Signals and Systems

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Version: 7. Februar 2022

1 Signals

1.1 Signal Definition: $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$

Continuous Time (CT): $x(t): t, \quad t \in \mathbb{R}, \quad x(t) \in \mathbb{R}$
Discrete Time (DT): $x[n]: n, \quad n \in \mathbb{Z}, \quad x[n] \in \mathbb{R}/\mathbb{Z}$
Sequence: $\{x[n]\} = \{\dots, x[-1], x[0], x[1], \dots\}$

1.2 Signal Classification

Even Signal:	$x[-n] = x[n]$	**Odd Signal:**	$x[-n] = -x[n]$

Periodic Signal:

$$x[n+N] = x[n] = x[n + m \cdot N], \quad N \in \mathbb{N}, m \in \mathbb{Z}$$

Sum of two periodic DT sequences is always periodic.

$$x[n] = \cos(\Omega_1 n + \varphi_1) + \cos(\Omega_2 n + \varphi_2)$$

Period: $N_0 = \frac{2\pi}{\Omega}$, $N = \frac{2\pi k}{\Omega}$

$$N_0 = \text{lcm}\left(\frac{2\pi}{\Omega_1}, \frac{2\pi}{\Omega_1}\right) = \text{lcm}(N_1, N_2)$$

Constant signal $N = 1$

1.3 Basic Signals

Unit Impulse:

$$\delta[n-k] = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}, \quad \delta[n] = s[n] - s[n-1]$$

Unit Step:

$$s[n-k] = \begin{cases} 1 & n \geq k \\ 0 & n < k \end{cases}, \quad s[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Complex Exponential:

$$x[n] = z^n = |z|^n e^{j\Omega_0 n} = |z|^n (\cos(\Omega_0 n) + j \sin(\Omega_0 n))$$

Sinusoidal:

$$x[n] = A\cos(\Omega_0 n + \theta)$$

1.4 Signal Discretization

CT → DT: Uniform Sampling

$$x[n] = x(nT_s)$$

T_s = sampling period. $f_s = \frac{1}{T_s}$ = sampling frequency

DT → CT: Zero Order Hold

$$x(t) = x[n], \quad nT_s \leq t < (n+1)T_s$$

Periodic Sampling: CT allways periodic with $T = \frac{2\pi}{|\omega|}$, $f = \frac{1}{T}$

$$x(t) = \cos(\omega t) = \cos(2\pi f t) \Rightarrow x[n] = \cos(\Omega n)$$

$x[n]$ only periodic if

$$\frac{\omega T_s}{2\pi} = \frac{\Omega}{2\pi} = \frac{m}{N}, \quad m, N \in \mathbb{N}$$

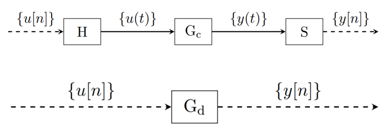
T_s = sampling time, $\Omega = \omega T_s$, N_0 fundamental period.
When sampling starts at t_0, we get $t = nT_s + t_0$.

2 Systems

2.1 System Definition

Operator wich transforms the input u to an output y.

$$y(t) = G_C u(t), \quad y[n] = G_d u[n] \quad G_d = SG_c H$$



H is a zero order hold, so $\dot{u}(t) = 0$, $u(t) = u[0]$ for $0 \leq t < T_s$.

2.2 System Discretization

2.2.1 Exact Discretization

Given CT System:

$$\dot{q}(t) = A_c \cdot q(t) + B_C \cdot u(t), \qquad q(0) = q[0]$$
$$y(t) = C_C \cdot q(t) + D_C \cdot u(t), \qquad u(0) = u[0]$$

As we have a zero order hold we can write:

$$\begin{bmatrix} \dot{q}(t) \\ \dot{u}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A_C & B_C \\ 0 & 0 \end{bmatrix}}_{=M} \begin{bmatrix} q(t) \\ u(t) \end{bmatrix} \Rightarrow \dot{r}(t) = Mr(t)$$

The DT system-response is the same if the CT-input is constant during the sampling period. If not the response differs.

Adjunkte:

$$\text{adj}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Limit of sum:

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, \quad \sum_{k=1}^{\infty} a^k = \frac{a}{1-a}$$

Solution of this ODE: $r(t) = e^{Mt} r(0)$

$$e^{Mt} = \sum_{k=0}^{\infty} \frac{(Mt)^k}{k!} = I + Mt + \frac{(Mt)^2}{2} + \dots$$

As we set $q(0) = q[0]$ and $u(0) = u[0]$ we get:

$$\begin{bmatrix} q(T_s^-) \\ u(T_s^-) \end{bmatrix} = F \begin{bmatrix} q(0) \\ u(0) \end{bmatrix} \Rightarrow F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = e^{MT_s}$$

$\Rightarrow F_{21} = 0$ and $F_{22} = I$.
Discretization: $q[n+1] = A_d q[n] + B_d u[n]$

$$A_d = F_{11}, \quad B_d = F_{12}, \quad C_d = C_c, \quad D_d = D_c$$

Exact Discrete System

$$q[n+1] = A_d q[n] + B_d u[n]$$
$$y[n] = C_d q[n] + D_d u[n]$$

2.2.2 Euler Forward Discretization

Approximating the Matrix Exponential we get:

$$e^{MT_s^-} \approx I + MT_s^- = \begin{bmatrix} I + A_C T_s^- & B_C T_s^- \\ 0 & I \end{bmatrix}$$

We therefore get (Stability is not guaranteed)

$$\begin{bmatrix} q[n+1] \\ u[n+1] \end{bmatrix} = \begin{bmatrix} I + A_C T_s^- & B_C T_s^- \\ 0 & I \end{bmatrix} \begin{bmatrix} q[n] \\ u[n] \end{bmatrix}$$

Other Derivation also possible with Euler Forward approximation.

$$\dot{q}(t) \approx \frac{q(t + T_s^-) - q(t)}{T_s^-}, \quad \text{with} t = n \cdot T_s^-$$

2.3 System Classification

- **Memoryless:** $y[n] = f_n(u[n])$, output depends only on the current input.
- **Causal:** $y[n] = f_k(u[k])$, $k \leq n$ output depends only ond past or present inputs.
  LTI Sys causal **iff** $h[n] = 0$ for $n < 0$
  If sys is causal and input is causal then output is also causal.
- **Linear:** $G\{\alpha_1 u_1[n] + \alpha_2 u_2[n]\} = \alpha_1 G\{u_1\} + \alpha_2 G\{u_2\}$
  Define $y_1 = G\{u_1\}$ and $y_2 = G\{u_2\}$
  now check if $G\{\alpha_1 u_1 + \alpha_2 u_2\} = \alpha_1 y_1 + \alpha_2 y_2$
- **Time-Invariant:** $u_2[n] = u_1[n-k]$, $y_1 = Gu_1$, $y_2 = Gu_2$
  $\to \{y_2[n]\} = \{y_1[n-k]\}, \forall n, k \in \mathbb{Z}$. Delay has no effect
- **BIBO Stable:** $|x[n] \leq M| \ \forall \ n, \quad |u[n]| \leq 1$
  Also when: $\sum |h[n]| < \infty$

2.4 LTI System Response

Every Signal can be represendet by:

$$\{x[n]\} = \sum_{k=-\infty}^{\infty} x[k]\{\delta[n-k]\}$$

Convolution:

$$x * h = \sum_{k=-\infty}^{\infty} x[k]\{h[n-k]\}$$

Properties:

- Commutative: $x * h = h * x$
- Associative: $(x * h_1) * h_2 = x * (h_1 * h_2)$
- Distributive: $x * (h_1 + h_2) = x * h_1 + x * h_2$

Impulse Response:

$$\{h[n]\} = G\{\delta[n]\}$$

Step Response:

$$\{r[n]\} = G\{s[n]\} = \{s[n]\} * \{h[n]\} = G\{\sum_{k=-\infty}^{n} h[k]\}$$

$$r[n] - r[n-1] = h[n]$$

Arbitrary Input Response: can also be swapped

$$\{y[n]\} = G\{u[n]\} = \{u[n]\} * \{h[n]\} = \sum_{k=-\infty}^{\infty} u[k]\{h[n-k]\}$$

Finite Impulse Respones FIR: $h[n] = 0, \forall n \geq N, N \in \mathbb{Z}$

Systems with a FIR response have all the poles at $z_p = 0$.
Systems of the following form allways have a FIR response:

$$y[n] = \sum_{k=0}^{M} b_k u[n-k], \quad M \in \mathbb{Z}$$

Causality:

$$h[n-k] = 0, k > n, \quad \iff \quad h[n] = 0, \forall n < 0$$

so we get:

$$y[n] = \sum_{k=0}^{n} h[k]u[n-k], \quad \forall n$$

System is stable iff:

$$\sum |h[n]| < \infty$$

2.5 Linear Constant Coefficient Difference Equations

Linear Constant Coefficient Difference Equations (LCCDE):

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k u[n-k], \qquad a_k, b_k \in \mathbb{R}$$

LCCDE to State Space: ($a_0 = 1$ and $b_k = 0$ for $k > 0$)

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ \vdots & \vdots & \vdots & & 1 \\ -a_N & -a_{N-1} & -a_{N-2} & \dots & -a_1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ b_0 \end{bmatrix}$$

$$C = \begin{bmatrix} -a_N & -a_{N-1} & -a_{N-2} & \dots & -a_1 \end{bmatrix}, D = [b_0]$$

Impulse Response of LTI System: ($q[0] = 0$, $u[n] = \delta[n]$)

$$h = \{D, CB, CAB, \dots, CA^{n-1}B, \dots\} \quad n \geq 0$$

2.5.1 FIR/IIR ↔ CCDE

Given an FIR the CCDE can be calculated really easily:

$$y[n] = \dots + h[-1]u[n+1] + h[0]u[n] + h[1]u[n-1] + \dots$$

A LTI system has a FIR **iff** it can be brought into the following form

$$y[n] = \sum_{k=0}^{M} b_k u[n-k]$$

3 Frequency Domain

Complex Number: $z = a + jb = |z| e^{j\Omega}, \quad \Omega \in (-\pi, \pi]$

3.1 Eigenfunction

for $u[n] = z^n$ the LTI System output is:

$$\{y[n]\} = G\{z^n\} = \sum_{k=-\infty}^{\infty} h[k]z^{-k}\{z^n\} = H(z)\{z^n\}$$

$H(z)$ is called the eigenvalue and $\{z^n\}$ is an eigenfunction

3.2 Z-Transform

Definition: Given a sequence $\{x[n]\}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \quad z \in \mathbb{C}$$

Properties:

- Linearity: $\alpha_1\{x_1[n]\} + \alpha_2\{x_2[n]\} \longleftrightarrow \alpha_1 X_1(z) + \alpha_2 X_2(z)$
- Time-Shifting: $\{x[n-k]\} \longleftrightarrow z^{-k}X(z)$
- Convolution: $\{x_1[n]\} * \{x_2[n]\} \longleftrightarrow X_1(z)X_2(z)$
- Accumulation: $\{\sum_{k=-\infty}^{n} x[k]\} \longleftrightarrow \frac{z}{z-1}X(z)$
- Differentiation: $\{nx[n]\} \longleftrightarrow -z\frac{d}{dz}X(z)$

Non-Uniqueness:
The z-Transform is not unique (causal and non-causal interpretation yield the same trans-from). Region of Convergence (ROC) should be included but will mostly be emitted as we mostly use the causal sequence.
Usefull Transforms:

$\delta[n]$	$\longleftrightarrow$	1	$a^n s[n]$	$\longleftrightarrow$	$\frac{z}{z-a}, \lvert\frac{a}{z}\rvert < 1$
$\delta[n-k]$	$\longleftrightarrow$	z^{-k}	$nx[n]$	$\longleftrightarrow$	$-z\frac{d}{dz}X(z)$
$x[-n]$	$\longleftrightarrow$	$X\left(\frac{1}{z}\right)$	$x^*[n]$	$\longleftrightarrow$	$X^*(z^*)$
$x[n-k]$	$\longleftrightarrow$	$z^{-k}X(z)$	$k^n x[n]$	$\longleftrightarrow$	$X\left(\frac{z}{k}\right)$

3.2.1 Transfer Function (z-transform of the impuls response)

From LCCDE:

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

From State Space: Inverse matrix noch reinschreiben.

$$H(z) = C(zI - A)^{-1}B + D = \frac{C \cdot \text{Adj}(zI - A) \cdot B}{\det(zI - A)} + D$$

From Impulse Response: causal / acausal

$$H(z) = \sum_{k=0}^{\infty} h[k] \cdot z^{-k}, \quad H(z) = \sum_{k=-\infty}^{-1} h[k] \cdot z^{-k}$$

For $h[n] = a^n$, $|z| > |a|$, causal($n \geq 0$) / acausal ($n < 0$)

$$H(z) = \frac{z}{z-a}, \quad H(z) = -\frac{z}{z-a}$$

3.2.2 Stability & Causality ($p_i \neq 1$)

Causal and Stable: iff p_i within unit circle $|p_i| < 1$
Acausal and Stable: p_i outside unit circle $|p_i| > 1$
If a causal and stable interpretation exists, then a anti-causal unstable interpretation exists.

3.2.3 Stable on knowledge of causality

- causal ⇒ Poles must lie inside unit circle
- anti-causal ⇒ Poles must lie outside unit circle
- if you dont know you cant guarantee stability

3.2.4 Transformations

Insert Transformation into "From Impulse Response".
$G(z) = H(-z) \to |G(\Omega)| = |H(\pi - \Omega)|$: Mirroring
$G(z) = H(z^{-1}) \to |G(\Omega)| = |H(\Omega)|$: Stays the same

4 Descrete Time Fourier Transform

DTFT	∞-long	abs. summable	Theory & LTI analysis
DFS	∞-long	periodic	Basis of DFT
DFT	finite	general	Used in Practice

Definition: (z-Transform at $z = e^{j\Omega}$)

$$X(\Omega) = \mathcal{F}x = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}, \quad \Omega \in (-\pi, \pi]$$

$X(\Omega)$ is finite for all Ω and it is continuous.
Fourier Spectra: $X(\Omega) = |X(\Omega)| e^{j\Theta_X(\Omega)}$ **Inverse:**

$$\{x[n]\} = \mathcal{F}^{-1}X := \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n} d\Omega\right\}$$

Properties:

- Linearity: $a_1\{x_1[n]\} + a_2\{x_2[n]\} = a_1 X_1(\Omega) + a_2 X_2(\Omega)$
- Convolution: $\{x_1[n]\} * \{x_2[n]\} = X_1(\Omega)X_2(\Omega)$
- Parseval's Theorem: $\sum_n |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega$
- $X(\Omega) = e^{ja\Omega} \to x[n] = \delta[n+a]$

For periodic signals we have diracs at the frequency Ω.

4.1 Frequency Response of LTI System

Definition:

$$H(\Omega) = H(z)|_{z=e^{j\Omega}} = |H(\Omega)|e^{j\Theta_H(\Omega)}$$

Output of an LTI System: $Y(\Omega) = H(\Omega)U(\Omega)$
$|Y(\Omega)| = |H(\Omega)| \, |U(\Omega)|$, $\angle Y(\Omega) = \angle H(\Omega) + \angle U(\Omega)$
Delay: $|H_2| = |H_1|$, $\angle H_2 \neq \angle H_1$

$$h_2[n] = h_1[n+d], \quad H_2[k] = e^{j \cdot d \cdot \Omega} H_1[k]$$

For a Real System:

$$|H(\Omega)| = |H(-\Omega)|, \quad \angle H(\Omega) = -\angle H(-\Omega)$$

From LCCDE:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \longleftrightarrow H(\Omega) = \frac{\sum_{k=0}^{M} b_k e^{-j\Omega k}}{\sum_{k=0}^{N} a_k e^{-j\Omega k}}$$

Response to Complex Exponential: $\{u[n]\} = \{e^{j\Omega_0 n}\}$

$$y[n] = |H(\Omega_0)| e^{j(\Omega_0 n + \Theta_H(\Omega_0))}$$

Response to Real Sinusoid: $\{u[n]\} = \cos(\Omega_0 n)$

$$y[n] = |H(\Omega_0)| \cos(\Omega_0 n + \Theta_H(\Omega_0))$$

Both are only valid if $n \in (-\infty, \infty)$.

5 Discrete Fourier Series

Definition: periodic signal with periodicity N

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{jk\frac{2\pi}{N}n}, \quad X[k] \in \mathbb{C}$$

DFS Coefficients: $X = \mathcal{F}_s x \iff x = \mathcal{F}_s^{-1} X$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}, \quad X[k+N] = X[k]$$

Properties:

- Linearity: $a_1\{x_1[n]\} + a_2\{x_2[n]\} = a_1 X_1[k] + a_2 X_2[k]$
- Parseval's Theorem: $\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$

DFS Coefficients for a Real Signal: $X[0]$ is allways real.

$$X[N - \gamma] = X^*[\gamma], \quad x[n] \in \mathbb{R}, \gamma \in \mathbb{Z}$$

For N even, $X[N/2]$ must be real.
LTI Response to a Perdiodic Signal: input periodic then output periodic

$$Y[k] = H\left(e^{jk\frac{2\pi}{N}}\right)U[k], \quad H\left(e^{jk\frac{2\pi}{N}}\right) = H(z)|_{z=e^{jk\frac{2\pi}{N}}}$$

DFS ⟷ DTFT:

$$X(\Omega) = \frac{2\pi}{N}\sum_{k=0}^{N-1} X[k]\delta(\Omega - k\frac{2\pi}{N})$$

6 Discrete Fourier Transform (DFT) / (Algorithm FFT)

Discrete Frequency: $\Omega_k = \frac{2\pi}{N_k} = \frac{2\pi}{N}k \to X[K]$
Signal Representation:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{jk\frac{2\pi}{N}n} \quad \text{for } n = 0, \dots, N-1$$

DFT Coefficients:

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n} \quad \text{for } k = 0, \dots, N-1$$

If the signal lenght is doubled the Coefficients are also doubeld.

$$\{U[k]\} = \{4, j, 1, -j\} \to \{U[k]\} = \{8, 0, 2j, 0, 2, -2j\}$$

6.1 Shifting

$$x_1[n] \text{ and } x_2[n] = x_1[n] \cdot e^{j\frac{2\pi}{N}an} \quad \Leftrightarrow \quad X_2[k] = X_1[k - a]$$

$$x_1[n] \text{ and } x_2[n] = x_1[n + a] \quad \Leftrightarrow \quad X_2[k] = X_1[k] \cdot e^{jk\frac{2\pi}{N}a}$$

6.2 Checks if Sequence can be DFS Coefficients of $x[n]$ which has period N

- DFS Coefficients have to be periodic with Period N
- $X[0] = \sum_{n=0}^{N-1} x[n]$
- if $x[n]$ is real it has to hold: $X[N-\gamma] = X^*[\gamma]$ $\Rightarrow |X[N-\gamma]| = |X[\gamma]|$

6.3 Aliasing

Multiple CT Frequencies map to the same DT Signal!

$$\cos((\Omega + 2\pi k)n) = \cos(\Omega), \quad \forall k \in \mathbb{Z}$$

New Frequency: $\Omega_{new} = 2\pi - \omega T_s$

Restrict the sampling frequency: $f_{max,signal} < \frac{1}{2} f_{sample}$

$$\frac{-\pi}{T_s} < \omega < \frac{\pi}{T_s}, \quad \rightarrow \quad |\omega| < \frac{\pi}{T_s} = f_s \pi$$

7 System Identification

Causal, Stable LTI System. u_e known input, u_d unknown input noise (white), y_d unknown output noise (white), $y_m = Gu_e + GU_d + y_d$ system output.

7.1 Impulse Response, $\{u_e[n]\} = \{\delta[n]\}$

7.1.1 Without White Noise

LTI System characterized by Impulse Response .

$$y_m = Gu_e, \quad H(\Omega) = \sum_{n=0}^{\infty} y_m[n] e^{-j\Omega n}$$

Usually the System are FIR not IIR. (Stop at N) $\Omega_k = \frac{2\pi k}{N}$

$$\widehat{H}(\Omega_k) := \underbrace{Y_m[k]}_{[0 \text{ to } N-1]} = \underbrace{H(\Omega_k)}_{[0 \text{ to } \infty]} - \underbrace{\sum_{n=N}^{\infty} h[n] e^{-j\Omega_k n}}_{H_N(\Omega_k)[N \text{ to } \infty]}$$

$H_N(\Omega) \to 0$ for $N \to \infty$. Larger N higher frequency resolution.

7.1.2 With White Noise

Given System:

$$y_m[n] = h[n] + y_d[n], \quad n = 0, 1, \dots, N-1$$

Frequency Response Estimation:

$$\widehat{H}(\Omega_k) = Y_m[k] = H(\Omega_k) - H_N(\Omega_k) + Y_d[k]$$

White Noise:

$$\mathbb{E}(y_d[n]) = 0 \quad \mathbb{E}(y_d[n]y_d[m]) = \sigma_y^2 \delta[n-m]$$

$$\Rightarrow \mathbb{E}(Y_d[k]) = 0 \quad \mathbb{E}(|Y_d[k]|^2) = N\sigma_y^2$$

Mean Error:

$$\Rightarrow \mathbb{E}\left(\widehat{H}(\Omega_k) - H(\Omega_k)\right) = -H_N(\Omega_k) \xrightarrow{N\to\infty} 0$$

Mean Error Squared:

$$\mathbb{E}\left(|\widehat{H}(\Omega_k) - H(\Omega_k)|^2\right) = H_N^2(\Omega_k) + N\sigma_y^2 \xrightarrow{N\to\infty} \infty$$

u_d has similar effect. Bigger N results in bigger error!!!
Solution:
Increase Amplitude (could lead to saturation or non-linear effects).

7.2 Sinusoidal Response

Robustness: Both Energys grow linear with N.
Energy of Input is concentrated to one Frequency.
Energy of Noise is spread across all Frequencies.
Transient Behaviour: Choose N_T to let Transient die down

System and Input: Base Frequency $= \Omega_l = \frac{2\pi}{N} l$.

$$y_m = Gu_e + y_d, \quad u_e[n] = e^{j\frac{2\pi}{N} ln}, \quad n \in [0, N_T + N - 1]$$

Output: $w[n]$ = transient

$$y_e[n] = H(\Omega_l) u_e[n] - w[n]$$

Take DFT: for $n \geq N_T$

$$\widehat{H}(\Omega_l) := \frac{Y_m[l]}{U_e[l]} = H(\Omega_l) - \frac{W[l]}{N} + \frac{Y_d[l]}{N}$$

$W[l]$ DFT of transient, $Y_d[l]$ DFT of Noise

Mean Error:

$$\mathbb{E}\left(\widehat{H}(\Omega_l) - H(\Omega_l)\right) = -\frac{W[l]}{N} \xrightarrow{N\to\infty} 0$$

Mean Error Squared:

$$\mathbb{E}\left(|\widehat{H}(\Omega_l) - H(\Omega_l)|^2\right) = \frac{W^2[l]}{N^2} + \frac{\sigma_y^2}{N} \xrightarrow{N\to\infty} 0$$

7.2.1 Experimental Procedure (same for closed loop)

- Chose N_T, N and A
- $u_e[n] = A\cos(\Omega_l n), \quad \Omega_l = \frac{2\pi l}{N}, \quad l \in [0, \frac{N-1}{2}]$
- Calculate $Y_m[l] = \sum_{n=N_t}^{N_T+N-1} y_m[n] e^{-j\Omega_l n}$, $U_e[l] = \frac{NA}{2}$
- Estimate $\hat{H}(\Omega_l) := \frac{Y_m[l]}{U_e[l]}$ and repeat for all l.

$\Omega = 0$: 1 equation, $\Omega \neq 0$: 2 equations $\to N = 2 \cdot l - 1$ equations

7.2.2 Identifying the Transfer Function

Transfer Function: A and B known

$$H(\Omega) = \frac{\sum_{k=0}^{B-1} b_k e^{-j\Omega k}}{1 + \sum_{k=1}^{A-1} a_k z^{-j\Omega k}}$$

Resulting Equation: $\hat{H}(\Omega_l) \overset{!}{=} H(\Omega_l)$

$$\left(1 + \sum_{k=1}^{A-1} a_k e^{-j\Omega_l k}\right) \widehat{H}(\Omega_l) = \sum_{k=0}^{B-1} b_k e^{-j\Omega_l k}$$

Splitting Real and Imaginary Part:

$$\text{Re}: R_l \cos(\phi_l) + \sum_{k=1}^{A-1} a_k R_l \cos(\phi_l - k\Omega_l) = \sum_{k=0}^{B-1} b_k \cos(k\Omega_l)$$

$$\text{Im}: R_l \sin(\phi_l) + \sum_{k=0}^{A-1} a_k R_l \sin(\phi_l - k\Omega_l) = \sum_{k=1}^{B-1} -b_k \sin(k\Omega_l)$$

Resulting Least Squares:

$F \cdot \Theta = g$ g is red part in upper equation

$$\text{argmin}((F\Theta - g)^T(F\Theta - g)) \Rightarrow \Theta^* = (F^T F)^{-1} F^T g$$

with

$$\Theta = [a_1\ a_2\ \dots\ a_{A-1}\ b_0\ b_1\ \dots\ b_{B-1}]^T$$

Weighted Least Squares:

$$W = \text{diag}(w_0, w_0, \dots, w_l, w_l) \in (2L) \times (2L)$$

$$\Theta^* = (F^\top W^\top W F)^{-1} F^\top W^\top W g$$

Least Squares has Solution for:

- $2L \geq A + B - 1$ if frequencies $\Omega_l = 0$ and $\Omega_l = \pi$ were not tested
- $2L \geq A + B$ if either $\Omega_l = 0$ or $\Omega_l = \pi$ were tested
- $2L \geq A + B + 1$ if both $\Omega_l = 0$ and $\Omega_l = \pi$ were tested

With A: number of a_k coefficients (including a_0), B: number of b_k coefficients.

8 Filtering

8.1 Probability Theory & Definition

Probability Function:

$$p(x), \quad \int_{-\infty}^{\infty} p(x)dx = 1, \quad p(x) \geq 0, \forall x \in \mathbb{R}$$

Expected Value and Variance:

$$\mathbb{E}(x) = \int_{\mathbb{R}} xp(x)dx, \quad \text{Var}(x) = \mathbb{E}\left((x - \mathbb{E}(x))^2\right)$$

Uniform and Normal Distribution:

$$p(x) = \begin{cases} \frac{1}{b-1} & a \leq x \leq b \\ 0 & else \end{cases}, \quad p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

White Noise: Signal with a flat and uncorrelated spectrum

$$\mathbb{E}(X[k]) = 0, \quad \mathbb{E}\left(X^*[k]X[q]\right) = N\delta[k-q]$$

White Noise can be generate with different PDF for Example uniform or Normal. For unit variance the amplitude of whitenoise has to be $\sqrt{3}$ (for unifrom dist.)

8.2 Non-Linear Filtering

Median Filter:

$$y[n] = \text{median}(u[n - M/2], \dots, u[n], \dots, u[n + M/2])$$

8.3 Non-Causal Filtering

Non-Causal Filters: Better but Computationally Expensive

$$\{u[n]\} \xrightarrow{DFT} \{U[k]\} \xrightarrow{FD-Manip} \{Y[k]\} \xrightarrow{(DFT)^{-1}} \{y[n]\}$$

Non-Causal Filtering with a Causal Filter:
$\tilde{y} = Gu$ with G real, causal, LTI filter with $H(z)$
let $\tilde{G}$ real, anti-causal, LTI filter with $H(z^{-1}) \Rightarrow y = \tilde{G}\tilde{y} = \tilde{G}Gu$

$$Y(e^{j\Omega}) = H(e^{-j\Omega})H(e^{j\Omega})U(e^{j\Omega}) = |H(e^{j\Omega})|^2 U(e^{j\Omega})$$

Because for Real Filters we have: $H^*(e^{j\Omega}) = H(e^{-j\Omega})$

9 Finite Impulse Response Filters (FIR)

Difference Equation of an FIR Filter:

$$y[n] = \sum_{k=0}^{M-1} b_k u[n-k], \quad b_k \in \mathbb{R}$$

M number of coefficients (filter length), $M-1$ order, FIR always stable $p_i = 0$.
A causal, LTI, FIR filter of order $M-1$ has at least $M-1$ poles at $z_p = 0$.
Filter length is equal to the length of the impulse response given by:

$$h = \{b_0, b_1, \dots, b_{M-1}\}$$

Frequency Response:

$$H(z) = \sum_{k=0}^{M-1} h[k] z^{-k} \xrightarrow{z = e^{j\Omega}} H(\Omega) = \sum_{k=0}^{M-1} b_k e^{-j\Omega k}$$

9.1 Moving Average Filter

Low-Pass (LP) FIR Filter:

$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} u[n-k] \quad \Rightarrow \quad b_k = \frac{1}{M}$$

Frequency Response:

$$H(\Omega) = \frac{1}{M}\sum_{k=0}^{M-1} e^{-j\Omega k} = \frac{1}{M}\frac{(1 - e^{-j\Omega M})}{(1 - e^{-j\Omega})}$$

$H(0) = 1$. $H(\Omega) = 0$ for $k \neq 0$ or $k \neq \lambda M$. $\Omega_{zero} = \frac{2\pi k}{M}$.

Magnitude: $\text{sinc}(x) = \frac{\sin(x)}{x}$

$$|H(\Omega)| = \left|\frac{\text{sinc}(\Omega M/2)}{\text{sinc}(\Omega/2)}\right| \approx \left|\text{sinc}\left(\frac{\Omega M}{2}\right)\right| \text{ for small } \Omega$$

Phase Response:

$$\angle H(\Omega) \approx -\frac{\Omega(M-1)}{2}$$

9.1.1 Fast Moving Average Filter

Fast MA Filter: Computationally more efficient

$$y[n] = y[n-1] + \frac{u[n] - u[n-M]}{M} \quad (+d[n])$$

Causion: Errors are summed up

$$Y(z) = \frac{1}{M}\left(\frac{1 - z^{-M}}{1 - z^{-1}}\right)U(z) + \left(\frac{1}{1 - z^{-1}}\right)D(z)$$

$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} u[n-k] + \sum_{k=0}^{n} d[n-k]$$

9.1.2 Weighted MA Filter

Definition: w_k is a decreasing function of k

$$y[n] = \frac{1}{S}\sum_{k=0}^{M-1} w_k u[n-k], \quad S: \sum_{k=0}^{M-1}\frac{w_k}{S} = 1$$

Common Choice:

$$w_k = (M-k), \quad \Rightarrow S = \frac{M(M+1)}{2}$$

Less emphasis on older inputs → less aggressive with smaller phase response.

9.1.3 Non-Causal MA Filter

Impules Response: M odd, includes all inputs

$$h = \{0, \frac{1}{M}, \dots, \underset{\uparrow}{\frac{1}{M}}, \dots, \frac{1}{M}, 0\}$$

Frequency Response:

$$H(\Omega) = \frac{1}{M}\sum_{k=0}^{M-1} e^{-j\Omega\left(k - \frac{M-1}{2}\right)} = e^{j\Omega\frac{M-1}{2}} H_{MA}(\Omega)$$

$H_{MA}(\Omega)$ = freq. resp. of causal Filter, Added Phase of $\Omega(\frac{M-1}{2})$ Magnitude is the same.

9.1.4 Non-Causal WMA Filter

$$h[n] = \frac{1}{S}\tilde{h}[n], \quad S = \sum_{k=-\infty}^{\infty}\tilde{h}[n]$$

No Phase Delay and Good Low Pass Behaviour.

$$\underbrace{H(z)}_{\text{causal}} \rightarrow \underbrace{H(z^{-1})}_{\text{anti-causal}} \rightarrow \underbrace{H(z)H(z^{-1})}_{\text{non-causal}}$$

$H(z)$	causal	M coeff.	phase shift
$H(z^{-1})$	anti-causal	M coeff.	phase shift
$H(z)H(z^{-1})$	non-causal	$2M-1$ coeff.	**no shift**

9.1.5 Phase

$-\angle H(\Omega)/\Omega \Rightarrow$ Number of Samples being delayed
Linear Phase (constand delay) is good for audio but not necessary for control.

9.1.6 Differentiation With FIR Filters

Differentiating the Input Signal $y(t) = \dot{u}(t)$

	Causal	A-Causal	N-Causal
$y(t)$	$\frac{u(t)-u(t-\tau)}{\tau}$	$\frac{u(t+\tau)-u(t)}{\tau}$	$\frac{u(t+\tau)-u(t-\tau)}{2\tau}$
$y[n]$	$\frac{u[n]-u[n-1]}{T_s}$	$\frac{u[n+1]-u[n]}{T_s}$	$\frac{u[n+1]-u[n-1]}{T_s}$

Frequency Response: $u(t) = e^{j\omega t}$, $\dot{u}(t) = j\omega e^{j\omega t}$

$$H_C(\Omega) = \frac{1 - e^{-j\Omega}}{T_s} = \frac{2je^{-j\Omega/2}}{T_s}\sin\frac{\Omega}{2}$$

$$H_A(\Omega) = \frac{2je^{j\Omega/2}}{T_s}\sin\frac{\Omega}{2}, \quad H_N(\Omega) = \frac{j}{T_s}\sin\Omega$$

10 Infinite Impulse Response Filter (IIR)

Difference Equation of a causal IIR Filter:

$$y[n] = \sum_{k=0}^{M-1} b_k u[n-k] - \sum_{k=1}^{N-1} a_k y[n-k], \quad a_k, b_k \in \mathbb{R}$$

Usually Infinite Lenght. Not Nescessarily Stable!!!
Order of filter: $\max(M-1, N-1)$ equals:

- the number of delay elements an implementation of the filter would require
- the size of the state in a state-space description of the system

Advantage: Meets Specification at lower order.
Transfer Function and Frequency Response:

$$H(z) = \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{1 + \sum_{k=1}^{N-1} a_k z^{-k}} \rightarrow H(\Omega) = \frac{\sum_{k=0}^{M-1} b_k e^{-j\Omega k}}{1 + \sum_{k=1}^{N-1} a_k z^{-j\Omega k}}$$

10.1 First Order Low-Pass Filter

Definition: $\alpha \to 1$ more constant, stable: $\alpha \in [0, 1)$, $H(0) = 1$.

$$y[n] = \alpha y[n-1] + (1-\alpha)u[n] \quad \rightarrow \quad H(z) = \frac{1-\alpha}{1 - \alpha z^{-1}}$$

Magnitude Response: second inequality holds for $\Omega \in [0, \pi]$

$$|H(\Omega)| = \frac{1-\alpha}{\sqrt{(1-\alpha\cos\Omega)^2 + \alpha^2\sin^2\Omega}}, \quad \frac{d|H(\Omega)|}{d\Omega} \leq 0$$

Phase Response: for $\Omega \in [0, \pi]$

$$\angle H(\Omega) = \arctan\left(\frac{-\alpha\sin\Omega}{1 - \alpha\cos\Omega}\right)$$

Phase Limit: $-\pi/2 < \angle H(\Omega) \leq 0$

10.1.1 Desing Considerations

α to Determine **Decay Time**. $y(T_0) = e^{-1}$

$$y[\alpha] = \alpha^n \overset{!}{=} e^{-1} \implies \alpha = e^{-\frac{1}{n}} = e^{-\frac{T_s}{T_0}} \approx \underbrace{1 - \frac{T_s}{T_0}}_{\text{1st Order}}$$

10.2 CT Butterworth Filter

Definition: K is the order. Butterworth is Maximally Flat

$$R(\omega) = \frac{1}{\sqrt{1+\omega^{2K}}}, \quad \frac{dR}{d\omega} < 0, \forall \omega > 0$$

Transfer Function: Steeper Slope = More Phase Delay

$$H(s) = \frac{1}{\prod_{k=1}^{K}\left(\frac{s}{\omega_c} - s_k\right)}, \quad s_k = exp\left[\frac{j(2k+K-1)\pi}{2K}\right]$$

10.3 Bilinear Transformation

Mapping: $z = e^{sT_s} = \frac{e^{s\frac{T_s}{r}}}{e^{-s\frac{T_s}{r}}} \approx \frac{1+s\frac{T_s}{2}}{1-s\frac{T_s}{2}}$

$$z = \frac{1 + s\frac{T_s}{2}}{1 - s\frac{T_s}{2}}, \quad s = \frac{2}{T_s}\left(\frac{z-1}{z+1}\right)$$

Left Imaginary Plane in s is mapped into Unit Circle in z(stability guaranteed).
Frequency Mapping: $s = j\omega$, $\omega \in (-\infty, \infty) \to \Omega \in (-\pi, \pi)$.

$$\Rightarrow \Omega = 2\arctan\left(\omega\frac{T_s}{2}\right) \approx \omega T_s \text{ for } \omega T_s \downarrow$$

10.3.1 Frequency Pre-Warping (Billinear Transformation)

Frequency Maping only good for Low ω!!! → Pre-Wrap Frequency

$$\bar{\omega}_c = \frac{2}{T_s}\tan\left(\frac{\omega_c T_s}{2}\right)$$

ω_c = Desired Frequency, $\bar{\omega}_c$ = Used Frequency

10.4 High-Pass Filter Design (CT/DT)

- Preserve Stability: left plane/inside circle to left plane/inside circle
- map $j\omega$-axis/unit circle to $j\omega$-axis/unit circle
- map $\omega = 0 \to \omega = \infty$ and $\omega = \infty \to \omega = 0$
- map $\Omega = 0 \to \Omega = \pi$ and $\Omega = \pi \to \Omega = 0$

$$s \to \frac{1}{s} \quad H_{HP}\left(\frac{s}{\omega_c}\right) = H_{LP}\left(\left(\frac{s}{\omega_c}\right)^{-1}\right)$$

$$z \to -z \quad H_{HP}(z) = H_{LP}(-z)$$

Causion: This shift Ω_c by π so $\Omega_{c,LP} = \pi - \Omega_{c,HP}$

10.5 Band-Pass Filter

CT Low-Pass and High-Pass to Band-Pass:

$$H_{BP}(s) = H_{LP}(s)H_{HP}(s) \quad \text{if } \omega_0/\omega_1 \gg 1$$

CT Low-Pass to Band-Pass: $s \to \frac{s^2+\omega_s^2}{s}$

$$H_{BP}\left(\frac{s}{\omega_c}\right) = H_{LP}\left(\frac{s^2+\omega_s^2}{s}\frac{1}{\omega_c}\right)$$

$\omega_c = \omega_1 - \omega_0$, $\omega_s = \sqrt{\omega_0\omega_1}$
DT Low-Pass to Band-Pass: $z \to -z^2$

10.6 Band-Stop Filter

$$H_{BS}(s) = H_{LP}(s) + H_{HP}(s) \quad \text{if } \omega_1/\omega_0 \gg 1$$

10.7 Notch Filter

Band-Stop Filter with very narrow band and $H_{NO}(\omega_c) = 0$.
Second Order Band-Stop Filter:

$$H_{NO}(s) = \frac{s^2+\omega_c^2}{s^2+\sqrt{2}s\omega_c+\omega_c^2}$$

Cosine and Sine

$$\cos(\omega) = \frac{e^{j\omega}+e^{-j\omega}}{2}, \quad \sin(\omega) = \frac{e^{j\omega}-e^{-j\omega}}{2j}$$

$$\cos\left(\omega - \frac{\pi}{2}\right) = \sin(\omega), \quad \sin\left(\omega + \frac{\pi}{2}\right) = \cos(\omega)$$