow$ $\uparrow$ difficult

$$P(s) = \begin{bmatrix} \dfrac{b_{11}(s)}{a_{11}(s)} & \dfrac{b_{1m}(s)}{a_{1m}(s)} \\ \\ \dfrac{b_{p1}(s)}{a_{p1}(s)} & \dfrac{b_{pm}(s)}{a_{m}(s)} \end{bmatrix}$$

MIMO System Representation

$$\begin{aligned} \dot{x}(t) &= A \cdot x(t) + B \cdot u(t), \quad x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m \\ y(t) &= C \cdot x(t) + D \cdot u(t), \quad y(t) \in \mathbb{R}^p \end{aligned} \tag{30}$$

Such a description is well-defined only if it is obtained by a modeling process based on "physical first laws."

If only the input/output (IO) behavior is known (say, by measuring the impulse responses of the $m \times p$ channels), then there are infinitely many sets of matrices $\{A, B, C, D\}$ that produce the same IO behavior.

Moreover, most of these sets will have more than the minimum number of states n required to reproduce the observed IO behavior.

The Laplace transformation of equations (30) yields

$$s \cdot X(s) = A \cdot X(s) + B \cdot U(s) \tag{31}$$

hence

$$X(s) = (s\,I - A)^{-1} \cdot B \cdot U(s) \tag{32}$$

and

$$Y(s) = \big[C \cdot (s\,I - A)^{-1} \cdot B + D\big] \cdot U(s) \tag{33}$$

If only the input/output (IO) behavior is of interest, the frequency domain representation

$$y(s) = \big[C(sI - A)^{-1}B + D\big] \cdot u(s) = P(s) \cdot u(s) \tag{34}$$

is sufficient. Notice that $P(s)$ contains only the controllable and observable parts of (30).

$$P(s) = \begin{bmatrix} P_{1,1}(s) & P_{1,2}(s) & \dots & P_{1,m}(s) \\ P_{2,1}(s) & P_{2,2}(s) & \dots & P_{2,m}(s) \\ \dots & \dots & \dots & \dots \\ P_{p,1}(s) & P_{p,2}(s) & \dots & P_{p,m}(s) \end{bmatrix} \tag{35}$$

$$P_{i,j}(s) = \frac{b_{m,i,j}s^m + \cdots + b_{1,i,j}s + b_{0,i,j}}{s^n + a_{n-1,i,j}s^{n-1} + \cdots + a_{1,i,j}s + a_{0,i,j}} = \frac{b_{i,j}(s)}{a_{i,j}(s)} \tag{36}$$

In the MIMO case the realization problem is more difficult to solve.

$$P(s) = \begin{bmatrix} \dfrac{2}{s+1} & \dfrac{3}{s+2} \\ \\ \dfrac{1}{s+1} & \dfrac{1}{s+1} \end{bmatrix} \tag{37}$$

has the following “naive realization”

$$\begin{aligned} \frac{d}{dt}x(t) &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot u(t) \\ y(t) &= \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot x(t) \end{aligned} \tag{38}$$

which is not minimal.

System Stability, Controllability and Observability Most system *analysis* results valid for SISO systems remain true for MIMO systems:

1. The stability properties of the system $\{A, B, C, D\}$ are determined by the eigenvalues of A.

2. The system $\{A, B, C, D\}$ is completely controllable iff the matrix

$$\mathcal{R}_n = [B, A\,B, \ldots, A^{n-1}B] \in I\!R^{n\times(n\cdot m)} \tag{39}$$

has full rank n.

3. The system $\{A, B, C, D\}$ is completely observable iff the matrix

$$\mathcal{O}_n = [C^T, A^T\,C^T, \ldots, (A^{n-1})^T C^T]^T \in I\!R^{(n\cdot p)\times n} \tag{40}$$

has full rank n.

Plant $\{A, B, C, 0\}$ and controller $\{F, G, H, 0\}$ connected in the standard feedback configuration. Closed-loop system is asymptotically stable iff all eigenvalues of the matrix

$$\begin{bmatrix} A & B\,H \\ -G\,C & F \end{bmatrix} \tag{41}$$

have stricly negative real parts.

Nyquist theorem for MIMO systems: closed-loop system stable iff

$$\mathcal{N} = \det[I + P(j\,\omega) \cdot C(j\,\omega)], \quad \omega \in [-\infty, +\infty] \tag{42}$$

encircles the origin $n_+ + n_0/2$ times.

System Poles and Zeros

The poles of $P(s)$ are the roots of the least common denominator of all minors of $P(s)$.

Example:

$$P(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} \tag{43}$$

The zeros and poles of the SISO entries are $\zeta = \infty$ and $\pi_1 = -1$, $\pi_2 = -2$. The minors of $P(s)$ are

$$\frac{2}{s+1}, \ \frac{3}{s+2}, \ \frac{1}{s+1}, \frac{1}{s+1}, \ \frac{1-s}{(s+1)^2(s+2)} \tag{44}$$

The least common denominator is

$$p(s) = (s+1)^2(s+2) \tag{45}$$

and the poles π_i of $P(s)$ are -2, -1, and -1. An internal description of the system has order $n \leq 3$.

Obviously, the internal description $\{A, B, C, D\}$ of the IO description $P(s)$ must be of minimal order, otherwise additional zeros and poles appear, which cancel out. Also, if $P(s)$ is square, the zeros of $P(s)$ are simply the poles of $P^{-1}(s)$. Moreover

The zeros of $P(s)$ are the roots of the greatest common divisor of the numerators of the maximum minors of $P(s)$ after normalization to have the pole polynomial of $P(s)$ as denominators.

Example, contd.: The only maximum minor of $P(s)$ is

$$\frac{1-s}{(s+1)^2(s+2)} \tag{48}$$

which is already normalized by the pole polynomial. Therefore, one zero at $\zeta = 1$. None of the entries of $P(s)$ had a finite zero. The zero found at 1 (a non-minimumphase zero!) is due to the MIMO structure of the system.

A MIMO system can have poles and zeros at the same frequency s without incurring a pole-zero cancellation!

$$P(s) = \begin{bmatrix} \frac{s+2}{s+1} & 0 \\ \\ 0 & \frac{s+1}{s+2} \end{bmatrix} \tag{49}$$

has a pole and a zero at $s = -1$ and $s = -2$. In fact, the minors of $P(s)$ are

$$\frac{s+2}{s+1}, \quad \frac{s+1}{s+2}, \quad \frac{(s+2)\cdot(s+1)}{(s+1)\cdot(s+2)} = 1$$

The lcd is $(s+1)\cdot(s+2)$. Therefore, one pole at $s = -1$ and one at $s = -2$. Since $P(s)$ is square, there is only one maximum minor equal to 1. After normalization to have the pole polynomial $(s+2)\cdot(s+1)$ as denominator, this maximum minor is defined by the fraction

$$\frac{(s+2)\cdot(s+1)}{(s+2)\cdot(s+1)}$$

such that the gcd of the numerator is $(s+2)\cdot(s+1)$. Accordingly, the system has $s = -2$ and $s = -1$ as zeros.

This seemingly contradictory result is a consequence of the fact that in MIMO systems a direction is associated with each pole and zero. A cancellation takes place only if the frequency *and* the direction of a pole and a zero coincide.

Directions $\delta_{\pi,i}^{in,out}$ associated with pole π_i are

$$P(s)|_{s=\pi_i} \cdot \delta_{\pi,i}^{in} = \infty \cdot \delta_{\pi,i}^{out} \tag{50}$$

where $\delta_{\pi,i}^{in}$ is the input and $\delta_{\pi,i}^{out}$ the output pole direction.

Directions $\delta_{\zeta,i}^{in,out}$ associated zero ζ_i are

$$P(s)|_{s=\zeta_i} \cdot \delta_{\zeta,i}^{in} = 0 \cdot \delta_{\zeta,i}^{out} \tag{51}$$

where $\delta_{\zeta,i}^{in}$ is the input and $\delta_{\zeta,i}^{out}$ the output zero direction.

An approach to compute the directions is to use the singular value decomposition introduced later. For square $m \times m$ systems, the directions may be obtained by

$$P(s)|_{s \to \pi_i + \varepsilon} = U \cdot \Sigma \cdot V^T \quad \Rightarrow \quad \delta_{\pi,i}^{in} = V(:,1), \quad \delta_{\pi,i}^{out} = U(:,1),$$

or

$$P(s)|_{s \to \zeta_i + \varepsilon} = U \cdot \Sigma \cdot V^T \quad \Rightarrow \quad \delta_{\zeta,i}^{in} = V(:,m), \quad \delta_{\zeta,i}^{out} = U(:,m),$$

respectively, where ε is an arbitrary small number.

Example, contd.

The pole directions associated with the three poles $\pi_1 = -1$, $\pi_2 = -1$, and $\pi_3 = -2$ are

$$\delta^{in}_{\pi,1} = \begin{bmatrix} 0.97 \\ 0.23 \end{bmatrix}, \qquad \delta^{out}_{\pi,1} = \begin{bmatrix} 0.85 \\ 0.53 \end{bmatrix}$$

$$\delta^{in}_{\pi,2} = \begin{bmatrix} -0.23 \\ 0.97 \end{bmatrix}, \qquad \delta^{out}_{\pi,2} = \begin{bmatrix} -0.53 \\ 0.85 \end{bmatrix}$$

$$\delta^{in}_{\pi,3} = \begin{bmatrix} 0.00 \\ -1.00 \end{bmatrix}, \qquad \delta^{out}_{\pi,3} = \begin{bmatrix} -1.00 \\ 0.00 \end{bmatrix}$$

The zero directions associated with the zero $\zeta = 1$ are

$$\delta^{in}_{\zeta} = \begin{bmatrix} -0.71 \\ 0.71 \end{bmatrix}, \qquad \delta^{out}_{\zeta} = \begin{bmatrix} 0.45 \\ -0.89 \end{bmatrix}$$

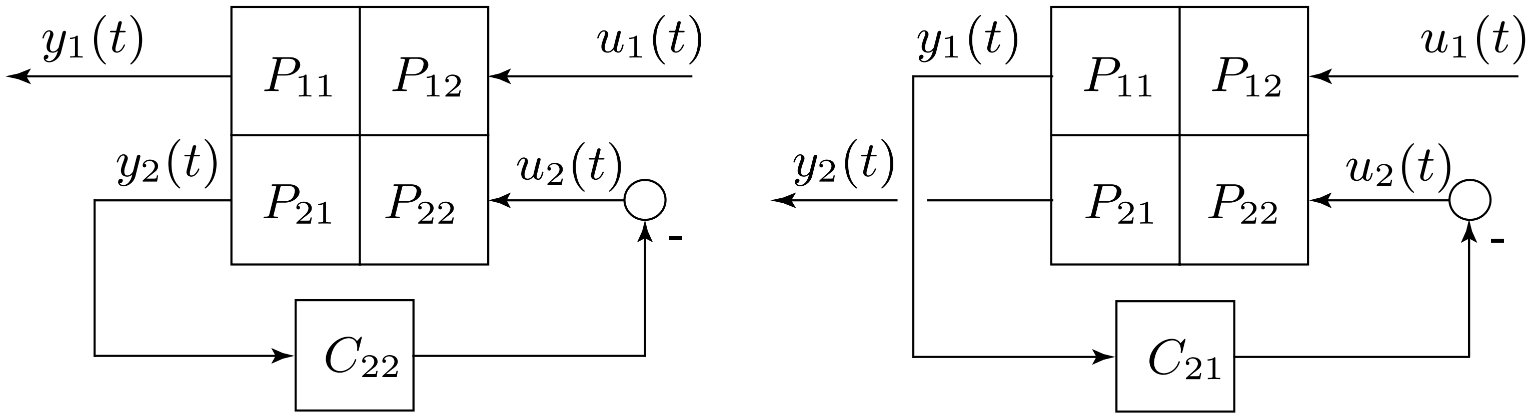
Lecture VI – Relative Gain Array, Singular Values and System Gains

Relative-Gain Array

Question: When can MIMO plants be controlled well by SISO controllers?

Answer: When their *relative-gain array matrix* $RGA(s)$ is close to I.

General idea of RGA explained with a 2×2 plant. To compute the $(1,1)$ elements of $RGA(s)$ close loop from y_2 to u_2 and compute transfer function from u_1 to y_1.



Two cases:

- First assume open loop conditions ($C_{2,2} = 0$): in this case $u_1 \to y_1$ is $P_{11}(s)$.
- Second assume high controller gains ($C_{2,2}P_{2,2} >> 1$): in this case $u_1 \to y_1$ is $\frac{P_{11}P_{22} - P_{21}P_{12}}{P_{22}}$.

The element $(1, 1)$ of the matrix $RGA(s)$ is then defined by

$$RGA_{1,1}(s) = P_{11}/\frac{P_{11}P_{22} - P_{21}P_{12}}{P_{22}} = \frac{P_{11}P_{22}}{P_{11}P_{22} - P_{21}P_{12}}$$

It's easy to see that $RGA_{2,2}(s) = RGA_{1,1}(s)$. These scalars are close to 1 iff $P_{12} \cdot P_{21} << P_{11} \cdot P_{22}$. If $RGA_{11} = RGA_{22}$ substantially differ from 1 the MIMO interactions are substantial and a SISO-similar approach is not recommended.

To compute the element $RGA_{2,1}(s)$ analyze the transfer function $u_1 \to y_2$. The result is

$$RGA_{2,1}(s) = \frac{-P_{12}P_{21}}{P_{11}P_{22} - P_{21}P_{12}}$$

It's easy to see that $RGA_{1,2}(s) = RGA_{2,1}(s)$. These scalars are close to 0 iff $P_{12} \cdot P_{21} << P_{11} \cdot P_{22}$. If $RGA_{12} = RGA_{21}$ substantially differ from 0 the MIMO interactions (as defined by P_{12} and P_{21}) are substantial and a SISO-similar approach is not recommended.

In general:

$$RGA(s) = P(s). \times P(s)^{-T} \tag{52}$$

where the operator $.\times$ denotes element-wise mutiplication (as $.*$ in Matlab) and $P^{-T} = (P^{-1})^T$ (transpose, not conjugate transpose!).

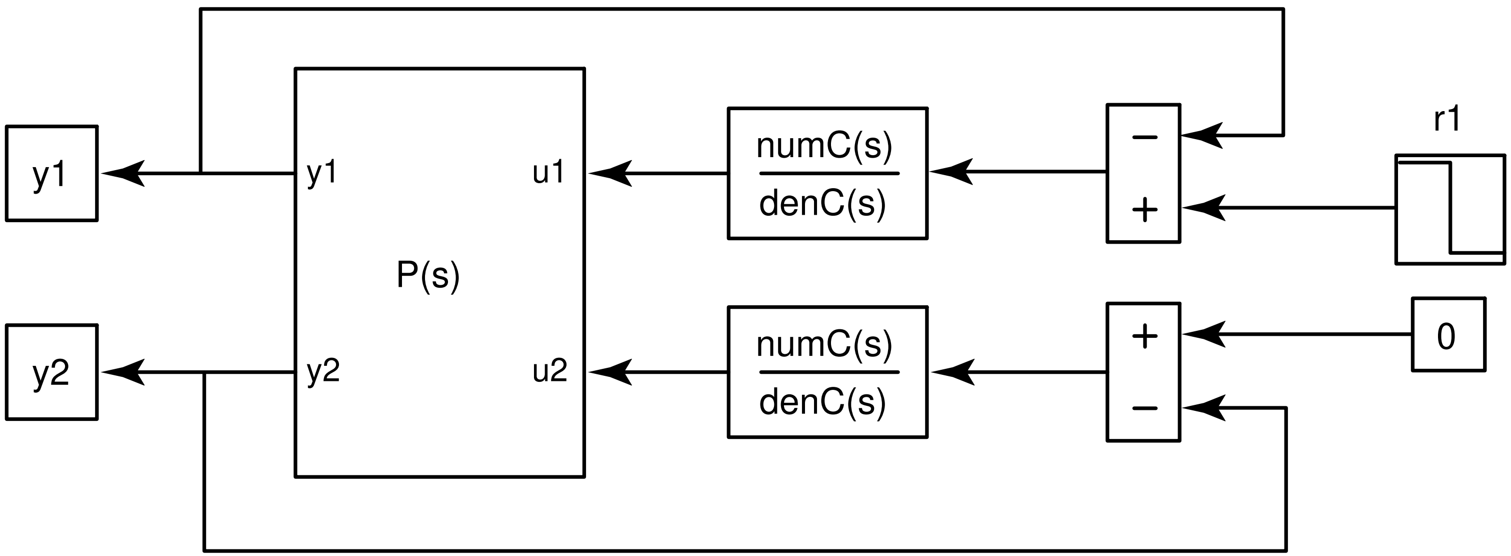
It can be shown that

- The columns and the rows of $RGA(s)$ always add up to 1.
- The RGA is invariant with respect to scaling, i.e., for any diagonal matrices D_i the equation $RGA(P(s)) = RGA(D_1 \cdot P(s) \cdot D_2)$ holds true.
- The RGA of a triangular matrix $P(s)$ is the identity matrix I.

The main result is: if $RGA(P(s))$ is substantially different from I for all frequencies s, the cross-coupling gains are important and MIMO approaches must be used to control the plant $P(s)$. If $RGA(P(s)) \approx I$ for all s, the individual channels can be controlled "one loop at the time" using m SISO controllers. In feedback control applications, the input/output pairing should be chosen such that input i is paired with output j when $RGA_{j,i}$ is close to 1.

Summary:

If $RGA(P(s)) \approx I$ then "one loop at the time" OK



If $RGA(P(s)) \neq I$ then "true MIMO design" necessary

For the example of the heat exchanger

$$RGA(s) = \begin{bmatrix} \frac{(\tau\, s+1)^2}{(\tau\, s+1)^2-\sigma^2} & \frac{-\sigma^2}{(\tau\, s+1)^2-\sigma^2} \\ \\ \frac{-\sigma^2}{(\tau\, s+1)^2-\sigma^2} & \frac{(\tau\, s+1)^2}{(\tau\, s+1)^2-\sigma^2} \end{bmatrix} \tag{53}$$

or, at $s = 0$ (static gain)

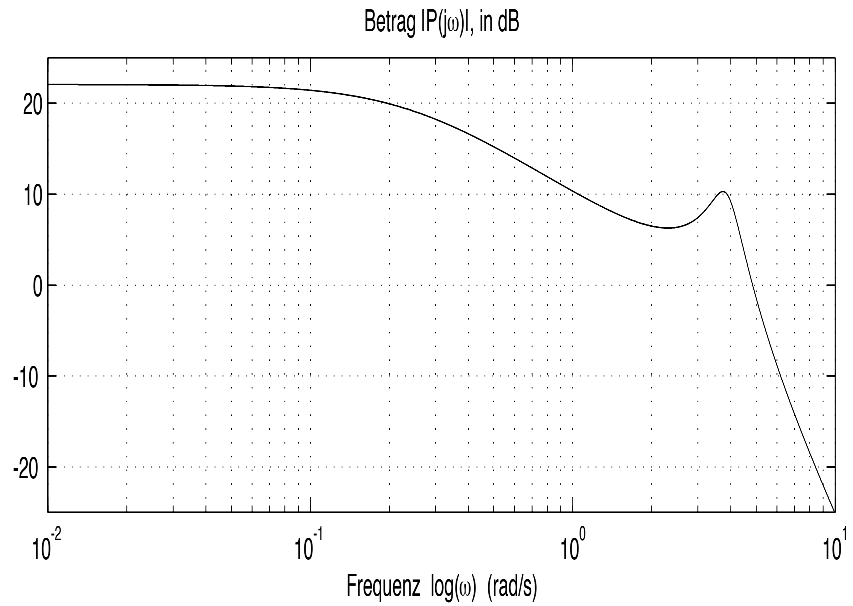
$$RGA(s) = \begin{bmatrix} \frac{1}{1-\sigma^2} & \frac{-\sigma^2}{1-\sigma^2} \\ \\ \frac{-\sigma^2}{1-\sigma^2} & \frac{1}{1-\sigma^2} \end{bmatrix} \tag{54}$$

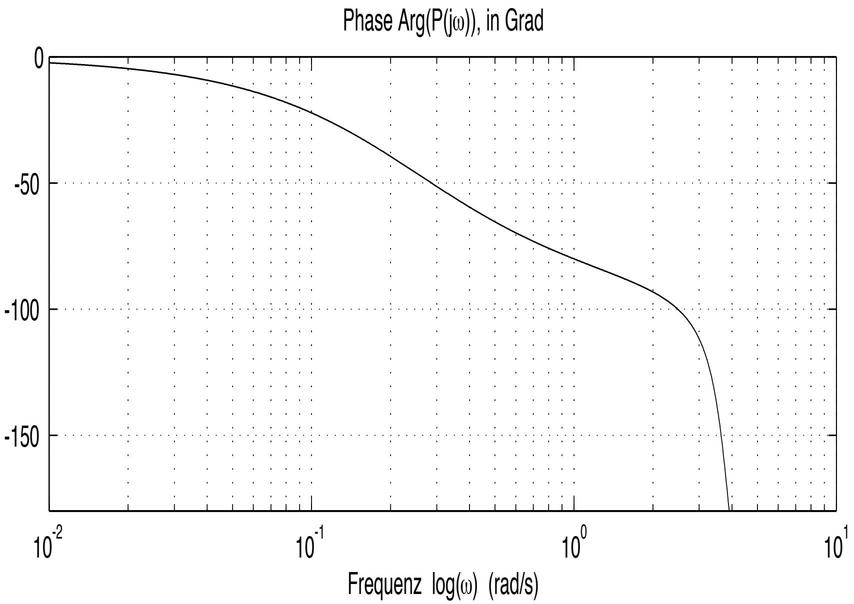
Pro memoria:

$$\sigma = \frac{k \cdot A}{\overset{*}{m} \cdot c + k \cdot A}, \quad 0 < \sigma < 1 \tag{55}$$

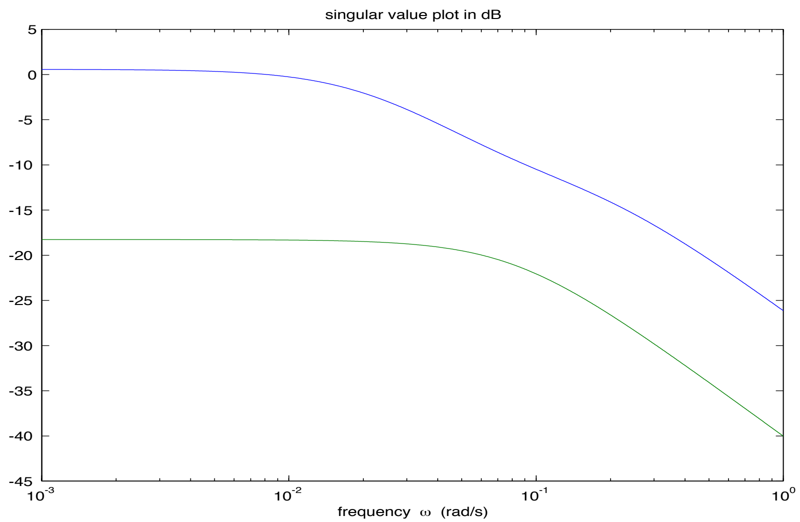
For small heat exchange capability ($k \cdot A << \overset{*}{m} \cdot c$) the parameter $\sigma \approx 0$ and the $RGA(P(s)) \approx I$, i.e., SISO control loops are OK. However, if ($k \cdot A >> \overset{*}{m} \cdot c$) the parameter $\sigma \approx 1$ and the $RGA(P(s))$ becomes very large. Therefore, the plant can only be controlled using a true MIMO approach.

SISO


Betrag |P(jω)|, in dB
20
10
0
-10
-20
10^{-2}
10^{-1}
10^{0}
10^{1}
Frequenz log(ω) (rad/s)


Phase Arg(P(jω)), in Grad
0
-50
-100
-150
10^{-2}
10^{-1}
10^{0}
10^{1}
Frequenz log(ω) (rad/s)

MIMO


singular value plot in dB
5
0
-5
-10
-15
-20
-25
-30
-35
-40
-45
10^{-3}
10^{-2}
10^{-1}
10^{0}
frequency ω (rad/s)

?

Matrix Norms and Singular Values

Induced norm of any linear operator M

$$y = M(u) \tag{56}$$

defined using a specific norm $||x||$ by

$$||M|| = \max_{u \neq 0} \frac{||y||}{||u||} = \max_{||u||=1} ||M(u)|| \tag{57}$$

Important example $u \in \Re^m$, $y \in \Re^p$, $M \in \Re^{p \times m}$ and $||x|| = \sqrt{x^T \cdot x}$, where $x = \{u, y\}$ ("inner product norm"). In this case

$$||M|| = \max_i \{\sigma_i\{M\}\} \tag{58}$$

where the singular values $\sigma_i\{M\}$ are the positive square roots of the eigenvalues of the matrix $M^T M$. Since $M^T M$ is symmetric (by construction), its eigenvalues all are real and non-negative.

Lagrange Method Optimizations with Constraints

$$J : \mathbb{R}^m \ \rightarrow \ \mathbb{R}_+, \quad f : \mathbb{R}^m \ \rightarrow \ \mathbb{R}, \quad u \in \mathbb{R}^m$$

Problem: Find u^* which maximizes $J(u^*) \geq J(u)$ and which simultaneously satisfies $f(u^*) = 0$.

Geometric interpretation:

.

"Proof"

Use Lagrange's method for constrained optimization. Objective function to be maximized:

$$J = ||y||^2 + \lambda \cdot (1 - ||u||^2) \tag{59}$$

$$= u^T \cdot M^T M \cdot u + \lambda \cdot (1 - u^T \cdot u) \stackrel{!}{=} \max \tag{60}$$

Necessary conditions for a local maximum

$$\frac{\partial J}{\partial u} = 0 \tag{61}$$

In this special case

$$\frac{\partial J}{\partial u} = 2 \cdot M^T M \cdot u - 2 \cdot \lambda \cdot u \tag{62}$$

Accordingly, the optimal solution u^* must satisfy the homogeneous equation

$$(\lambda \cdot I - M^T M) \cdot u^* = 0 \tag{63}$$

and a non-trivial u^* exists iff

$$\det\left\{\lambda \cdot I - M^T M\right\} \stackrel{!}{=} 0 \tag{64}$$

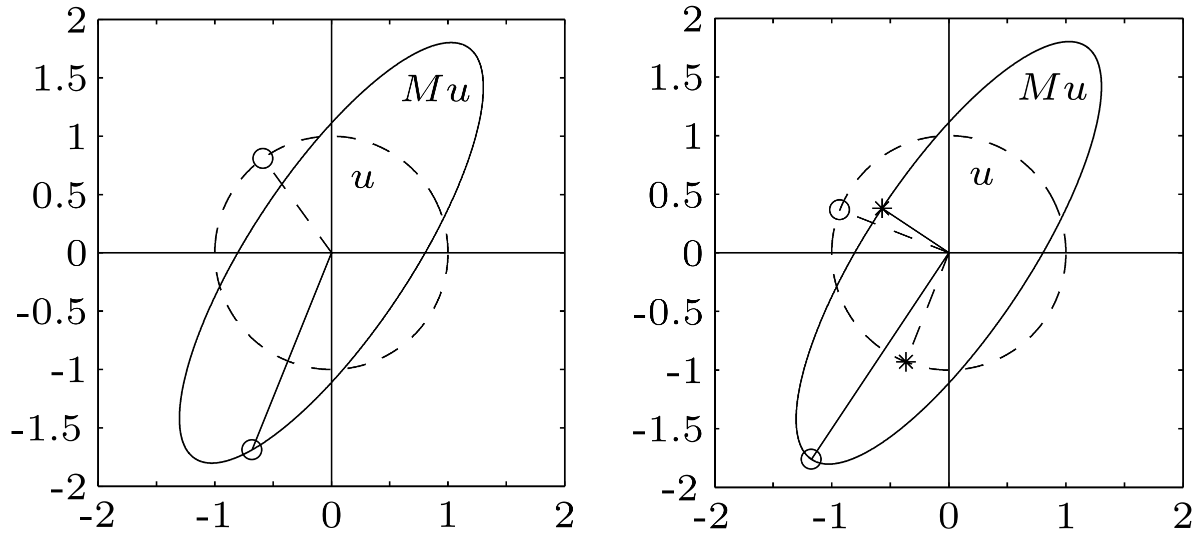
The scalars $\lambda_i = \sigma_i^2$ that satisfy this equation are the eigenvalues of $M^T M$. Since $M^T M$ is a symmetric and positive semi-definite matrix (by construction), its eigenvalues λ_i are real and non-negative, i.e., their square root σ_i is real and non-negative as well.

With this result and using the definition of the induced norm it's easy to see that

$$||M||^2 = \max_{||u||\neq 0} \frac{||y||^2}{||u||^2} = \max_{||u||\neq 0} \frac{u^T \cdot M^T M \cdot u}{u^T \cdot u} = \max_i \left\{\sigma_i^2\right\} \tag{65}$$

Geometric interpretation for the case $u, y \in \Re^2$ and

$$M = \begin{bmatrix} 1.3 & 0.1 \\ 1.5 & -1 \end{bmatrix}$$



Obviously, repeating the analysis shown above for the minimum gain yields

$$\sigma_{\min}\{M\} \leq \frac{||y||}{||u||} \leq \sigma_{\max}\{M\}, \quad ||u|| \neq 0 \tag{66}$$

where it is assumed that $\sigma_{\min}\{M\}$ is the minimum and $\sigma_{\max}\{M\}$ the maximum singular value of M.

Geometric properties of linear mappings represented by the matrix M

$$M = U \cdot \Sigma \cdot V^T \tag{67}$$

where $U \in I\!R^{p \times p}$ and $V \in I\!R^{m \times m}$ are unitary matrices, i.e.

$$U \cdot U^T = I_{p \times p}, \quad V \cdot V^T = I_{m \times m}$$

and where the only non-zero elements of $\Sigma \in I\!R^{p \times m}$ are the singular values of M, i.e.,

$$[\Sigma]_{i,j} = 0 \ \forall \ i \in [1, p] \neq j \in [1, m]$$

and

$$[\Sigma]_{i,j} = \sigma_k \ \forall \ i = j = k \in [1, \min\{p, m\}]$$

The geometric interpretation associated with this decomposition is that every mapping represented by M can be decomposed into first a isometric transformation[a] represented by V^T, then a scaling projection represented by Σ, and finally another isometry represented by U. Since fast and robust numerical algorithms exist for the computation of the singular value decomposition of high-order matrices M, this concept has many useful applications in system theory.

[a]Such transformations are essentially rotations and, therefore, preserve the length of the vectors transformed. In general, reflections through the origin are possible as well.

Generalization to the case of $M \in \mathbb{C}^{p \times m}$ using definition of the Euclidean norm of a complex scalar $z = a + j\, b$ ($a,\, b \in \mathbb{R}$)

$$||z||^2 = a^2 + b^2 = (a - j\, b) \cdot (a + j\, b) = \bar{z} \cdot z \tag{68}$$

where $\bar{z}$ denotes complex conjugation.

For a complex vector $v \in \mathbb{C}^n$

$$||v||^2 = \sum_{i=1}^{n} a_i^2 + b_i^2 = \sum_{i=1}^{n} (a_i - j\, b_i) \cdot (a_i + j\, b_i) = \bar{v}^T \cdot v \tag{69}$$

For the linear mapping represented by the complex matrix M

$$y = M \cdot u, \quad u \in \mathbb{C}^m, \; y \in \mathbb{C}^p, \; M \in \mathbb{C}^{p \times m} \tag{70}$$

Repeating the analysis yields

$$||M|| = \sigma_{\max}\{M\} = \max_i \sqrt{\lambda_i\{\bar{M}^T \cdot M\}} \tag{71}$$

In particular, if $||u|| = 1$ the norm of y must satisfy the inequality

$$\sigma_{\min}\{M\} \leq ||y|| \leq \sigma_{\max}\{M\} \tag{72}$$

Since the matrix $Q = \bar{M}^T \cdot M$ is Hermitian[a] by definition, its eigenvalues are all non-negative real numbers. Therefore, equation (72) makes sense. The eigenvectors of $Q = \bar{M}^T \cdot M$ are always linearly independent, even if multiple eigenvalues exist. However, in general they consist of complex-valued entries.

[a]A matrix Q is Hermitian if $Q = \bar{Q}^T$, i.e., if it is equal to its complex conjugate and transpose.

Lecture VII – Frequency Response of MIMO Systems

Assume a BIBO stable 2×2 system $P(s)$ is driven by the input

$$u(t) = \begin{bmatrix} u_1(t) \\ \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \mu_1 \cdot \cos(\omega\,(t + \varphi_1/\omega)) \cdot h(t) \\ \\ \mu_2 \cdot \cos(\omega\,(t + \varphi_2/\omega)) \cdot h(t) \end{bmatrix}$$

Laplace transformation of $u(t)$ yields (use "shift law")

$$U(s) = \begin{bmatrix} \mu_1 \cdot \frac{s}{s^2+\omega^2} \cdot e^{(\varphi_1/\omega)\cdot s} \\ \\ \mu_2 \cdot \frac{s}{s^2+\omega^2} \cdot e^{(\varphi_2/\omega)\cdot s} \end{bmatrix} = \frac{s}{s^2+\omega^2} \cdot \begin{bmatrix} e^{(\varphi_1/\omega)\cdot s} & 0 \\ \\ 0 & e^{(\varphi_2/\omega)\cdot s} \end{bmatrix} \cdot \begin{bmatrix} \mu_1 \\ \\ \mu_2 \end{bmatrix}$$

or more compactly

$$U(s) = \frac{s}{s^2+\omega^2} \cdot e^{\Phi \cdot s/\omega} \cdot \mu, \quad \Phi = \mathrm{diag}(\varphi_1, \varphi_2), \quad \mu = [\mu_1, \mu_2]^T$$

The system is linear and BIBO stable. Trefore, the output will be

$$y(t) = y_t(t) + y_\infty(t)$$

where $y_t(t) \to 0$ for $\lim t \to \infty$. The steady-state output will have the form

$$Y_\infty(s) = \frac{s}{s^2 + \omega^2} \cdot e^{\Psi \cdot s/\omega} \cdot \nu, \quad \Psi = \mathrm{diag}(\psi_1, \psi_2), \quad \nu = [\nu_1, \nu_2]^T$$

and since $Y(s) = P(s) \cdot U(s)$ this yields

$$e^{j\Psi} \cdot \nu = \begin{bmatrix} P_{11}(j\,\omega) & P_{12}(j\,\omega) \\ \\ P_{21}(j\,\omega) & P_{22}(j\,\omega) \end{bmatrix} \cdot e^{j\Phi} \cdot \mu$$

Application to MIMO frequency responses

$$y(j\,\omega) = P(j\,\omega) \cdot u(j\,\omega) \tag{73}$$

requires generalization to complex vectors $u(j\,\omega), y(j\,\omega)$ and matrices $M = P(j\,\omega)$

$$||P(j\,\omega)|| = \max_{||u(j\,\omega)||\neq 0} \frac{||y(j\,\omega)||}{||u(j\,\omega)||} \tag{74}$$

In this case the norms are defined by

$$||u(j\,\omega)|| = \sqrt{\bar{u}(j\,\omega)^T u(j\,\omega)} \tag{75}$$

where $\bar{x}$ denotes complex conjugate of x, i.e., for $x = a + j\,b$, $\bar{x} = a - j\,b$.

In analogy to the real case, the singular values are now defined as

$$\sigma_i\{P(j\,\omega_0)\} = \sqrt{\mathrm{eig}\{\bar{P}(j\,\omega_0)^T \cdot P(j\,\omega_0)\}} \tag{76}$$

The singular values $\sigma_i\{P(j\,\omega_0)\}$ are again positive non negative real numbers. This is always the case because the matrix $\bar{P}(j\,\omega_0)^T \cdot P(j\,\omega_0)$ is, by construction, a "positive semi-definite" Hermitian matrix.

Note that the $\sigma_i\{P(j\,\omega_0)\}$ are functions of the frequency ω, i.e., for each *fixed* frequency ω_0 a singular-value problem must be solved. Matlab provides dedicated commands (for instance "sigma," use "help sigma" for more information).

If you want to write your own software, note that in Matlab the operator ' indicates transpose *and* conjugate complex.

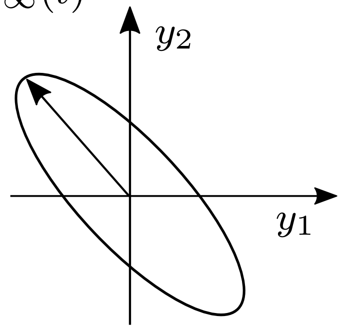
Frequency Response

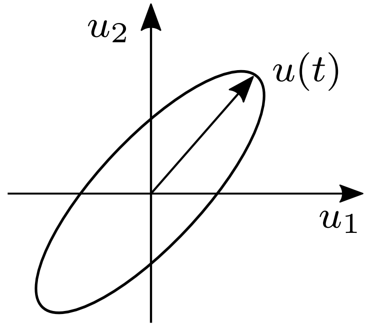
$$y_{1t}(t) + \nu_1 \cdot \cos(\omega t + \psi_1) = y_1 \quad \longleftarrow \quad \boxed{\dot{x} = Ax + Bu} \quad \longleftarrow \quad u_1(t) = \mu_1 \cdot \cos(\omega t + \varphi_1)$$

$$\underbrace{y_{2t}(t)}_{\substack{\text{transient}\\\text{complicated}}} + \underbrace{\nu_2 \cdot \cos(\omega t + \psi_2)}_{y_\infty(t)} = y_2 \quad \longleftarrow \quad \boxed{y = Cx + Du} \quad \longleftarrow \quad u_2(t) = \mu_2 \cdot \cos(\omega t + \varphi_2)$$

$$\mu_1^2 + \mu_2^2 = 1$$

astab





$$Y_t(s) + \frac{s}{s^2+\omega^2} \begin{bmatrix} e^{\psi_1 s/\omega} & 0 \\ 0 & e^{\psi_2 s/\omega} \end{bmatrix} \cdot \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} = Y(s) \quad \longleftarrow \quad \boxed{P(s)} \quad \longleftarrow \quad U(s) = \frac{s}{s^2+\omega^2} \begin{bmatrix} e^{\frac{\varphi_1 s}{\omega}} & 0 \\ 0 & e^{\frac{\varphi_2 s}{\omega}} \end{bmatrix} \cdot \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

$$t \to \infty \qquad y_t(t) \to 0 \qquad s \to j\omega$$

$$\underbrace{\begin{bmatrix} e^{j\psi_1} & 0 \\ 0 & e^{j\psi_2} \end{bmatrix} \cdot \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}}_{\xi \in \mathbb{C}^2} \quad \longleftarrow \quad \boxed{P(j\omega)} \quad \longleftarrow \quad \underbrace{\begin{bmatrix} e^{j\varphi_1} & 0 \\ 0 & e^{j\varphi_2} \end{bmatrix} \cdot \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}}_{\xi \in \mathbb{C}^2}$$

$P(j\omega) \in \mathbb{C}^{2\times 2}$

General case: BIBO stable $m \times m$ MIMO-Plants $P(s)$ and input

$$u(t) = \left[\begin{array}{c} \cos(\omega \cdot t + \varphi_1) \cdot \mu_1 \\ \cos(\omega \cdot t + \varphi_2) \cdot \mu_2 \\ \cdots \\ \cos(\omega \cdot t + \varphi_m) \cdot \mu_m \end{array} \right] = \text{diag}\left\{\cos(\omega \cdot t + \varphi_i)\right\} \cdot \mu$$

where $\mu = [\mu_1,\, \mu_2,\, \ldots, \mu_m]^T$, will have a steady state response

$$y_\infty(t) = \left[\begin{array}{c} \cos(\omega \cdot t + \psi_1(\omega)) \cdot \nu_1(\omega) \\ \cos(\omega \cdot t + \psi_2(\omega)) \cdot \nu_2(\omega) \\ \cdots \\ \cos(\omega \cdot t + \psi_p(\omega)) \cdot \nu_m(\omega) \end{array} \right] = \text{diag}\left\{\cos(\omega \cdot t + \varphi_i(\omega))\right\} \cdot \nu(\omega)$$

where $\nu(\omega) = [\nu_1(\omega),\, \nu_2(\omega),\, \ldots, \nu_m(\omega)]^T$.

If $||\mu|| = 1$, the "output magnitude vector" ν will always satisfy the constraints

$$\min_i \sigma_i\{P(j\,\omega)\} \le ||\nu(\omega)|| \le \max_i \sigma_i\{P(j\,\omega)\} \tag{77}$$

At each fixed frequency ω_0, the maximum and minimum singular values $\sigma_{\max}(\omega_0)$ and $\sigma_{\min}(\omega_0)$ indicate the limits within which the norm of the *amplitude vector* $\nu(\omega_0)$ of the output $y(t)$ must lie in *steady-state conditions*.

Most important drawback: phase information lost! Each channel has a different phase lag and there is no neat way to extract a characteristic lag information that is useful, e.g., to assess the system stability.

Accordingly, the condition (77) only yields "worst worst-case" conditions, see example below.

Example Heat exchanger, now with different mass flows and volumes in the hot and cold leg ($\overset{*}{m}_1 = 0.5\,kg/s$, $\overset{*}{m}_2 = 0.2\,kg/s$, $V_1 = 0.01\,m^3$, $V_2 = 0.02\,m^3$). Therefore, $\tau_1 \neq \tau_2$ $\beta_1 \neq \beta_2$, and $\sigma_1 \neq \sigma_2$

$$A = \begin{bmatrix} -1/\tau_1 & \sigma_1/\tau_1 \\ \\ \sigma_2/\tau_2 & -1/\tau_2 \end{bmatrix}, \quad B = \begin{bmatrix} \beta_1/\tau_1 & 0 \\ \\ 0 & \beta_2/\tau_2 \end{bmatrix} \tag{78}$$

Taking $\omega_0 = 0.1\ rad/s$ yields

$$P(j\,\omega_0) \quad = \quad C \cdot [j\,\omega_0 \cdot I - A]^{-1} \cdot B + D \tag{79}$$

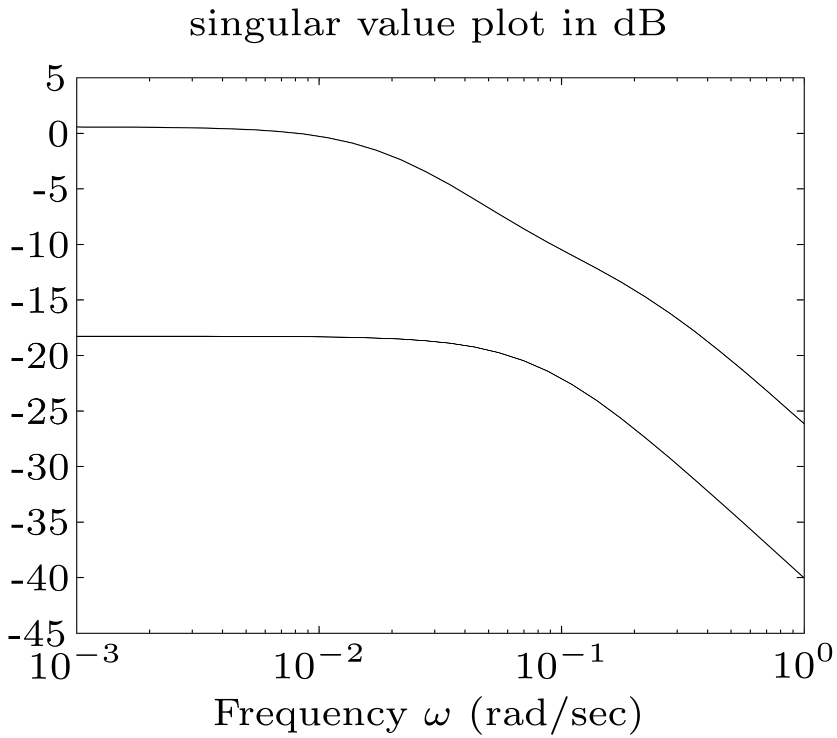
$$= \begin{bmatrix} 0.186 - j\,0.199 & -0.013 - j\,0.043 \\ \\ -0.033 - j\,0.107 & 0.0252 - j\,0.079 \end{bmatrix} \tag{80}$$

The singular values are

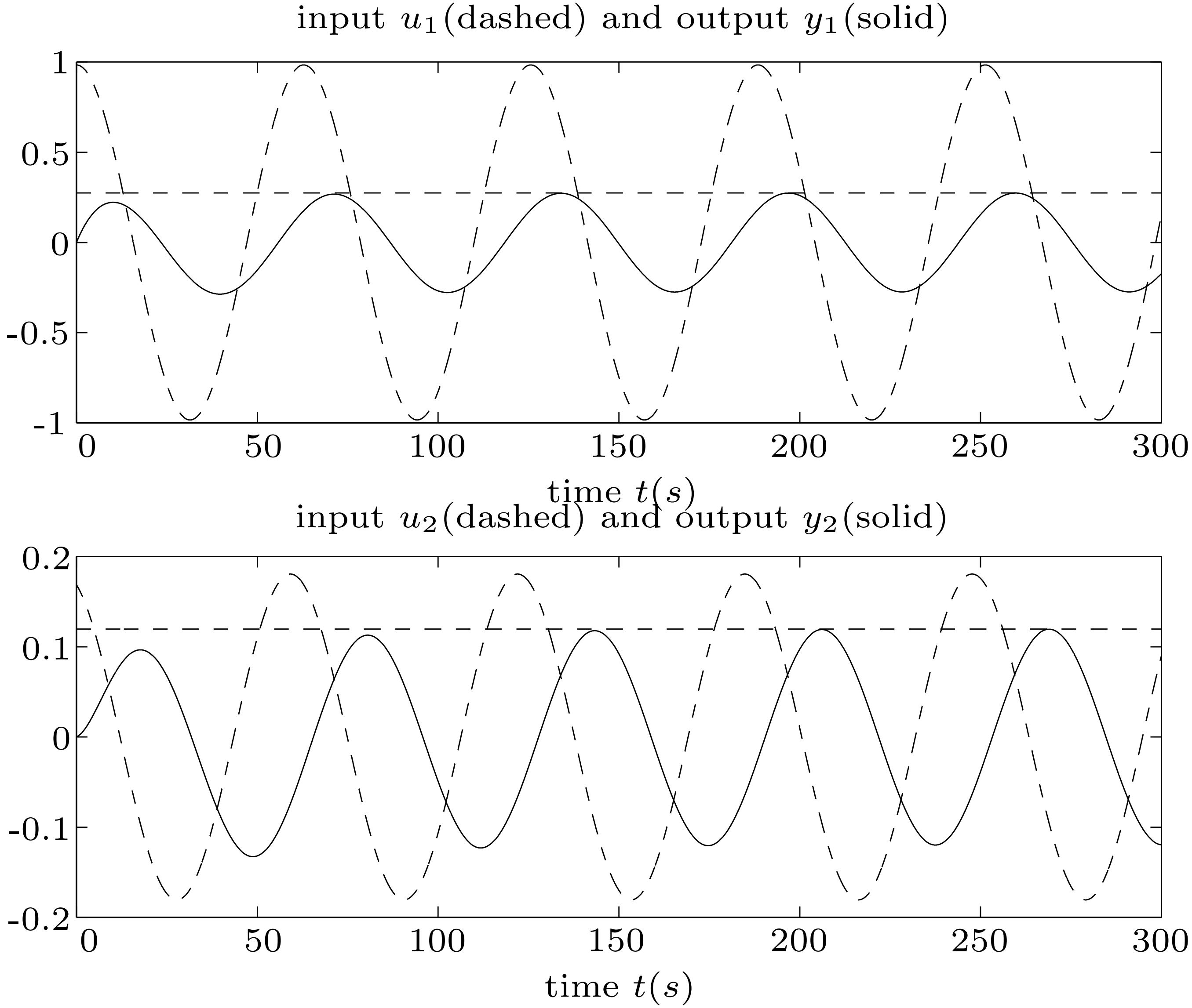
$$\sqrt{\mathrm{eig}\{\bar{P}(j\,\omega_0)^T \cdot P(j\,\omega_0)\}} \approx \{0.300, 0.079\} \tag{81}$$

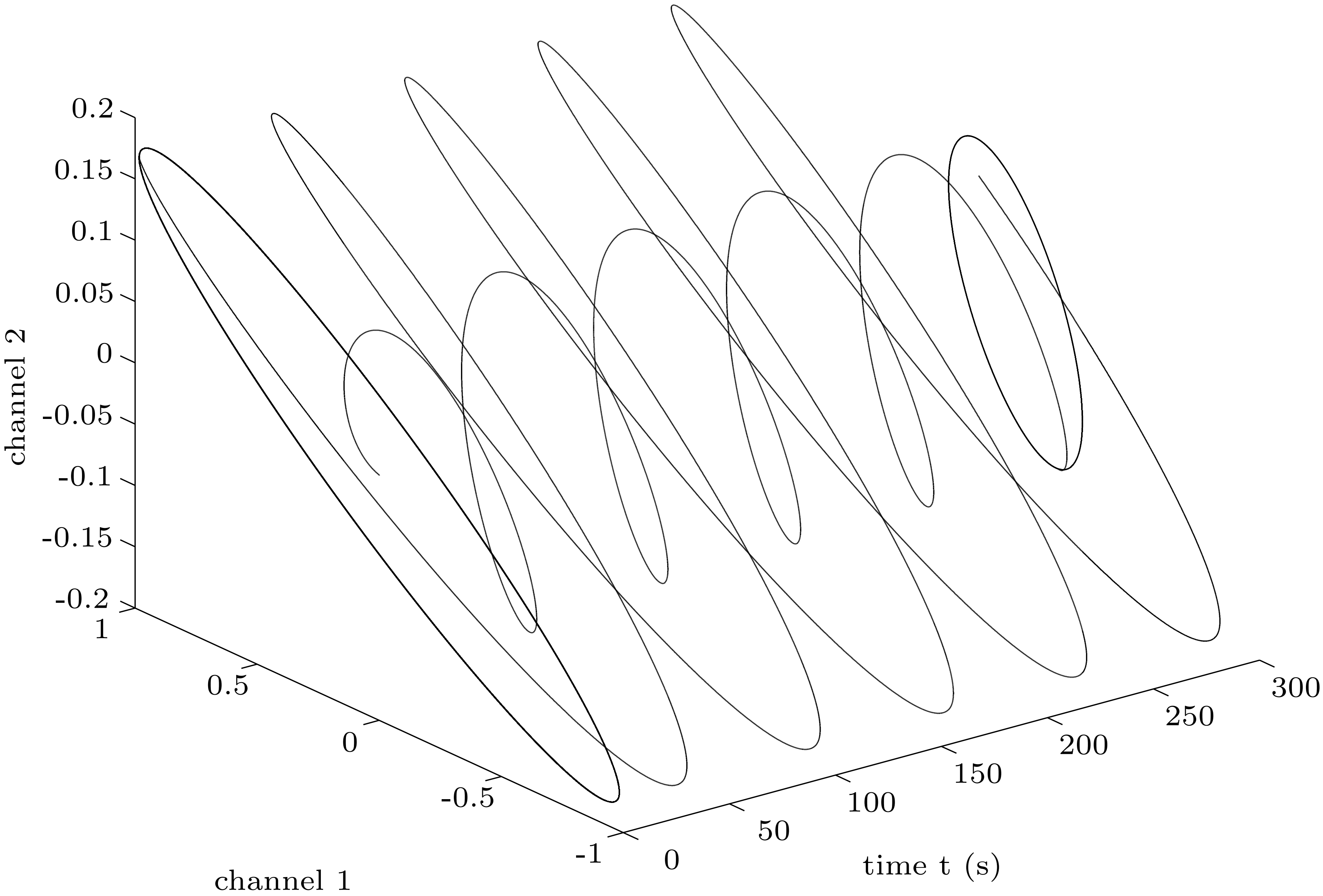
and the corresponding eigenvectors of $P(j\,\omega_0)^T \cdot P(j\,\omega_0)$ are

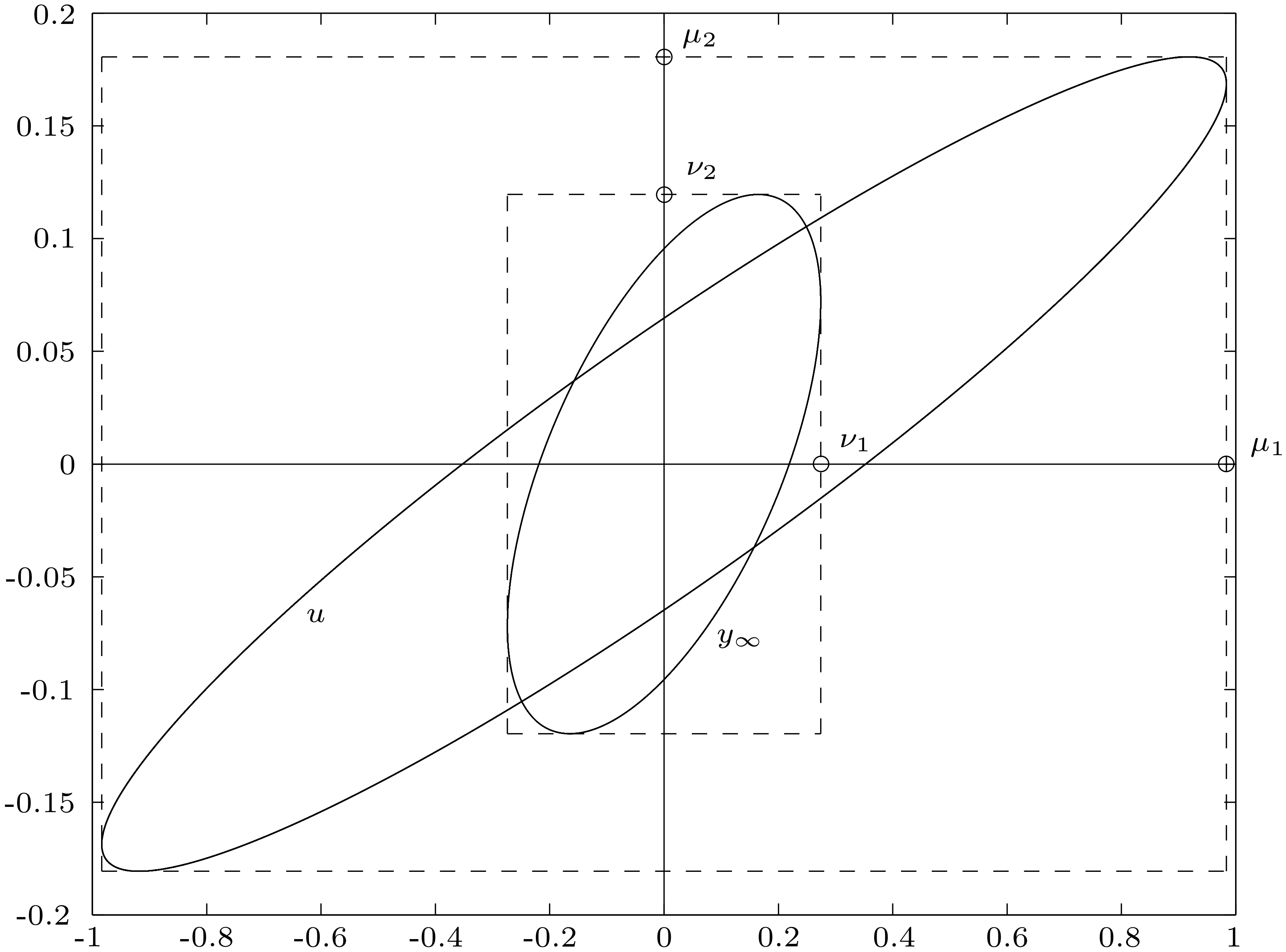
$$\zeta_1 = \begin{bmatrix} e^{-j\,0.37} & 0 \\ 0 & e^0 \end{bmatrix} \cdot \begin{bmatrix} 0.984 \\ 0.181 \end{bmatrix}, \quad \zeta_2 = \begin{bmatrix} e^{j\,2.77} & 0 \\ 0 & e^0 \end{bmatrix} \cdot \begin{bmatrix} 0.181 \\ 0.984 \end{bmatrix}$$



Input ζ_1, output amplitudes $\nu = [0.273,\, 0.123]$

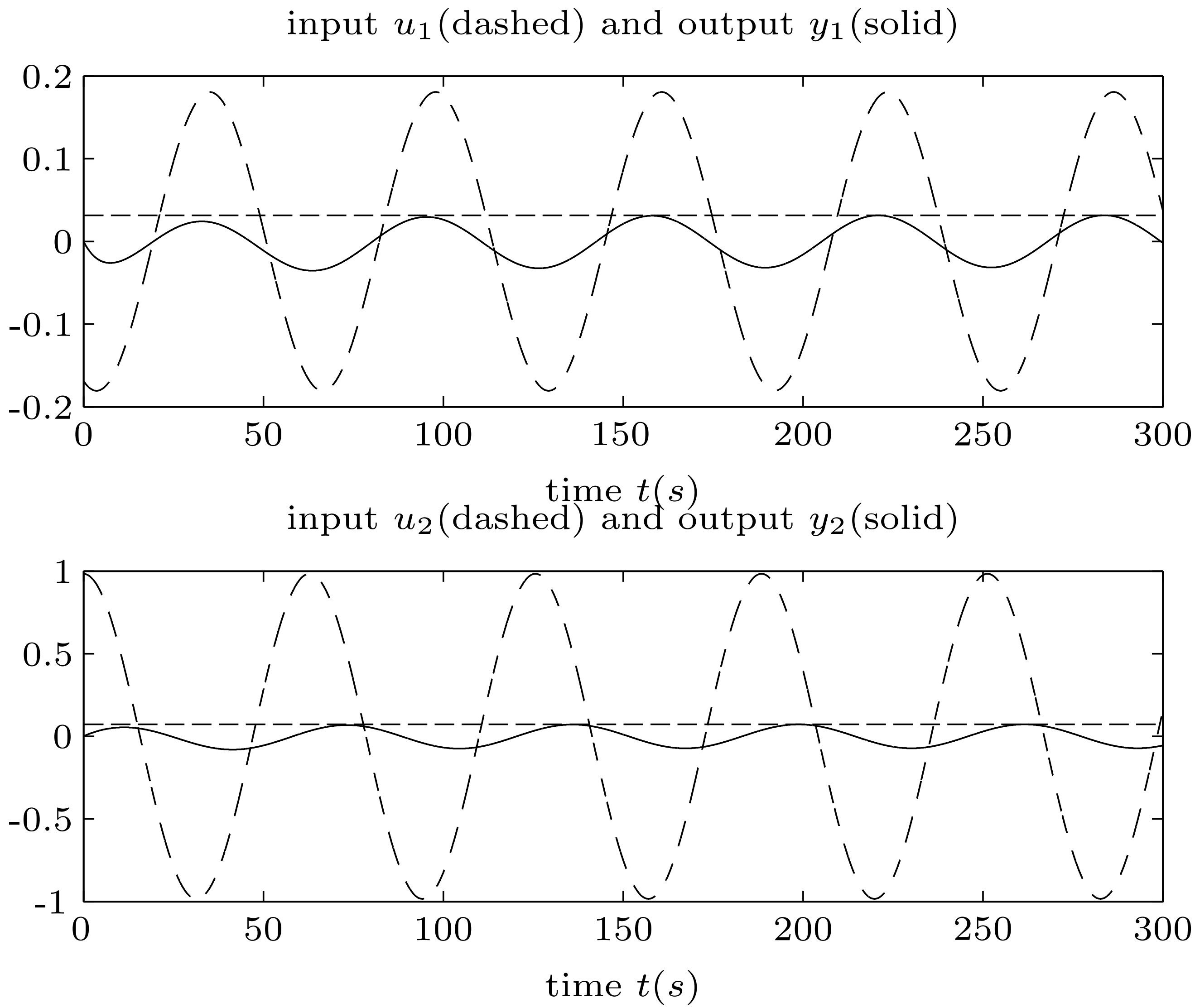



0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
channel 2
1
0.5
0
-0.5
-1
channel 1
0
50
100
150
200
250
300
time t (s)



0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
μ_2
ν_2
ν_1
μ_1
u
y_∞

Input ζ_2, output amplitudes $\nu = [0.045, 0.067]$



Physical interpretation:

- Having the two inputs “in phase” (the temperature variations have the same sign), yields a stronger output temperature variation than in the case where the hot input temperature rises when the cold input temperature falls.
- The gain from the hot to the cold leg ($P_{2,1} = -0.033 - j\,0.107$) is larger that the gain from the cold to the hot leg ($P_{1,2} = -0.013 - j\,0.043$). Accordingly, since the total temperature variations are limited by the constraint $||u|| = 1$, the amplitude of u_1 is chosen larger than that of u_2 if a large effect has to be achieved, as can be seen in ζ_1.
- The same arguments show that in the case where the effect has to be as small as possible, almost all input action needed to satisfy $||u|| = 1$ is placed in u_2 which has the smaller effect on the hot leg.

Summary: In the MIMO case there are no immediate counterparts to the SISO Nyquist or Bode diagrams. The only frequency response tools are singular value plots. These plots are magnitude plots and they contain "worst-case" information only. No phase information is available.

Besides the transfer function of the plant, other transfer functions can be mapped by singular-value plots:

$$T(s) = [I - P(s) \cdot C(s)]^{-1} \cdot P(s) \cdot C(s) \tag{83}$$

$$S(s) = [I - P(s) \cdot C(s)]^{-1} \tag{84}$$

$$D(s) = I + P(s) \cdot C(s) \tag{85}$$

These relations use a loop-breaking point at the controller input. Contrary to the SISO case, for MIMO systems the choice of loop-breaking point is relevant.

Design of MIMO Systems in the FD

One of the main reasons why the $W_1(s)$, $W_2(s)$ formalism is so widely used is that it can be easily generalized to MIMO systems. In this case, all assertions related to the maximum magnitude of a transfer function must be replaced by constraints on the maximum *singular value* of a transfer function matrix.

For instance the condition

$$||S(s) \cdot W_1(s)||_\infty = \max_\omega \{|S(j\,\omega) \cdot W_1(j\,\omega)|\} < 1 \tag{86}$$

valid for SISO systems can be written in the MIMO case as follows

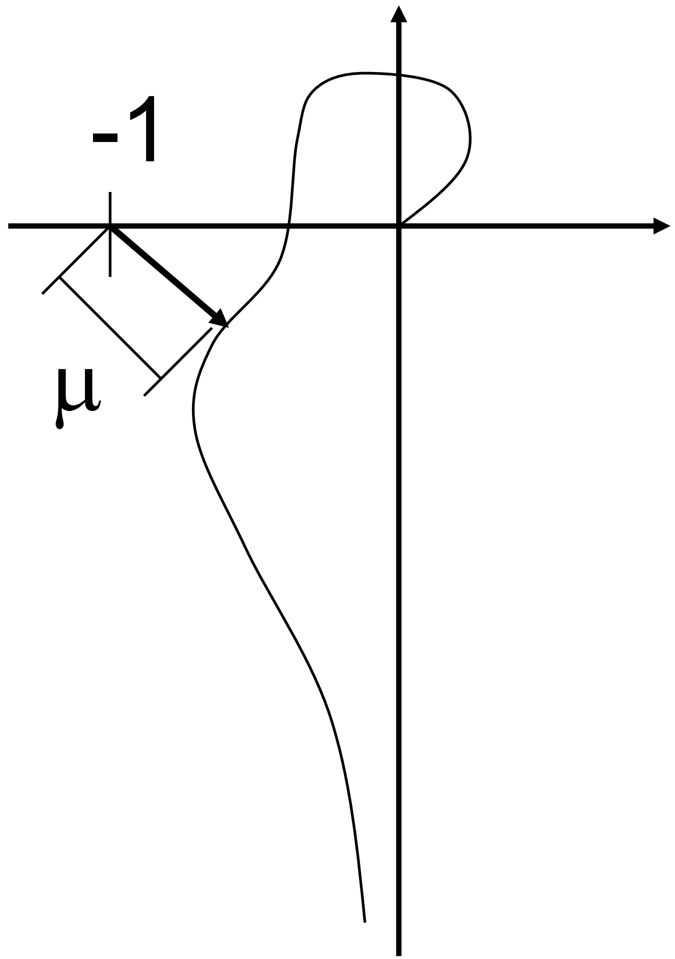
$$||S(s) \cdot W_1(s)||_\infty = \max_\omega \{\sigma_{\max} \{S(j\,\omega) \cdot W_1(j\,\omega)\}\} < 1 \tag{87}$$

(similar for all other constraints and equations).

SISO

$$1 + L(j\omega)$$

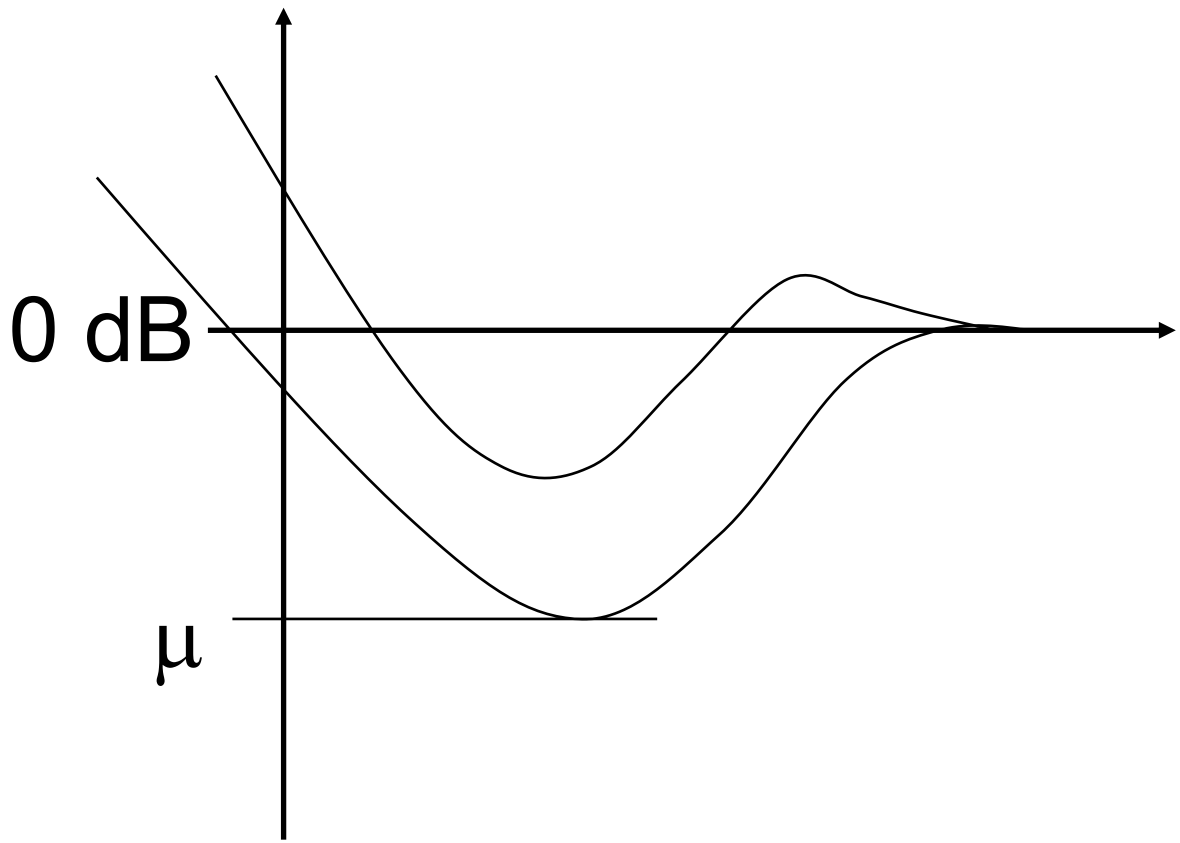
$$\mu = \min_{\omega}\{|1 + L(j\omega)|\}$$



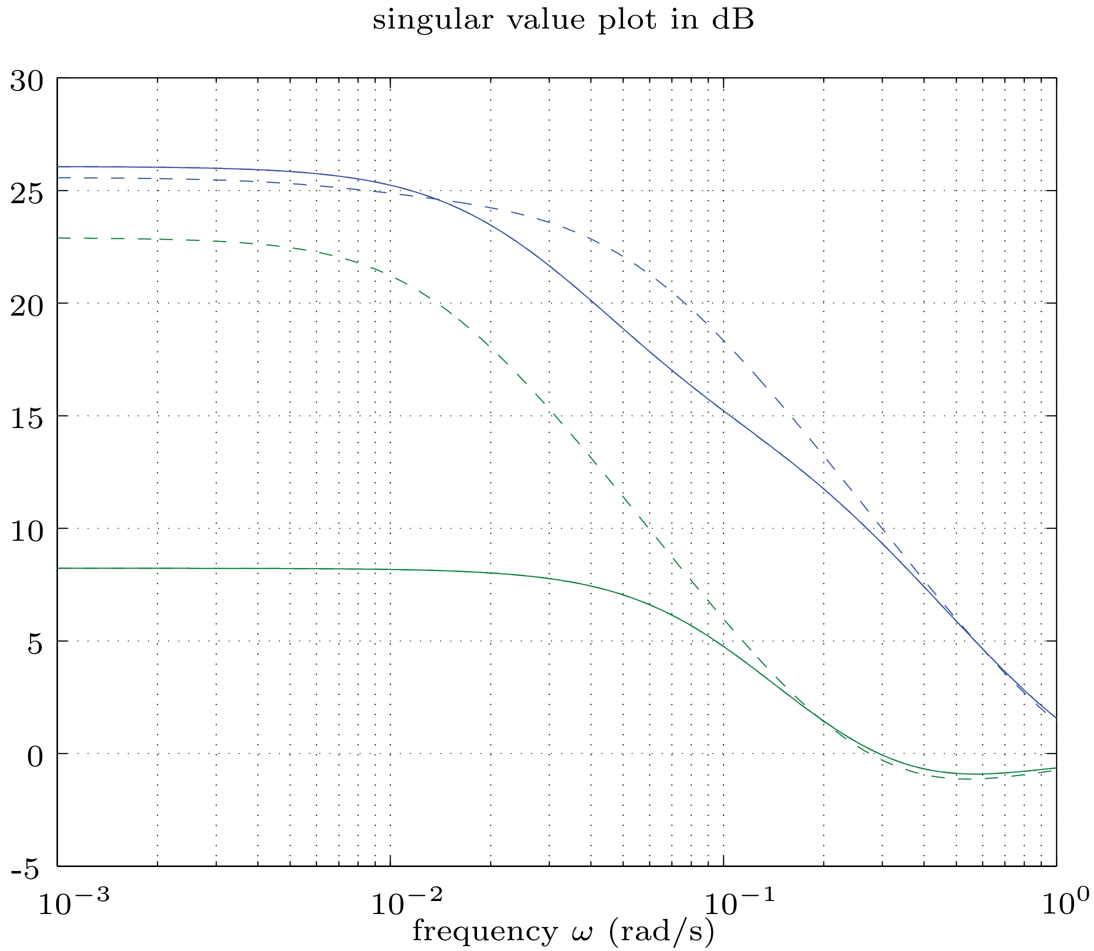
MIMO

$$I + L(j\omega)$$

$$\mu = \min_{\omega}\left\{\sigma_{\min}\left\{I + L(j\omega)\right\}\right\}$$



Example: singular values of the return difference of the heat exchanger (LQG controller, will be introduced later)



Solid $k = 2000\,W/(m^2\,K)$, dashed $k = 100\,W/(m^2\,K)$

Useful time-domain relationships

$$\text{plant}: \quad \frac{d}{dt}x(t) = A\,x(t) + B\,u(t), \quad y(t) = C\,x(t) \tag{88}$$

$$\text{controller}: \quad \frac{d}{dt}z(t) = F\,z(t) + G\,e(t), \quad u(t) = H\,z(t) \tag{89}$$

Open loop gain $L(s)$:

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A & B\,H \\ 0 & F \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} e, \quad y = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} \tag{90}$$

Return difference $I + L(s)$:

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A & B\,H \\ 0 & F \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} e, \tag{91}$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + I\,e$$

Complementary sensitivity $T(s)$:

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A & B\,H \\ -G\,C & F \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} r, \tag{92}$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}$$

Sensitivity $S(s)$:

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A & B\,H \\ -G\,C & F \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ -G \end{bmatrix} d, \tag{93}$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + I\,d \tag{94}$$

Lecture VIII – Synthesis of MIMO Control System, Pole Placement, LQR Approach

State-Feedback Controllers

General problem formulation of a *regulator problem*

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t), \quad x(0) = x_0 \neq 0 \tag{95}$$

Find a *state-feedback* control signal

$$u(t) = f(x(t)) \tag{96}$$

such that

$$\lim_{t \to \infty} x(t) = 0 \tag{97}$$

Several methods available for the solution of this problem:

- eigenvalue placement
- linear-quadratic optimal control

Eigenvalue Placement

Objective: use feedback to place eigenvalues of feedback-controlled system at desired places.

Problem easy to solve in the state-feedback case (provided that the system $\{A, B\}$ is stabilizable) by using

$$u(t) = -K \cdot x(t) \tag{98}$$

Choose matrix $K \in \Re^{m \times n}$ such that all eigenvalues of $A - B \cdot K$ have negative real parts and are located at some desired places (this is always possible under these assumptions).

Example ($n=3$): SISO, $\{A, b\}$ in controller-canonical form (obviously completely controllable)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad k = \begin{bmatrix} k_0 & k_1 & k_2 \end{bmatrix}$$

Closed-loop system

$$A - b \cdot k = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(a_0 + k_0) & -(a_1 + k_1) & -(a_2 + k_2) \end{bmatrix}$$

Choose any desired real constants $\{\alpha_0, \alpha_1, \alpha_2\}$ yielding a Hurwitz polynomial (all roots in the negative complex plane)

$$\alpha(s) = s^3 + \alpha_2 \cdot s^2 + \alpha_1 \cdot s + \alpha_0$$

Comparing this polynomial to the closed-loop characteristic polynomial

$$\det[s \cdot I - (A - b \cdot k)] = \det \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ (a_0 + k_0) & (a_1 + k_1) & s + (a_2 + k_2) \end{bmatrix}$$

$$= s^3 + (a_2 + k_2) \cdot s^2 + (a_1 + k_1) \cdot s + (a_0 + k_0)$$

yields the controller coefficients

$$k_0 = \alpha_0 - a_0,\ k_1 = \alpha_1 - a_1,\ k_2 = \alpha_2 - a_2$$

What happens if the system $\{A, b\}$ has this state-space description?

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a_1 & -a_2 & -a_3 & 1 \\ 0 & 0 & 0 & -a_4 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

General Case, SISO Systems

Many approaches known, for instance the "Formula of Ackermann." First, choose a desired characteristic polynomial

$$\alpha(s) = s^n + \alpha_{n-1} \cdot s^{n-1} + \ldots + \alpha_1 \cdot s + \alpha_0 \tag{99}$$

Then form the vector q as follows

$$q = [0, 0, \ldots 0, 1] \cdot \mathcal{R}^{-1} \tag{100}$$

The state-feedback gain k that places the eigenvalues of $A - b \cdot k$ at the locations defined by the solutions of $\alpha(s) = 0$ is given by

$$k = q \cdot \alpha(A) \tag{101}$$

where

$$\alpha(A) = A^n + \alpha_{n-1} \cdot A^{n-1} + \ldots + \alpha_1 \cdot A + \alpha_0 \cdot I \tag{102}$$

(for a proof see [Kailath]). QC: Apply Ackermanns formula to the 3×3 example shown above.

Remarks:

- Obviously, complete controllability ($\det\{\mathcal{R}\} \neq 0$) is a necessary and sufficient condition to be able to arbitrarily place the eigenvalues of $A - b \cdot k$.
- This result holds true also in the MIMO case.
- Controller designs using eigenvalue placement approaches are tricky. In the SISO case this approach can lead to acceptable results but only with some care (robustness can be small).
- In the MIMO case this approach often fails and is *not* recommended (the situation where the different channels work against each other is difficult to avoid).

MIMO Synthesis?

Model-based optimization!

H_∞, FD, “modern”

⇨ later

H_2 (LQG), TD, “classic”

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t)$$

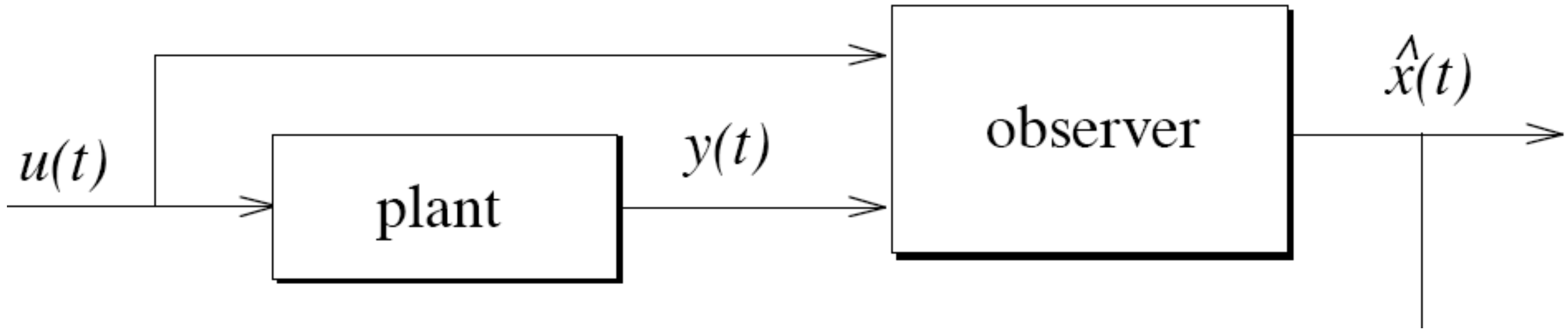
$$y(t) = C \cdot x(t) + D \cdot u(t)$$

“a-posteriori” interpretation in FD!

new page numbers!

Plan:

1. $u(t) = f(x(t), t)$

2. 

3.

4. Many improvements

Example “regulator problem”?

All other control objectives (reference tracking, disturbance rejection, …) can be reformulated in this framework

Many solutions $u(t) = f(x(t), t)$ possible

Find the one that minimizes

$$J(u) = \int_0^\infty \left[x^T(u(t)) \cdot Q \cdot x(u(t)) + u^T(t) \cdot R \cdot u(t) \right] dt$$

$$Q = Q^T \in \mathbb{R}^{n \times n},\ Q \geq 0, \quad \text{and} \quad R = R^T \in \mathbb{R}^{m \times m},\ R > 0$$

Interpretation:

Special Case: SISO and $Q = c^T \cdot c$

Objective function:

$$J(u) = \int_0^\infty \left[y^2(u(t)) + r \cdot u^2(t) \right] dt$$

$$E_y = \int_0^\infty y^2(u(t))dt \qquad E_u = \int_0^\infty u^2(t)dt$$

Interpretation:

$$J(u) = E_y + \qquad E_u$$

cheap expensive

control

State Feedback Controllers

Solution to optimization problem:

$$u(t) = -K \cdot x(t), \quad \text{where} \quad K = R^{-1} \cdot B^T \cdot \Phi$$

Riccati equation

$$\Phi \cdot B \cdot R^{-1} \cdot B^T \cdot \Phi - \Phi \cdot A - A^T \cdot \Phi - Q = 0$$

obviously $\Phi = \Phi^T$

Solution Φ>0 exists if:

c1: the pair $\{A, B\}$ is completely controllable; and
c2: the pair $\{A, \bar{C}\}$ is completely observable.

$$Q = \bar{C}^T \cdot \bar{C}$$

Remarks:

$$u(t) = -K \cdot x(t)$$

- is a *linear* feedback law
- is a *time-invariant* feedback law
- $A - B \cdot K$ is a Hurwitz matrix

Remarks:

- Q and R are the "tuning knobs"

- First guess

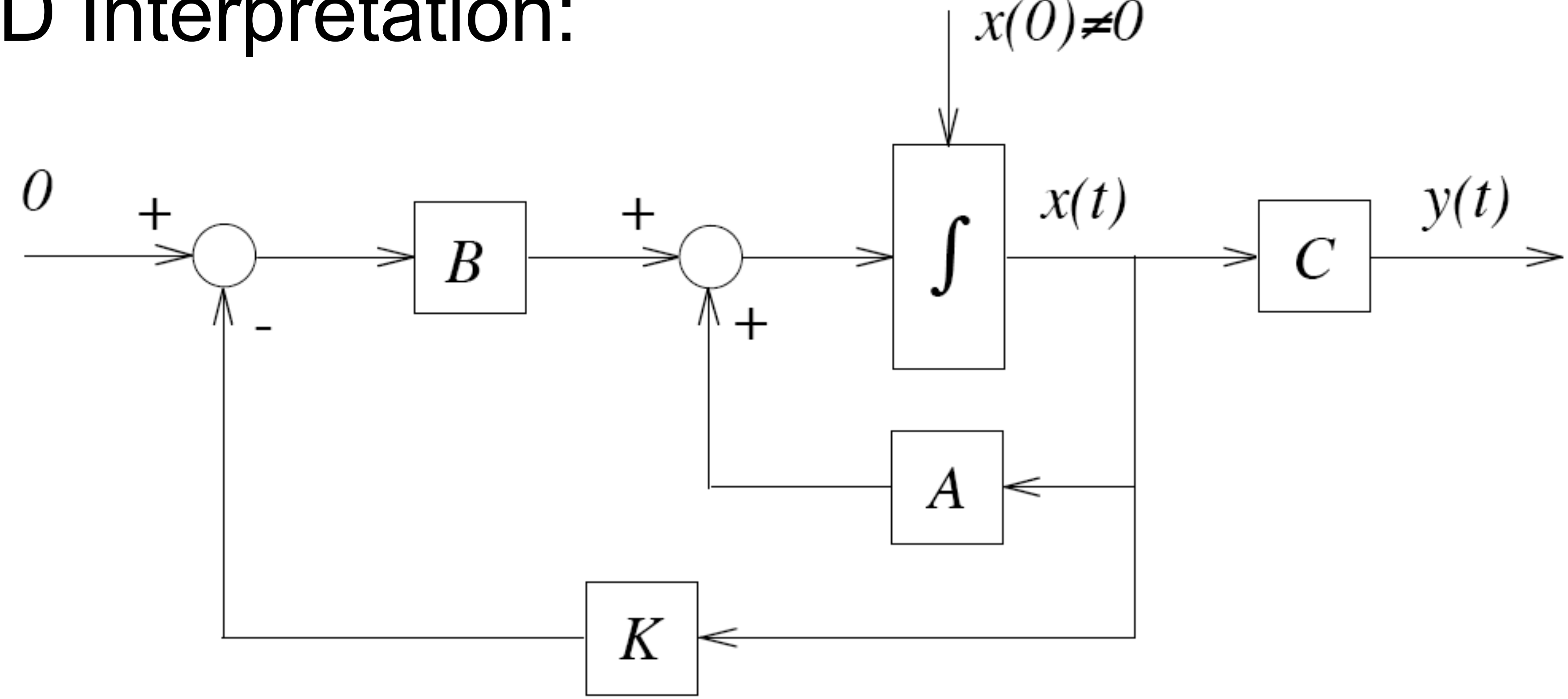
$$R = r \cdot I_{m \times m}, \quad r > 0$$

$$Q = C^T \cdot C$$

- CACSD `K=lqr(A,B,Q,R)`

- Proof:
  - sufficiency easy (see handouts)
  - necessity difficult

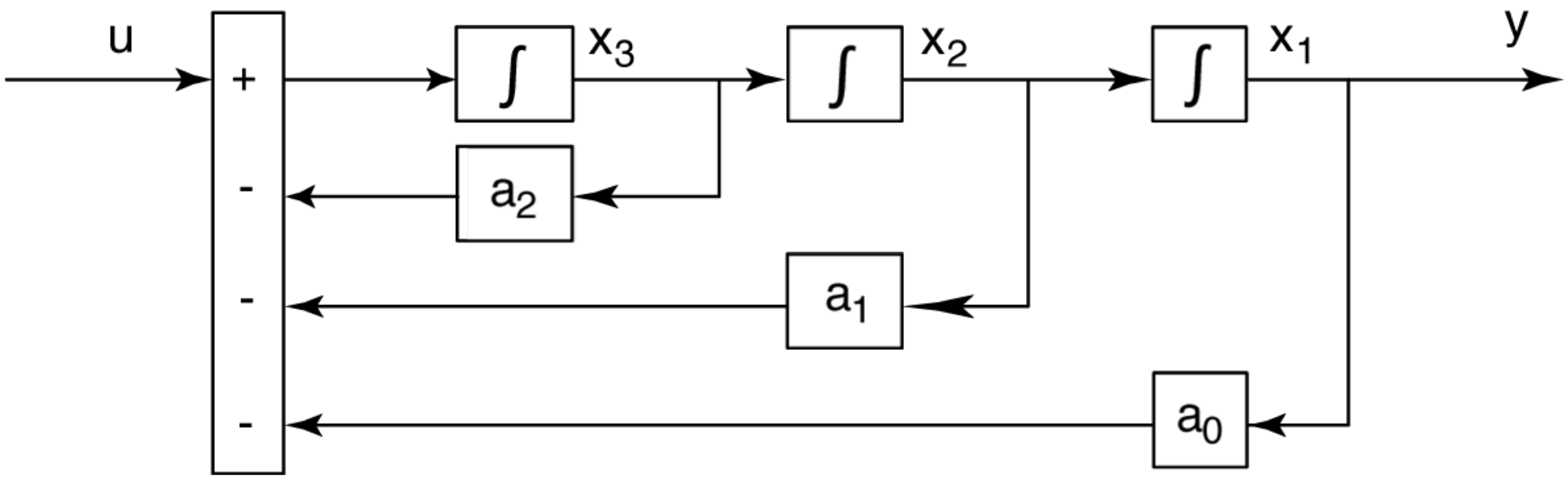
FD Interpretation:



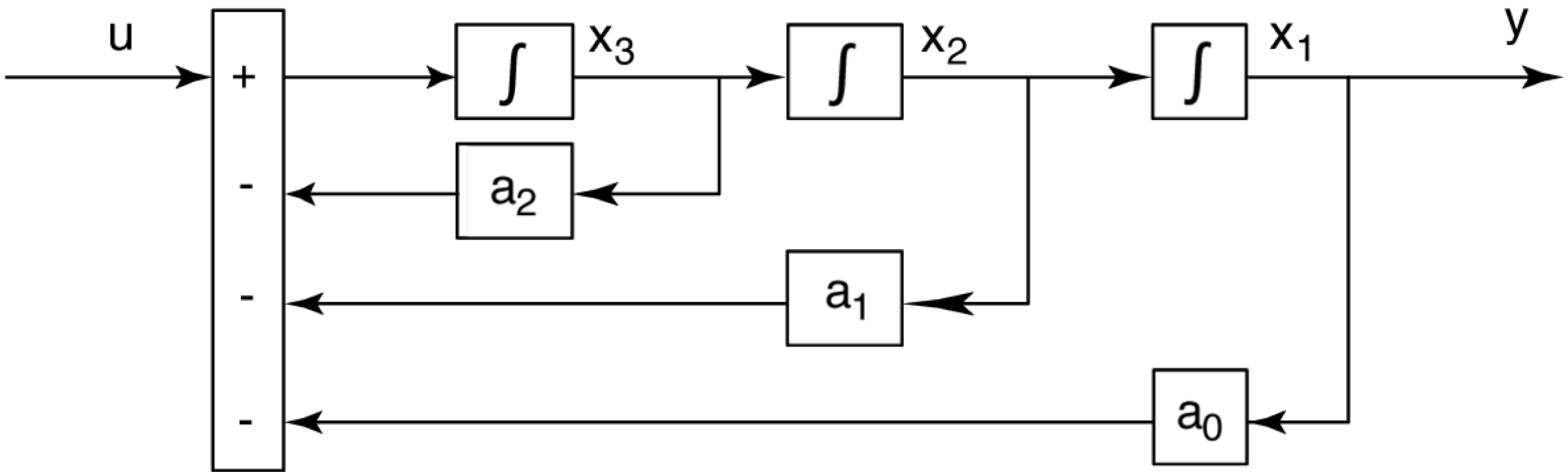
$$T_{LQR}(s) = C \cdot [s\,I - (A - B \cdot K)]^{-1} \cdot B$$

$$L_{LQR}(s) = K \cdot [s\,I - A]^{-1} \cdot B$$

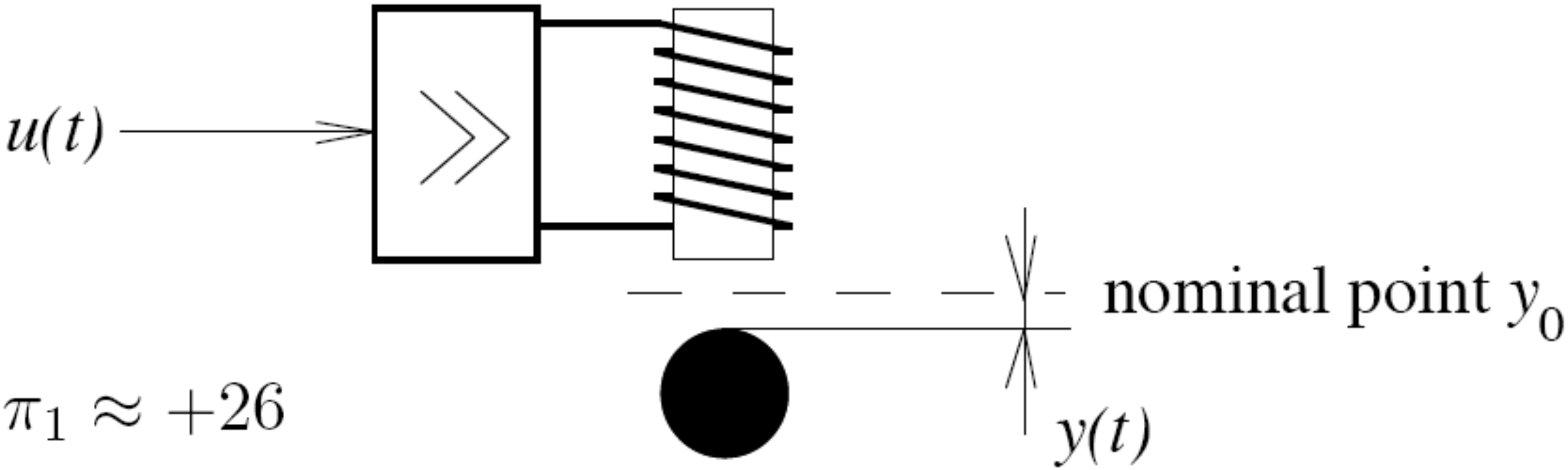
Question: Block diagram of SISO LQR controller $u(t)=-k\, x(t)$ (example $n=3$, $y(t)=x_1(t)$)?



Question: SISO LQR controller u(t)=-k x(t) similar to which „classical“ controller?



Case Study: Levitating Sphere



$\pi_1 \approx +26$

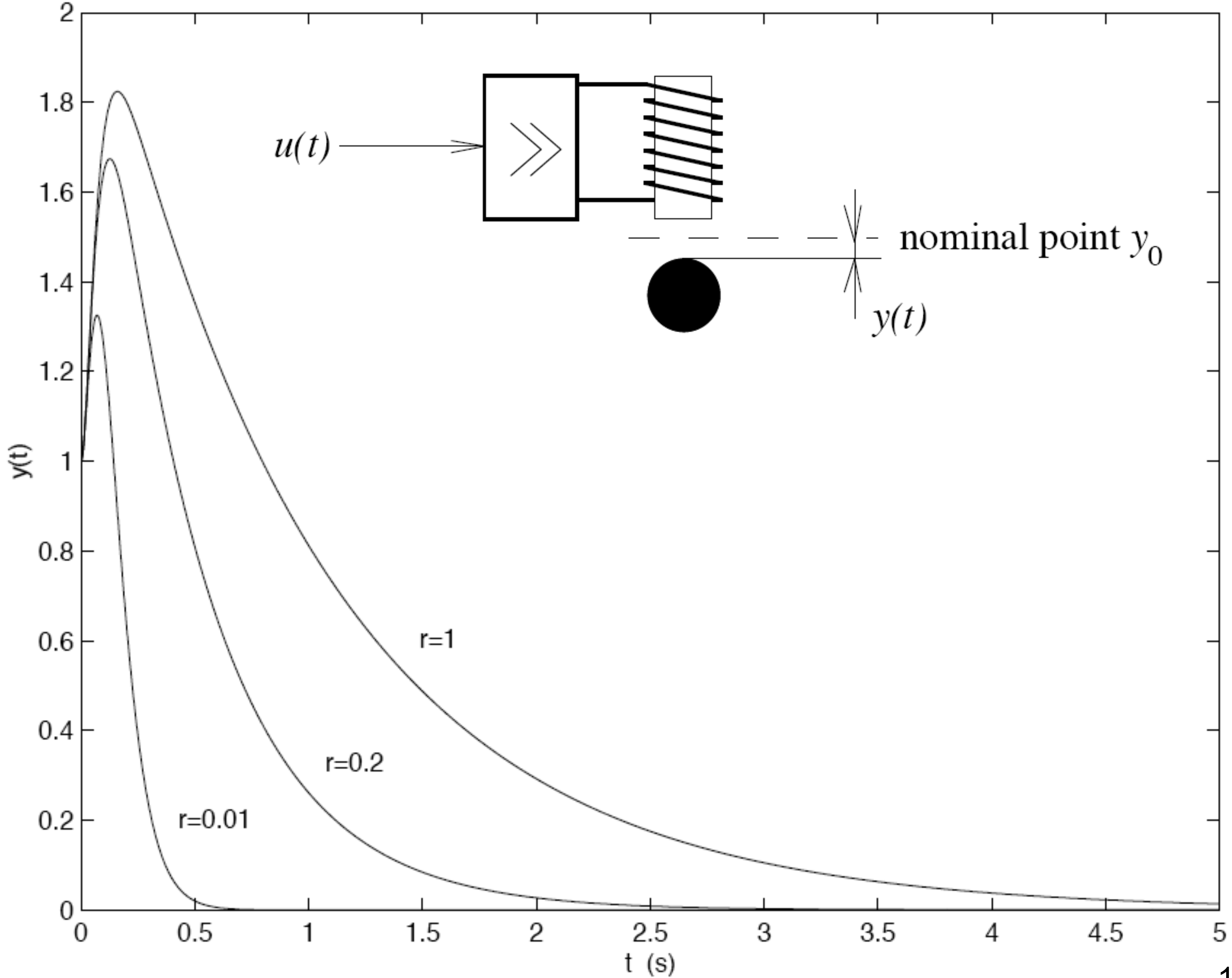
$$\frac{d}{dt}x(t) = \begin{bmatrix} 0 & 1 & 0 \\ 700 & 0 & -700 \\ 0 & 0 & -0.2 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \cdot x(t)$$

$$J(u) = \int_0^\infty \left[y^2(u(t)) + r \cdot u^2(t) \right] dt$$

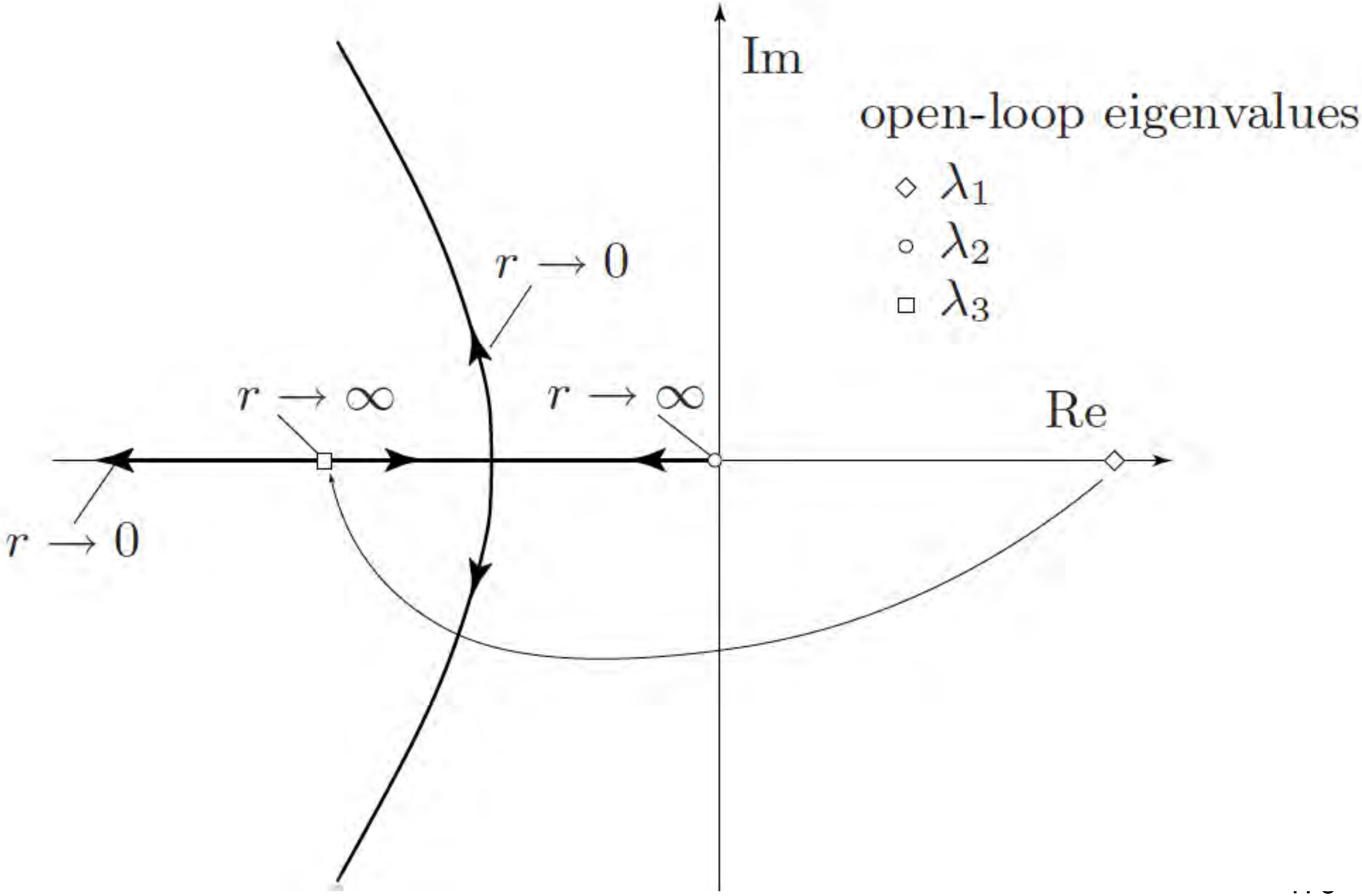
$$u(t) = -[k_1(r), k_2(r), k_3(r)] \cdot x(t)$$

$r = 1$	$E_u = 59$	$E_y = 2.17$
$r = 0.2$	$E_u = 62$	$E_y = 1.01$
$r = 0.01$	$E_u = 79$	$E_y = 0.27$



u(t)
nominal point y0
y(t)
r=1
r=0.2
r=0.01
y(t)
t (s)

eigenvalues of $A - b \cdot k(r)$ for $r \in (0, \infty)$

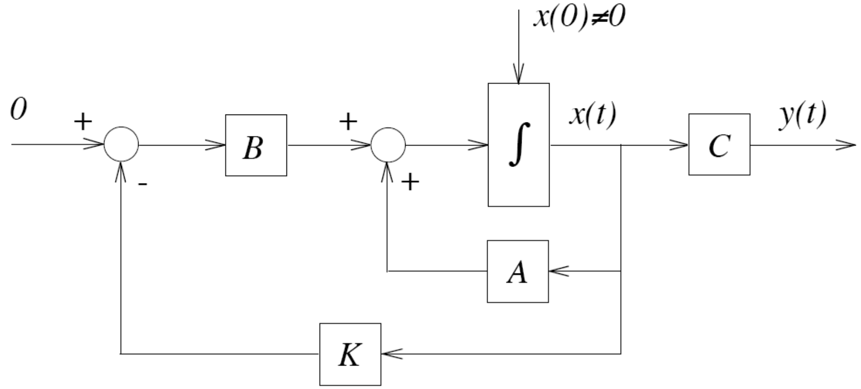


FD Interpretation

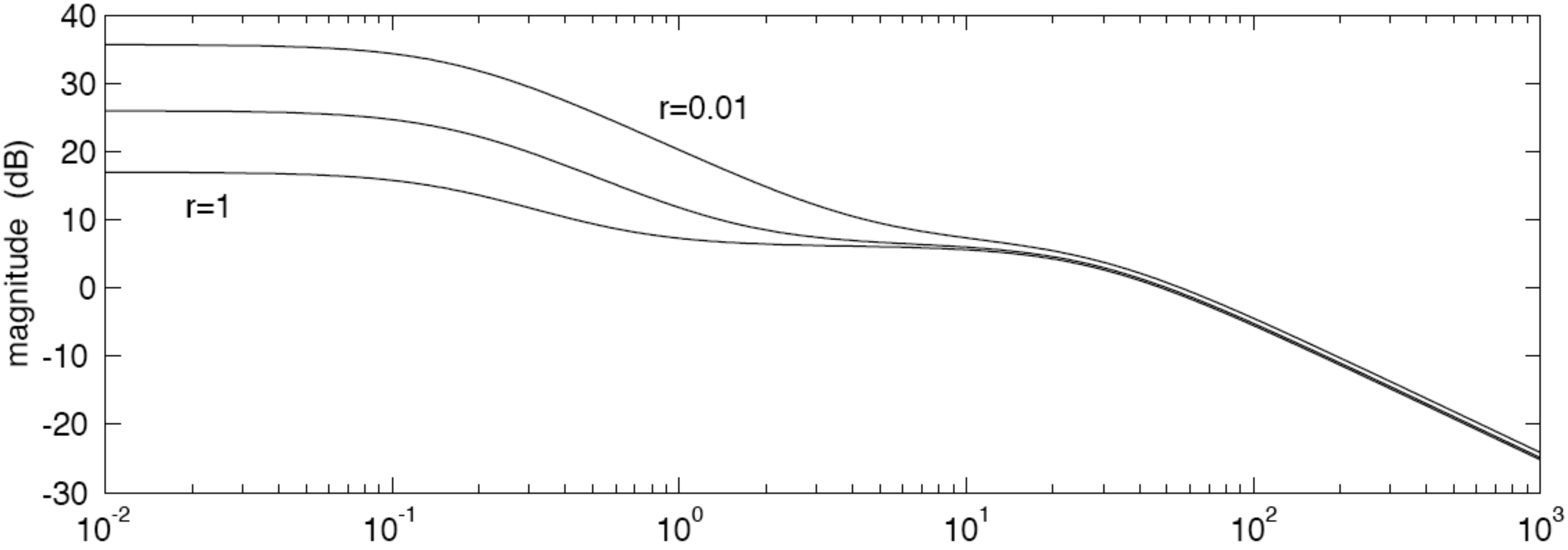
$$L_{LQR}(s) = k \cdot [s\,I - A]^{-1} \cdot b$$

$$= \frac{k \cdot \mathrm{Adj}(s\,I - A) \cdot b}{\det(s\,I - A)}$$

$$= \frac{b(s)}{a(s)}$$



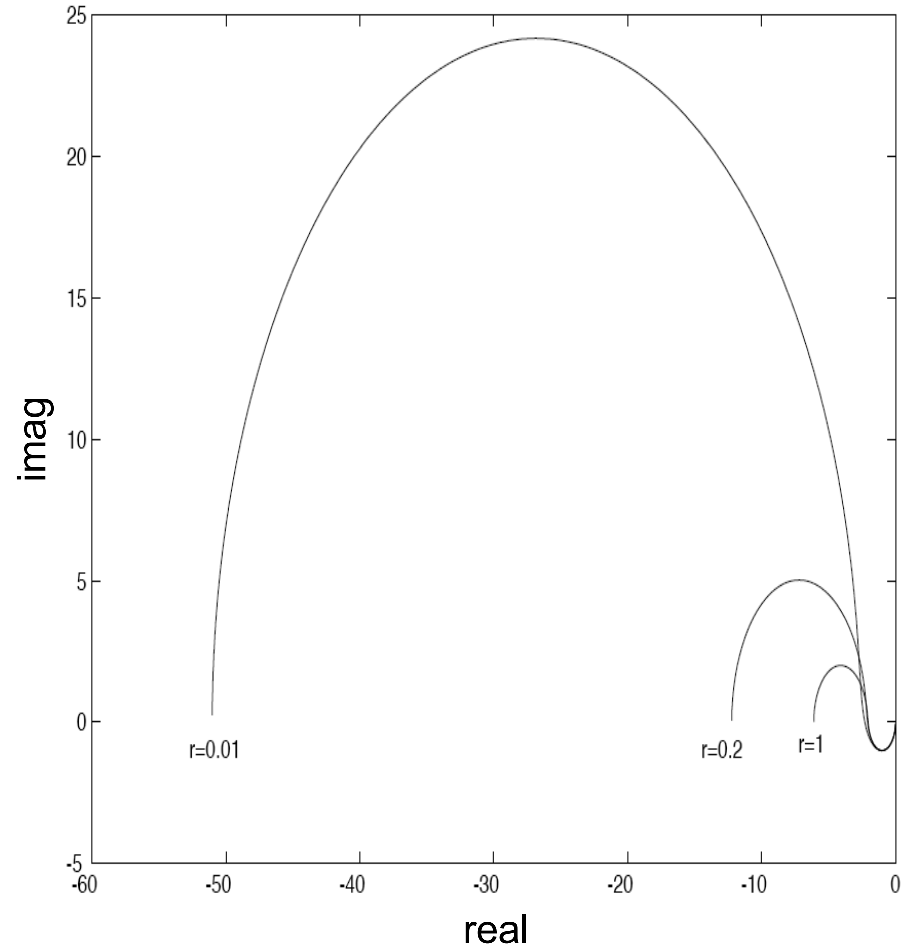
Bode Diagram (magnitude):

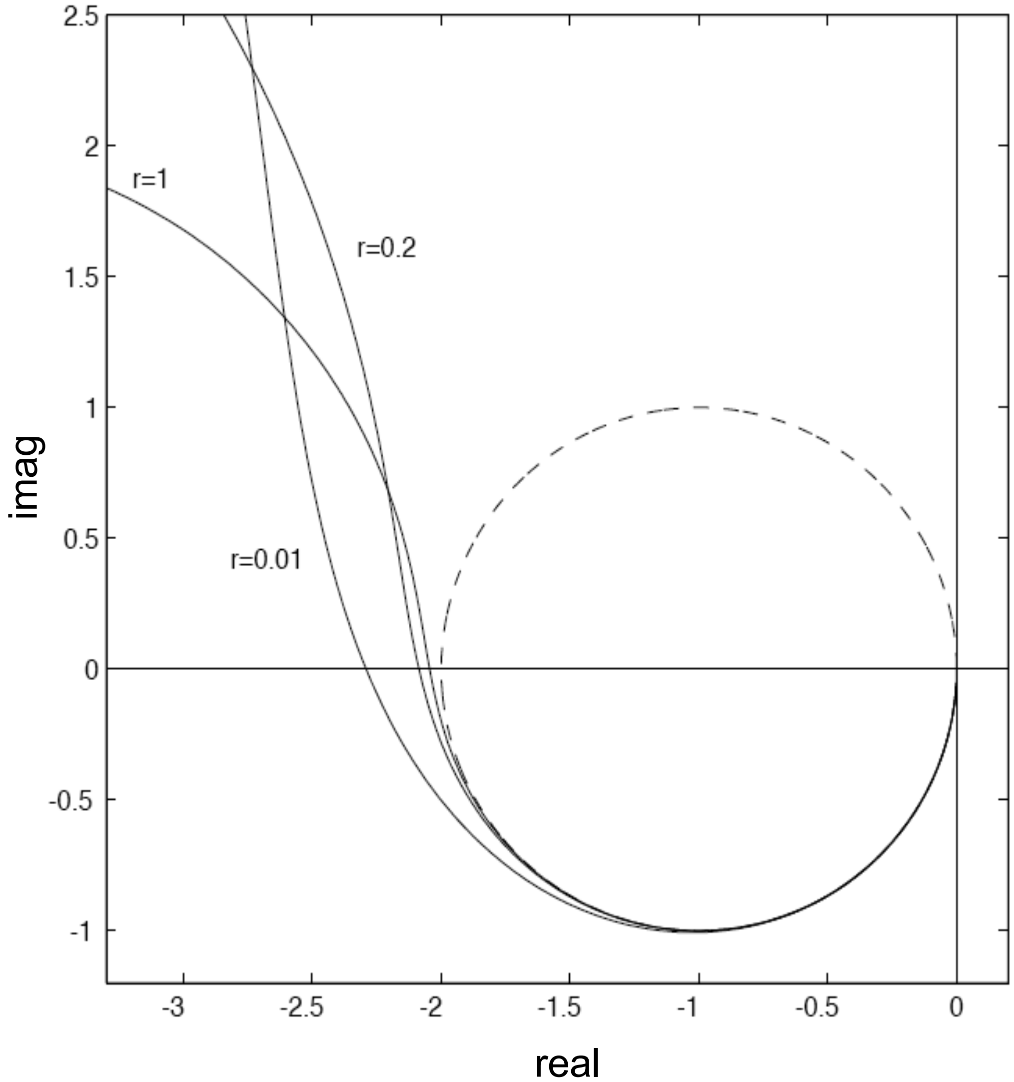


Obviously, relative degree = 1 (*k* = "full")

$$L_{LQR}(s) = \frac{b_{n-1} \cdot s^{n-1} + \ldots + b_0}{s^n + \ldots + a_0}$$

Nyquist Diagram of $L_{LQR}(s) = k \cdot [s\,I - A]^{-1} \cdot b$





Main result:

$$|1 + L_{LQR}(j\,\omega, r)| \geq 1, \quad \forall\, \omega \in \mathbb{R} \text{ and } r \in \mathbb{R}_+$$

Quick Check:

What are the gain and phase margins for SISO LQR loops?

Properties of LQR Controllers

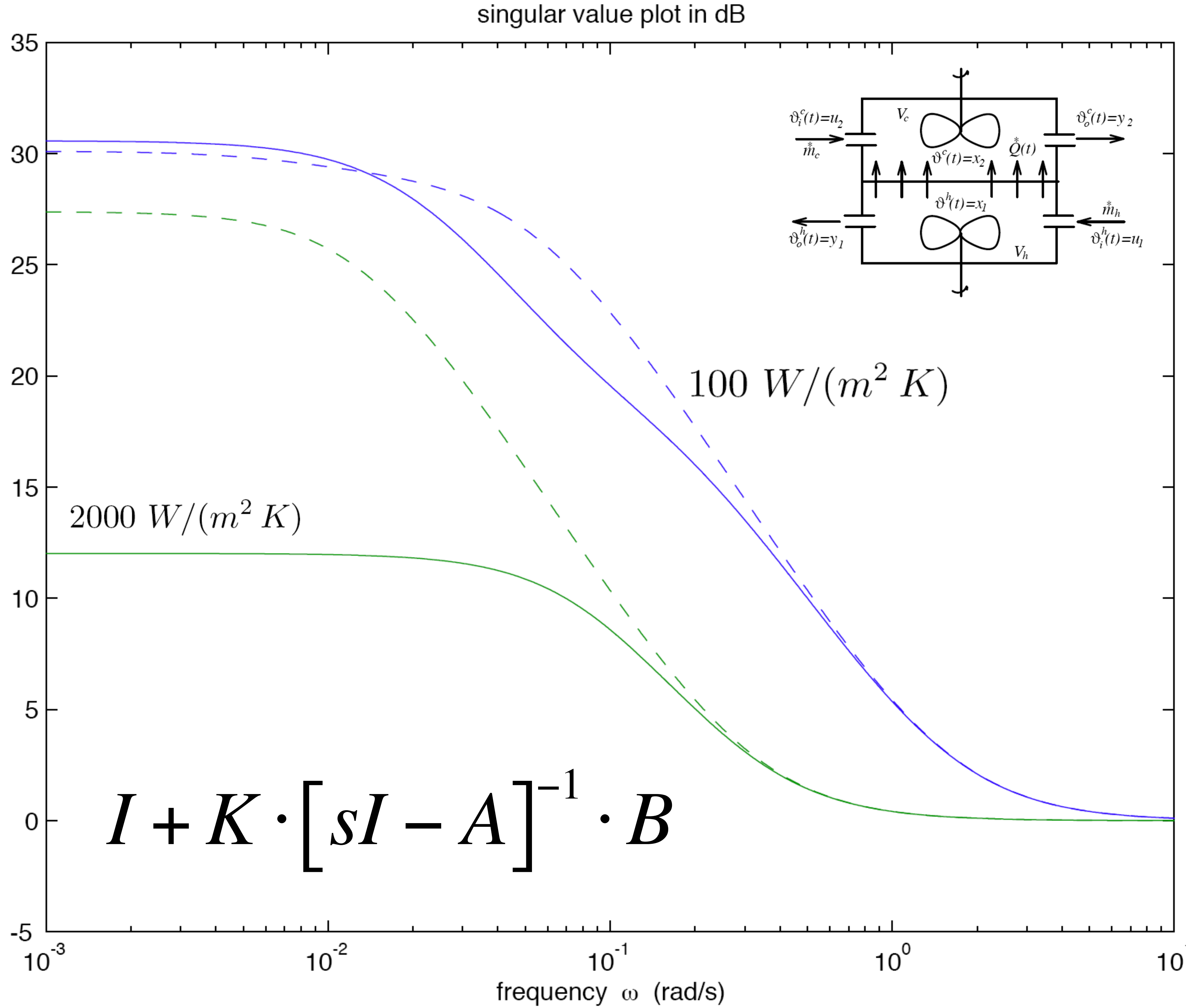
$$A - B \cdot K \text{ is always a Hurwitz matrix}$$

For $R = r \cdot I$ (arbitrary $Q = Q^T \in \mathbb{R}^{n \times n},\ Q \geq 0$)

$$\mu_{LQR} = \min_{\omega} \sigma_{\min}\{I + L_{LQR}(s)\} \geq 1$$

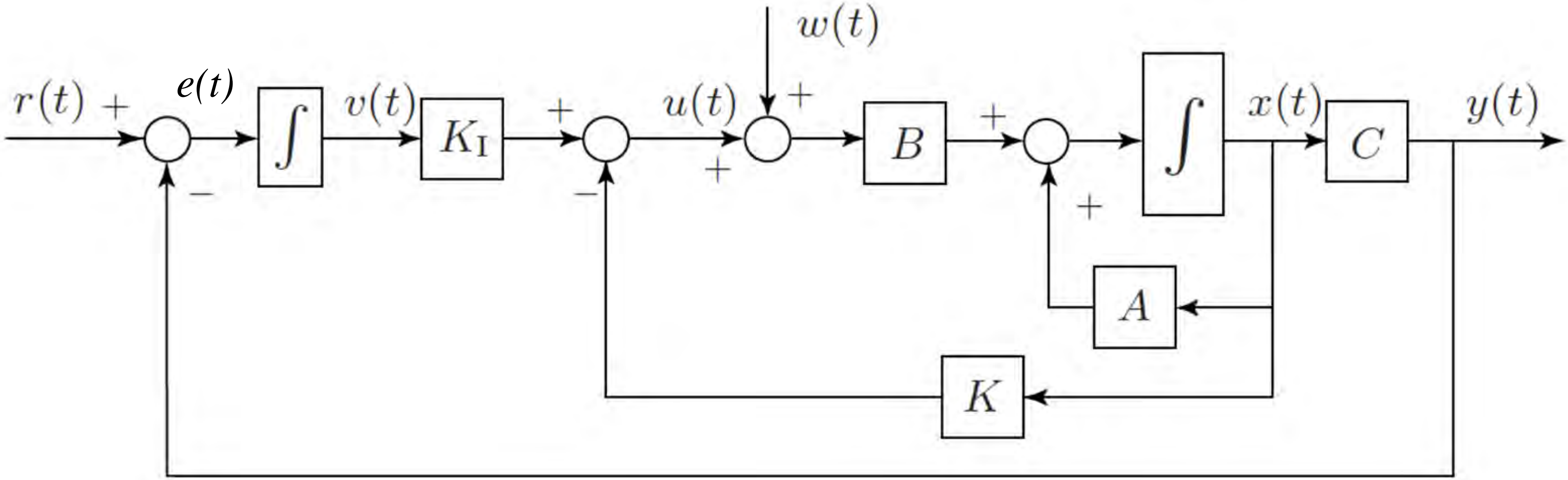
$$\max_{\omega} \sigma_{\max}\{S(j\,\omega)\} \leq 1$$

$$\max_{\omega} \sigma_{\max}\{T(j\,\omega)\} \leq 2$$


singular value plot in dB
35
30
25
20
15
10
5
0
-5
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
frequency ω (rad/s)
$100\ W/(m^2\ K)$
$2000\ W/(m^2\ K)$
$I + K \cdot [sI - A]^{-1} \cdot B$
$\vartheta_i^c(t)=u_2$
$\overset{*}{m}_c$
V_c
$\vartheta^c(t)=x_2$
$\overset{*}{Q}(t)$
$\vartheta_o^c(t)=y_2$
$\vartheta^h(t)=x_1$
$\overset{*}{m}_h$
$\vartheta_o^h(t)=y_1$
$\vartheta_i^h(t)=u_1$
V_h

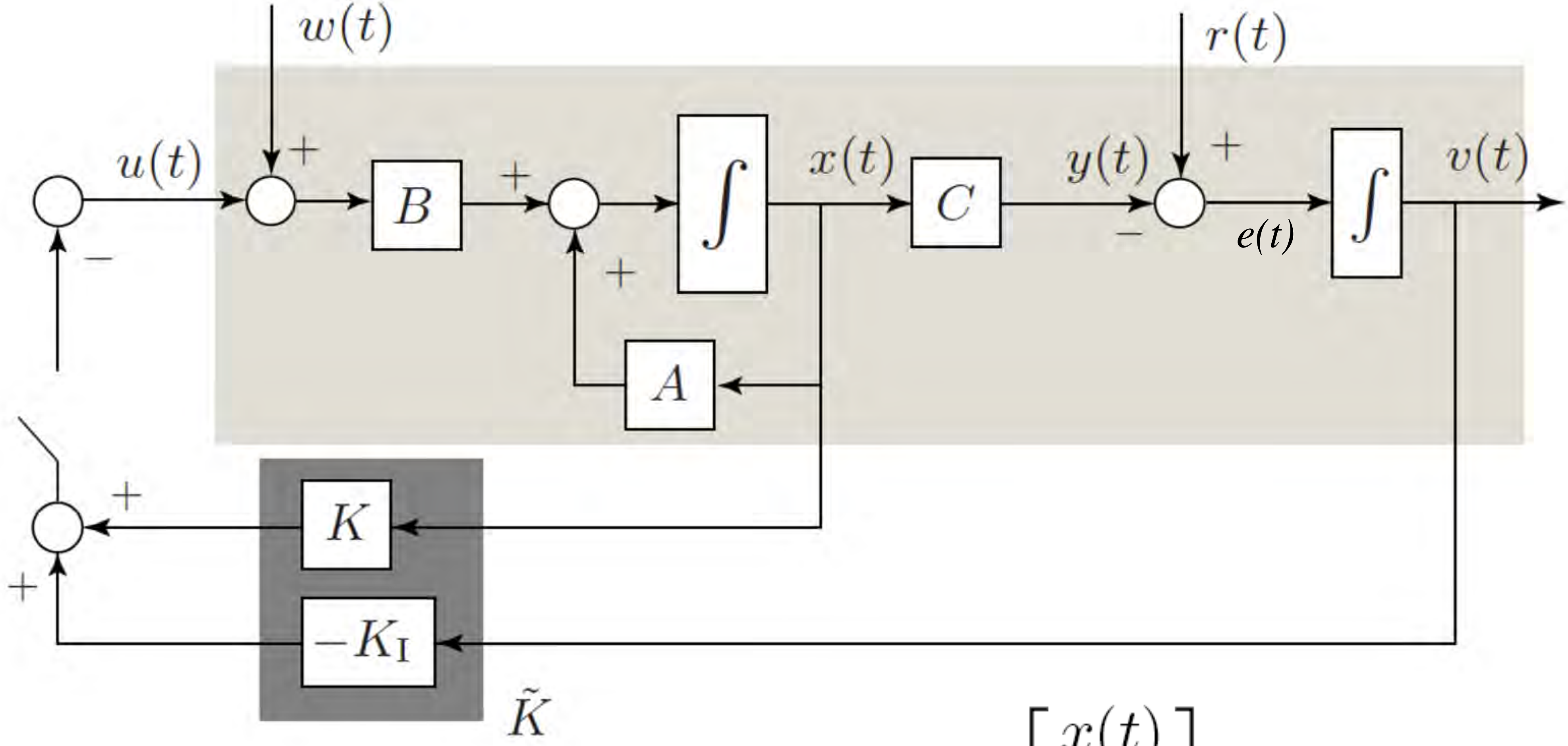
Lecture IX – Extensions of LQR Control Systems, State Observers

LQR-I Controller



Other structures possible

Re-arrange as standard LQR problem



$$\widetilde{x}(t) = \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} \in \mathbb{R}^{n+m}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}(t) = \widetilde{A} \cdot \widetilde{x}(t) + \widetilde{B}_u \cdot u(t) + \widetilde{B}_r \cdot r(t) + \widetilde{B}_w \cdot w(t)$$

$$\widetilde{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}, \quad \widetilde{B}_u = \widetilde{B}_w = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \widetilde{B}_r = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$\widetilde{C} = \begin{bmatrix} C & 0 \\ 0 & \gamma I \end{bmatrix} \gamma > 0 \text{ is a new tuning parameter}$$

Solve standard LQR problem for the extended System. Result:

$$\widetilde{K} = \begin{bmatrix} K & -K_I \end{bmatrix}$$

Robustness Enhancement

$$\frac{1}{\beta}\Phi_\beta \cdot B \cdot R^{-1} \cdot B^T \cdot \Phi_\beta - \Phi_\beta \cdot A - A^T \cdot \Phi_\beta - Q = 0 \qquad (3.40)$$

$$\beta > 1$$

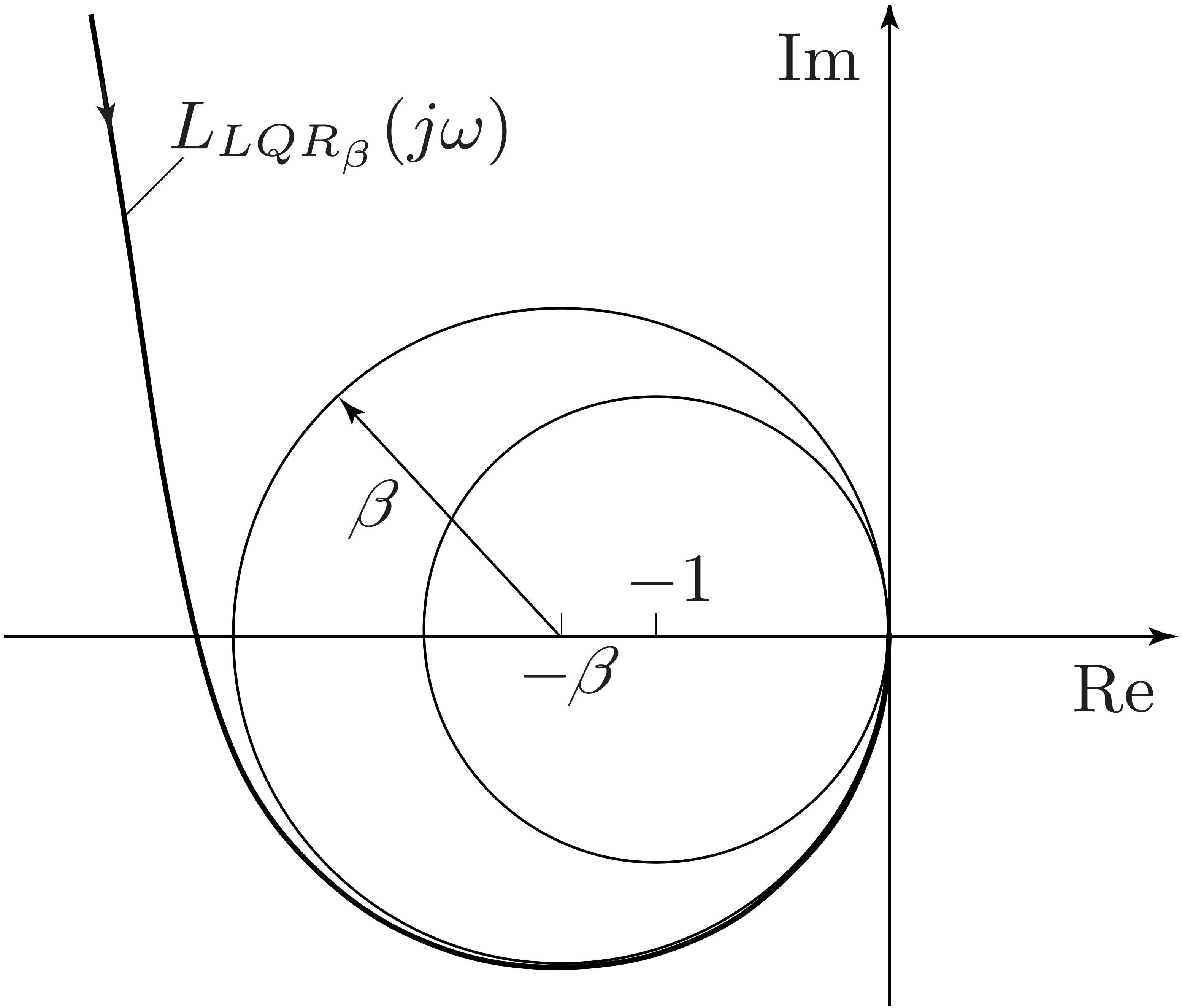
$$\mu_\beta = \min_\omega \sigma_{\min}\{\beta\, I + L(j\,\omega)\} \geq \beta$$

In the limit case $\beta \to \infty$

$$-\Phi_\infty \cdot A - A^T \cdot \Phi_\infty - Q = 0$$

$L_\infty(s) = K_\infty \cdot [s\,I - A]^{-1} \cdot B$ strictly positive real

Φ_∞ exists iff A is a Hurwitz matrix


Im
$L_{LQR_\beta}(j\omega)$
β
-1
$-\beta$
Re

Finite Horizon LQR

$$\dot{x}(t) = A(t) \cdot x(t) + B(t) \cdot u(t)$$

$$J(u) = x^T(t_b) \cdot P \cdot x(t_b) +$$

$$\int_{t_a}^{t_b} \left[x^T(u(t)) \cdot Q(t) \cdot x((u(t)) + u^T(t) \cdot R(t) \cdot u(t) \right] dt$$

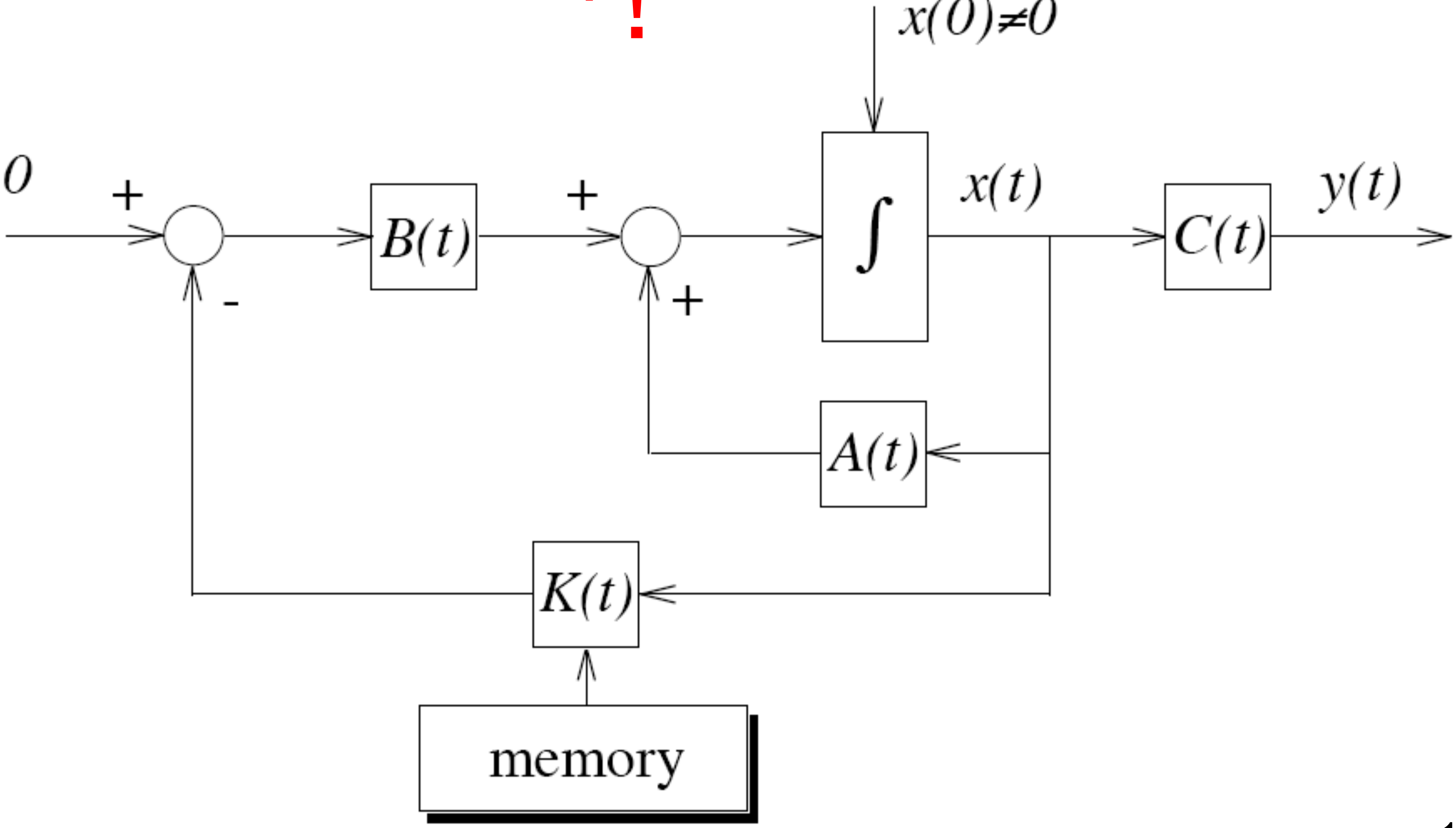
$$u(t) = -K(t) \cdot x(t)$$

$$K(t) = R^{-1}(t) \cdot B^T(t) \cdot \Phi(t)$$

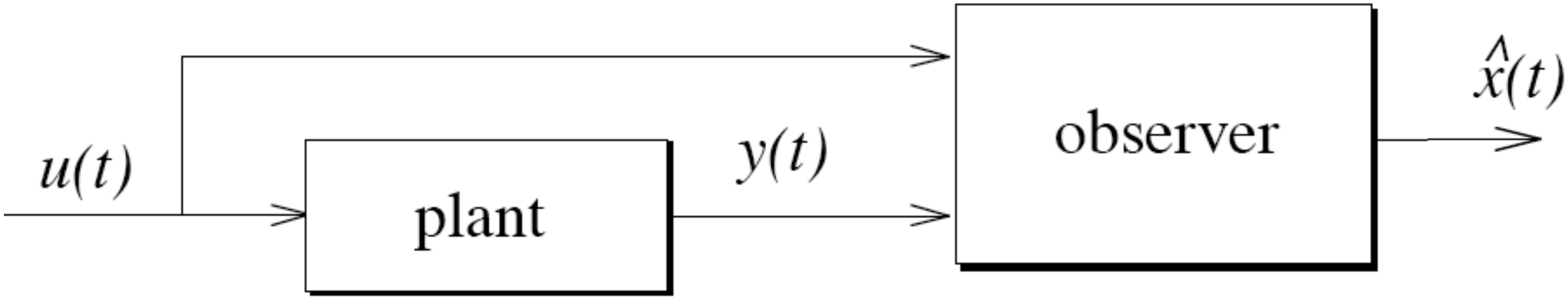
$$\frac{d}{dt}\Phi(t) = \Phi(t)\cdot B(t)\cdot R^{-1}(t)\cdot B^T(t)\cdot\Phi(t) - \Phi(t)\cdot A(t) - A^T(t)\cdot\Phi(t) - Q(t)$$

$$\Phi(t_b) = P$$

!

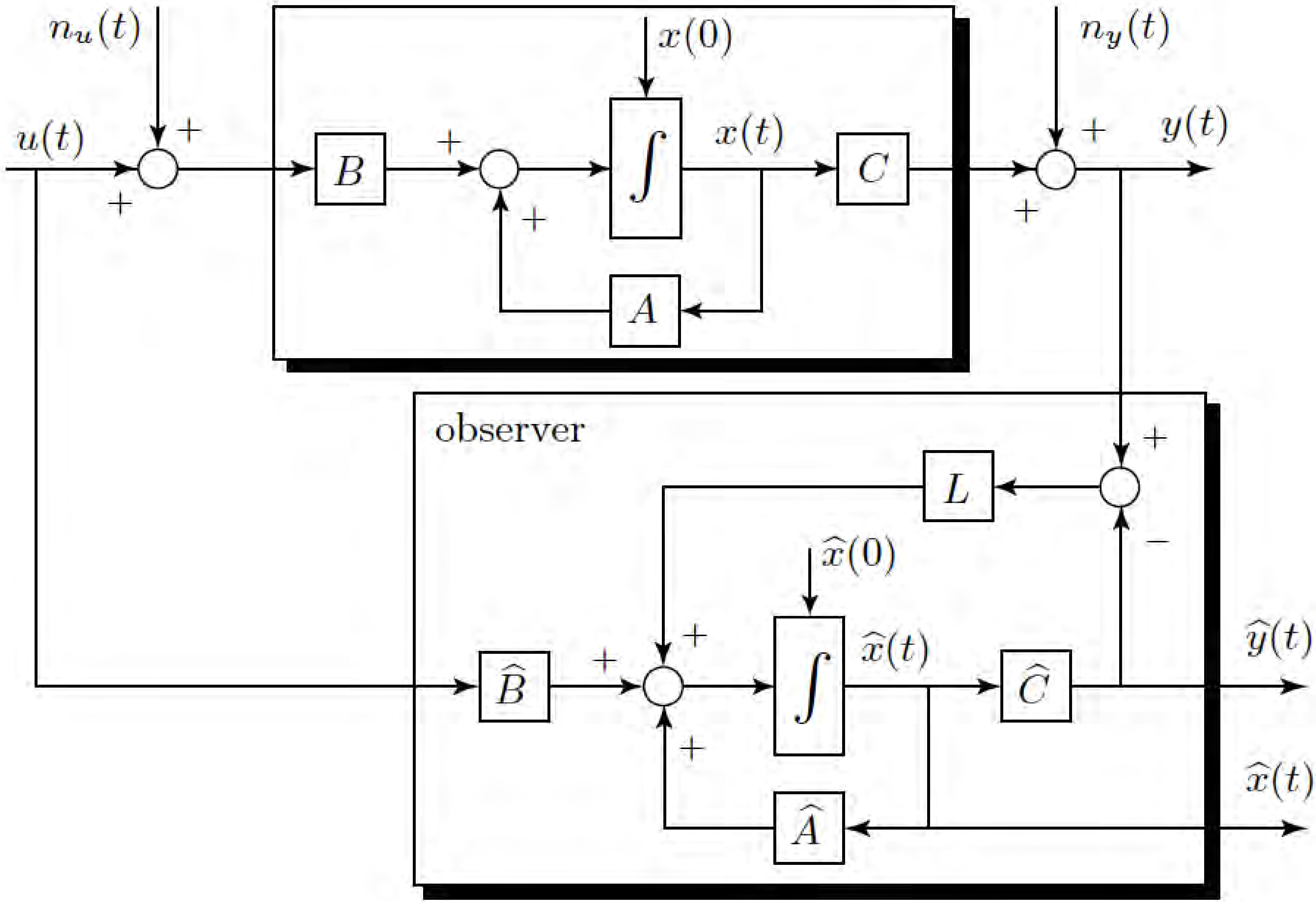


State Observers



Known: signals: $u(t)$ and $y(t)$
plant model $\{A,B,C,D\}$

Find: Estimation $\hat{x}(t)$ of $x(t)$



$n_u(t)$
$x(0)$
$n_y(t)$
$u(t)$
B
$x(t)$
C
$y(t)$
A
observer
L
$\widehat{x}(0)$
$\widehat{x}(t)$
$\widehat{B}$
$\widehat{C}$
$\widehat{y}(t)$
$\widehat{A}$
$\widehat{x}(t)$

Plant

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t)$$

$$y(t) = C \cdot x(t)$$

Observer

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{x}(t) = \widehat{A} \cdot \widehat{x}(t) + \widehat{B} \cdot u(t) + L \cdot (y(t) - \widehat{y}(t))$$

$$\widehat{y}(t) = \widehat{C} \cdot \widehat{x}(t)$$

Observation errors

$$\bar{x}(t) = x(t) - \hat{x}(t) \in \mathbb{R}^n$$

Ideal case: $A = \hat{A},\ B = \hat{B},\ \text{and}\ C = \hat{C}$

$$n_u(t) = n_y(t) = 0$$

Error dynamics

$$\begin{aligned}
\frac{d}{dt}\bar{x}(t) &= \frac{d}{dt}x(t) - \frac{d}{dt}\hat{x}(t) \\
&= A \cdot x(t) + B \cdot u(t) - [A \cdot \hat{x}(t) + B \cdot u(t) + L \cdot (y(t) - \hat{y}(t))] \\
&= A \cdot (x(t) - \hat{x}(t)) - L \cdot C \cdot (x(t) - \hat{x}(t)) \qquad (3.46) \\
&= [A - L \cdot C] \cdot \bar{x}(t), \qquad \bar{x}(0) = x(0) - \hat{x}(0) \neq 0
\end{aligned}$$

How to compute L

Dual LQR problem

$$A \longrightarrow \boxed{A^T}$$

$$B \longrightarrow \boxed{C^T}$$

$$Q = \bar{C}^T \cdot \bar{C} \longrightarrow \boxed{\bar{B} \cdot \bar{B}^T}$$

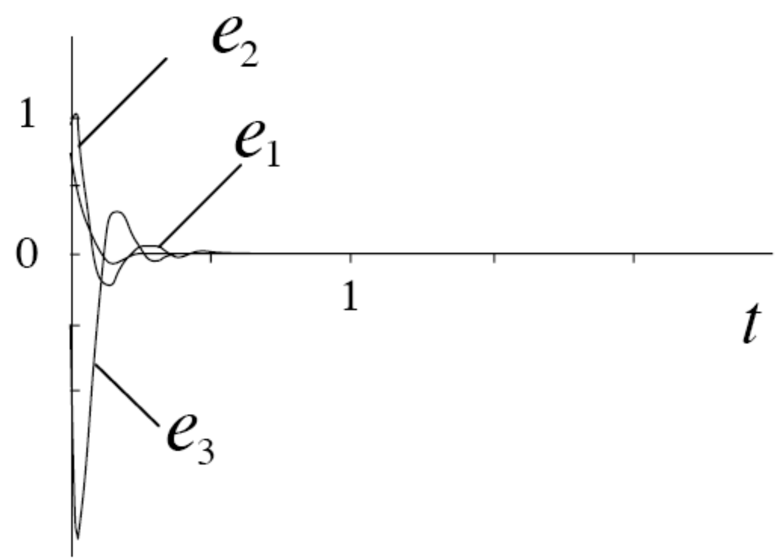
$$R = r \cdot I \longrightarrow \boxed{q \cdot I}$$

$$L^T = \frac{1}{q} \cdot C \cdot \Psi$$

$$\frac{1}{q} \cdot \Psi \cdot C^T \cdot C \cdot \Psi - \Psi \cdot A^T - A \cdot \Psi - \bar{B} \cdot \bar{B}^T = 0$$

One might be tempted to choose the eigenvalues of $A - L \cdot C$ much "faster" than those of A (or later $A - B \cdot K$). However, two unavoidable complications impose limitations:

- The observer is synthesized using a model $\{\hat{A}, \hat{B}, \hat{C}\}$ of the true plant $\{A, B, C\}$. Of course this model is never perfect. These and other modeling errors impose limits on the loop gain of the error dynamics (3.46).
- The input signals to the observer will always be corrupted by some noise, i.e., $n_u(t) \neq 0$ and $n_y(t) \neq 0$. This fact imposes limits on the loop gain.

<table>
<tr><th></th><th>“slow” observer</th><th>“fast” observer</th></tr>
<tr><td>no noise</td><td>


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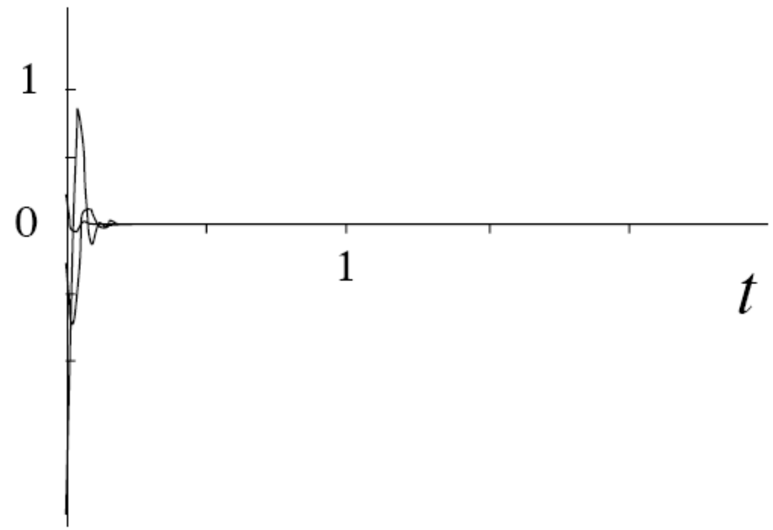
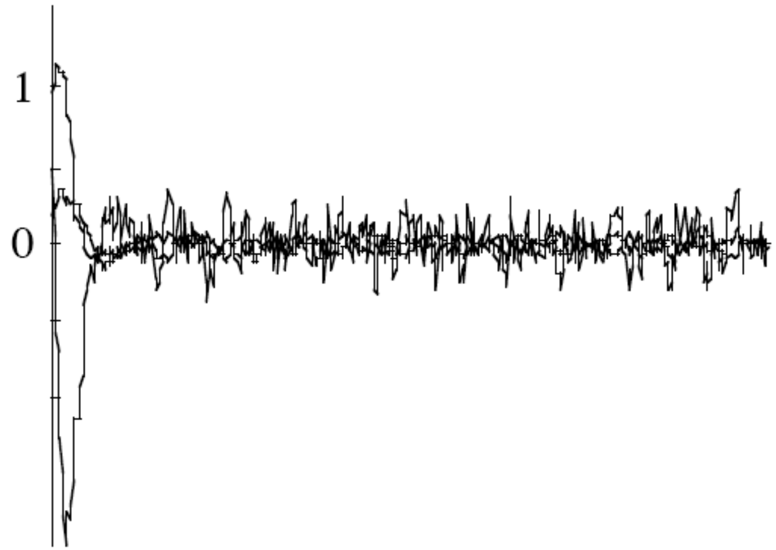
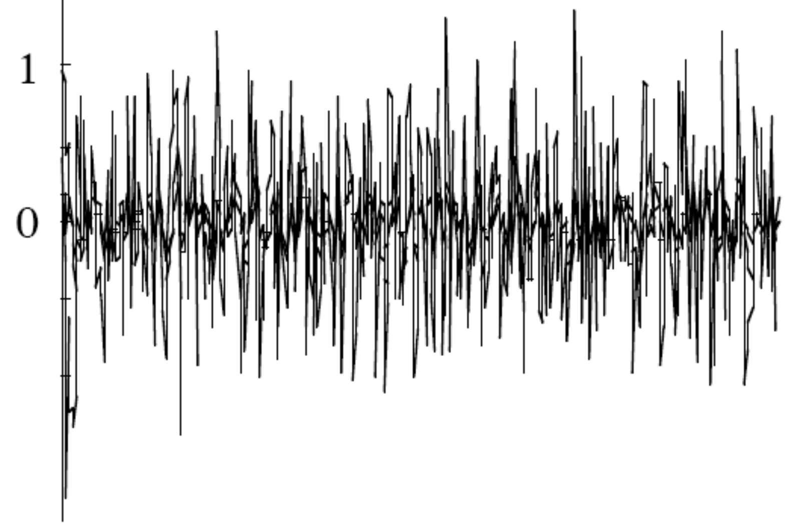


</td></tr>
<tr><td>some noise</td><td>


</td><td>


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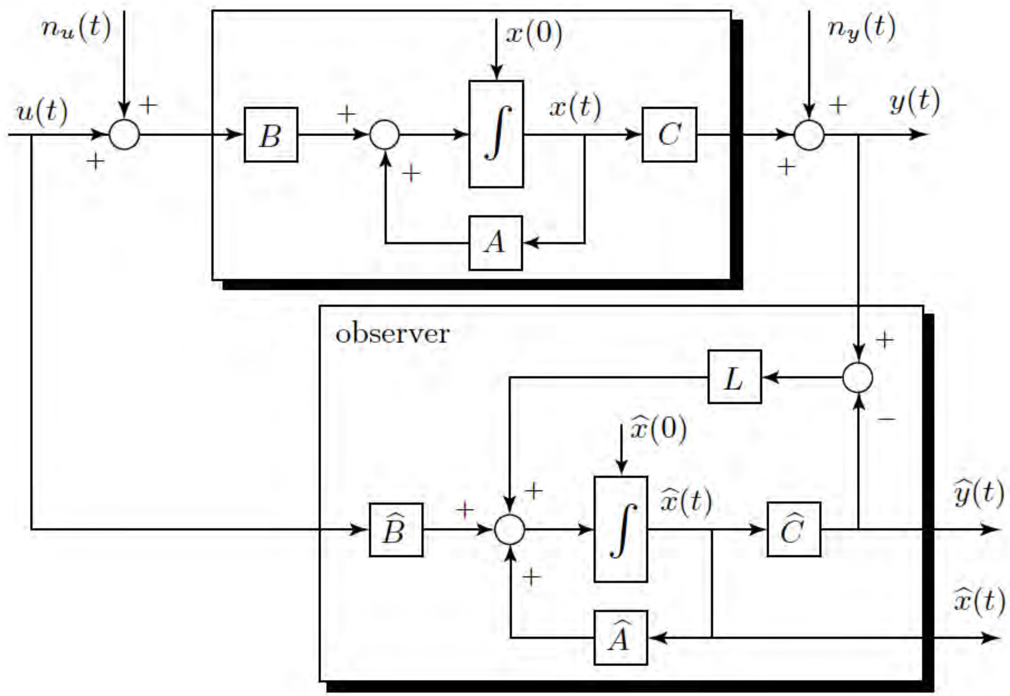
</table>

Kalman Filters

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = A \cdot x(t) + B \cdot (u(t) + n_u(t))$$

$$y_o(t) = C \cdot x(t) + n_y(t)$$



In the simplest case, the two noise signals n_u and n_y are assumed to be uncorrelated *white noise signals*. Without entering into the mathematical details of stochastic signals, a white noise signal $n(t)$ is defined as a signal whose *spectrum* is constant for all frequencies.

The spectrum $\phi_n(\omega)$ of a scalar signal $n(t) \in I\!R$ is obtained using its Fourier transform

$$n(t) = \int_{-\infty}^{+\infty} A(\omega) \cos\left(\omega t + \phi(\omega)\right) d\omega \tag{3.55}$$

by the operation

$$\phi_n(\omega) = A^2(\omega) \tag{3.56}$$

If the spectrum is constant for all frequencies, i.e., if

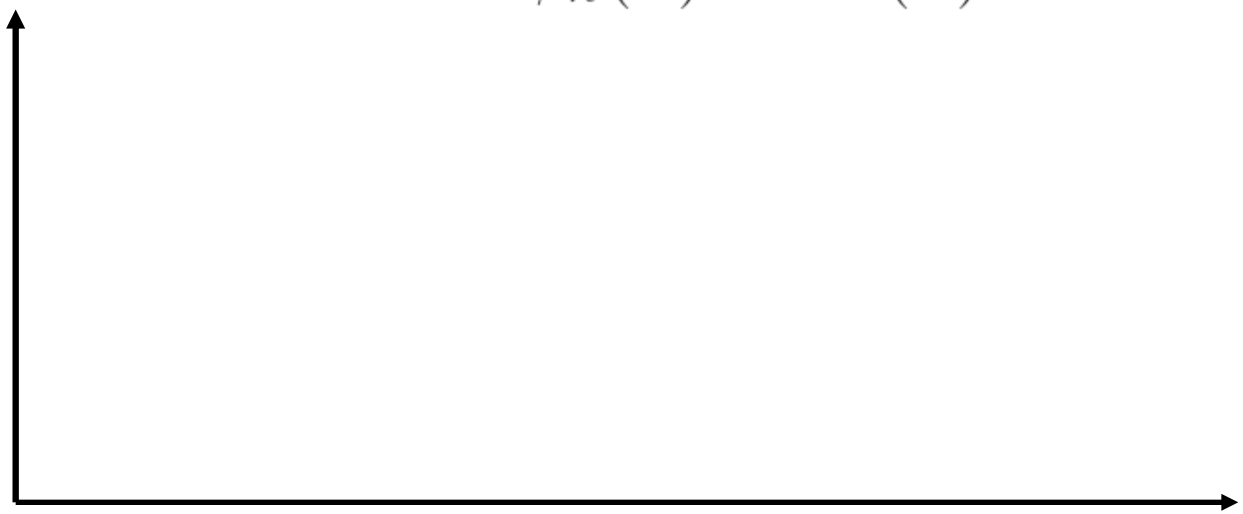
$$\phi_n(\omega) = r_n > 0 \in I\!R \tag{3.57}$$

then the signal $n(t)$ is a white noise signal.

$\phi_n(\omega)$ of a scalar signal $n(t) \in I\!R$ is

$$n(t) = \int_{-\infty}^{+\infty} A(\omega) \cos\left(\omega t + \phi(\omega)\right) \, d\omega$$

$$\phi_n(\omega) = A^2(\omega)$$



Interpretation:

These definitions can be easily extended to the case where $n(t) \in \mathbb{R}^m$ is a vector signal. In this case the spectrum is a Hermitian $m \times m$ matrix $\Phi_n(\omega)$, which, for white noise signals, is constant for all frequencies

$$\Phi_n(\omega) = R_n = R_n^T \geq 0 \in \mathbb{R}^{m \times m} \tag{3.58}$$

This matrix describes the "intensity" of a noisy signal. Below, it is assumed that both $R_u \geq 0$, associated to $n_u(t)$, and $R_y > 0$, associated to $n_y(t)$, are known.

L_K that minimizes the expectation of the estimation error $\bar{x}(t)$. This optimal gain is defined by

$$L_K = P \cdot C^T \cdot R_y^{-1} \tag{3.59}$$

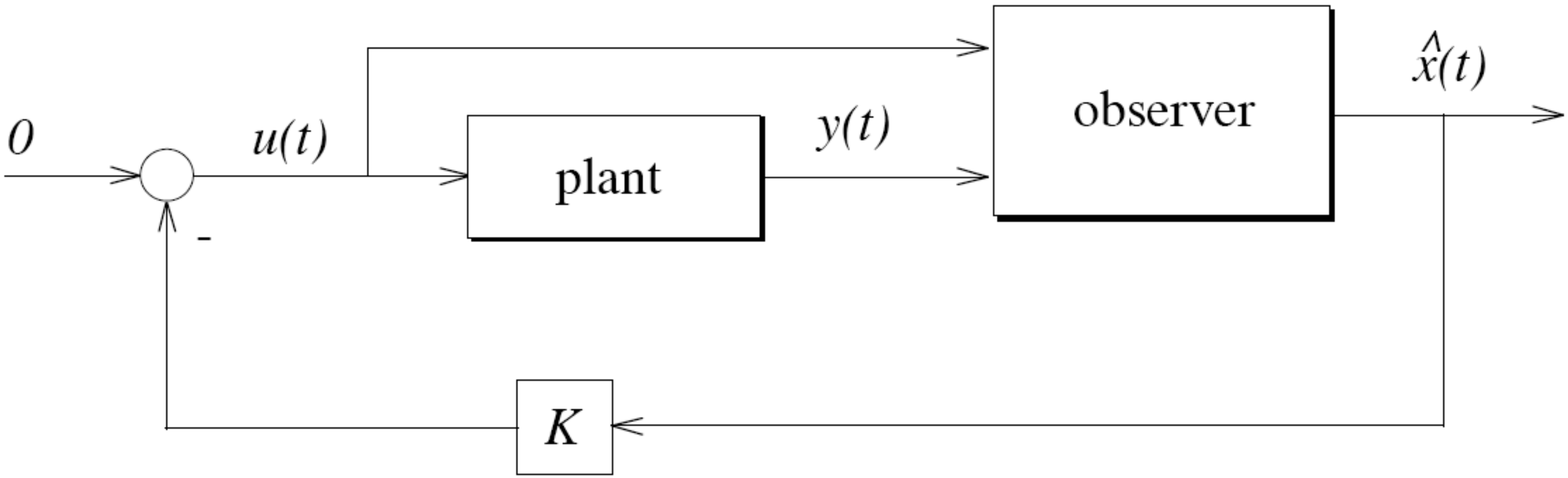
where the matrix $P = P^T \in \mathbb{R}^{n \times n}$ is the positive (semi-)definite solution of the Riccati equation

$$A \cdot P + P \cdot A^T - P \cdot C^T \cdot R_y^{-1} \cdot C \cdot P + B \cdot R_u \cdot B^T = 0 \tag{3.60}$$

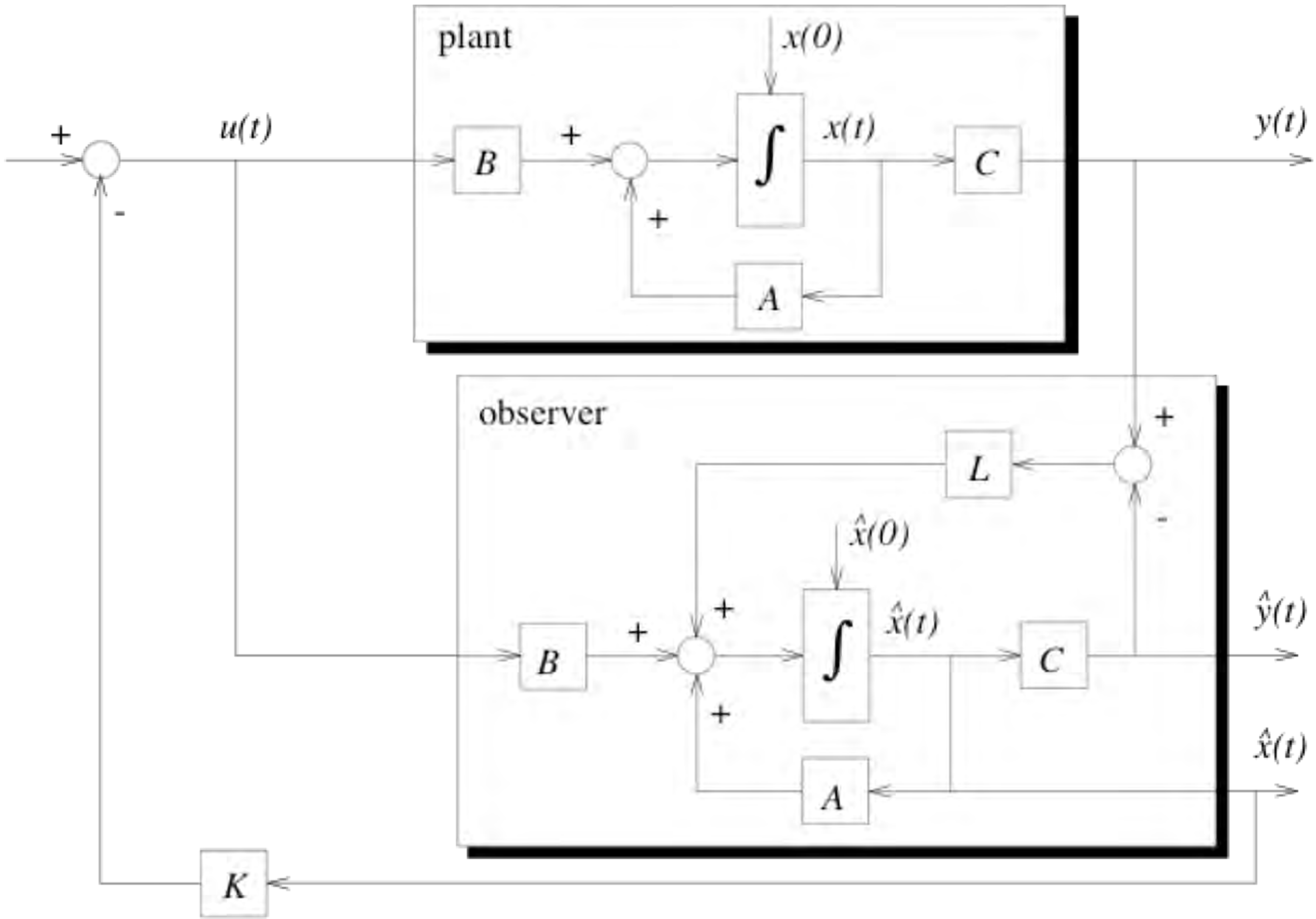
Lecture X – Observer-Based Output Feedback Control

Key idea:

$$u(t) = -K \cdot \hat{x}(t)$$



LQG(aussian): observer gain computed using dual LQR approach


plant
x(0)
+
u(t)
B
+
∫
x(t)
C
y(t)
-
+
A
observer
+
L
-
x̂(0)
+
+
∫
x̂(t)
C
ŷ(t)
B
+
x̂(t)
A
K

LQG = Output-Feedback, i.e., realizable!

State vector

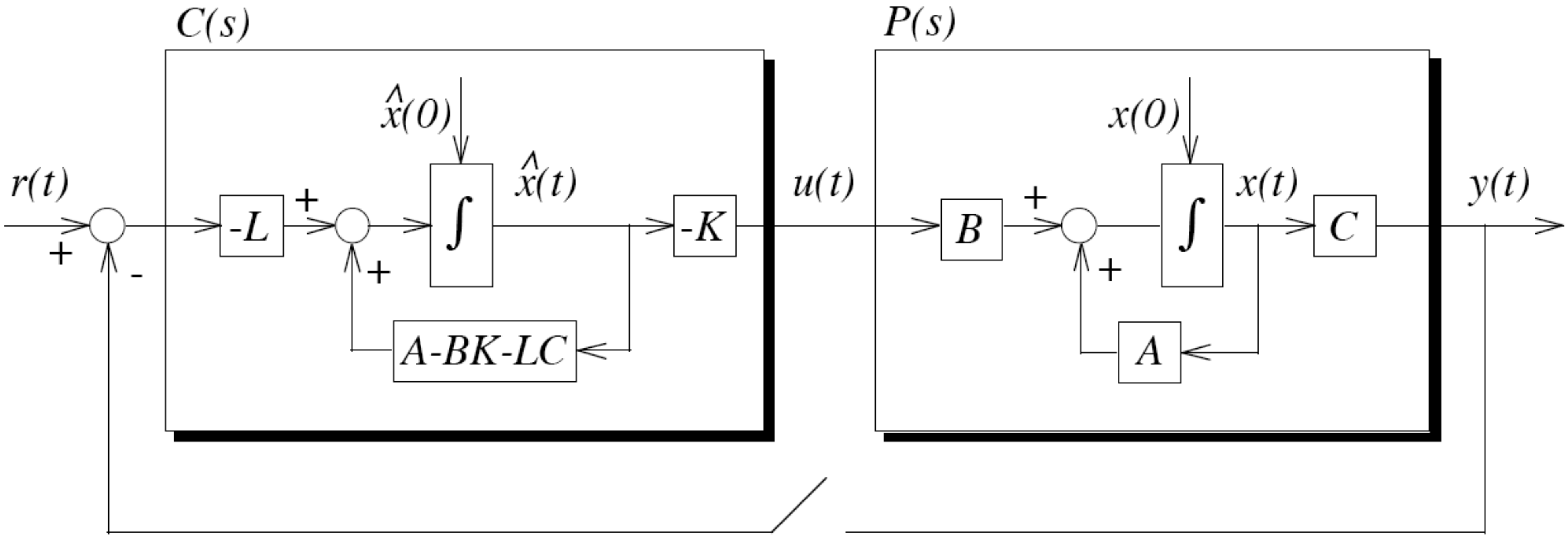
$$\widetilde{x}(t) = \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

System description:

a) open loop $\Rightarrow$ robustness

b) closed loop $\Rightarrow$ stability

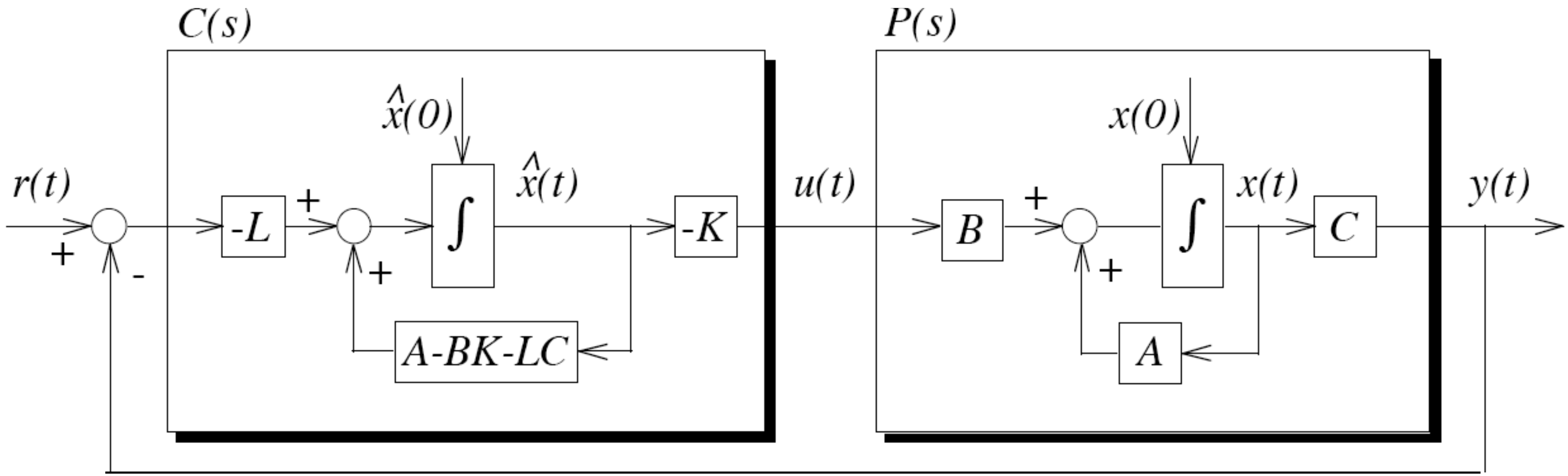
$$\frac{d}{dt}\widetilde{x}(t) = \widetilde{A}_{ol} \cdot \widetilde{x}(t) + \widetilde{B} \cdot r(t), \quad y(t) = \widetilde{C} \cdot \widetilde{x}(t)$$



$$\widetilde{A}_{ol} = \begin{bmatrix} A & -B \cdot K \\ 0 & A - B \cdot K - L \cdot C \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} 0 \\ -L \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} C & 0 \end{bmatrix}$$

$$L_{LQG}(s) = C \cdot [s\,I - A]^{-1} \cdot B \cdot K \cdot [s\,I - (A - B \cdot K - L \cdot C)]^{-1} \cdot L$$

$$\frac{d}{dt}\widetilde{x}(t) = \widetilde{A}_{cl}\cdot\widetilde{x}(t) + \widetilde{B}\cdot r(t), \quad y(t) = \widetilde{C}\cdot\widetilde{x}(t)$$



$$\widetilde{A}_{cl} = \begin{bmatrix} A & -B\cdot K \\ L\cdot C & A - B\cdot K - L\cdot C \end{bmatrix}$$

In particular, two points must be verified:

1. the stability of the closed-loop system (3.58); and
2. the robustness of the open-loop system (3.56).

later!

The Separation Principle

Assume

$$A - B \cdot K \text{ and } A - L \cdot C \text{ are both Hurwitz}$$

then $\widetilde{A}_{cl}$ is Hurwitz as well and its eigenvalues are those of $A - B \cdot K$ and $A - L \cdot C$

The Separation Principle

$$\widetilde{A}_{cl} = \begin{bmatrix} A & -B \cdot K \\ L \cdot C & A - B \cdot K - L \cdot C \end{bmatrix}$$

$\widetilde{x} = T \cdot \widetilde{z}$ with

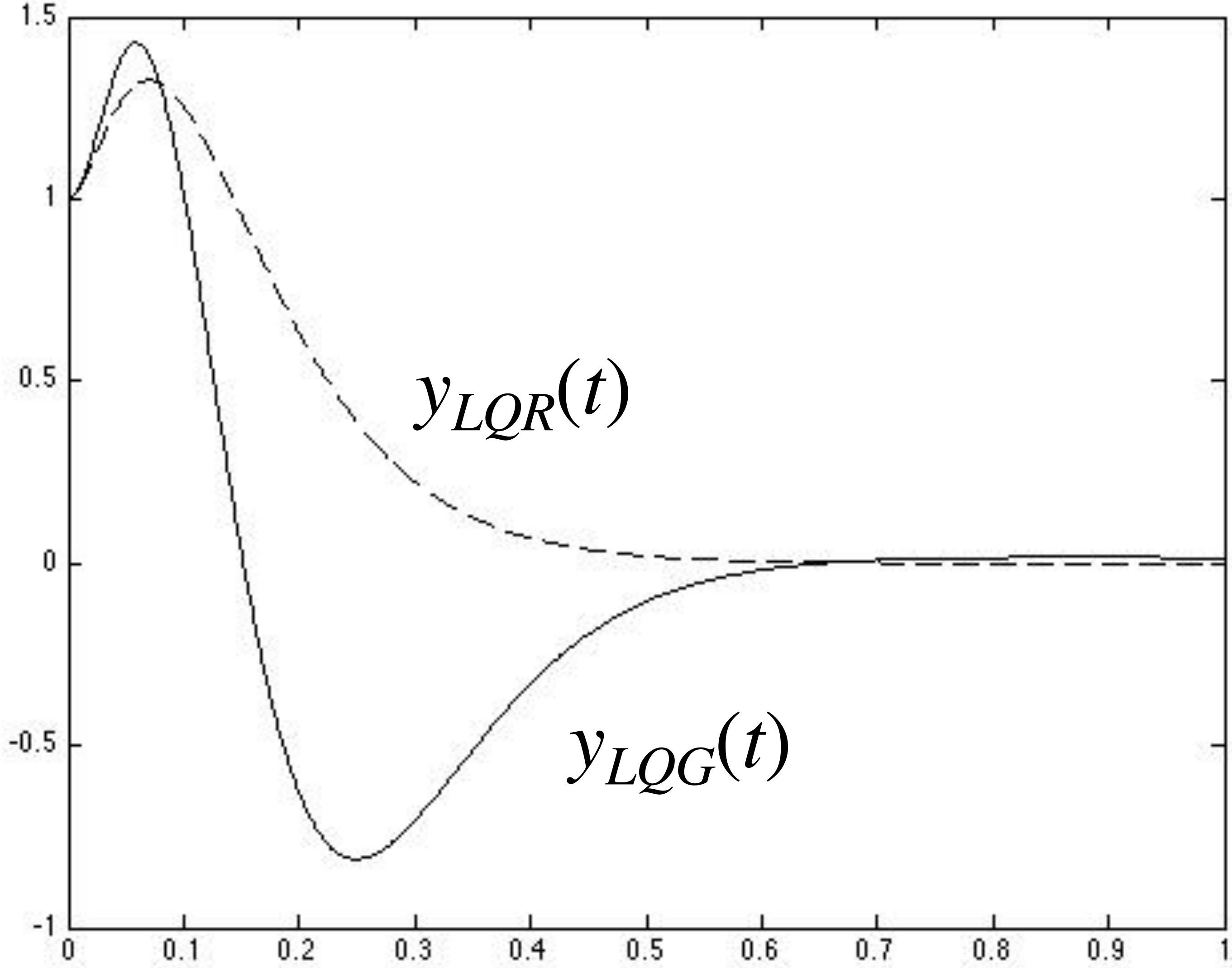
$$T = \begin{bmatrix} I_{n\times n} & 0_{n\times n} \\ I_{n\times n} & -I_{n\times n} \end{bmatrix} = T^{-1}$$

$$T^{-1} \cdot \widetilde{A}_{cl} \cdot T = \begin{bmatrix} A - B \cdot K & B \cdot K \\ 0_{n\times n} & A - L \cdot C \end{bmatrix}$$

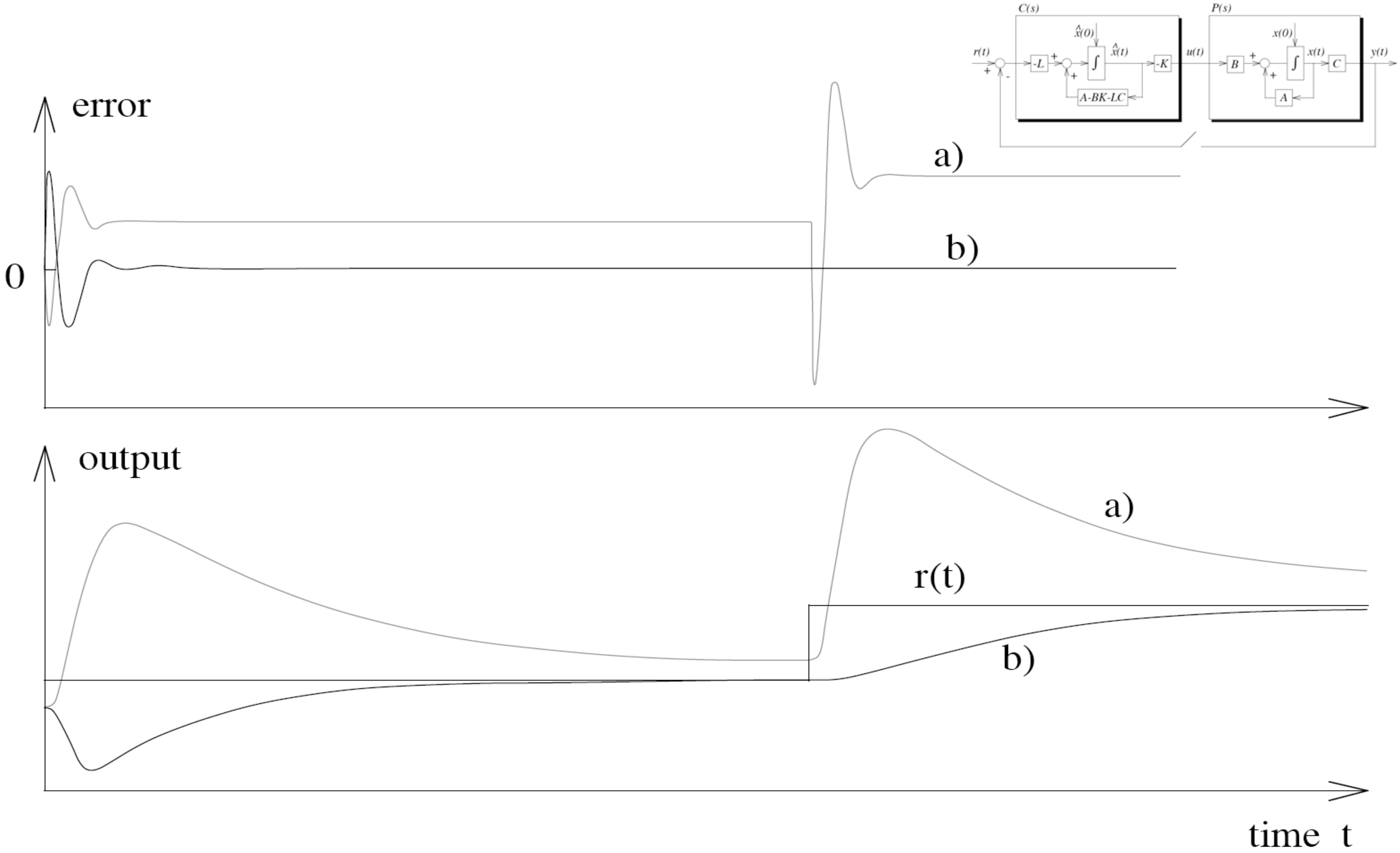
Levitating Sphere – TD Behavior

$$x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

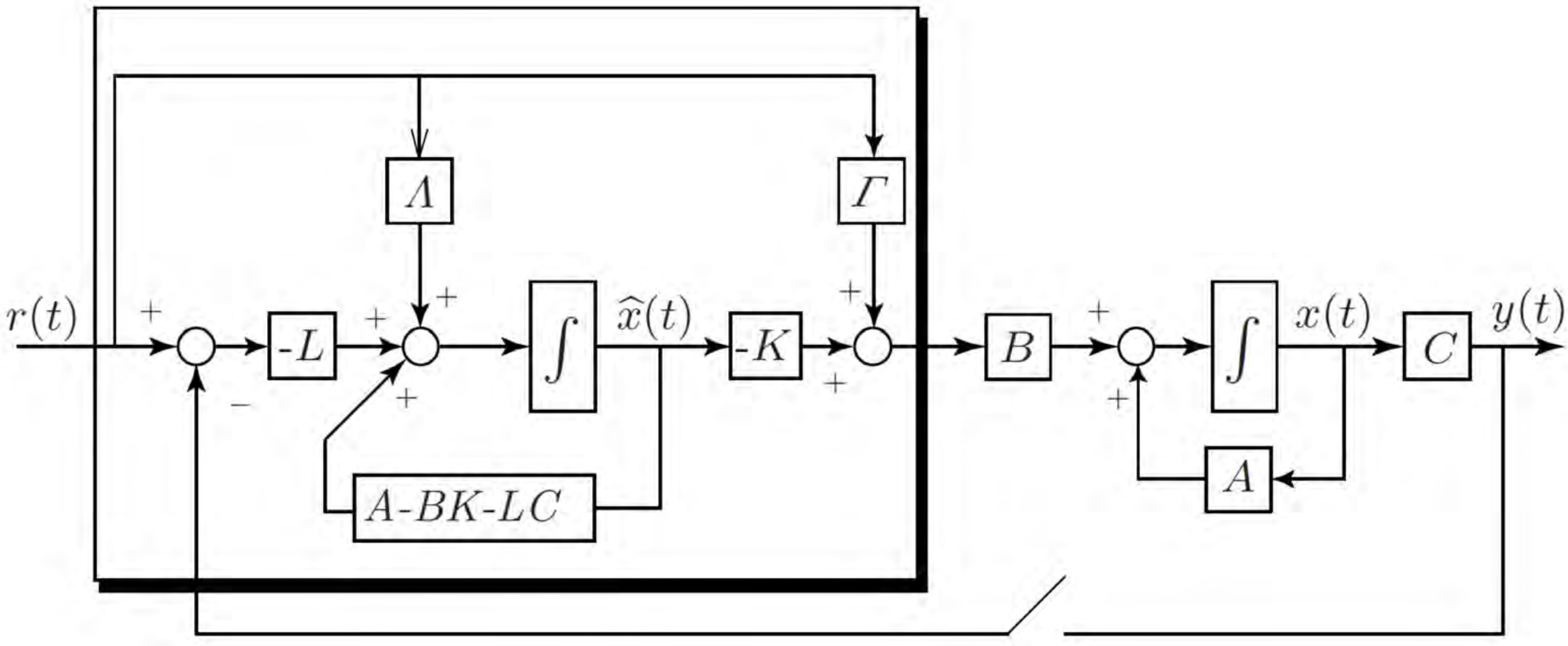


LQG for Reference Tracking



$$1: \quad \frac{\partial}{\partial r}\left\{\frac{d}{dt}e(t)\right\} = 0$$

$$2: \quad \lim_{t\to\infty} y(t) = r(t), \quad \text{for } r(t) = \begin{bmatrix} r_1 \\ \dots \\ r_m \end{bmatrix} \cdot h(t)$$

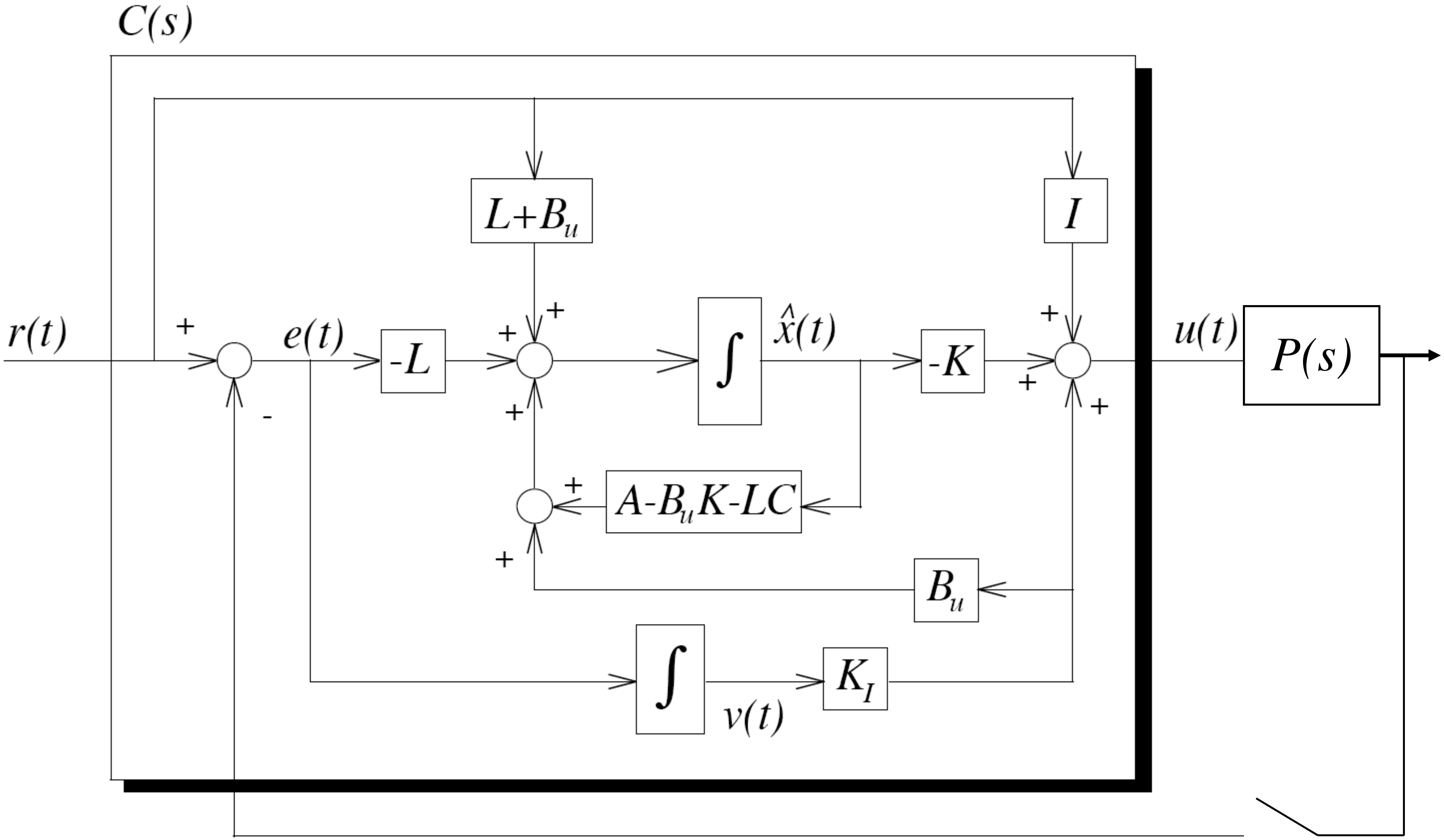


$$\Lambda = L + B \cdot \Gamma, \qquad \Gamma = -\left(\widetilde{C} \cdot \widetilde{A}_{\mathrm{cl}}^{-1} \cdot \widetilde{B}_r\right)^{-1}$$

$$\frac{d}{dt}\widetilde{x}(t) = \frac{d}{dt}\begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} = \widetilde{A}_{cl} \cdot \widetilde{x}(t) + \widetilde{B}_r \cdot \Gamma \cdot r(t), \quad y(t) = \widetilde{C} \cdot \widetilde{x}(t)$$

$$\widetilde{A}_{cl} = \begin{bmatrix} A & -B\,K \\ L\,C & A - B\,K - L\,C \end{bmatrix}, \quad \widetilde{B}_r = \begin{bmatrix} B \\ B \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} C & 0 \end{bmatrix}$$

LQG Controllers with Integral Action



Open-Loop Description

$$\frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \\ v \end{bmatrix} = \begin{bmatrix} A & -B_u K & B_u K_I \\ 0 & A - B_u K - L C & B_u K_I \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ -L \\ I \end{bmatrix} e$$

$$y = \begin{bmatrix} C & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \\ v \end{bmatrix}$$

Important for frequency-domain analysis

Closed-Loop Description

$$\frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \\ v \end{bmatrix} = \begin{bmatrix} A & -B_u K & B_u K_I \\ LC & A - B_u K - LC & B_u K_I \\ -C & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \\ v \end{bmatrix} + \begin{bmatrix} B_u & B_d \\ B_u & 0 \\ I & 0 \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix}$$

$$y = \begin{bmatrix} C & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \\ v \end{bmatrix}$$

Important for time-domain analysis

A typical design procedure consists of the following three steps:

1. Form the extended system $\{\widetilde{A}, \widetilde{B}_u, \widetilde{C}\}$ following the approach outlined in Section 3.4. With that data, design suitable state-feedback control gains $\widetilde{K} = [K, \ -K_I]$. Start with simple weights ($R = r \cdot I$, etc.) and iterate on r and γ; if that is not sufficient use more general weights until the desired system behavior is achieved.
2. Design an observer gain L for the original system $\{A, B_u, C\}$ using the duality approach introduced in (3.47).
3. Add the feedforward (3.74) which is straightforward because the closed-loop feedback system is known to have a DC gain of I.

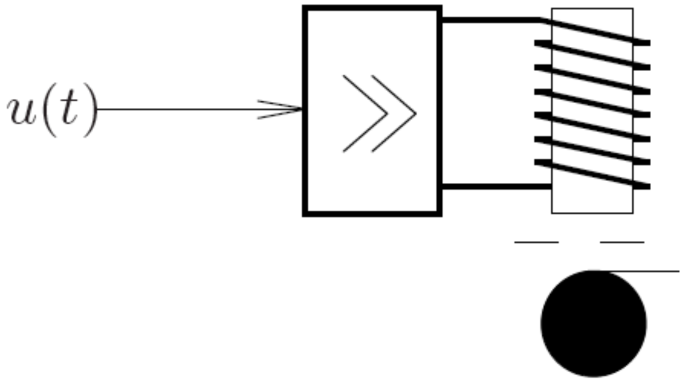
Robustness of LQG-Loops

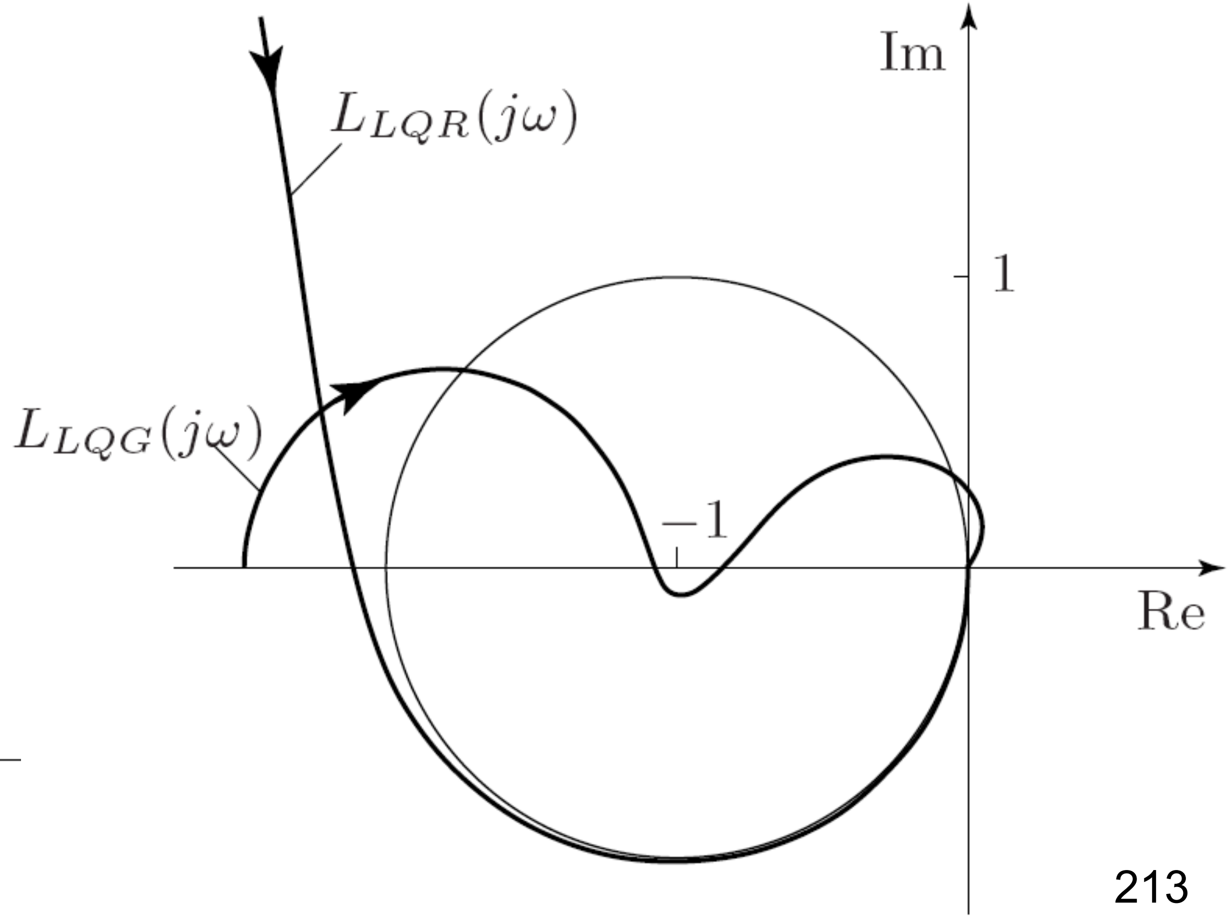
$$L_{LQG}(s) = C \cdot [sI - A]^{-1} \cdot B \cdot K \cdot [sI - (A - B \cdot K - L \cdot C)]^{-1} \cdot L$$

$$\mu_{LQG} = \min_{\omega} \{\sigma_{\min}\{I + L_{LQG}(j\omega)\}\}$$

can be arbitrarily close to 0!

Example:





Robustness Recovery

- First design a suitable state-feedback LQR controller that is known to yield "nice" loop gains.
- Set up a design procedure for an output-feedback controller with one iteration parameter (the "LTR gain").
- Increase the LTR gain until the loop gain $L(s)$ of the output-feedback controller sufficiently approaches the loop gain $L_{LQR}(s)$ of the state-feedback controller without violating the limits imposed by non-minimumphase zeros, noise and model uncertainty.

A "Naive" Approach to LTR

Given: simple plant (SISO, miniphase)

$$P(s) = c \cdot (s\,I - A)^{-1} \cdot b \;=\; \frac{c \cdot \mathrm{Adj}(sI - A) \cdot b}{\det(sI - A)}$$

Desired loop gain

$$C(s) \cdot P(s) \equiv L_{LQR}(s)$$

$$L_{LQR}(s) = k \cdot (sI - A)^{-1} \cdot b = \frac{k \cdot \mathrm{Adj}(sI - A) \cdot b}{\det(sI - A)}$$

Therefore, controller

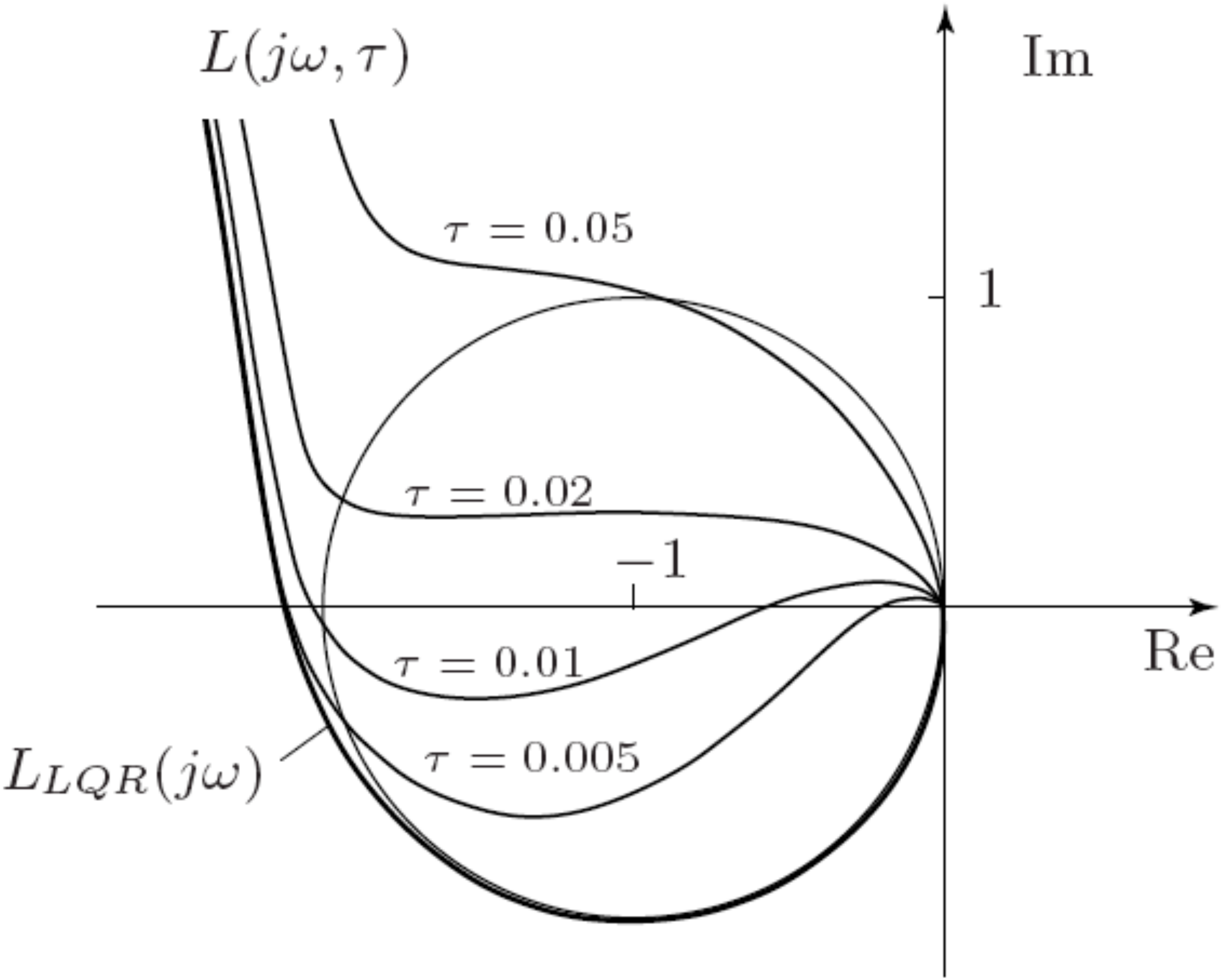
$$C(s) = \frac{k \cdot \mathrm{Adj}(sI - A) \cdot b}{c \cdot \mathrm{Adj}(sI - A) \cdot b \cdot (\tau s + 1)^{\kappa}}$$

Loop gain

$$L(s) = \frac{k \cdot \mathrm{Adj}(sI - A) \cdot b}{\det(sI - A) \cdot (\tau s + 1)^{\kappa}}$$

The corner frequency $1/\tau$ determines the bandwidth of the loop transfer recovery. On one hand, this LTR frequency may not be higher than the limits imposed by the model uncertainty and the noise present in the system. On the other hand, the LTR frequency must be sufficiently high to compensate for plant instabilities (see Figure 4.9).

QC: is the closed-loop system guaranteed to be asymptotically stable? Separation principle?



QC: What problems do you expect for $\tau \to 0$?

1. Derive a linearized and normalized system model $\{A, B_u, C\}$ of the plant.
2. Design a suitable state-feedback control gain K as a solution of the LQR problem defined by

```
K = lqr(A, B_u, Q, r*eye(m,m));
```

3. Design an observer gain L for the standard system $\{A, B_u, C\}$ using the duality approach

```
L = lqr(A', C', B_u*B_u', q*eye(m,m))';
```

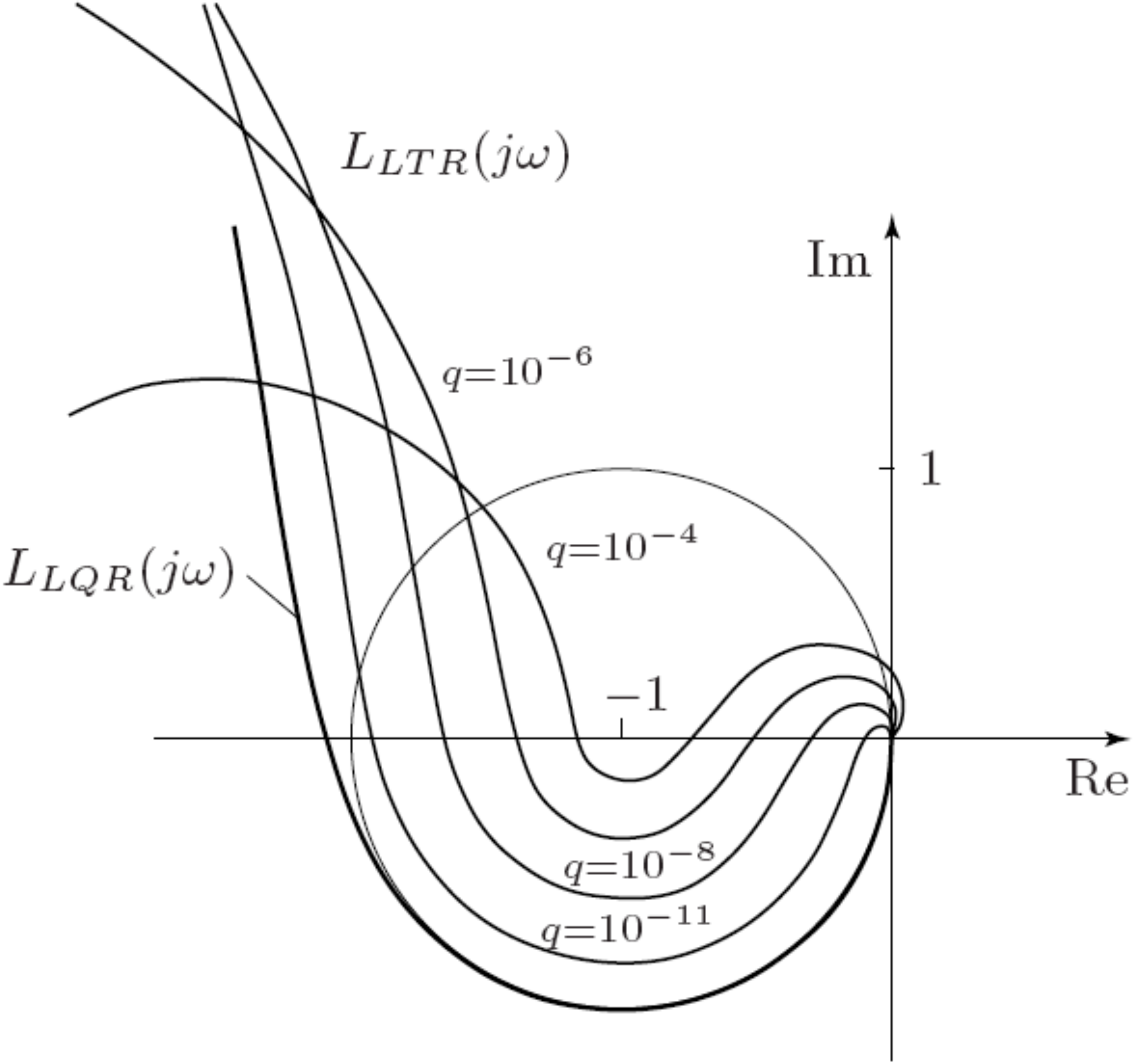
   Note: the choice $B_u \cdot B_u^T$ for the weight penalizing the dual states is necessary.
4. Analyze the resulting open-loop frequency-domain (singular values of the return difference) and closed-loop time-domain (disturbance steps) system behavior.
5. Repeat steps 3 and 4 with *decreasing* values of q (the numerical value of q has no immediate interpretation).

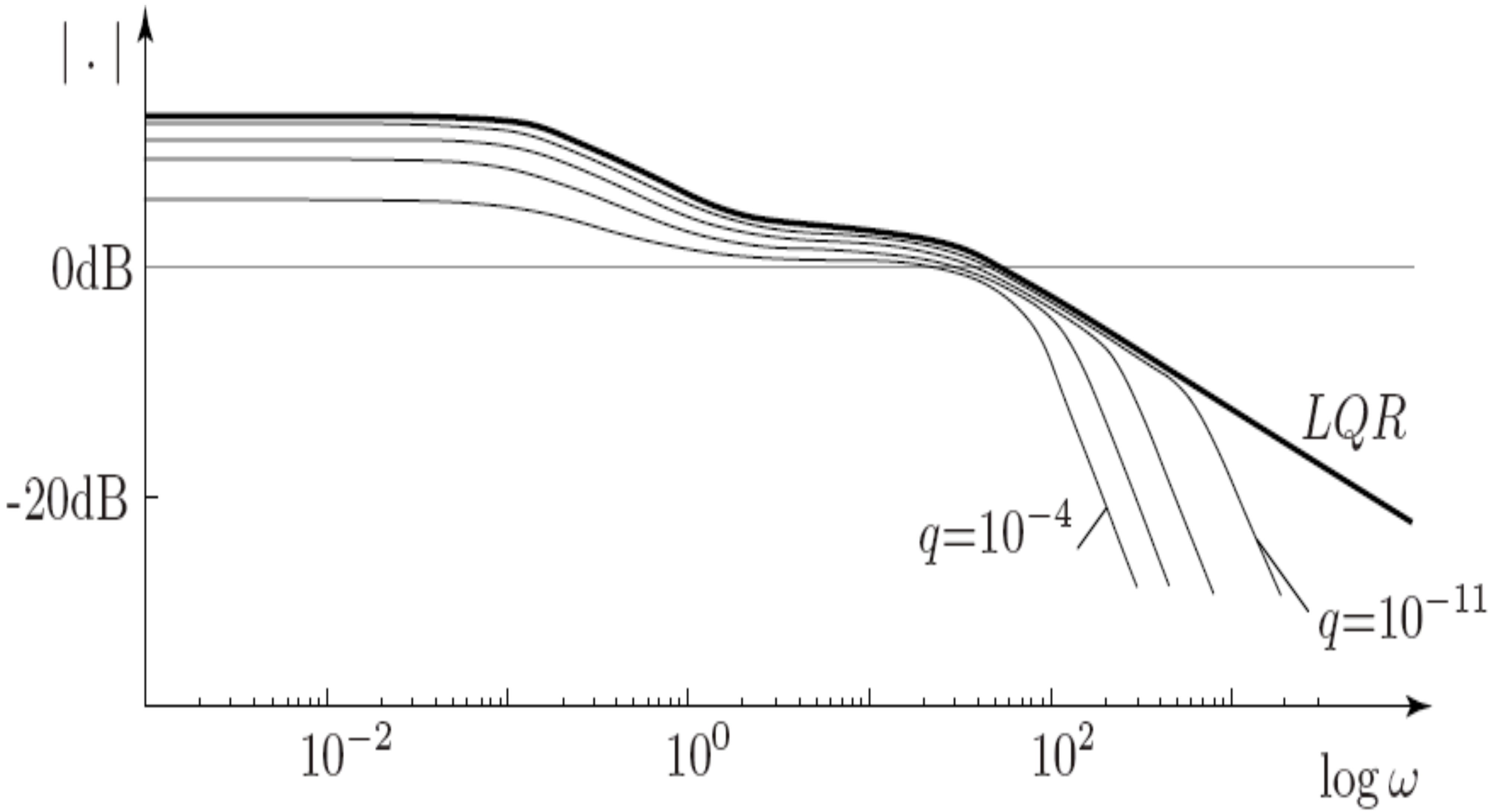
Some Remarks

In general, perfect LTR is *not* recommended. Full LTR, if possible, yields high-gain controllers at high frequencies that may cause noise amplification and robustness problems.

If the plant is non-minimumphase, the LTR procedure often yields an output-feedback controller that produces a loop gain that approaches the LQR loop gain to the extent possible.

The control system enhancements discussed in the previous chapters (integral action and feedforward parts) can be used in the LTR framework as well.


$L_{LTR}(j\omega)$
Im
1
$q{=}10^{-6}$
$q{=}10^{-4}$
$L_{LQR}(j\omega)$
-1
Re
$q{=}10^{-8}$
$q{=}10^{-11}$

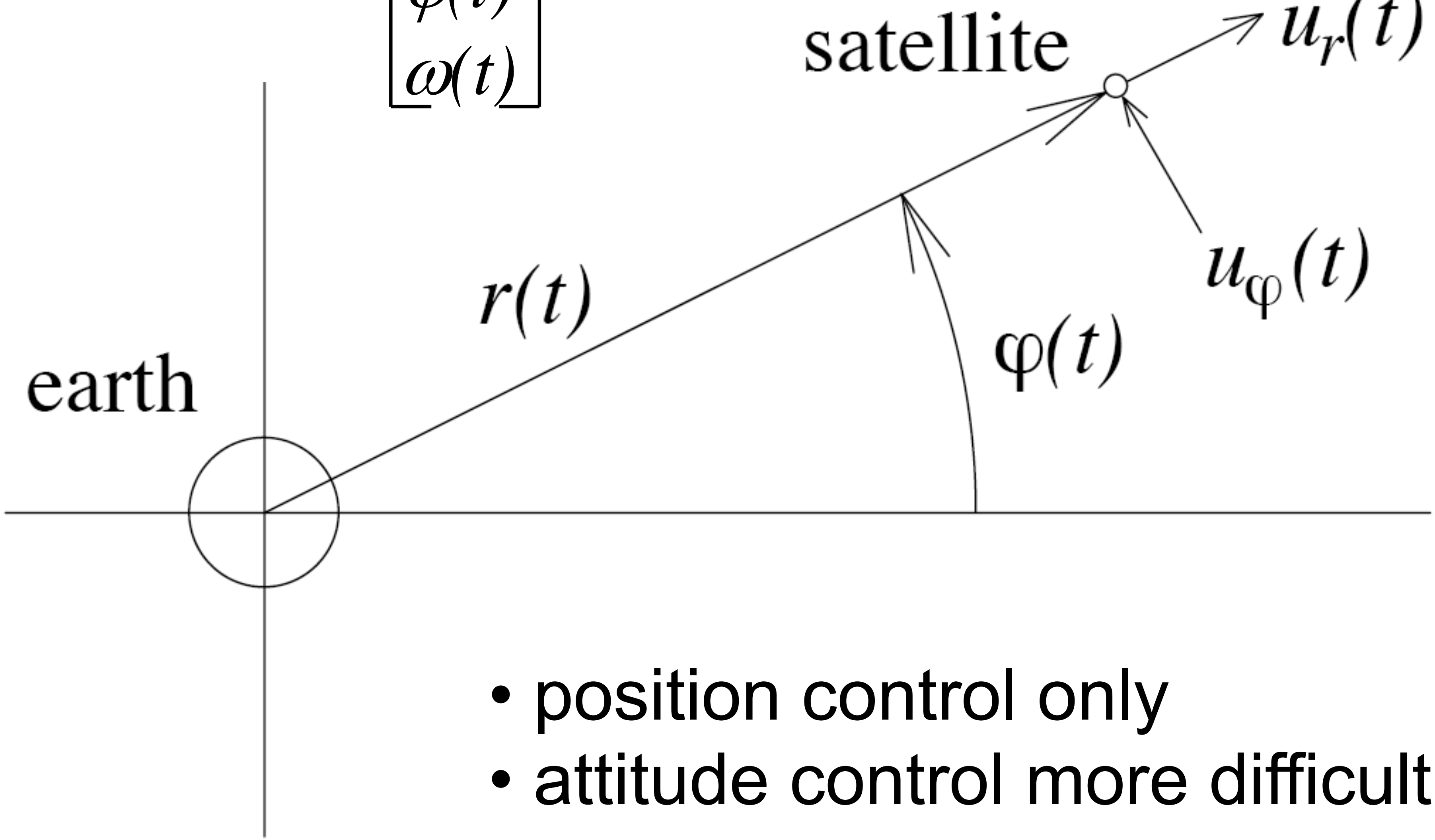


|.|
0dB
-20dB
LQR
q=10^{-4}
q=10^{-11}
10^{-2}
10^{0}
10^{2}
log ω

Case Study: Geostationary Satellite

SAR
Navigation
Communication

$$\text{state } x(\mathrm{t}) = \begin{bmatrix} r(t) \\ v(t) \\ \varphi(t) \\ \omega(t) \end{bmatrix}$$



- position control only
- attitude control more difficult

circular orbit at $r_0 \approx 4.22 \cdot 10^7 \ m$

sidereal angular velocity $\omega_0 \approx 7.29 \cdot 10^{-5} \ rad/s$

simplified and linearized dynamics

$$\frac{d}{dt}x(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\,\omega_0^2 & 0 & 0 & 2\,r_0 \cdot \omega_0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\,\omega_0/r_0 & 0 & 0 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1/r_0 \end{bmatrix} \cdot u(t)$$

$$y(t) = \begin{bmatrix} 1/r_0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot x(t)$$

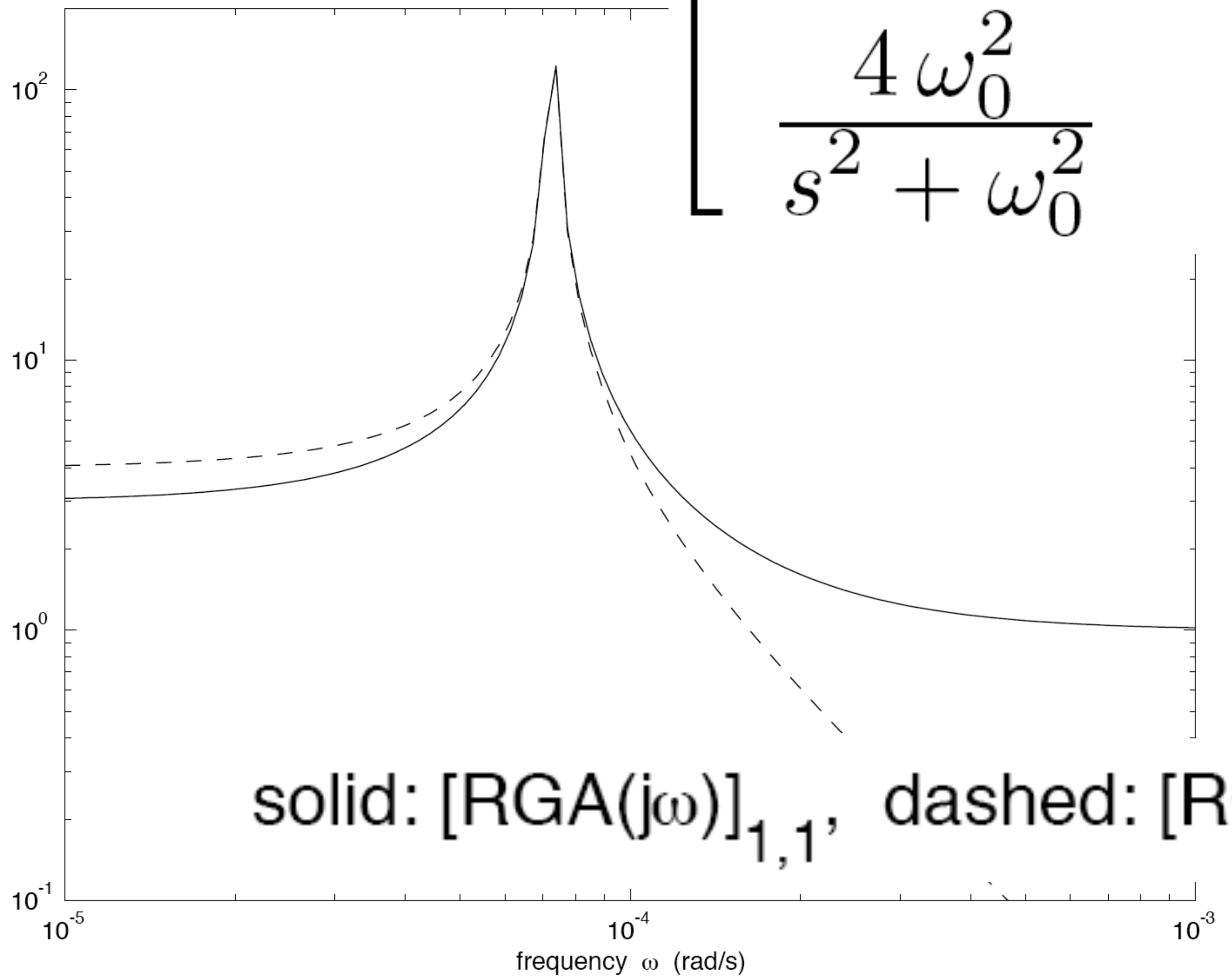
Transfer function

$$P(s) = \begin{bmatrix} \dfrac{1}{r_0 \cdot (s^2 + \omega_0^2)} & \dfrac{2\,\omega_0}{r_0 \cdot s \cdot (s^2 + \omega_0^2)} \\[2ex] \dfrac{-2\,\omega_0}{r_0 \cdot s \cdot (s^2 + \omega_0^2)} & \dfrac{s^2 - 3\,\omega_0^2}{r_0 \cdot s^2 \cdot (s^2 + \omega_0^2)} \end{bmatrix}$$

SISO not easy to control (only *$P_{22}(s)$* is c.c./o., but not minimum phase!)

MIMO no finite zeros!

$$RGA(s)=\begin{bmatrix} \dfrac{s^2-3\,\omega_0^2}{s^2+\omega_0^2} & \dfrac{4\,\omega_0^2}{s^2+\omega_0^2} \\ \\ \dfrac{4\,\omega_0^2}{s^2+\omega_0^2} & \dfrac{s^2-3\,\omega_0^2}{s^2+\omega_0^2} \end{bmatrix}$$

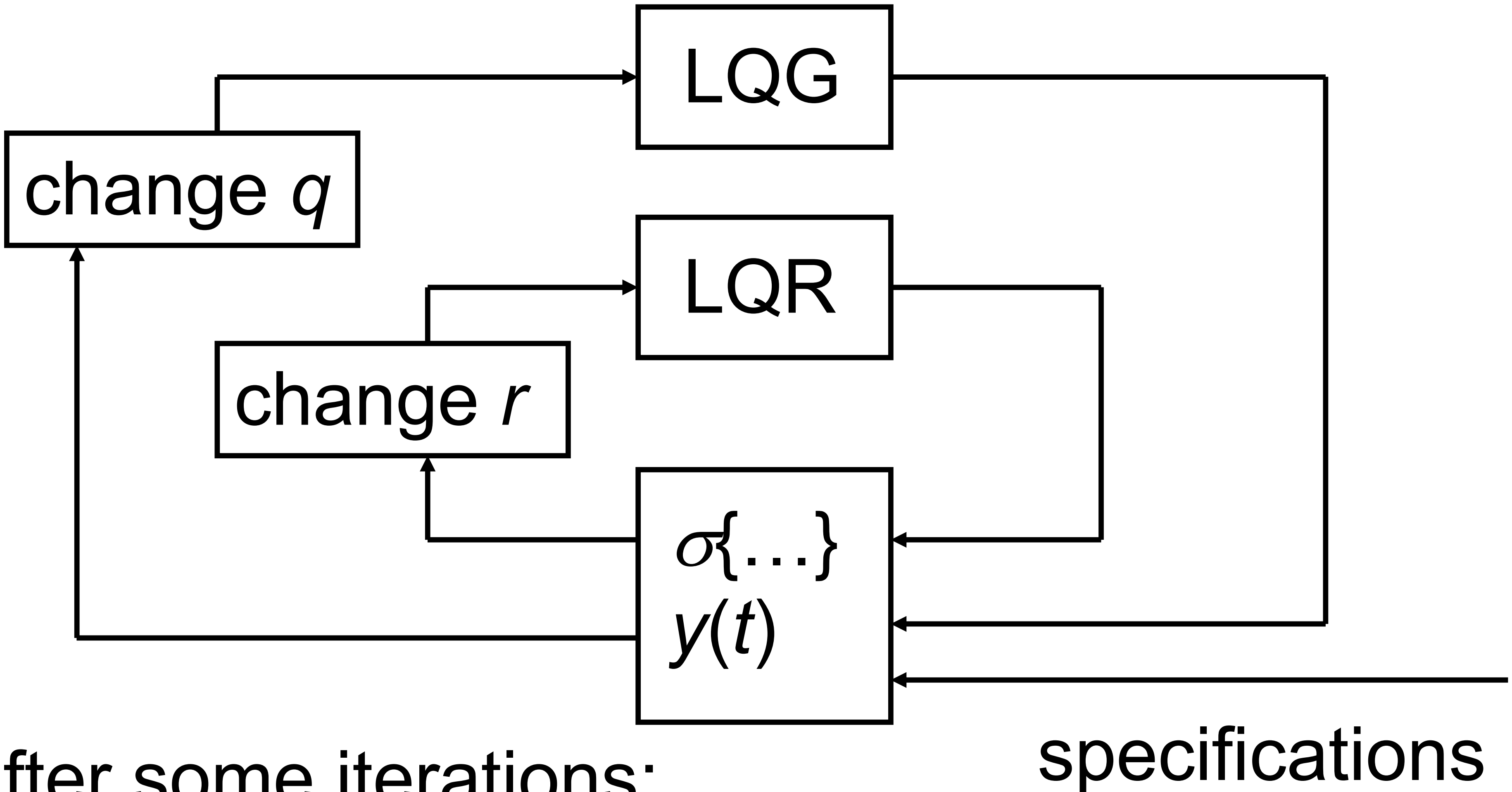


solid: $[\mathrm{RGA}(j\omega)]_{1,1}$, dashed: $[\mathrm{RGA}(j\omega)]_{1,2}$

Specifications

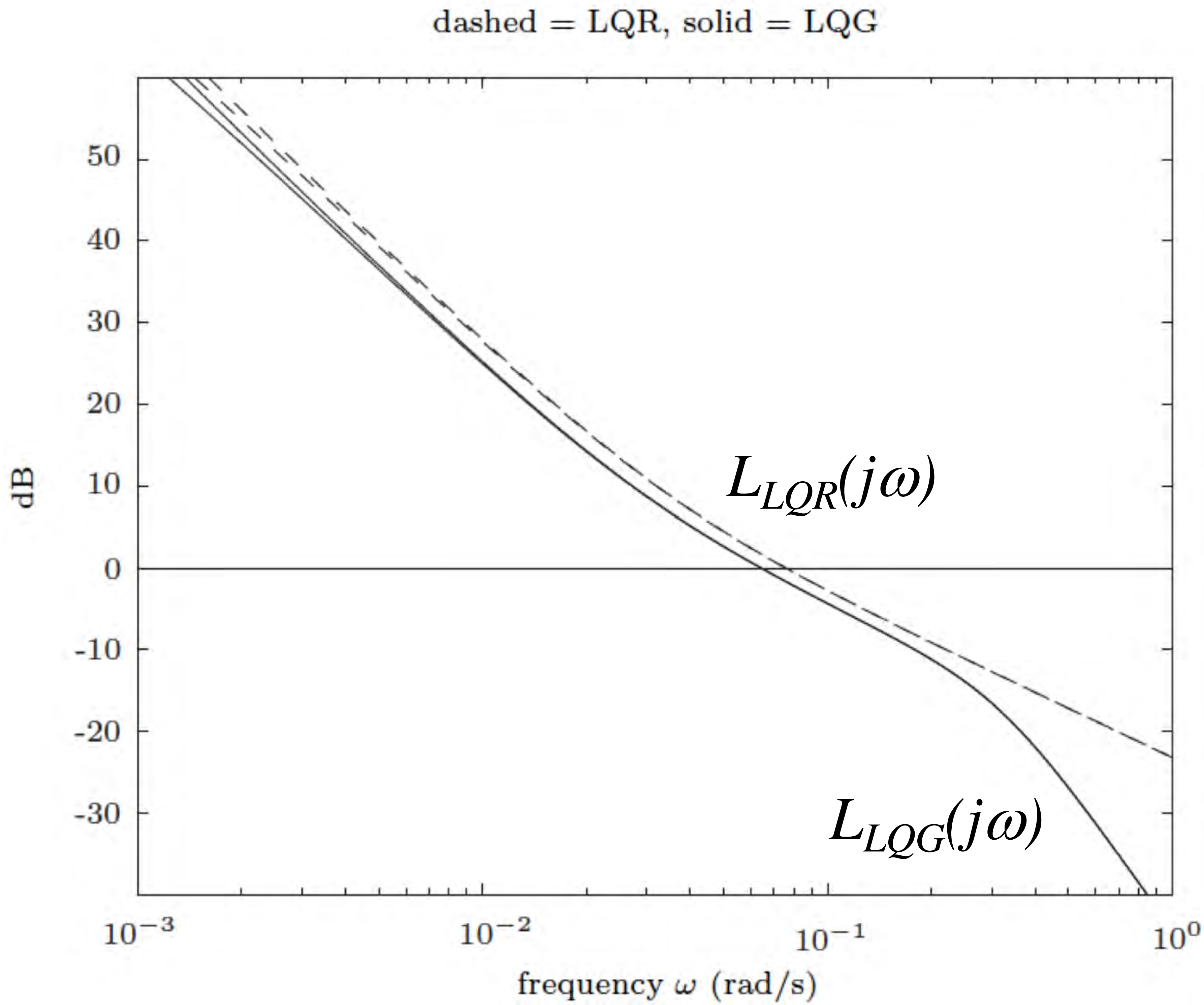
- The system must be stabilized robustly. In particular, the condition $\mu_{LQG} \geq 0.7$ must be satisfied, where μ_{LQG} is defined in (3.71).
- The crossover frequency must satisfy the bounds $5 \cdot 10^{-2} \leq \omega_c \leq 10^{-1}$ rad/s.
- All disturbances up to $10 \cdot \omega_0$ must be attenuated by at least -40 dB.
- The disturbance amplification may not be larger than 3 dB at any frequency.
- The noise attenuation must be larger than -40 dB for all frequencies higher that 10 rad/s.

```
K = lqr(A,B,C'*C,r*eye(2,2));
L = lqr(A,C',B*B',q*eye(2,2))';
```

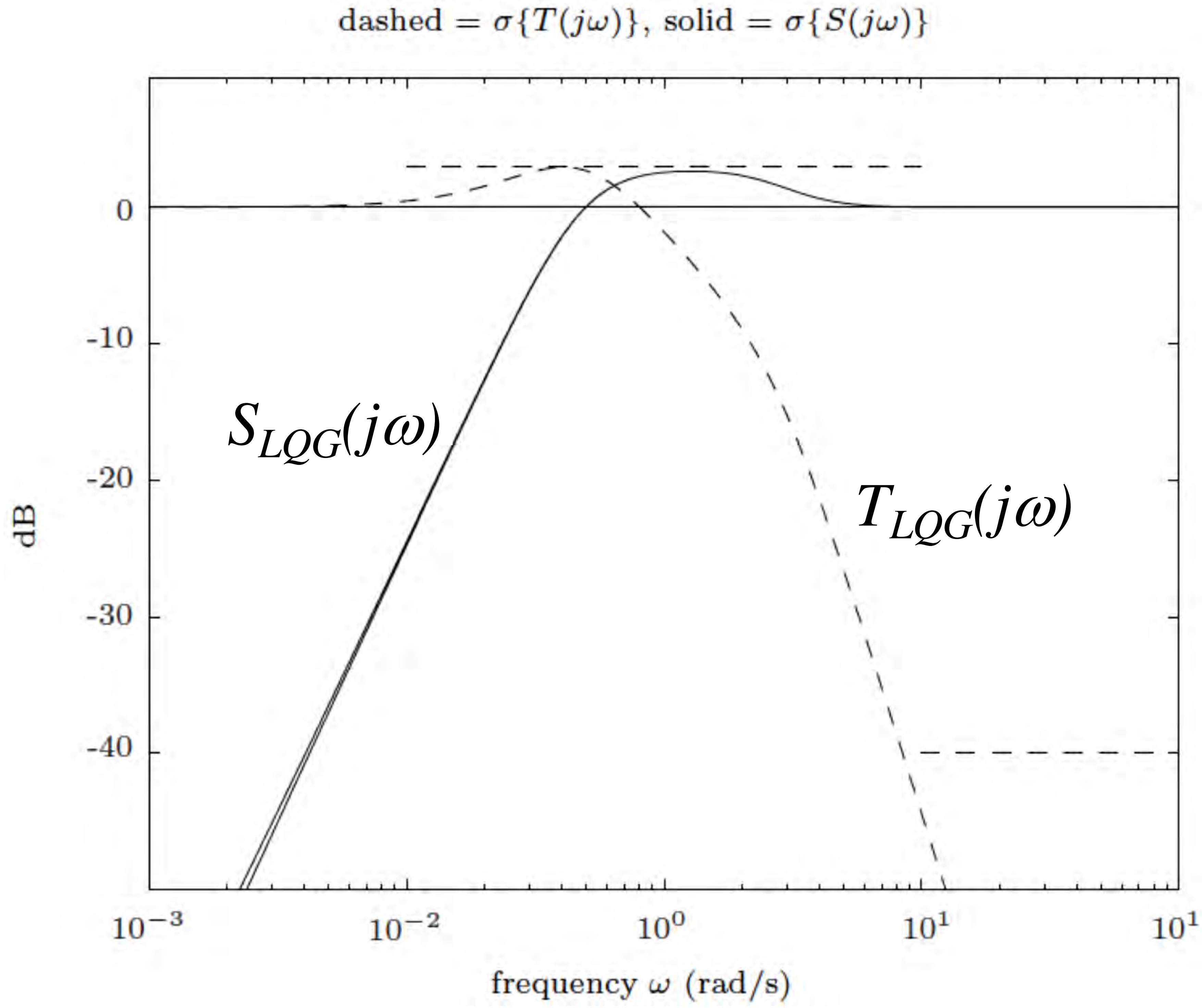


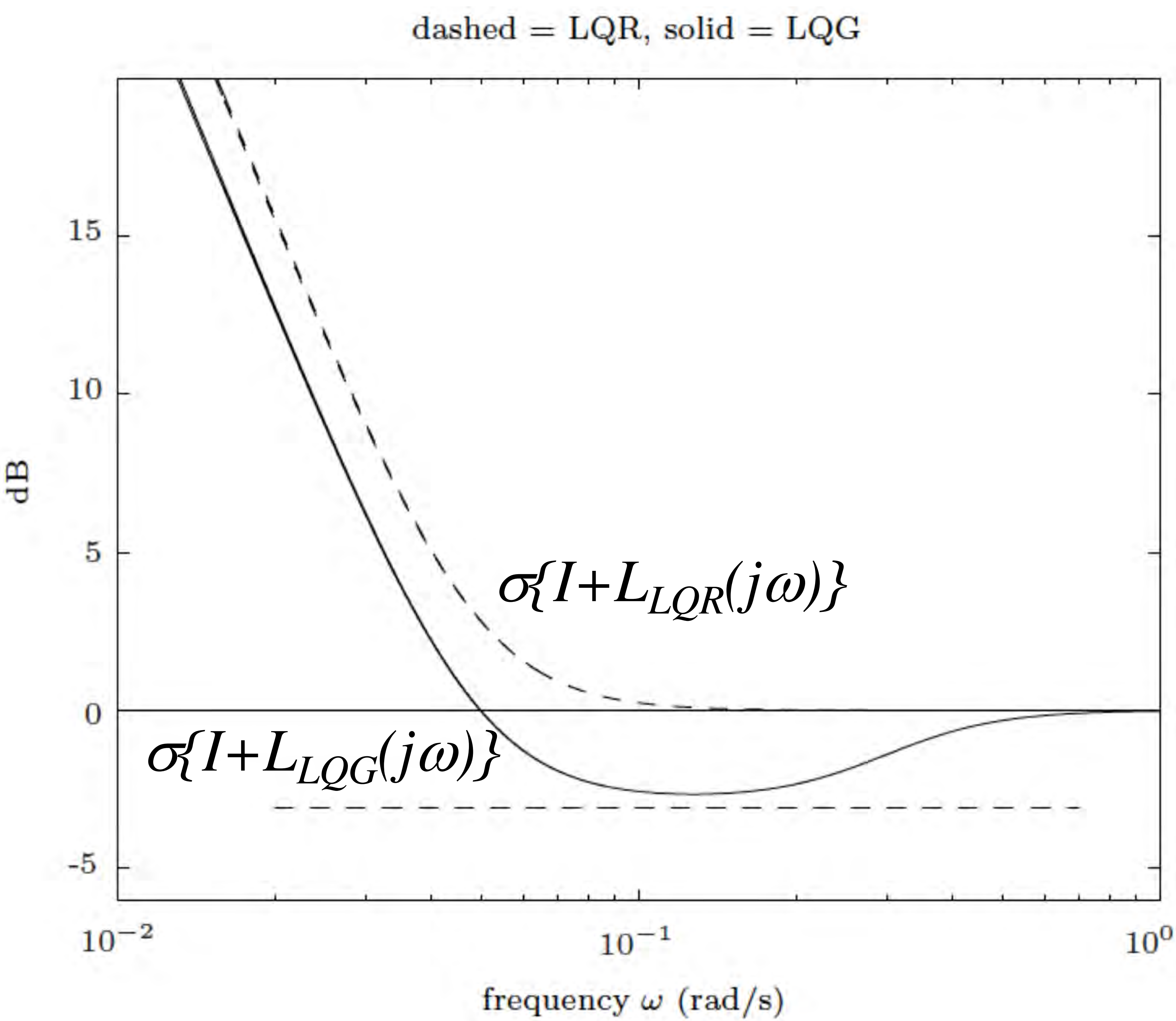
After some iterations:

$$r = 4 \cdot 10^{-10}, \quad q = 1 \cdot 10^{-13}$$

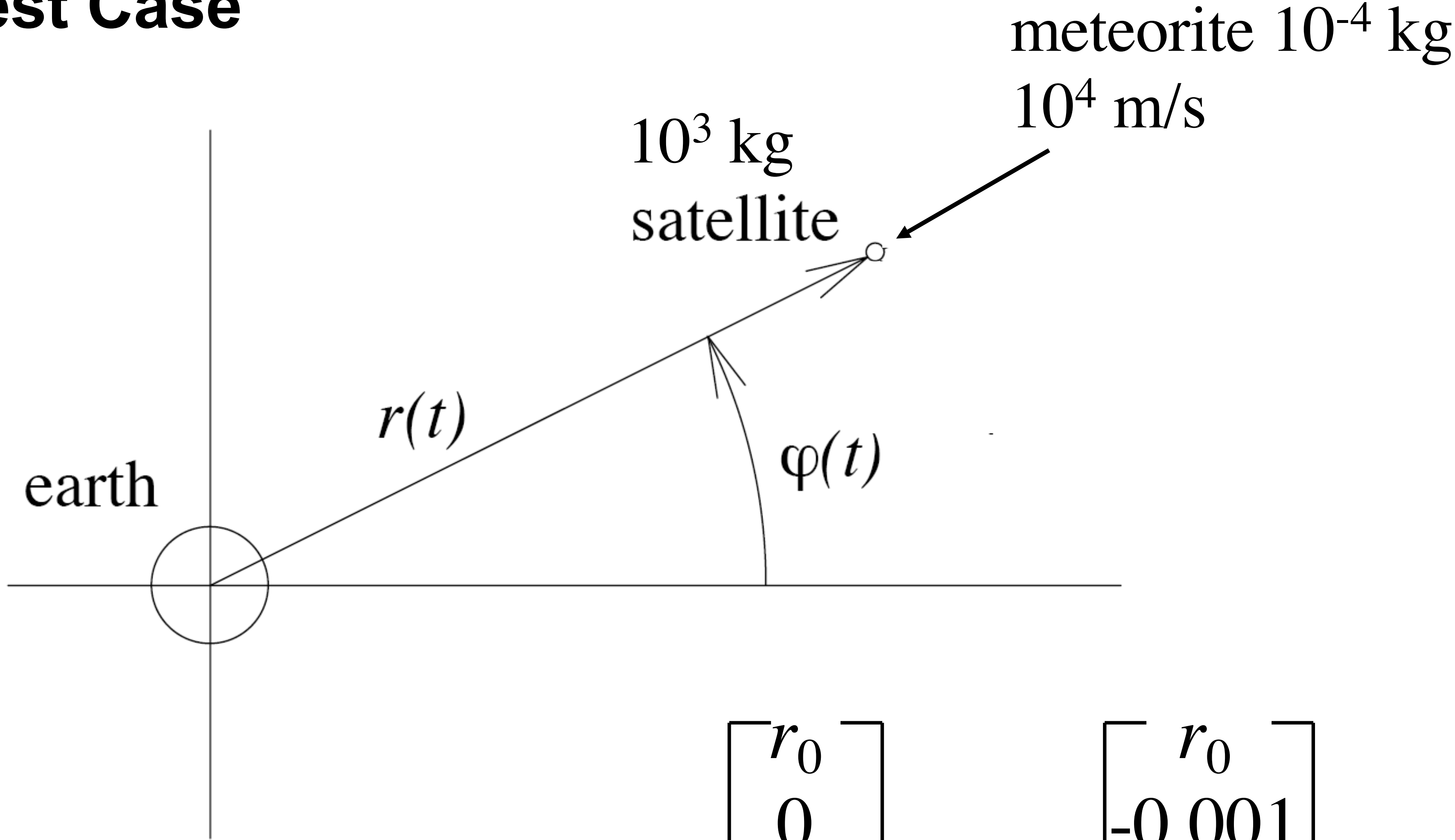


dashed = LQR, solid = LQG
50
40
30
20
10
0
-10
-20
-30
dB
$L_{LQR}(j\omega)$
$L_{LQG}(j\omega)$
10^{-3}
10^{-2}
10^{-1}
10^{0}
frequency ω (rad/s)

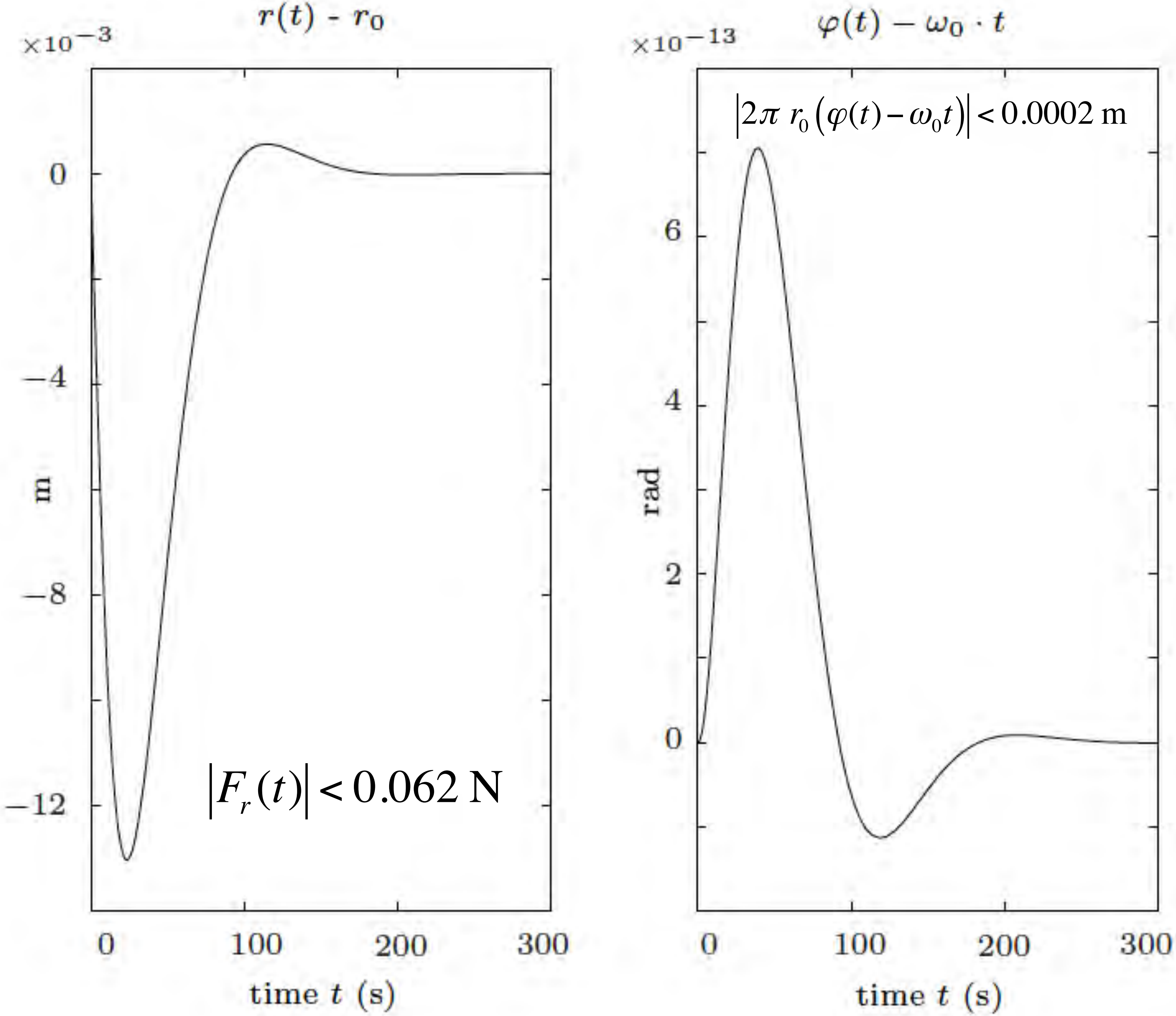

dashed = $\sigma\{T(j\omega)\}$, solid = $\sigma\{S(j\omega)\}$
0
-10
-20
-30
-40
dB
$S_{LQG}(j\omega)$
$T_{LQG}(j\omega)$
10^{-3}
10^{-2}
10^{0}
10^{1}
10^{1}
frequency ω (rad/s)


dashed = LQR, solid = LQG
dB
15
10
5
0
-5
10^{-2}
10^{-1}
10^{0}
frequency ω (rad/s)
$\sigma\{I+L_{LQR}(j\omega)\}$
$\sigma\{I+L_{LQG}(j\omega)\}$

Test Case



Before & after impact: $x(0_-)=\begin{bmatrix} r_0 \\ 0 \\ 0 \\ \omega_0 \end{bmatrix}$ $x(0_+)=\begin{bmatrix} r_0 \\ -0.001 \\ 0 \\ \omega_0 \end{bmatrix}$



r(t) - r0
×10−3
0
−4
−8
−12
m
|Fr(t)| < 0.062 N
0
100
200
300
time t (s)
φ(t) − ω0 · t
×10−13
|2π r0 (φ(t) − ω0t)| < 0.0002 m
6
4
2
0
rad
0
100
200
300
time t (s)

Lecture XI – Outlook: Glover-McFarlane and $\mathcal{H}_\infty$ Methods

Glover-McFarlane Method

In a nutshell, the Glover-McFarlane method consists of the following preliminary steps:

1 A model $P(s) = C \cdot [sI - A]^{-1} \cdot B$ of the $m \times m$ MIMO plant is the starting point. The RGA method introduced in Section 2.6 is used to obtain the best possible input/output pairing.

2 The specifications are formulated by choosing a cross-over frequency ω_c at which all singular values of the loop have to have be approximately 1.[17]

---

[17] Obviously, this scenario corresponds to requiring that all channels have approximately the same bandwidth. If there is a large separation of the bandwidths, several SISO loops designed following the cascaded-control paradigm might be the better choice.

3 A first controller $K_0(s)$ is designed such that the loop has the desired basic frequency-domain characteristics: high (infinite) gain at low (zero) frequencies, moderate (-20 db/dec) roll-off around crossover, rapid roll-off at high frequencies. Typically this controller has the form

$$K_0(s) = \mathrm{diag}\left(k_{\mathrm{p,l}} \cdot \left(1 + \frac{1}{T_{\mathrm{i,l}} \cdot s}\right) \cdot \frac{1}{\tau_l \cdot s + 1}\right), \quad l = 1, \ldots m \tag{7.1}$$

4 Two real constant $m \times m$ matrices K_1 and K_3 are used to further modify the loop gain such that the desired crossover frequency, a better decoupling of the control channels and a narrow singular value band at crossover are attained. The resulting transfer function of the extended plant

$$\widetilde{P}(s) = K_3 \cdot P(s) \cdot K_0(s) \cdot K_1, \tag{7.2}$$

which is realized using the matrices $\{\widetilde{A}, \widetilde{B}, \widetilde{C}\}$, forms the basis for the subsequent Glover-McFarlane method.

The core of the Glover-McFarlane approach is quite similar to the standard LQG design method, i.e., two Riccati equations must be solved, and the controller uses an observer-based state feedback. However, the method includes an additional tuning parameter $\alpha \in (1, \infty)$ that can be used to choose a desired trade-off between performance and robustness.

To be more specific, a controller $K_2\widetilde{(s)}$ is computed solving the following set of equations

$$\begin{aligned} &\Phi \cdot \widetilde{B} \cdot \widetilde{B}^T \cdot \Phi - \Phi \cdot \widetilde{A} - \widetilde{A}^T \cdot \Phi - \widetilde{C}^T \cdot \widetilde{C} = 0 \\ &\Psi \cdot \widetilde{C}^T \cdot \widetilde{C} \cdot \Psi - \widetilde{A} \cdot \Psi - \Psi \cdot \widetilde{A}^T - \widetilde{B} \cdot \widetilde{B}^T = 0 \end{aligned} \qquad (3.89)$$

Two positive definite solutions to these two Riccati equations exist under the usual conditions. The real positive scalar $\lambda_{\max}$, which is the largest eigenvalue of the matrix $\Phi \cdot \Psi$, is used to form a second real positive scalar

$$\gamma = \sqrt{1 + \lambda_{\max}} \cdot \alpha \tag{3.90}$$

where the influence of the design parameter $\alpha \in (1, \infty)$ will become clear shortly. Using the parameter γ the matrix

$$G = I - \frac{1}{\gamma^2}(I + \Psi \cdot \Phi) \tag{3.91}$$

is formed yielding the state feedback gain

$$\widetilde{K} = \widetilde{B}^T \cdot \Phi \tag{3.92}$$

and the observation error feedback gain

$$\widetilde{L} = G^{-1} \cdot \Psi \cdot \widetilde{C}^T$$

All necessary information is now available to build the controller $K_2(s)$

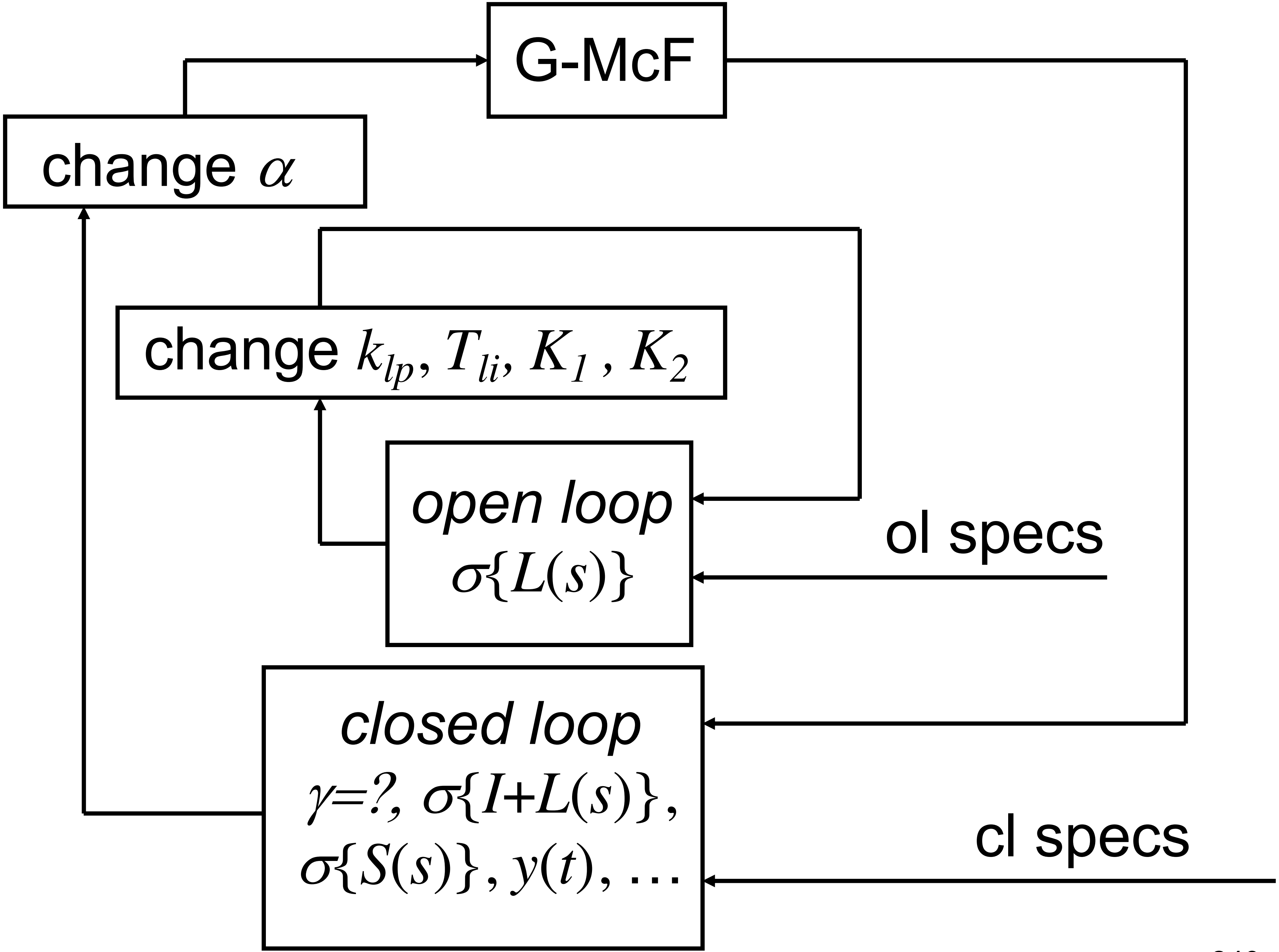
$$K_2(s) = \widetilde{K} \cdot \left(sI - (\widetilde{A} - \widetilde{B} \cdot \widetilde{K} - \widetilde{L} \cdot \widetilde{C})\right)^{-1} \cdot \widetilde{L}$$

The final controller can now be formed using the equation

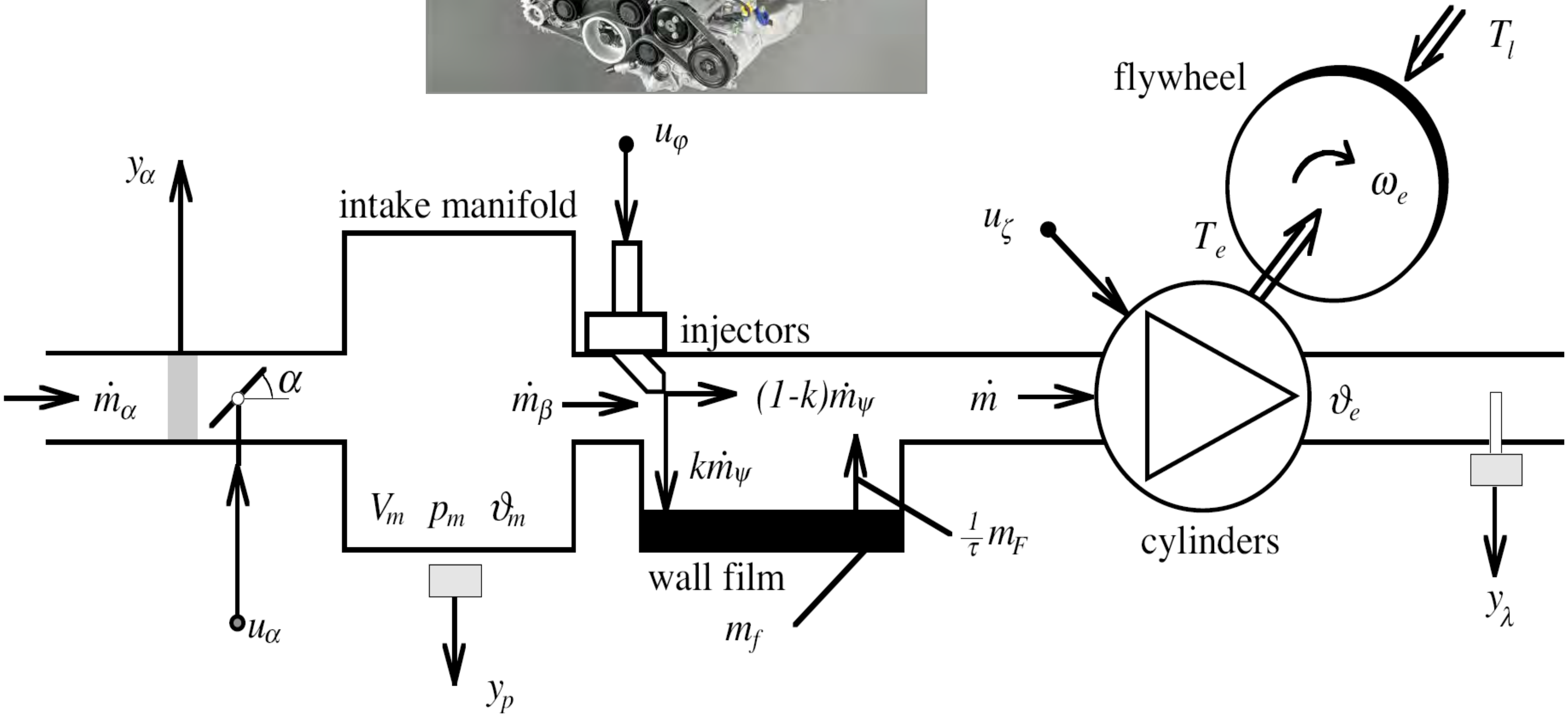
$$C(s) = K_0(s) \cdot K_1 \cdot K_2(s) \cdot K_3$$

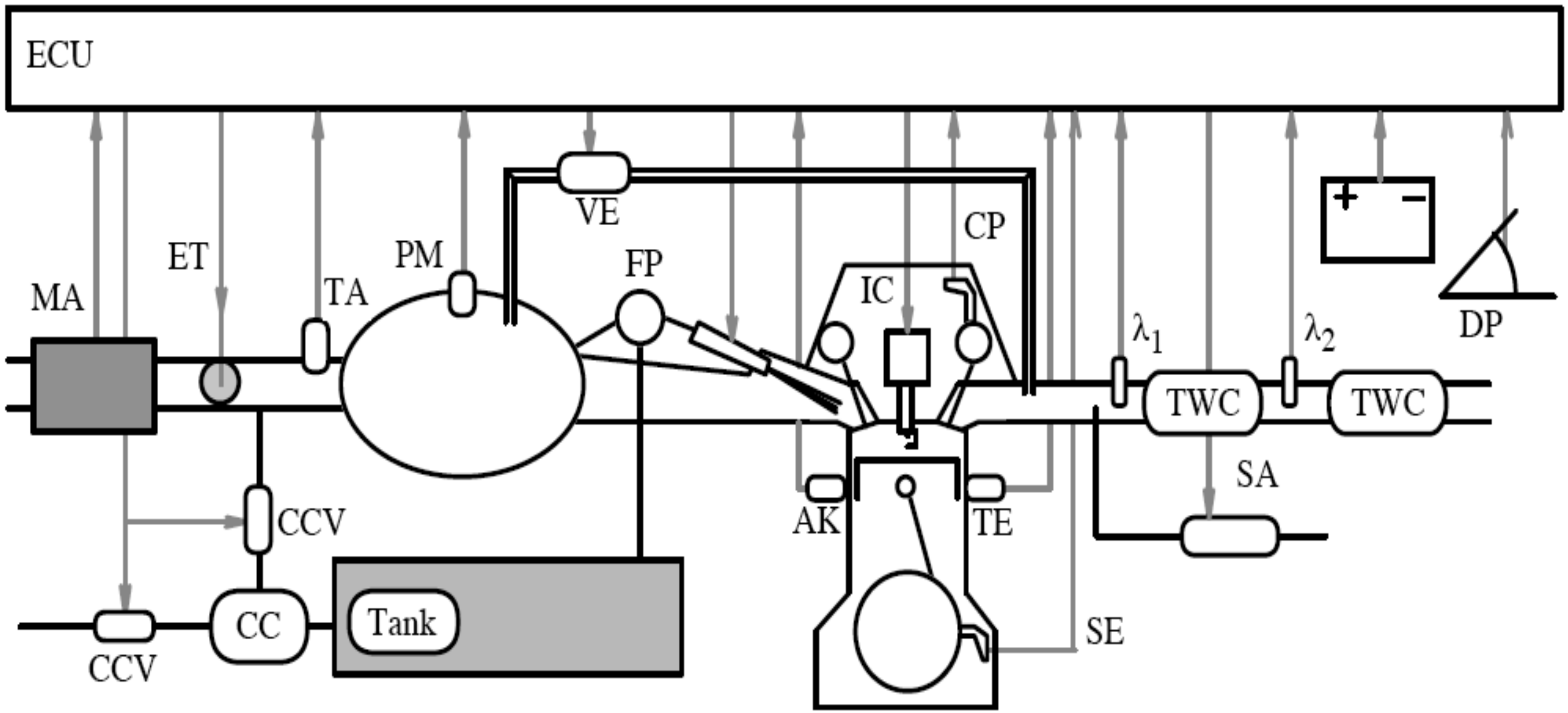
This controller is guaranteed to yield a stable closed-loop system (Separation Theorem). For $\alpha \rightarrow \infty$ the controller (3.94) simply represents the standard LQG controller obtained using (3.10) with $R = I$ and $Q = \widetilde{C}^T \cdot \widetilde{C}$, and using (3.51) with $q = 1$. By decreasing α the robustness of the loop is increased. It can be shown that for $\alpha \rightarrow 1$ the ability of the loop to tolerate modeling errors is maximized [8]. Of course, this reduces the performance of the loop, i.e., the time constants of the closed-loop response become larger.

Experience has shown that a value of $\gamma \approx 2$ indicates a very good robustness, whereas a value of $\gamma > 4$ is a sign of a rather fragile design, and further iterations on the compensators $K(s)$, K_1, and K_2 are recommended to decrease that value.

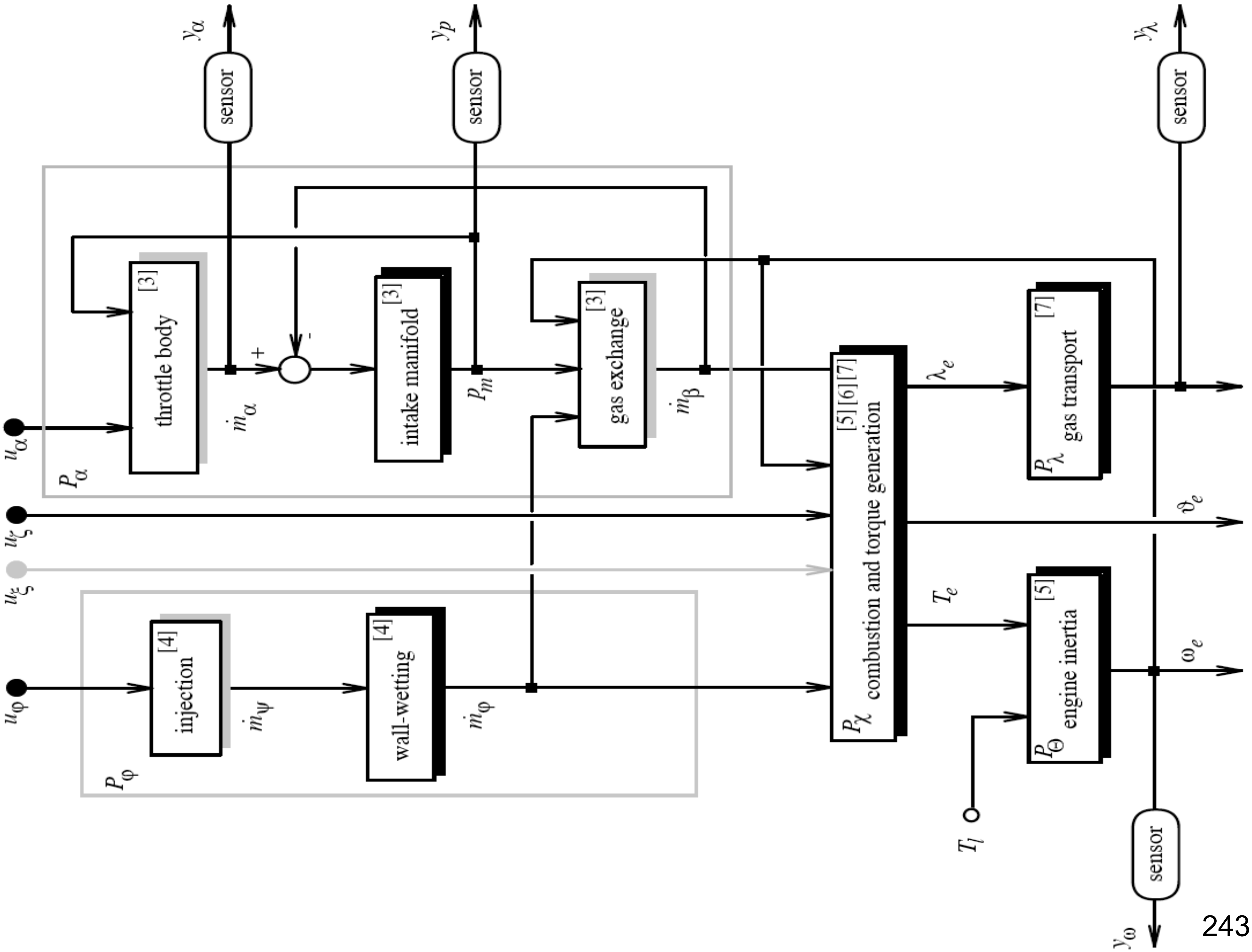

G-McF
change α
change k_{lp}, T_{li}, K_1, K_2
open loop
$\sigma\{L(s)\}$
ol specs
closed loop
$\gamma=?, \sigma\{I+L(s)\},$
$\sigma\{S(s)\}, y(t), \ldots$
cl specs

4.3 Case Study: Engine Speed and Air/Fuel Ratio Control





AK	knock sensor	PM	manifold pressure sensor	VE	EGR valve
CP	camshaft sensor	ET	electronic throttle	SA	secondary air valve
IC	ignition command	TA	intake air temperature sensor	TWC	3-way catalyst
MA	air mass-flow sensor	TE	cooling water temperature sensor	ECU	controller
SE	engine speed sensor	CC	active carbon canister	CCV	CC control valves
FP	fuel pressure control	$\lambda_{1,2}$	air/fuel ratio sensors	DP	driver pedal



sensor
sensor
sensor
sensor
y_α
y_p
y_λ
y_ω
P_α
throttle body [3]
intake manifold [3]
gas exchange [3]
u_α
$\dot{m}_\alpha$
p_m
$\dot{m}_\beta$
+
-
u_ζ
u_ξ
u_φ
P_φ
injection [4]
wall-wetting [4]
$\dot{m}_\psi$
$\dot{m}_\varphi$
P_χ combustion and torque generation [5] [6] [7]
λ_e
ϑ_e
T_e
P_λ gas transport [7]
P_Θ engine inertia [5]
ω_e
T_l

The system is of order four with the four state variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \text{throttle valve position} & (1 = 1^\circ) \\ \text{intake manifold pressure} & (1 = 0.05 \text{ bar}) \\ \text{engine speed} & (1 = 200 \text{ rpm}) \\ \text{air/fuel ratio} & (1 = 0.05\ -) \end{bmatrix}$$

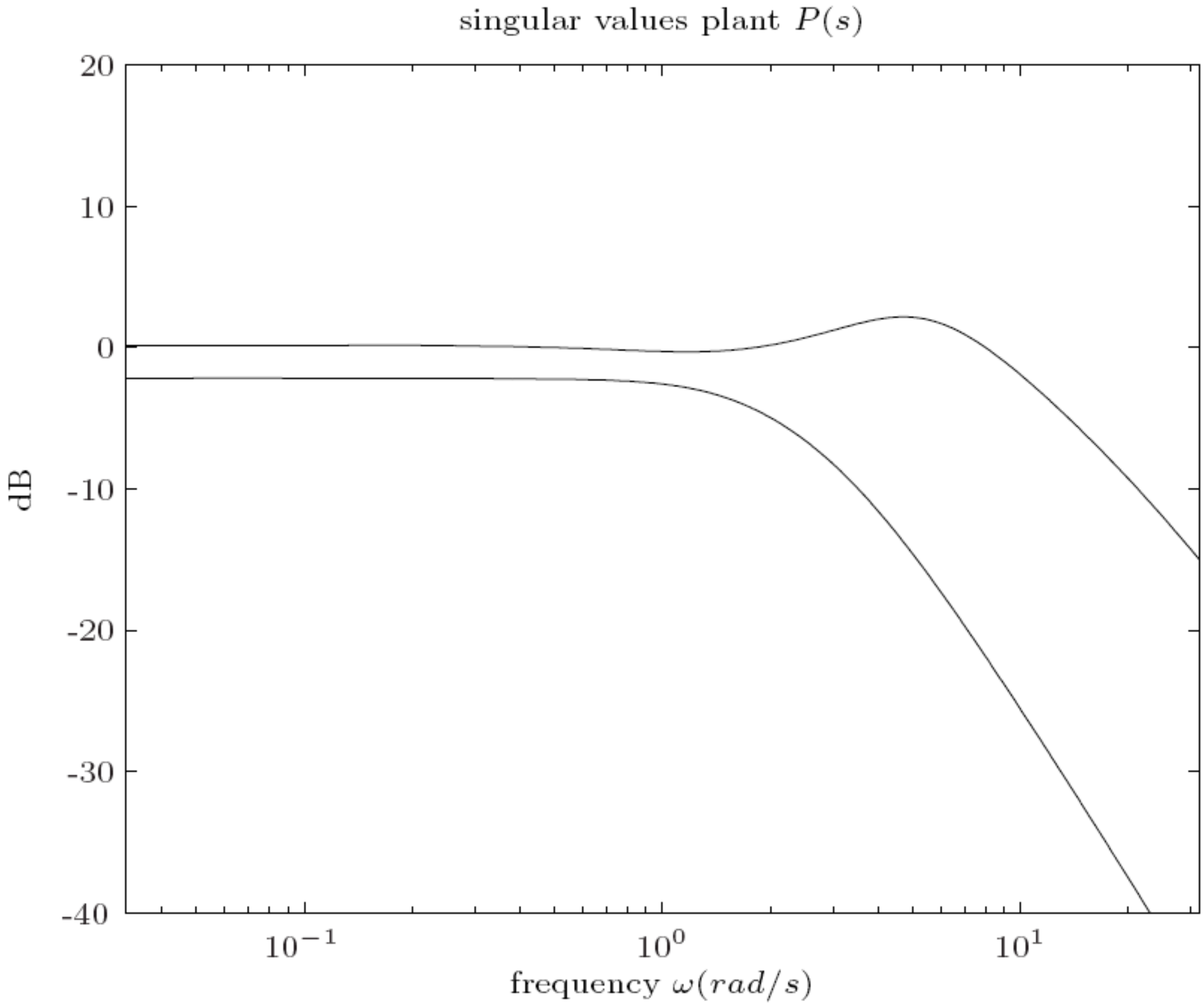
$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \text{throttle valve command } (1 = 1^\circ) \\ \text{air/fuel ratio command } (1 = 0.05\ -) \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \text{engine speed } (1 = 200 \text{ rpm}) \\ \text{air/fuel ratio } (1 = 0.05\ -) \end{bmatrix}$$

$$\left[\begin{array}{cc} A & B \\ C & D \end{array}\right] = \left[\begin{array}{cccc|cc} -25.00 & 0 & 0 & 0 & 25.00 & 0 \\ 5.22 & -4.00 & -8.24 & 0 & 0 & 0 \\ 3.09 & 1.91 & -3.07 & 0 & 0 & 0.85 \\ -7.68 & 5.89 & 12.10 & -2.10 & 0 & -2.10 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\frac{d}{dt}x(t) = A \cdot x(t) + B \cdot u(t) + b_d \cdot d(t) \quad \Bigg| \quad b_d = \left[\begin{array}{c} 0 \\ 0 \\ -7.6 \\ 0 \end{array}\right]$$

external disturbance is the load torque $d(t)$ $(1 = 40 \text{ Nm})$


singular values plant $P(s)$
20
10
0
-10
-20
-30
-40
dB
10^{-1}
10^{0}
10^{1}
frequency $\omega(rad/s)$

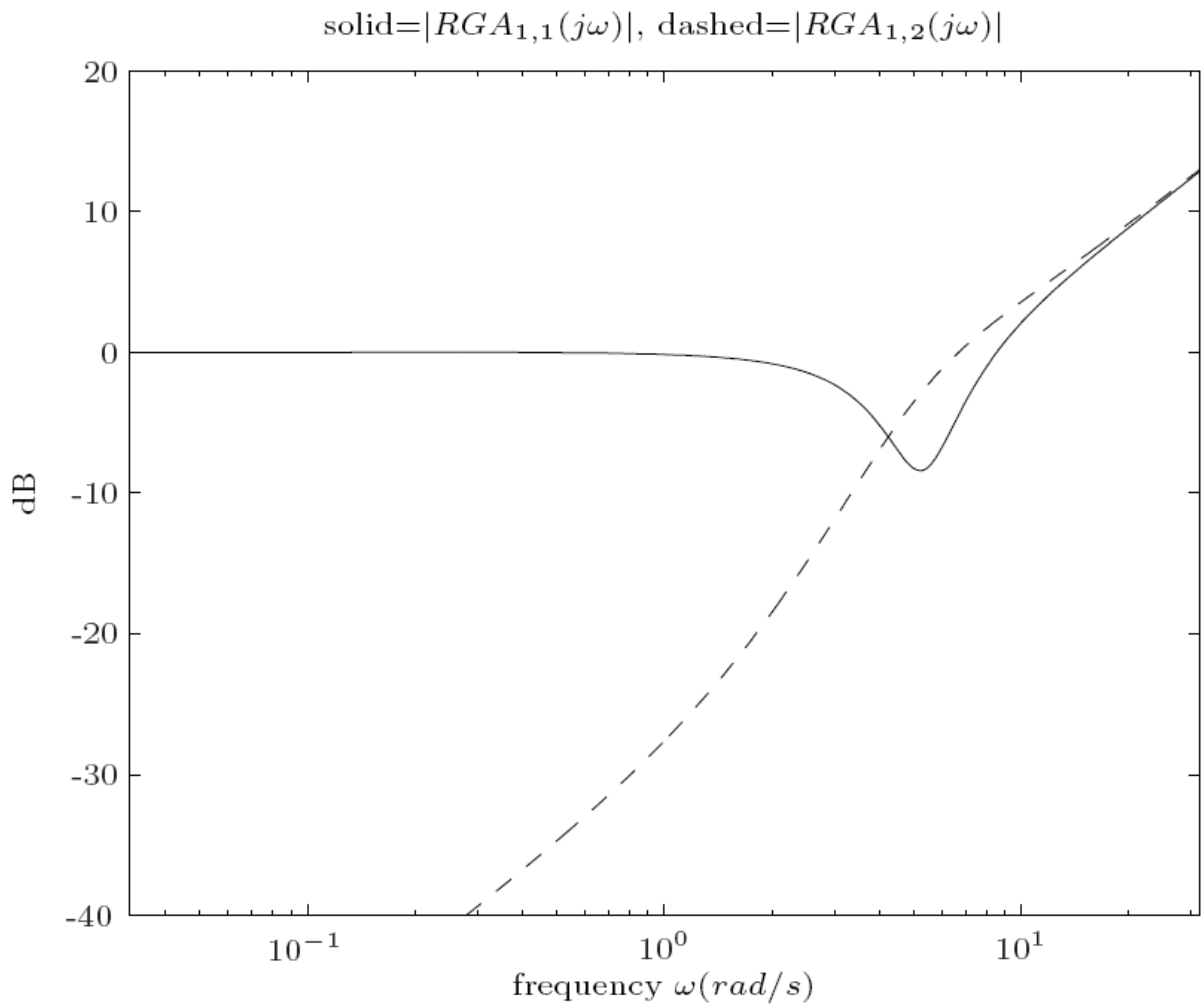

solid=$|RGA_{1,1}(j\omega)|$, dashed=$|RGA_{1,2}(j\omega)|$
20
10
0
-10
-20
-30
-40
dB
10^{-1}
10^{0}
10^{1}
frequency $\omega(rad/s)$

Figure 4.18 shows the singular values of the plant and Figure 4.19 the two relevant components of the plant's $RGA(j\omega)$ matrix. Obviously, the plant has finite gain at low frequencies (type-zero system) and it is rather well decoupled up to a frequency of approximately 1 rad/s. This frequency is, therefore, chosen as the desired crossover frequency, yielding expected settling times of the closed-loop system in the order of slightly below 2 s.

According to the Glover-McFarlane procedure, in the first step the plant is connected to a compensator

$$\frac{d}{dt}x_K(t) = A_K \cdot x(t) + B_K \cdot u_K(t), \tag{4.17}$$

$$u(t) = C_K \cdot x_K(t) + D_K \cdot e(t) \tag{4.18}$$

such that the main desired loop characteristics are obtained. In this case a PI element must be used in each channel to obtain the desired high gains at low frequencies

$$\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & \frac{k_{p,1}}{T_{i,1}} & 0 \\ 0 & 0 & 0 & \frac{k_{p,1}}{T_{i,1}} \\ \hline 1 & 0 & k_{p,1} & 0 \\ 0 & 1 & 0 & k_{p,2} \end{array}\right] \tag{4.19}$$

After some iterations, the following parameters have been chosen

$$k_{p,1} = k_{p,2} = 0.5, \quad T_{i,1} = T_{i,2} = 0.15 \text{ s}$$

Of course, these parameters are not the best possible choice. However, they yield reasonable results as shown below.

In addition to this compensator, the Glover-McFarlane procedure uses two constant matrices K_1 and K_3 to narrow the singular-value distribution around the crossover frequency. In this case study the choice $K_1 = K_3 = I$ is made because the distribution is considered to be satisfactory without any additional correcting terms.

How to choose K_i? Not trivial, physical intuition and some guidelines ⇒ [Ljung]

The next step in the Glover-McFarlane procedure is to solve the Equations (7.3) through (7.7) for the augmented system (7.2). The solution is the controller $K_2(s)$ (7.8), which in this case study has the order 6. The design parameter α is chosen to be 1.01, i.e., the design is pushed towards maximum robustness. The resulting gain γ is approximately 2.07, i.e., a very good result is achieved. This can also be seen in Figure 8.26, which shows that the minimum singular value of the return difference $I + L(s)$ is always larger than -2.5 dB (≈ 0.75).

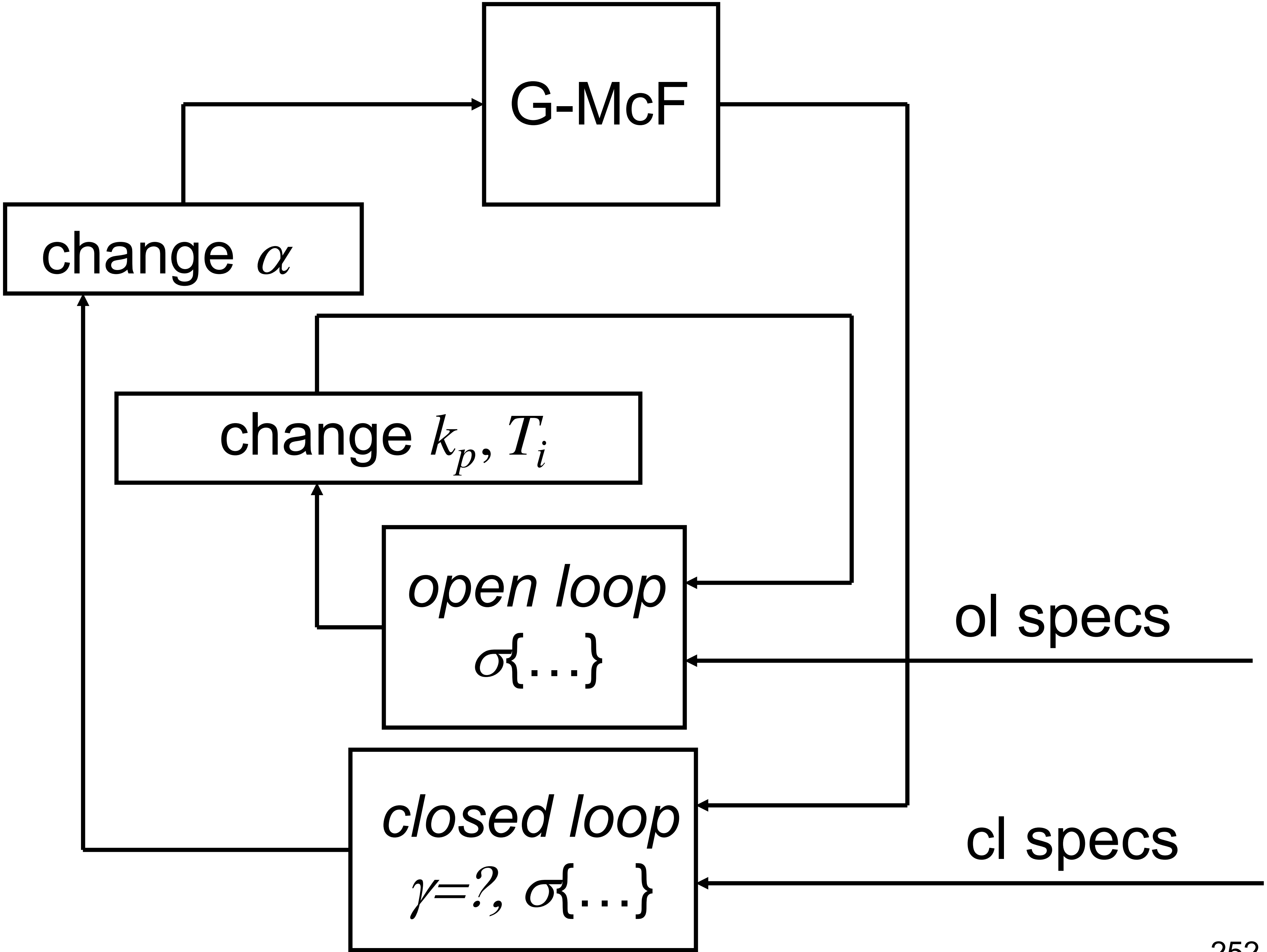
$$\Phi \cdot \widetilde{B} \cdot \widetilde{B}^{\mathrm{T}} \cdot \Phi - \Phi \cdot \widetilde{A} - \widetilde{A}^{\mathrm{T}} \cdot \Phi - \widetilde{C}^{\mathrm{T}} \cdot \widetilde{C} = 0$$

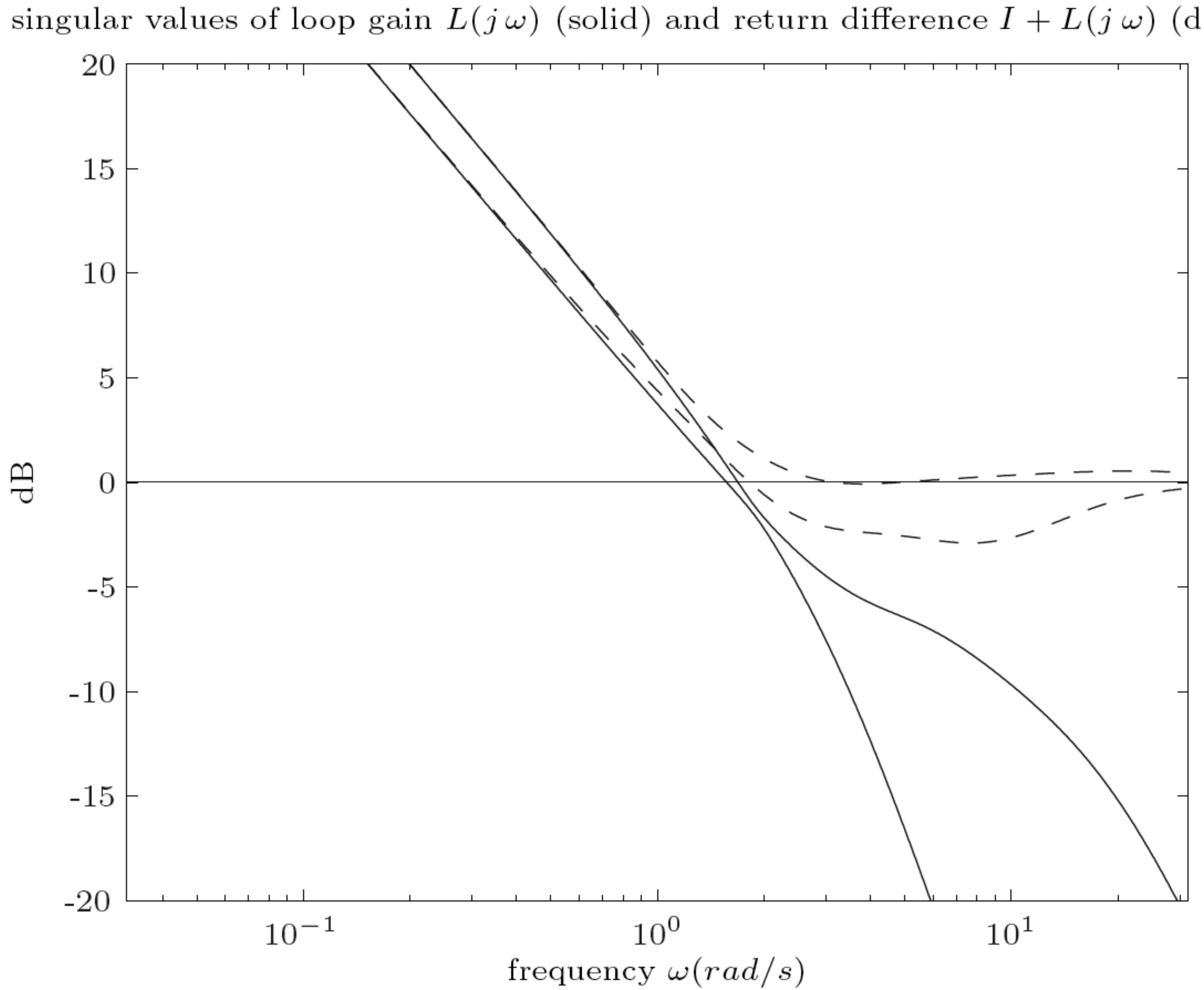
$$\Psi \cdot \widetilde{C}^{\mathrm{T}} \cdot \widetilde{C} \cdot \Psi - \widetilde{A} \cdot \Psi - \Psi \cdot \widetilde{A}^{\mathrm{T}} - \widetilde{B} \cdot \widetilde{B}^{\mathrm{T}} = 0$$

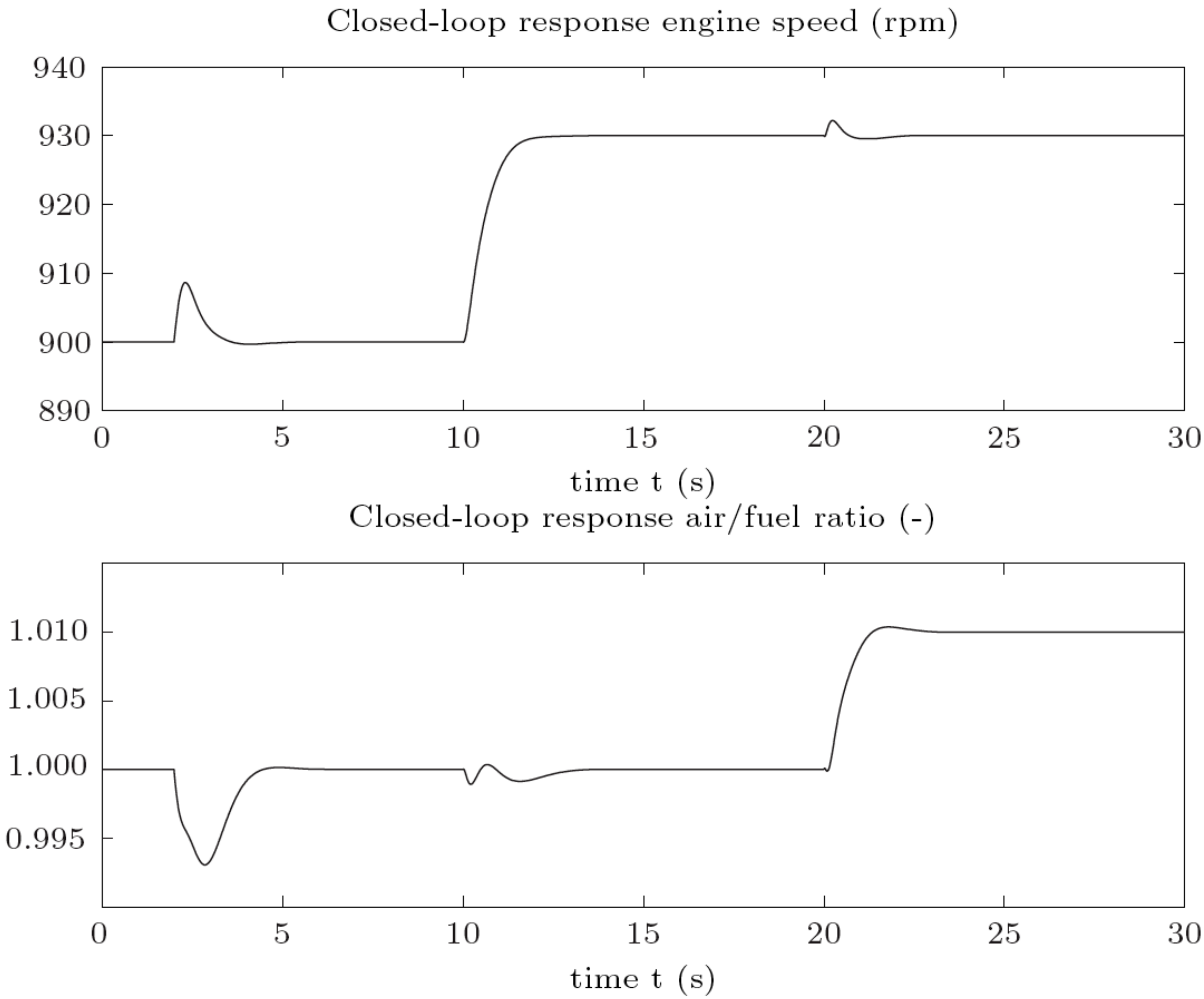
$$\gamma = \sqrt{1 + \lambda_{\max}} \cdot \alpha \qquad G = I - \frac{1}{\gamma^2}(I + \Psi \cdot \Phi) \qquad \widetilde{K} = \widetilde{B}^{\mathrm{T}} \cdot \Phi \qquad \widetilde{L} = G^{-1} \cdot \Psi \cdot \widetilde{C}^{\mathrm{T}}$$

$$K_2(s) = \widetilde{K} \cdot \left(sI - (\widetilde{A} - \widetilde{B} \cdot \widetilde{K} - \widetilde{L} \cdot \widetilde{C}) \right)^{-1} \cdot \widetilde{L}$$

$$C(s) = K_0(s) \cdot K_1 \cdot K_2(s) \cdot K_3$$


G-McF
change α
change k_p, T_i
open loop
$\sigma\{\ldots\}$
ol specs
closed loop
$\gamma=?, \sigma\{\ldots\}$
cl specs


singular values of loop gain $L(j\,\omega)$ (solid) and return difference $I + L(j\,\omega)$ (d
20
15
10
5
0
-5
-10
-15
-20
dB
10^{-1}
10^{0}
10^{1}
frequency $\omega(rad/s)$


Closed-loop response engine speed (rpm)
940
930
920
910
900
890
0
5
10
15
20
25
30
time t (s)
Closed-loop response air/fuel ratio (-)
1.010
1.005
1.000
0.995
0
5
10
15
20
25
30
time t (s)

Introduction Mixed-Sensitivity Approach

- The $\mathcal{H}_\infty$ control approach formulates the control system design as a mathematical optimization problem.
- The $\mathcal{H}_\infty$ method assures the stability of the closed loop system if the optimization problem can be solved.
- Loop shaping as well as robust control are combined in the optimization problem.
- The $\mathcal{H}_\infty$ controller can be used to minimize the effects of plant modeling errors on control system performance and robustness.
- The $\mathcal{H}_\infty$ approach is applicable to SISO systems as well as to MIMO systems with cross-couplings.
- The approach tries to minimize the maximal amplification, expressed by the infinity norm, of certain input signals. In this sense it is a worst-case analysis.
- In contrast to classical iterative loop shaping techniques, the $\mathcal{H}_\infty$ approach allows to impose specifications directly in the frequency domain as a part of the the controller design.

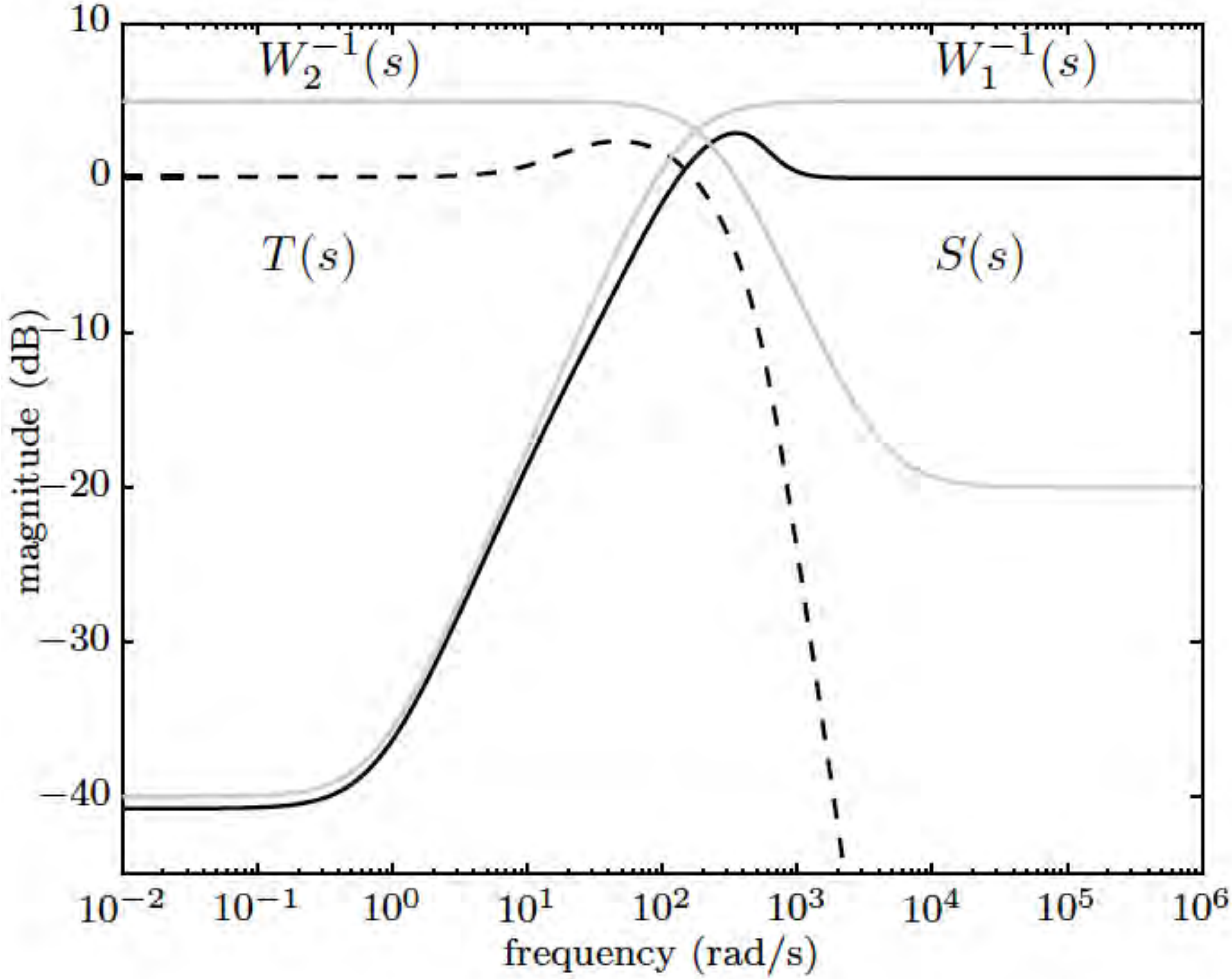
Full problem not directly solvable

$$\xi(C(s)) = \|\, |W_1(s) \cdot S(s)| + |W_2(s) \cdot T(s)| \,\|_\infty < 1$$

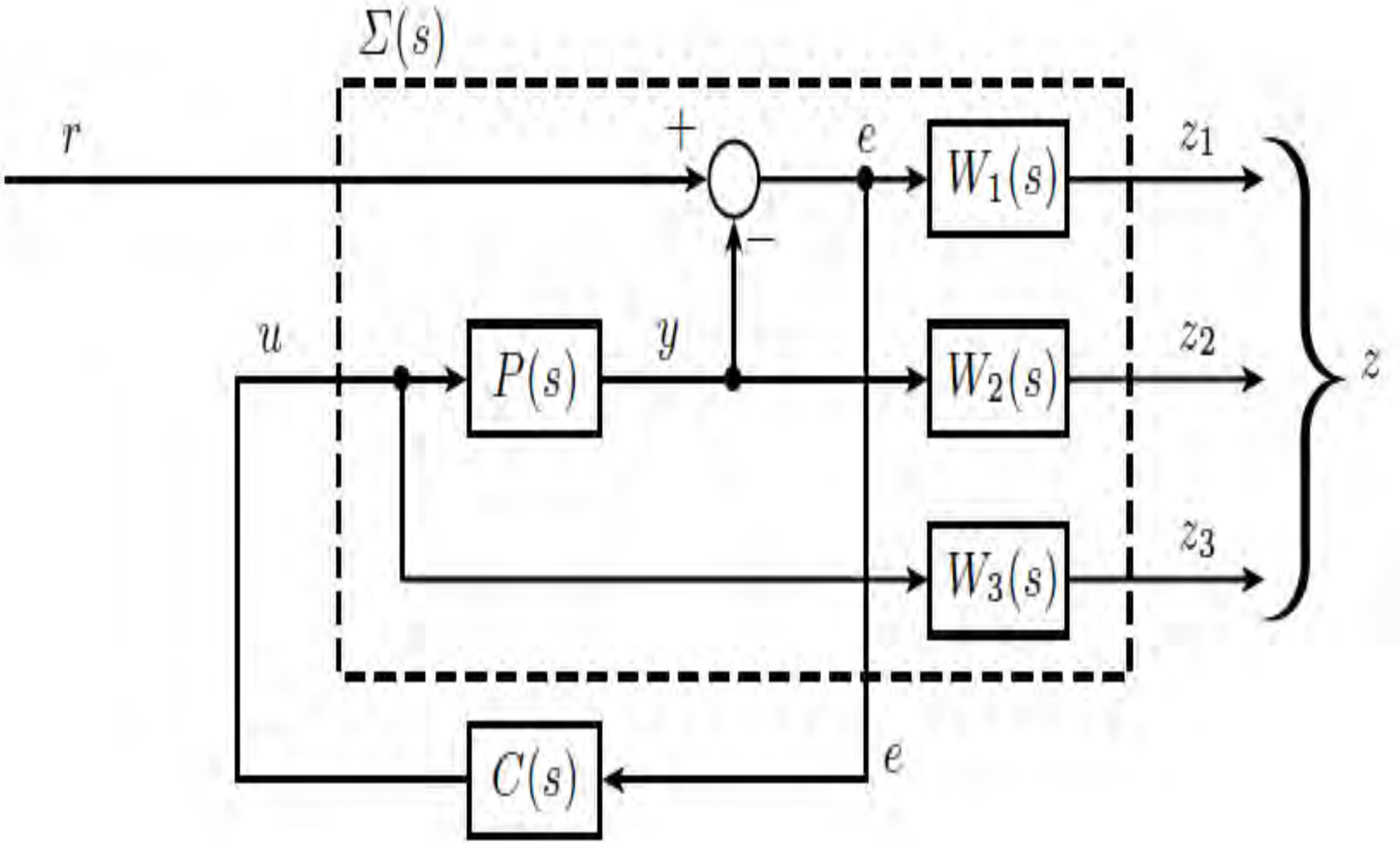
Relaxed problem solvable

$$\|W_1(s) \cdot S(s)\|_\infty < 1$$

$$\|W_2(s) \cdot T(s)\|_\infty < 1$$

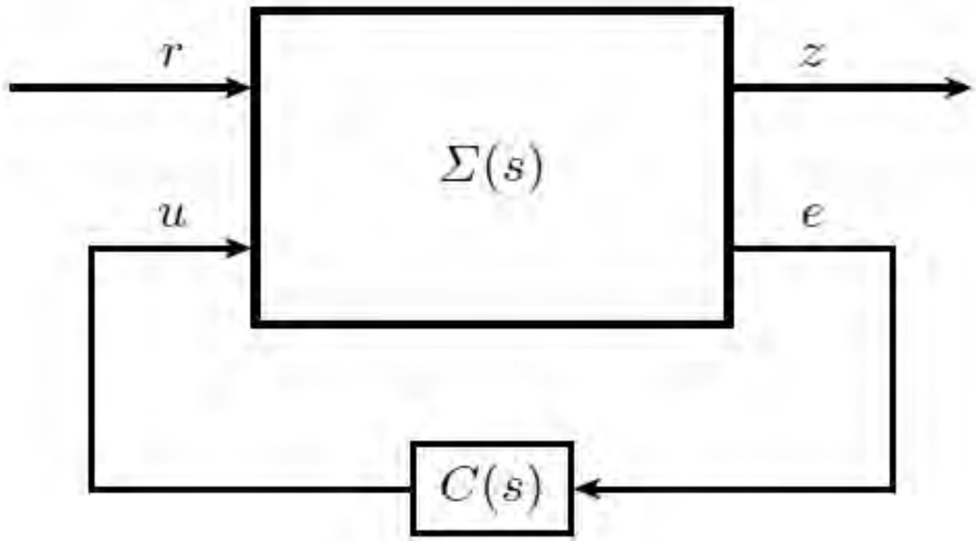


$W_2^{-1}(s)$
$W_1^{-1}(s)$
$T(s)$
$S(s)$
magnitude (dB)
frequency (rad/s)
10
0
−10
−20
−30
−40
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}



$\Sigma(s)$
r
$+$
$-$
e
$W_1(s)$
z_1
u
$P(s)$
y
$W_2(s)$
z_2
z
$W_3(s)$
z_3
$C(s)$
e

Problem



For system

$$\begin{bmatrix} Z(s) \\ E(s) \end{bmatrix} = \Sigma(s) \cdot \begin{bmatrix} R(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} \Sigma_{11}(s) & \Sigma_{12}(s) \\ \Sigma_{21}(s) & \Sigma_{22}(s) \end{bmatrix} \cdot \begin{bmatrix} R(s) \\ U(s) \end{bmatrix}$$

Find control

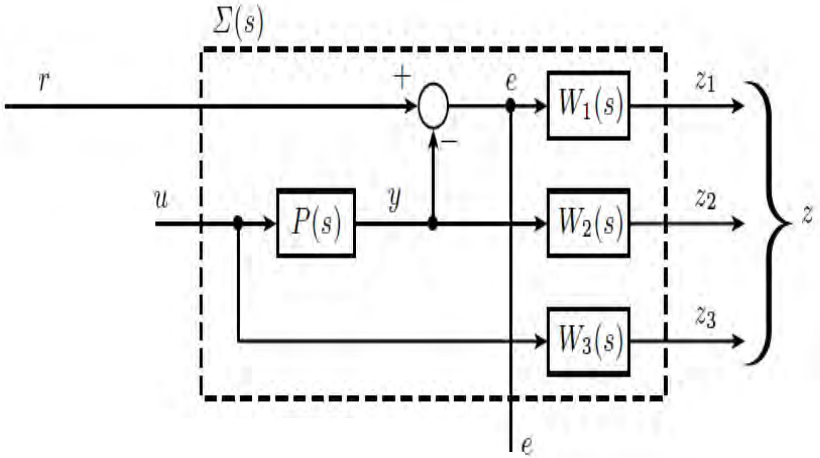
$$U(s) = C(s) \cdot E(s)$$

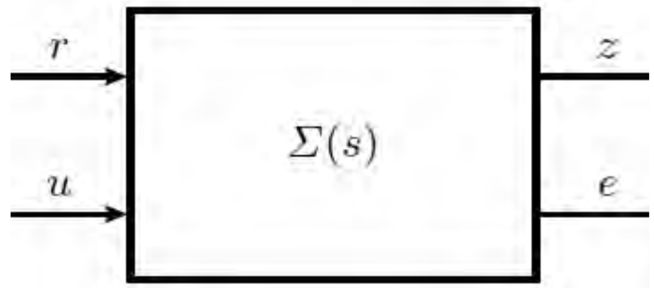
Such that

$$\|\mathcal{F}(\Sigma(s), C(s)\|_{\infty} < 1$$

Where

$$\mathcal{F}(\Sigma(s), C(s)) = \begin{bmatrix} W_1(s) \cdot S(s) \\ W_2(s) \cdot T(s) \\ W_3(s) \cdot C(s) \cdot S(s) \end{bmatrix}$$





$$\left(\begin{array}{c|cc} A_{\text{aug}} & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right)$$

```
Sigma = augw(P,W1,W2,W3)
```

1. Fix a large value for γ.
2. Find a quadratic real-valued matrix X such that $X \geq 0$ is a solution to the algebraic Riccati equation

$$0 = XA_{\text{aug}} + A_{\text{aug}}^{\text{T}} X + C_1^{\text{T}} C_1 + X(\frac{1}{\gamma^2} B_1 B_1^{\text{T}} - B_2 B_2^{\text{T}})X \tag{7.27}$$

and $\text{Re}\left[\lambda_i(A_{\text{aug}} + (\frac{1}{\gamma^2} B_1 B_1^{\text{T}} - B_2 B_2^{\text{T}})X)\right] < 0, \ \forall i.$[3]

3. Find a quadratic real-valued matrix Y such that $Y \geq 0$ is a solution to the algebraic Riccati equation

$$0 = A_{\text{aug}} Y + YA_{\text{aug}}^{\text{T}} + B_1 B_1^{\text{T}} + Y(\frac{1}{\gamma^2} C_1^{\text{T}} C_1 - C_2^{\text{T}} C_2) \tag{7.28}$$

and $\text{Re}\left[\lambda_i(A_{\text{aug}} + Y(\frac{1}{\gamma^2} C_1^{\text{T}} C_1 - C_2^{\text{T}} C_2)\right] < 0, \ \forall i.$

4. The spectral radius[4] ρ of the matrix product $X \cdot Y$ must satisfy the following inequality

$$\max_i |\lambda_i(XY)| = \rho(XY) \leq \gamma^2 \tag{7.29}$$

5. Reduce γ until no solution is found.[5].

`hinfsyn`

Solution: Observer-based state feedback

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) = A_\infty \cdot \hat{x}(t) - Z \cdot L \cdot y(t)$$

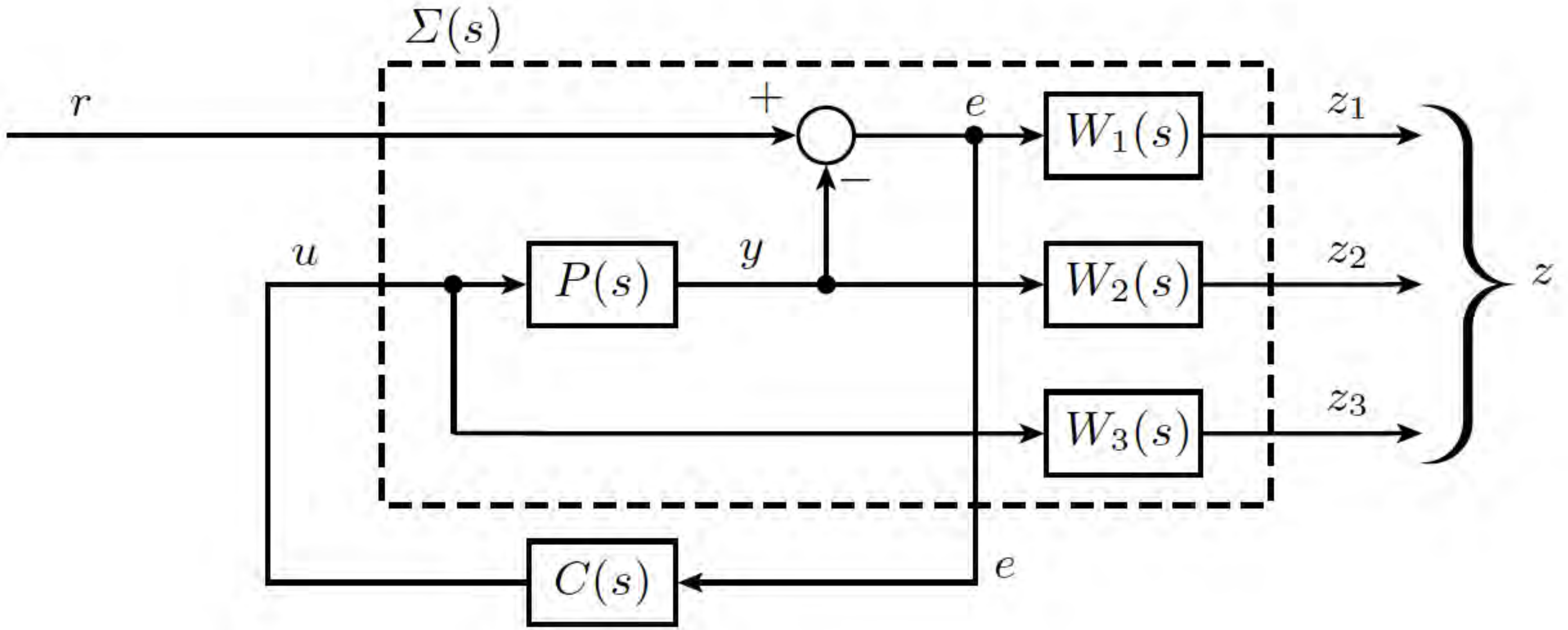
$$u(t) = F \cdot \hat{x}(t)$$

$$A_\infty = A_{\mathrm{aug}} + \tfrac{1}{\gamma^2} B_1 B_1^{\mathrm{T}} X + B_2 F + Z L C_2$$

$$F = -B_2^{\mathrm{T}} X,\; L = -Y C_2^{\mathrm{T}}$$

$$Z = (I - \gamma^{-2} Y X)^{-1}$$

Case Study: Levitating Sphere ($\mathcal{H}_\infty$ controller)



$$W_3(s) = 15 \cdot 10^4$$

$$W_1^{-1}(s) = k \cdot \frac{T \cdot s + 1}{\alpha \cdot T \cdot s + 1}$$

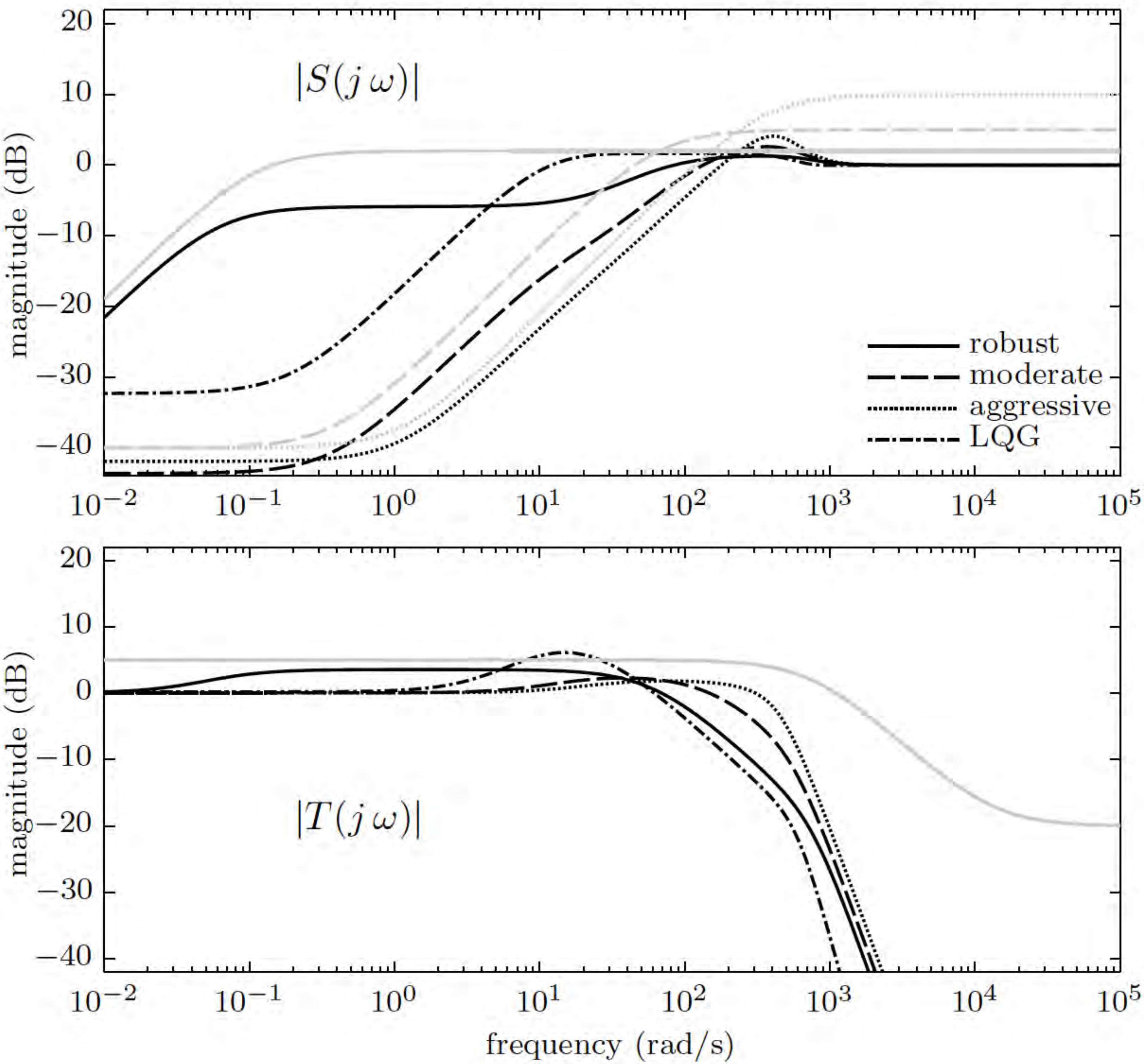
	Robust	Moderate	Aggressive
k	-40 dB	-40 dB	-40 dB
k/α	2 dB	5 dB	10 dB
$\hat{\omega}_1 = (T \cdot \sqrt{\alpha})^{-1}$	0.01 rad/s	5 rad/s	20 rad/s

$$W_2^{-1}(s) = k \cdot \frac{T \cdot s + 1}{\alpha \cdot T \cdot s + 1}$$

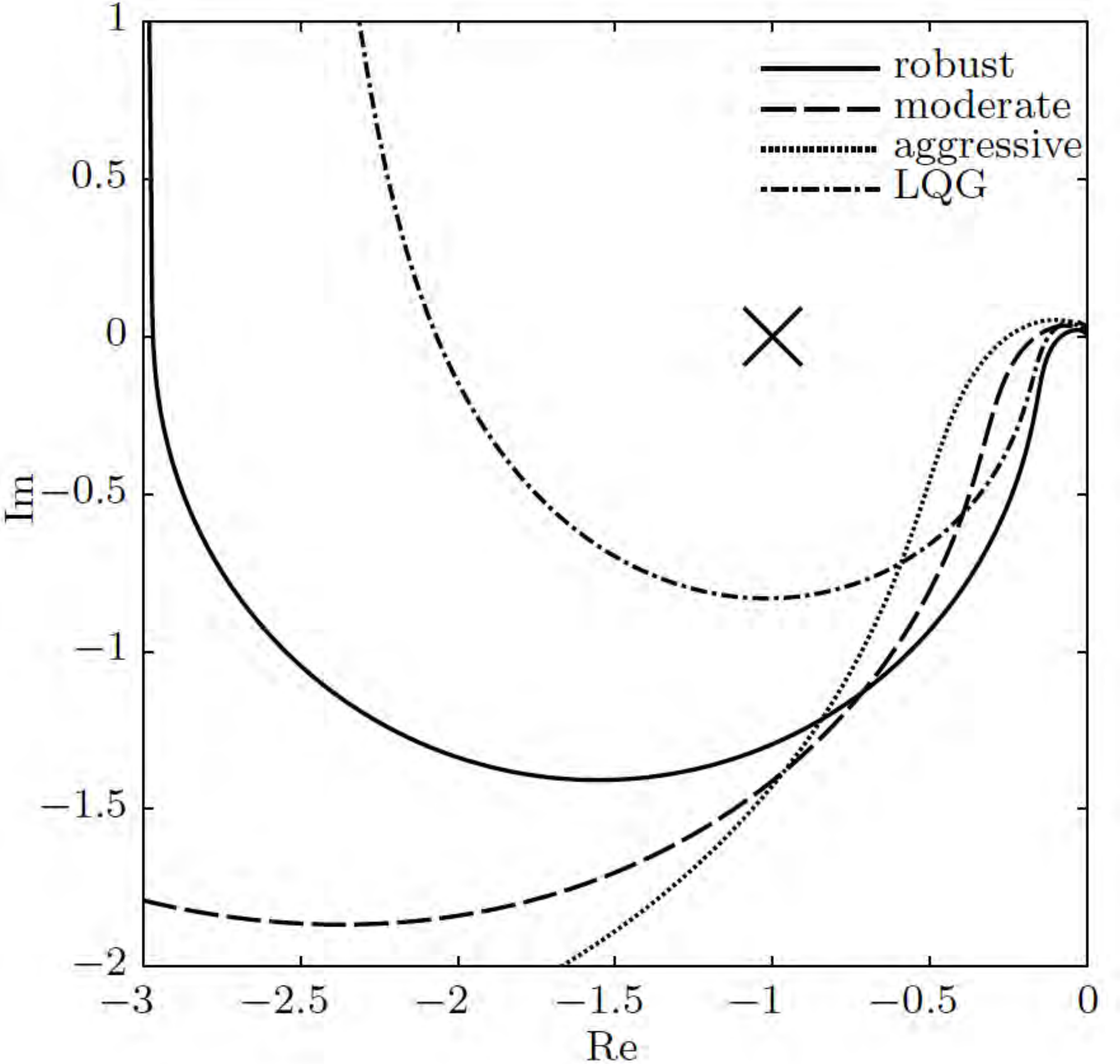
$$\lim_{s \to 0} W_2^{-1}(s) = k = 1.7 = 5\,\mathrm{dB}$$

$$\lim_{s \to \infty} W_2^{-1}(s) = \frac{k}{\alpha} = 0.1 = -20\,\mathrm{dB}$$

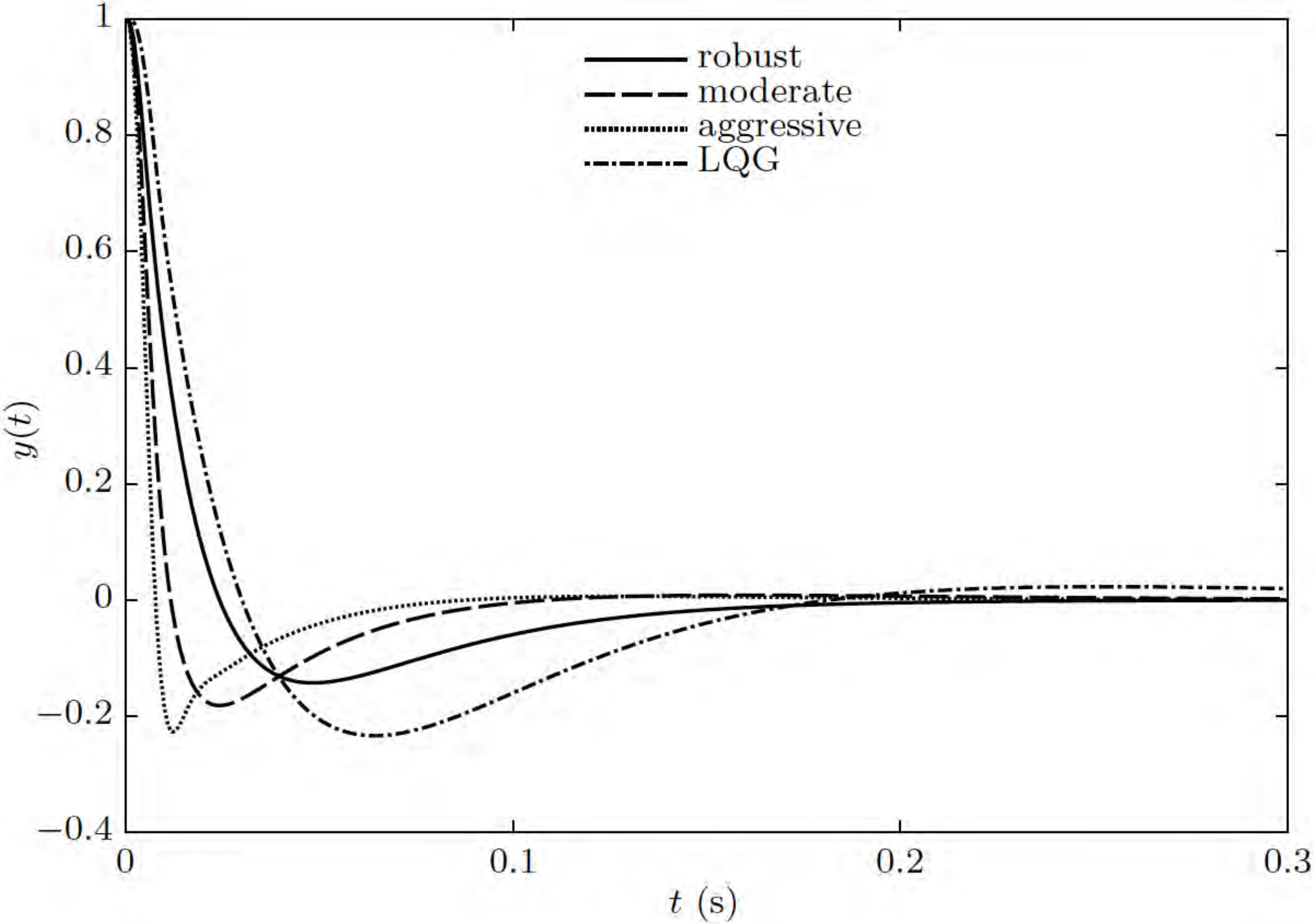
$$\hat{\omega}_2 = (T \cdot \sqrt{\alpha})^{-1} = 2 \cdot \pi \cdot 500\, \mathrm{rad}/\mathrm{s}$$



$|S(j\,\omega)|$
magnitude (dB)
robust
moderate
aggressive
LQG
$|T(j\,\omega)|$
frequency (rad/s)



robust
moderate
aggressive
LQG
Im
Re
1
0.5
0
−0.5
−1
−1.5
−2
−3
−2.5
−2
−1.5
−1
−0.5
0


robust
moderate
aggressive
LQG
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
y(t)
0
0.1
0.2
0.3
t (s)

H-infinity „Recipe“

1. Formulate plant dynamics $P(s)$
2. Find useful weights $W_i(s)$
3. Build standard system description *augw*
4. Find solution with *hinfsyn*
5. Check for resulting γ^*, if $\gamma^*>1$ relax weights and repeat step 3 ➻ 5 until $\gamma^*\leq 1$
6. Check if robust performance $\xi\leq 1$ satisfied, if not relax weights and repeat step 3 ➻ 6
7. Check time domain behavior

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