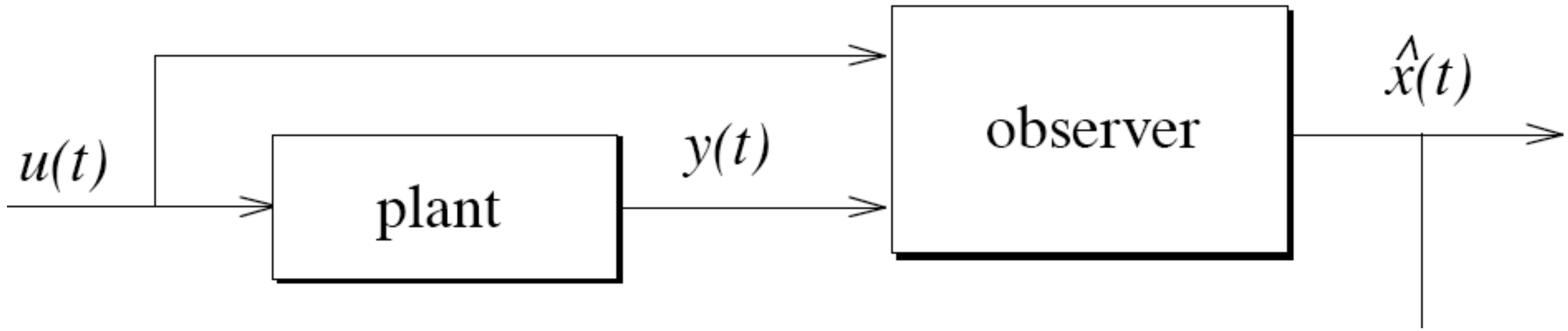
Plan:

1. $u(t) = f(x(t), t)$

2. 

3.

4. Many improvements

Use *u(t)=-k x(t)*

First idea: Place Eigenvalues of *A-b k* at desired locations

$$\alpha(s) = s^n + \alpha_{n-1} \cdot s^{n-1} + \ldots + \alpha_1 \cdot s + \alpha_0$$

$$q = [0, 0, \ldots 0, 1] \cdot \mathcal{R}^{-1}$$

$$k = q \cdot \alpha(A)$$

Remarks:

- Obviously, complete controllability ($\det\{\mathcal{R}\} \neq 0$) is a necessary and sufficient condition to be able to arbitrarily place the eigenvalues of $A - b \cdot k$.

- Controller designs using eigenvalue placement approaches are tricky. In the SISO case this approach can lead to acceptable results but only with some care (robustness can be small).

- In the MIMO case this approach often fails and is *not* recommended (the situation where the different channels work against each other is difficult to avoid).

Linear-Quadratic Optimal Controllers

An approach well suited for MIMO systems that will prove to be very useful even in output-feedback designs is obtained when the design problem is formulated as an *optimization problem.* Starting with the plant description

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t), \quad x(0) = x_0 \neq 0$$

a *state-feedback* control signal

$$u(t) = f(x(t))$$

is sought such that

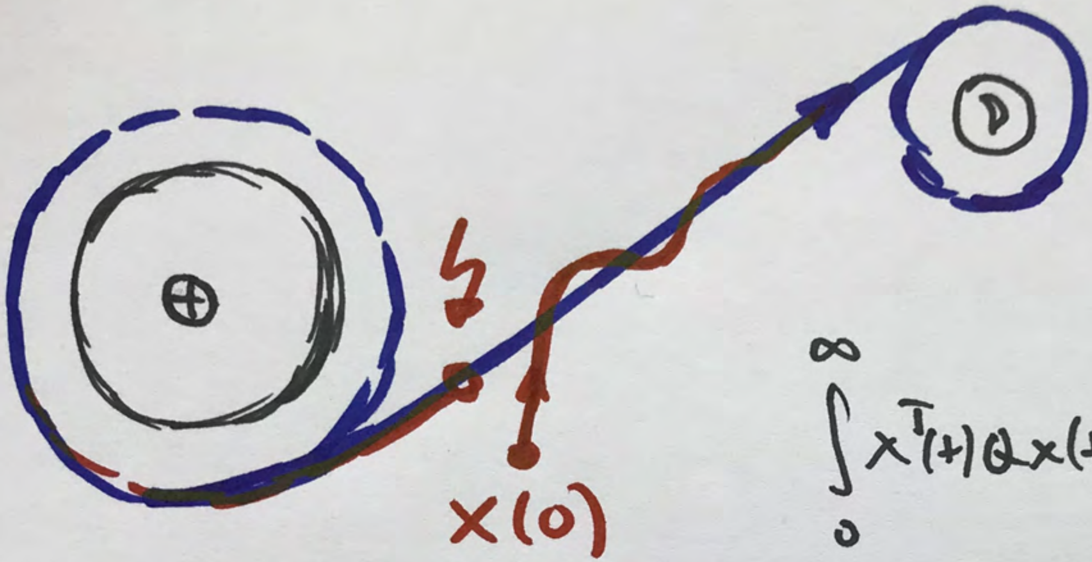
$$\lim_{t \to \infty} x(t) = 0$$

and such that this transient minimizes a *quadratic* objective function

$$J(x(t), u(t)) = \int_{t=0}^{\infty} x^T(t) \cdot Q \cdot x(t) + u(t)^T \cdot R \cdot u(t)\, dt$$

with

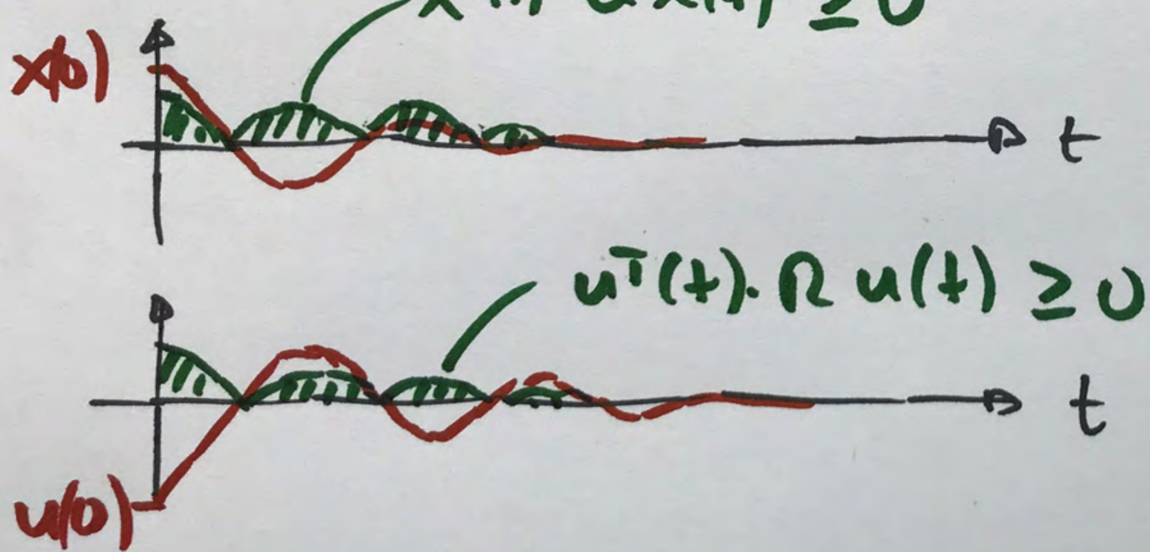
$$Q = Q^T \geq 0, \quad \text{and} \quad R = R^T > 0$$



$$\int_0^\infty x^T(t)\, Q\, x(t) + u^T(t) \cdot R \cdot u(t)\, dt$$

$= J$ soll minimal sein

$J \in \mathbb{R}_+$



Solution

$$u(t) = -K \cdot x(t), \quad \text{where} \quad K = R^{-1} \cdot B^T \cdot \Phi$$

The matrix Φ is the only positive definite and stabilizing solution of the matrix Riccati equation

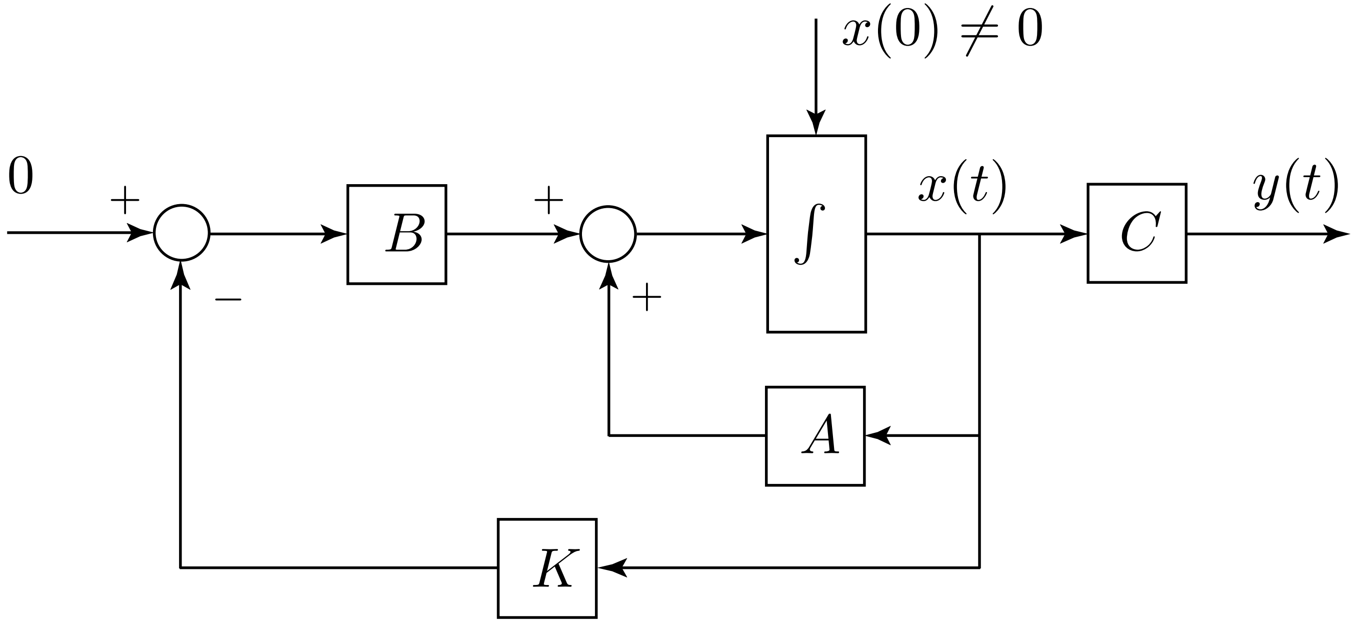
$$\Phi \cdot B \cdot R^{-1} \cdot B^T \cdot \Phi - \Phi \cdot A - A^T \cdot \Phi - Q = 0$$

Note: $\Phi = \Phi^T$ by construction. There are many solutions to that equation but only one positive definite one.

Key points:

- the controller is a *linear* function of the state variable $x(t)$;
- the controller is *time invariant*; and
- the controller requires a matrix-quadratic algebraic equation to be solved (requires numerical procedures, “lqr” in Matlab).

MIMO structure of the control system



Loop gain (breaking at plant input)

$$L_{LQR}(s) = K \cdot [s\,I - A]^{-1} \cdot B$$

Return difference (breaking at plant input)

$$I + L_{LQR}(s) = I + K \cdot [s\,I - A]^{-1} \cdot B$$

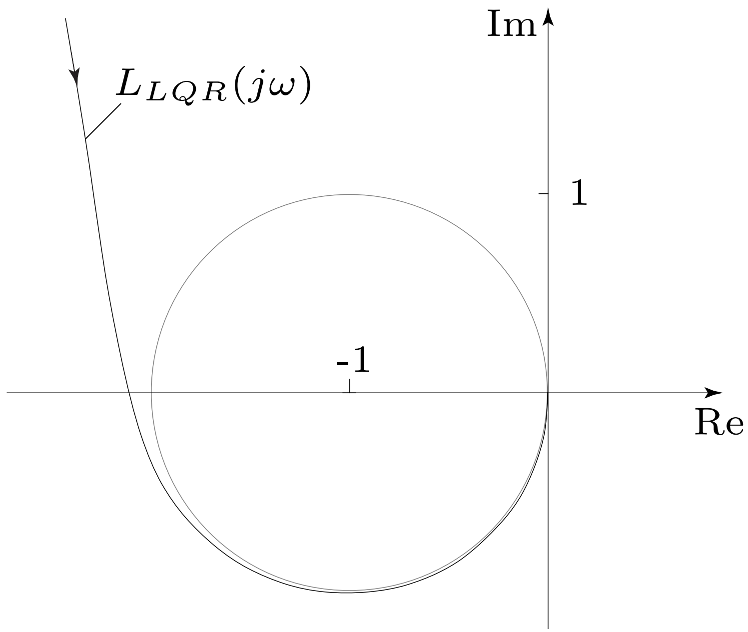
Key properties:

1) Closed-loop system always asymptotically stable, i.e., all eigenvalues of A-$B\,K$ have negative real parts (A-$B\,K$ is Hurwitz).

2) Robustness properties of LQR controllers (SISO case):

$$\mu_{LQR} = \min_{\omega} |1 + L_{LQR}(j\,\omega)| = 1$$

Graphical interpretation:



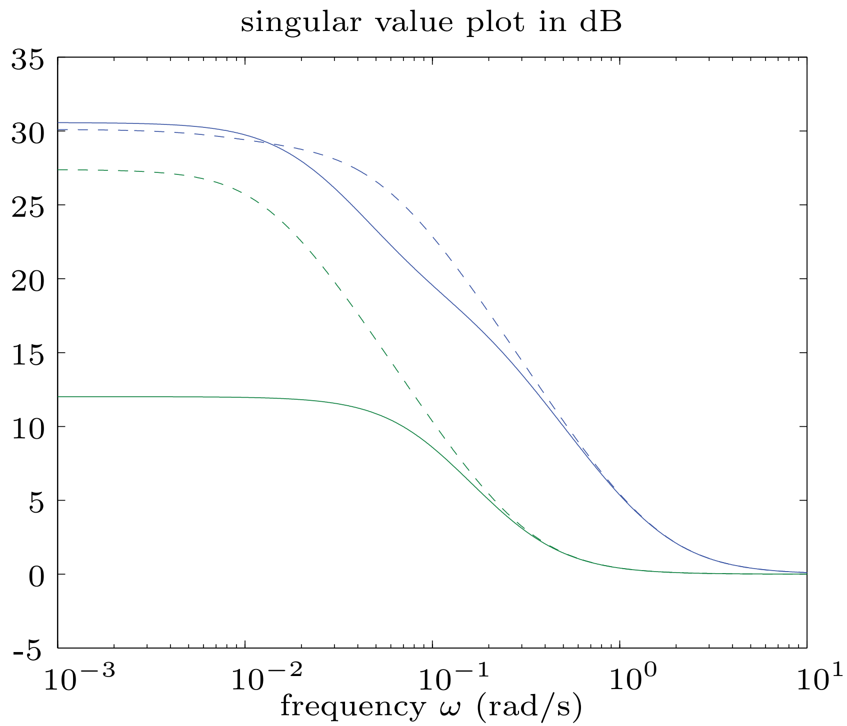
Robustness properties of LQR controllers in the MIMO case

$$\mu_{LQR} = \min_{\omega} \sigma_{\min}\{I + L_{LQR}(s)\} = 1$$

From this it follows that

$$\max_{\omega} \sigma_{\max}\{S(j\,\omega)\} = 1 \quad \text{and} \quad \max_{\omega} \sigma_{\max}\{T(j\,\omega)\} = 2$$

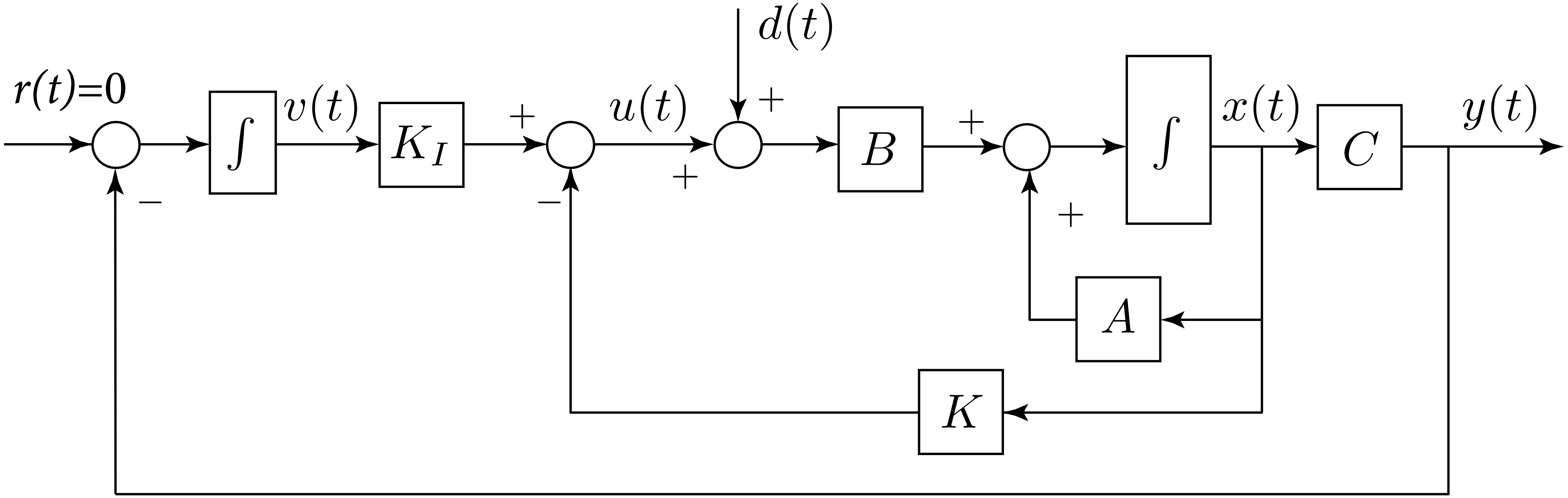
Singular-values of loop gains of heat exchanger example ($k = 2000$ (–) and 100 (- -) W/m^2)



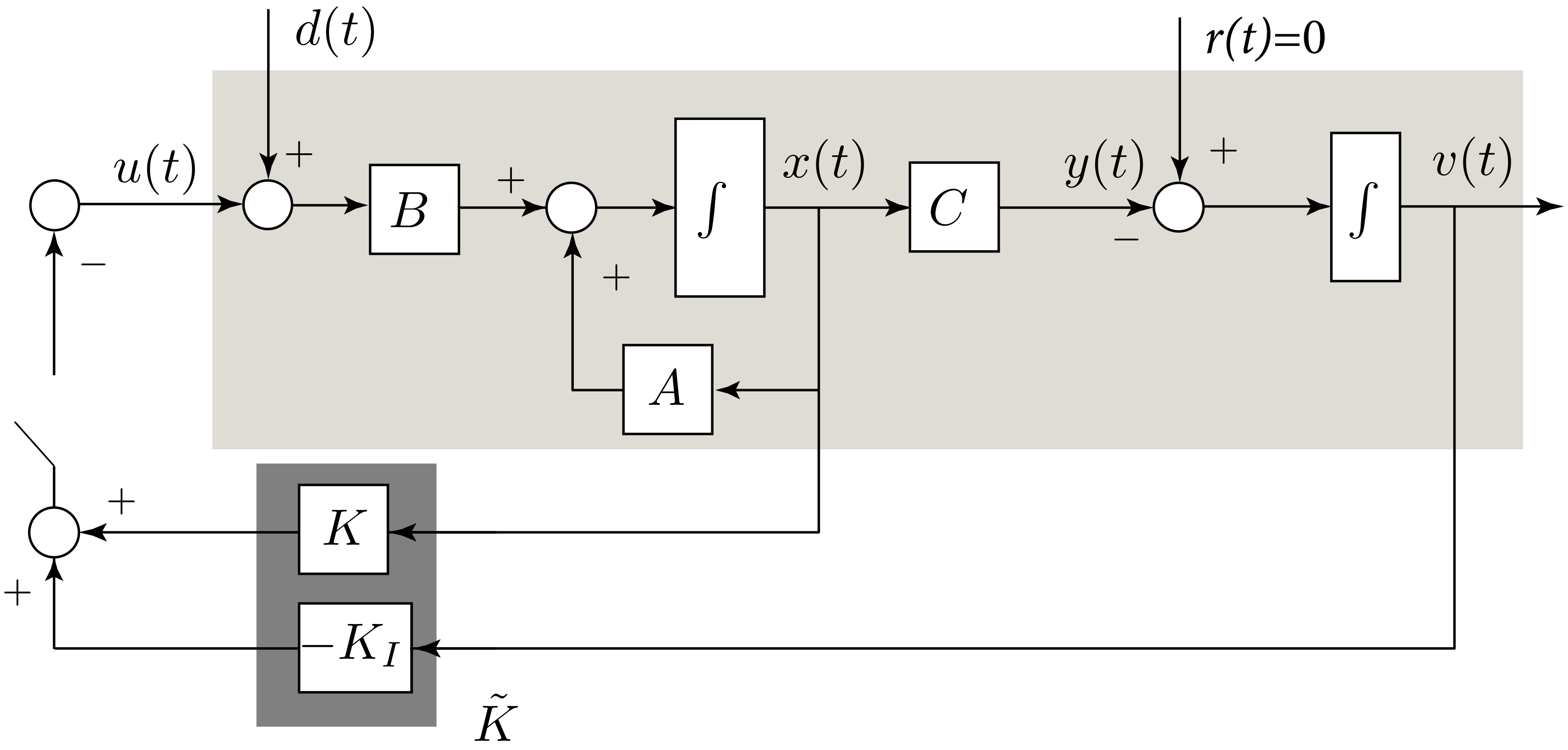
Lecture IX – Extensions of LQR Control Systems, State Observers

LQR-I Controllers (more see "Theorieblätter", here main idea)

The LQR controllers introduced so far are essentially PD^{n-1} controllers (assume the system to be in observability canonical form). No integral action is included and, hence, persistent unmeasurable disturbances cannot fully be compensated for (unless the plant itself includes integral action in each channel). Therefore, additional integrators must be included.



Redraw the previous block diagram to better see how this problem can be reformulated as a regular LQR problem



Define the new extended system state variable

$$\widetilde{x}(t) = \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$$

Then the open-loop system is described by

$$\frac{d}{dt}\widetilde{x}(t) = \widetilde{A}\,\widetilde{x}(t) + \widetilde{B}_u\,u(t) + \widetilde{B}_r\,r(t) + \widetilde{B}_d\,d(t), \quad y(t) = \widetilde{C}\,\widetilde{x}(t)$$

where

$$\widetilde{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}, \quad \widetilde{B}_u = \widetilde{B}_d = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \widetilde{B}_r \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \widetilde{C} \begin{bmatrix} C & 0 \\ 0 & \gamma\, I \end{bmatrix}$$

and the full-state feedback is $u(t) = \widetilde{K}\,\widetilde{x}(t)$ where $\widetilde{K}$ is obtained solving a standard LQR problem for the system $\{\widetilde{A}, \widetilde{B}, \widetilde{C}\}$. Using the partition

$$\widetilde{K} = \begin{bmatrix} K & -K_I \end{bmatrix}$$

the solution to the original problem is obtained (γ is a new tuning parameter).

Robustness Enhancement

$$\frac{1}{\beta}\Phi_\beta \cdot B \cdot R^{-1} \cdot B^T \cdot \Phi_\beta - \Phi_\beta \cdot A - A^T \cdot \Phi_\beta - Q = 0$$

$$\beta > 1$$

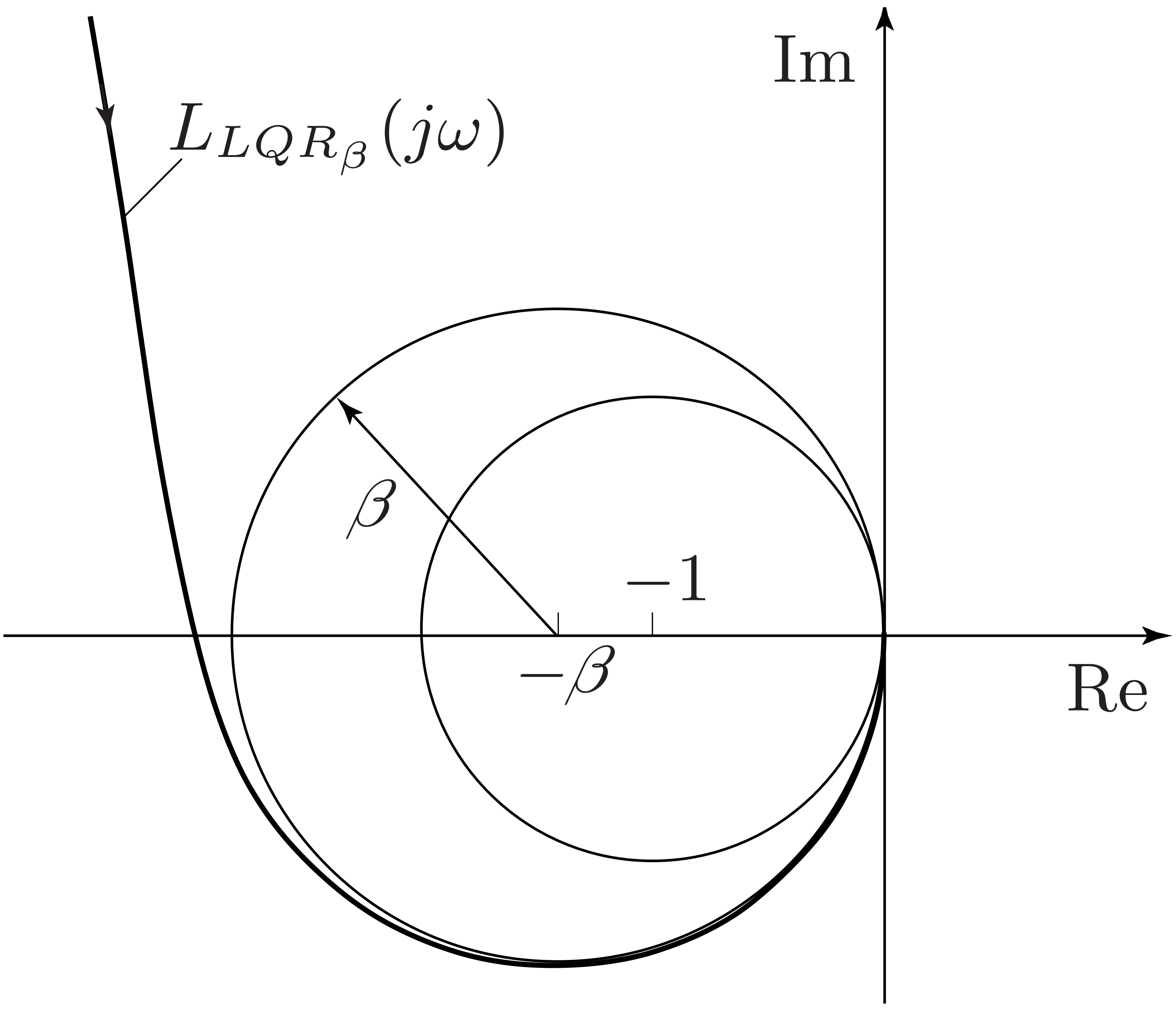
$$\mu_\beta = \min_\omega \sigma_{\min}\{\beta\, I + L(j\,\omega)\} \geq \beta$$

In the limit case $\beta \to \infty$

$$-\Phi_\infty \cdot A - A^T \cdot \Phi_\infty - Q = 0$$

$L_\infty(s) = K_\infty \cdot [s\,I - A]^{-1} \cdot B$ strictly positive real, but

Φ_∞ exists iff A is a Hurwitz matrix


Im
$L_{LQR_\beta}(j\omega)$
β
-1
$-\beta$
Re

Finite Horizon LQR

$$\dot{x}(t) = A(t) \cdot x(t) + B(t) \cdot u(t) \qquad \boldsymbol{x(t_a)=x_a}$$

$$J(u) = x^T(t_b) \cdot P \cdot x(t_b) +$$

$$\int_{t_a}^{t_b} \left[x^T(u(t)) \cdot Q(t) \cdot x((u(t)) + u^T(t) \cdot R(t) \cdot u(t) \right] dt$$

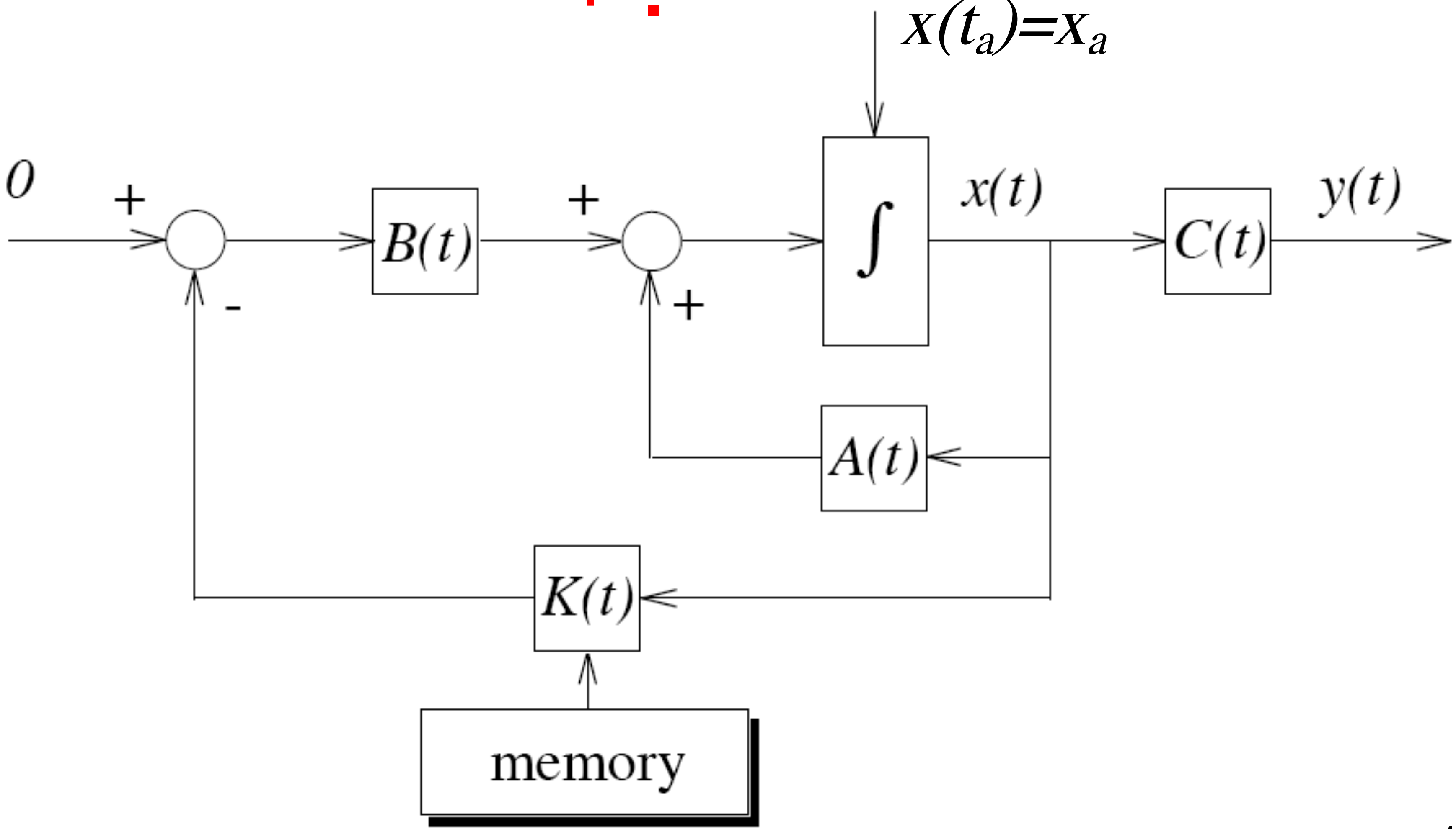
$$u(t) = -K(t) \cdot x(t)$$

$$K(t) = R^{-1}(t) \cdot B^T(t) \cdot \Phi(t)$$

$$\frac{d}{dt}\Phi(t) = \Phi(t)\cdot B(t)\cdot R^{-1}(t)\cdot B^T(t)\cdot\Phi(t) - \Phi(t)\cdot A(t) - A^T(t)\cdot\Phi(t) - Q(t)$$

$$\Phi(t_b) = P$$

!

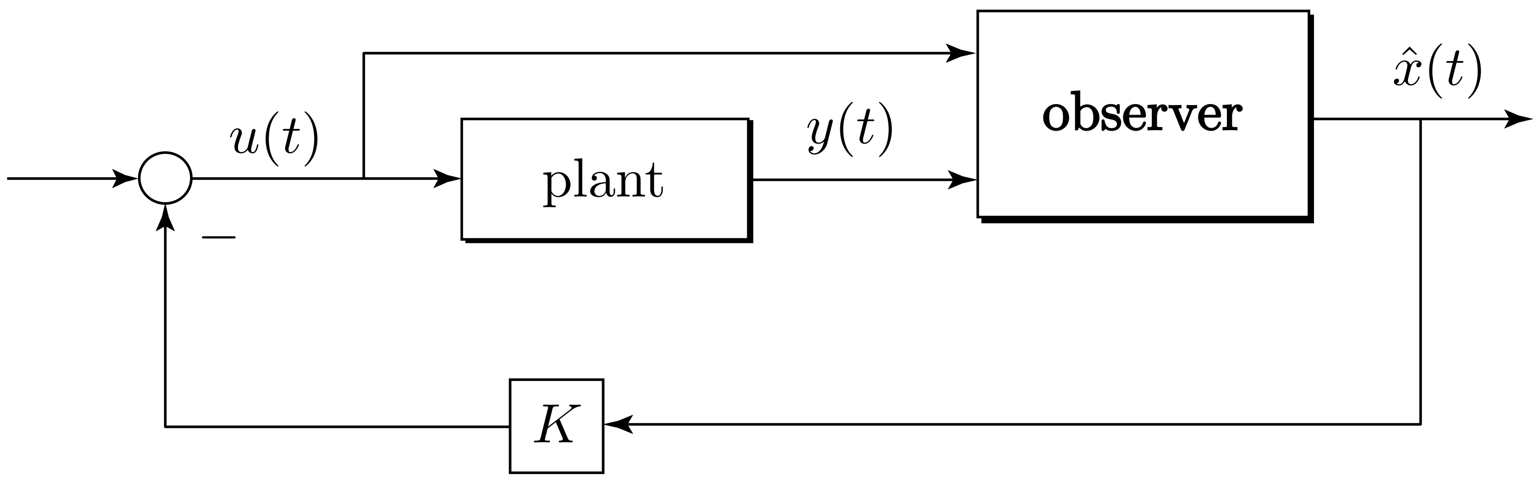


State Observers

State-feedback controllers cannot be realized in practice, only the system output (and its input) are available for further processing in control algorithms.

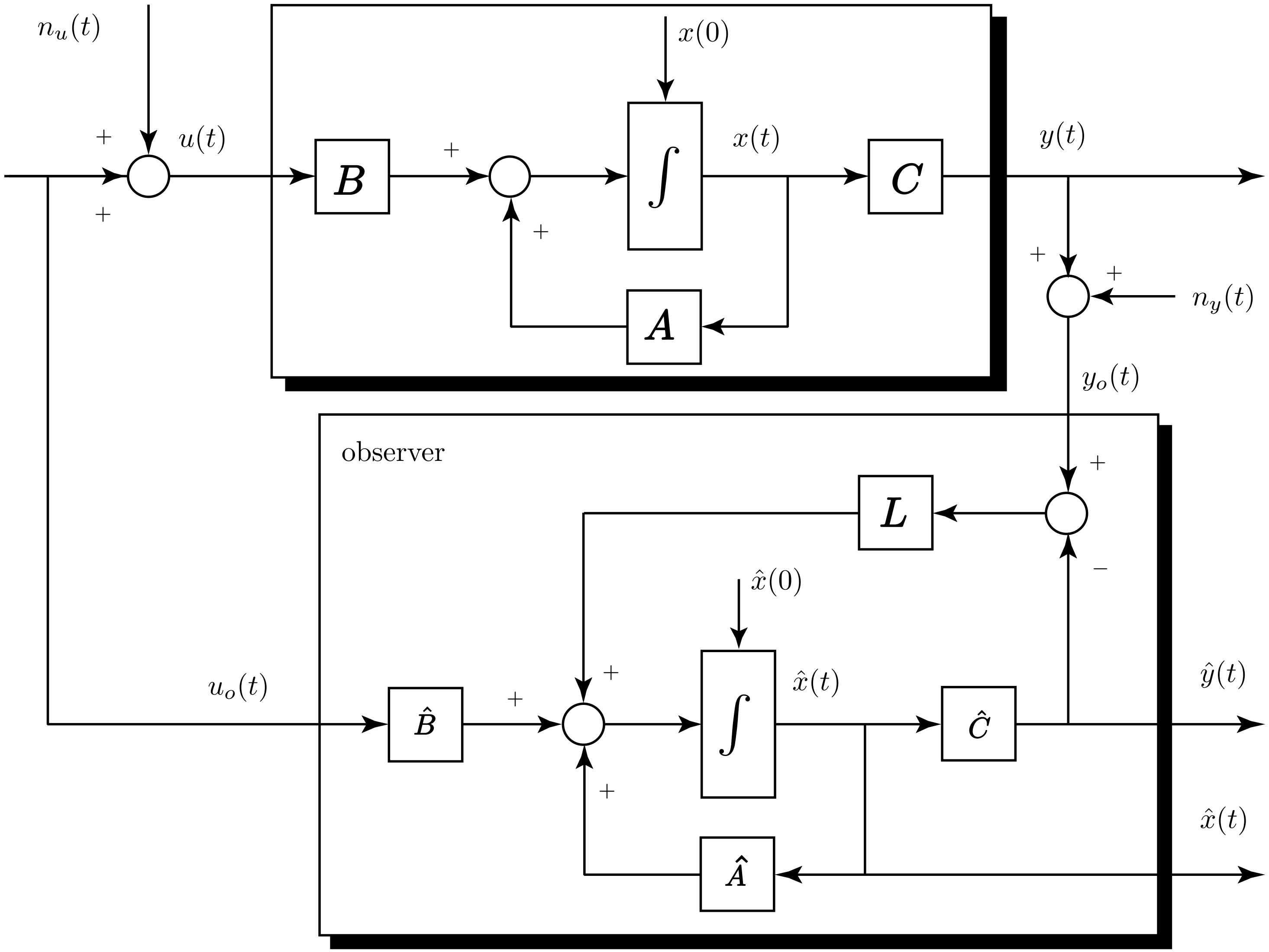
Nevertheless, the state-feedback approach is very powerful when combined with a filter that is able to produce an *estimate* $\hat{x}(t)$ of the system's true state variables $x(t)$ using the system's input and output signals only.

The key idea of the observer-based state feedback approach is illustrated below.



Key question: how to design the "**observer**"?

Main idea: Copy plant and use "input injection"



Plant

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t)$$

$$y(t) = C \cdot x(t)$$

Observer

$$\frac{d}{dt}\hat{x}(t) = \hat{A} \cdot \hat{x}(t) + \hat{B} \cdot u_o(t) + L \cdot (y_o(t) - \hat{y}(t))$$

$$\hat{y}(t) = \hat{C} \cdot \hat{x}(t)$$

Observation errors

$$\bar{x}(t) = x(t) - \hat{x}(t) \in I\!R^n$$

Simplified case: $A = \hat{A}$, $B = \hat{B}$, and $C = \hat{C}$

$$n_u(t) = n_y(t) = 0$$

Error dynamics

$$\begin{aligned}
\frac{d}{dt}\bar{x}(t) &= \frac{d}{dt}x(t) - \frac{d}{dt}\hat{x}(t) \\
&= A \cdot x(t) + B \cdot u(t) - [A \cdot \hat{x}(t) + B \cdot u(t) + L \cdot (y(t) - \hat{y}(t))] \\
&= A \cdot (x(t) - \hat{x}(t)) - L \cdot C \cdot (x(t) - \hat{x}(t)) \\
&= [A - L \cdot C] \cdot \bar{x}(t), \qquad \bar{x}(0) = x(0) - \hat{x}(0) \neq 0
\end{aligned}$$

How to compute L

Dual LQR problem

$$A \longrightarrow \boxed{A^T}$$

$$B \longrightarrow \boxed{C^T}$$

$$Q = \bar{C}^T \cdot \bar{C} \longrightarrow \boxed{\bar{B} \cdot \bar{B}^T}$$

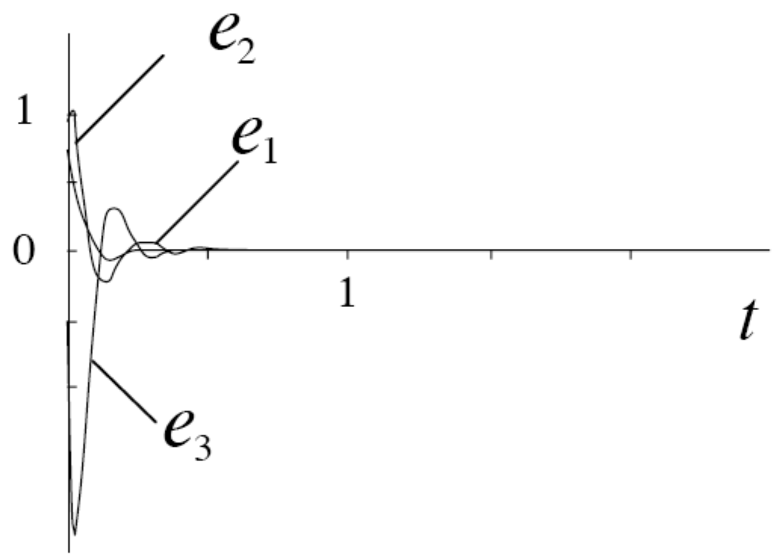
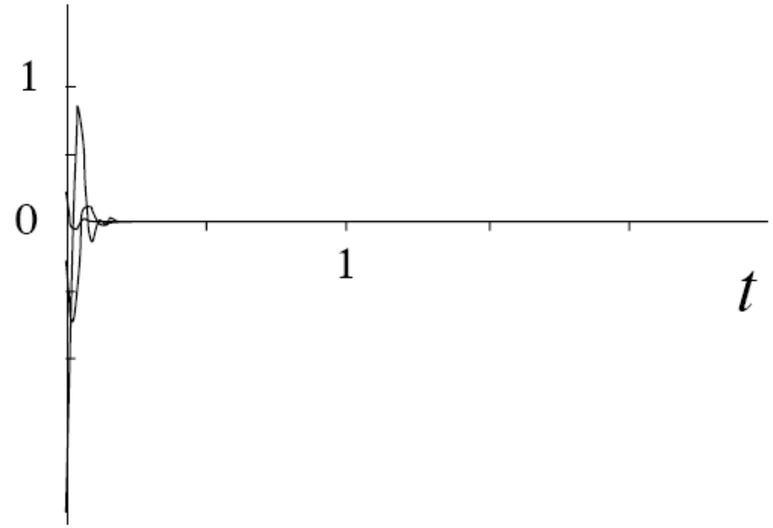
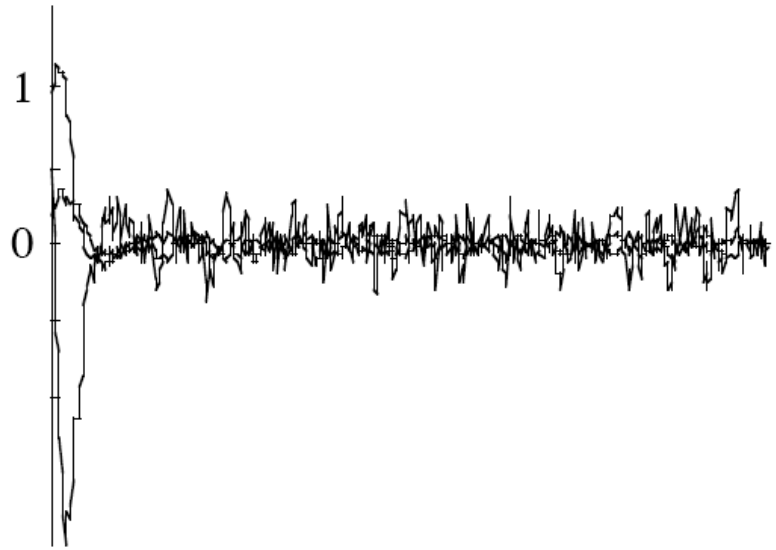
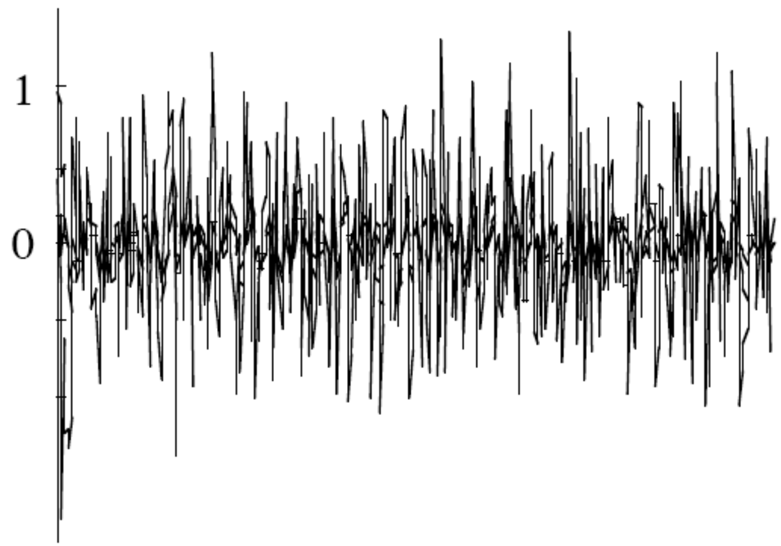
$$R = r \cdot I \longrightarrow \boxed{q \cdot I}$$

$$L^T = \frac{1}{q} \cdot C \cdot \Psi$$

$$\frac{1}{q} \cdot \Psi \cdot C^T \cdot C \cdot \Psi - \Psi \cdot A^T - A \cdot \Psi - \bar{B} \cdot \bar{B}^T = 0$$

One might be tempted to choose the eigenvalues of $A - L \cdot C$ much "faster" than those of A (or later $A - B \cdot K$). However, two unavoidable complications impose limitations:

- The observer is synthesized using a model $\{\hat{A}, \hat{B}, \hat{C}\}$ of the true plant $\{A, B, C\}$. Of course this model is never perfect. These and other modeling errors impose limits on the loop gain of the error dynamics (3.46).
- The input signals to the observer will always be corrupted by some noise, i.e., $n_u(t) \neq 0$ and $n_y(t) \neq 0$. This fact imposes limits on the loop gain.

<table>
<tr><th></th><th>“slow” observer</th><th>“fast” observer</th></tr>
<tr><td>no noise</td><td>
</td><td>
</td></tr>
<tr><td>some noise</td><td>
</td><td>
</td></tr>
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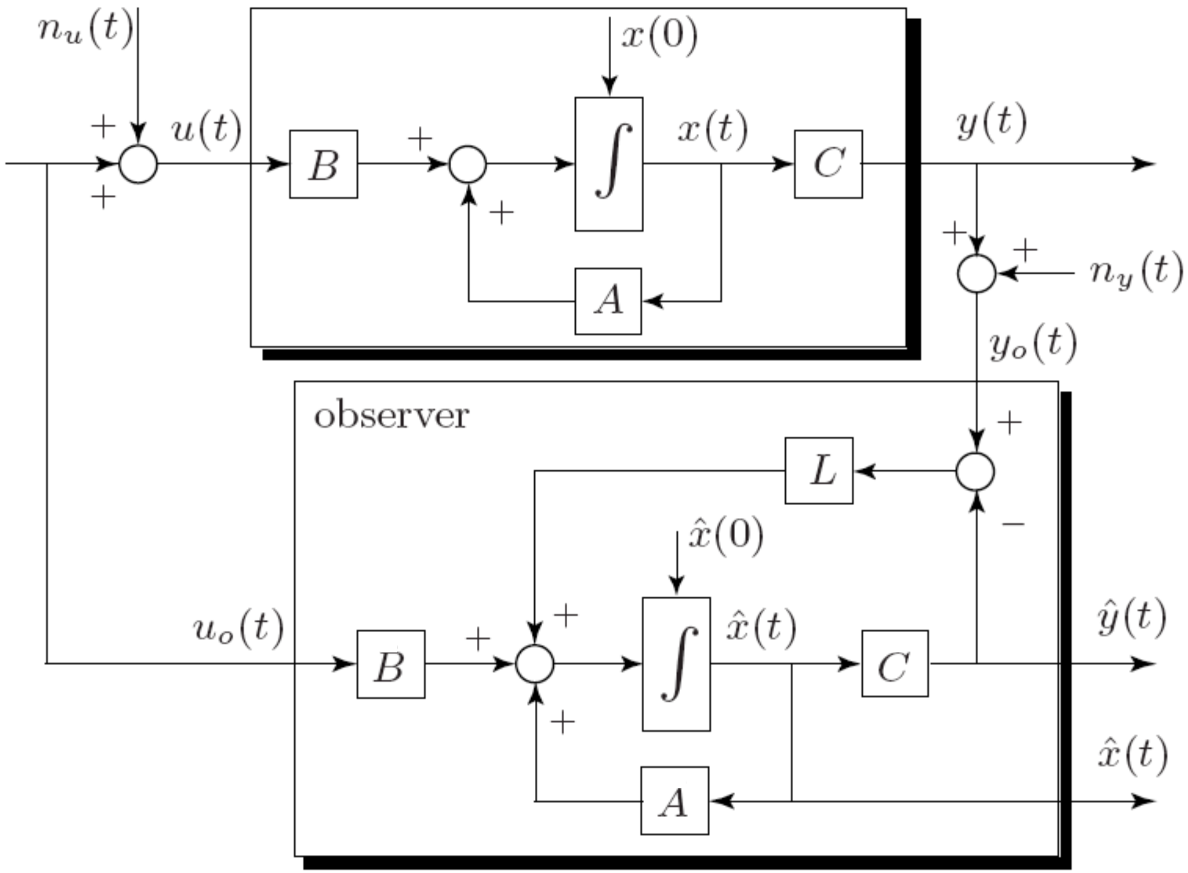
3.7 Kalman Filters

$$\frac{d}{dt}x(t) = A \cdot x(t) + B \cdot (u_o(t) + n_u(t))$$

$$y_o(t) = C \cdot x(t) + n_y(t)$$



In the simplest case, the two noise signals n_u and n_y are assumed to be uncorrelated *white noise signals*. Without entering into the mathematical details of stochastic signals, a white noise signal $n(t)$ is defined as a signal whose *spectrum* is constant for all frequencies.

The spectrum $\phi_n(\omega)$ of a scalar signal $n(t) \in I\!R$ is obtained using its Fourier transform

$$n(t) = \int_{-\infty}^{+\infty} A(\omega) \cos\left(\omega t + \phi(\omega)\right) \, d\omega$$

by the operation

$$\phi_n(\omega) = A^2(\omega)$$

If the spectrum is constant for all frequencies, i.e., if

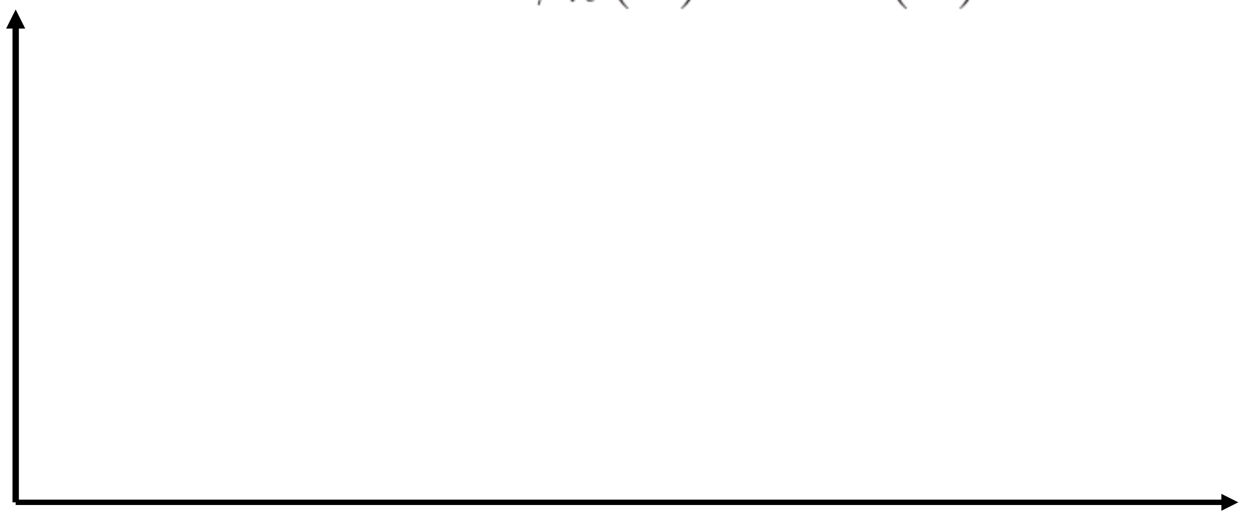
$$\phi_n(\omega) = r_n > 0 \in I\!R$$

then the signal $n(t)$ is a white noise signal.

$\phi_n(\omega)$ of a scalar signal $n(t) \in I\!R$ is

$$n(t) = \int_{-\infty}^{+\infty} A(\omega) \cos\left(\omega t + \phi(\omega)\right)\, d\omega$$

$$\phi_n(\omega) = A^2(\omega)$$



Interpretation:

These definitions can be easily extended to the case where $n(t) \in \mathbb{R}^m$ is a vector signal. In this case the spectrum is a Hermitian $m \times m$ matrix $\Phi_n(\omega)$, which, for white noise signals, is constant for all frequencies

$$\Phi_n(\omega) = R_n = R_n^T \geq 0 \in \mathbb{R}^{m \times m}$$

This matrix describes the "intensity" of a noisy signal. Below, it is assumed that both $R_u \geq 0$, associated to $n_u(t)$, and $R_y > 0$, associated to $n_y(t)$, are known.

L_K that minimizes the expectation of the estimation error $\bar{x}(t)$. This optimal gain is defined by

$$L_K = P \cdot C^T \cdot R_y^{-1}$$

where the matrix $P = P^T \in \mathbb{R}^{n \times n}$ is the positive (semi-)definite solution of the Riccati equation

$$A \cdot P + P \cdot A^T - P \cdot C^T \cdot R_y^{-1} \cdot C \cdot P + B \cdot R_u \cdot B^T = 0$$