System Modeling

Jorit Geurts, jgeurts@student.ethz.ch

Version: 4. Februar 2022

1 Basic Modeling

1.1 System Models

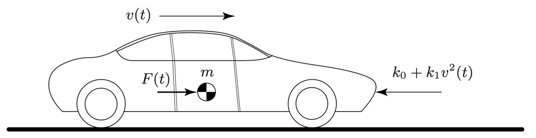
White Box Model:
Everything is known in form of ODE/PDEs.
Grey Box Model:
Physics is knwon but some Parameters are unknown and we need experiments.
Black Box Model:
Nothing is known and has to be derived from experiments.

1.2 Parametric and Nonparametric

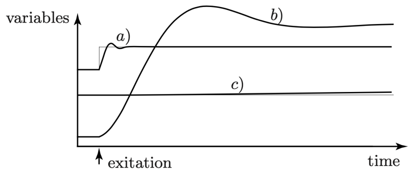
Parametric Model:
System Description through the Parameters and Physics (ODE, PDE, TF)
Nonparametric Model:
System Description through a known system response.

1.3 Forward and Backwards

Parametric Models can be:
Forward:
Regular Causality. (E.g. Given $F(t)$ what is $v(t)$.)
Backwards:
Inverted Causality. (E.g. Given $v(t)$ what is the needed $F(t)$.)



1.4 System Dynamics



(a)	Algebraic	fast
(b)	Dynamic	relevant
(c)	Static	slow

State Variable is Static

$$\frac{d}{dt}x(t) = 0$$

Solve algebraic equation

1.4.1 Causality Diagramms

Graphical representation of the systems equation. There are multiple ways to draw a causality diagramm.

1.5 Reservoir Based Approach

Reservoir:
Accumulative Elements (e.g. mass, heat, energy). Only systems with reservoirs have dynamic behaviour.
Every reservoir is associated with a level variable (state variable).
Flows:
Flow of the quantity between the elements. (e.g. massflow, heatflow). Are driven by differences in reservoir leves.

Precedure:

1. Define System Boundaries: what can be controled, what can be measured
2. Identify the relevant reservoirs and corresponding state variables
3. Formulate conservation laws for each reservoir
$$\frac{d}{dt}(\text{Reservoir}) = \sum \text{Inflows} - \sum \text{Outflows}$$
4. Formulate the Algebraic Relations that describe the Flows
5. Solve the Implicit Algebraic Loops
6. Identify Unknonw System Parameters with Experiments
7. Validate Model with Experiments

2 Extras

2.1 Algebraic Stability

Polynomial $p(s) = a_n s^n + \dots + a_1 s + a_0$
Hurwitz Matrix

$$H_n = \begin{bmatrix} a_n-1 & a_n & 0 & \dots & \dots & 0 \\ a_n-3 & a_n-2 & a_n-1 & a_n & \dots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \dots & a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & \dots & 0 & a_0 & a_1 & a_2 \\ 0 & \dots & \dots & \dots & 0 & a_0 \end{bmatrix}$$

H_i : square $i \times i$ matrix aligned to top left

$$\begin{aligned} d_i &= \det(H_i) \\ d_1 &= a_{n-1} \\ d_2 &= a_{n-1}a_{n-2} - a_n a_{n-3} \\ d_3 &= d_2 \cdot a_{n-3} - a_{n-1}(a_{n-1}a_{n-4} - a_n a_{n-5}) \end{aligned}$$

Hurwitz Criterion

Roots p_i all have $\mathrm{Re} < 0$ iff all $\det$ strictly positive

2.2 Trigonometrie

Shift by one quarter period	Shift by one half period[10]	Shift by full periods[11]	Period
$\sin(\theta \pm \frac{\pi}{2}) = \pm\cos\theta$	$\sin(\theta + \pi) = -\sin\theta$	$\sin(\theta + k\cdot 2\pi) = +\sin\theta$	2π
$\cos(\theta \pm \frac{\pi}{2}) = \mp\sin\theta$	$\cos(\theta + \pi) = -\cos\theta$	$\cos(\theta + k\cdot 2\pi) = +\cos\theta$	2π
$\tan(\theta \pm \frac{\pi}{4}) = \frac{\tan\theta \pm 1}{1 \mp \tan\theta}$	$\tan(\theta + \frac{\pi}{2}) = -\cot\theta$	$\tan(\theta + k\cdot\pi) = +\tan\theta$	π

2.3 Control Systems

Transfer Function

$$P(s) = C(s\mathbb{I} - A)^{-1}B = \frac{C\mathrm{Adj}(s\mathbb{I}-A)B}{\det(s\mathbb{I}-A)}$$

2.3.1 MIMO

Poles

The poles of $P(s)$ are the roots of the least common denominator of all minors of $P(s)$

Zeros

The zeros of $P(s)$ are the roots of the greatest common divisor of the numerator of the maximum minors of $P(s)$ after normalization to have the pole polynomial of $P(s)$ as denominators

2.4 Matrix Math

2X2 Inverse

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(M)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Positive Definite Symmetric and all Eigenvalues positive or if $x^\top Ax > 0 \ \forall x \neq 0$
Minors determinants of all square submatrices
Scalar-by-vector derivative:

$$\frac{\partial f}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial f}{\partial \vec{x}_1} & \frac{\partial f}{\partial \vec{x}_2} & \dots & \frac{\partial f}{\partial \vec{x}_n} \end{bmatrix}$$

Hessematrix:

$$\frac{\partial^2 f}{\partial \vec{x}^2} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

positiv semi-definite $\left(\frac{\partial^2 f}{\partial \vec{x}^2} \geq 0\right)$ if all eigenvalues $\lambda_i \geq 0$.

2.5 Analysis

Chainrule:

$$f(x(t)) \quad \rightarrow \quad \frac{df(x(t))}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt}$$

DGL first order solution:

$$\frac{dy}{dt} = A + By(t), \quad y(0) = y_0$$

$$y(t) = -\frac{A}{B} + \left(\frac{A}{B} + y_0\right)e^{B\cdot t}$$

2.5.1 1 Ordnung

$$\dot{y} + a(t) \cdot y = b(t)$$

Lösung:

$$y(t) = \left(\int b(t) \cdot e^{A(t)} dt + K\right) \cdot e^{-A(t)}$$

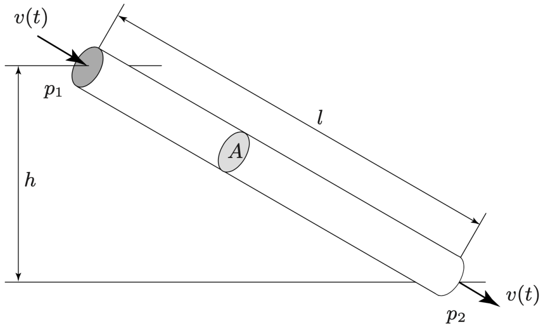
3 Hydraulic Systems

3.1 Bernoulli

Incompressible Flow:

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g z_2$$

3.2 Hydraulic Ducts



Velocity: (incompressible with Newtons Third Law)

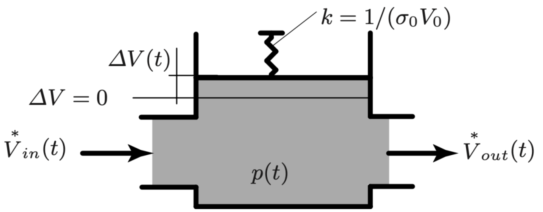
$$m\frac{d}{dt}v(t) = \rho l A \frac{d}{dt}v(t) = A(p_1(t) - p_2(t) + \rho g h) - F_f(t)$$

Friction Force: (λ form the Moody Diagramm)

$$F_f(t) = A\lambda(v(t))\frac{l}{d}\frac{\rho}{2}\text{sign}(v(t))v^2(t)$$

3.3 Compressibility

All Compressibilty put into one **Lumped Parameter**.



Volume leaving not the same as entering so the Spring (Compressibility) moves up and down.

Volume:

$$\frac{d}{dt}V(t) = \dot{V}_{\text{in}} - \dot{V}_{\text{out}}$$

Compressibility:

$$p(t) = \frac{\Delta V}{\sigma_0 V_0} + p_{\text{stat}}, \quad \Delta V = V(t) - V(0)$$

σ_0 = compressibility constant
V_0 = nominal volume (determined experimentally).

4 Electromagnetic Systems

4.1 RLC Networks

Resistance: R Static Block

$$U_R(t) = R \cdot I(t), \quad W_R = UI = RI^2$$

Inductance: L Dynamic Block → Level Variable $I(t)$

$$L\frac{d}{dt}I(t) = U_L(t), \quad W_M = \frac{1}{2}LI^2(t)$$

Capacitance: C Dynamic Block → Level Variable $U(t)$

$$C\frac{d}{dt}U_C(t) = I(t), \quad W_E = \frac{1}{2}CU_C^2(t)$$

Element	Capacitance	Inductance
Energy	$W_E = \frac{1}{2}CU^2(t)$	$W_M = \frac{1}{2}LI^2(t)$
Level Variable	$U(t)$	$I(t)$
Conservation	$C\frac{d}{dt}U(t) = I(t)$	$L\frac{d}{dt}I(t) = U(t)$

Energy Conservation:

$$\frac{d}{dt}E(t) = P(t), \quad P(t) = U(t)I(t)$$

Kirchhoff's Laws:

$$\sum I_k = 0, \quad \text{in each node}$$
$$\sum U_k = 0, \quad \text{in a closed loop}$$

Subtract Energies

If $U_{min} = \frac{1}{2}U_{max}$:

$$E_{useful} = \frac{1}{2}C \cdot U_{max}^2 - \frac{1}{2}C \cdot U_{min}^2 = \frac{3}{8}C \cdot U_{max}^2$$

Electric Losses

Electric losses are due to resistance, so if there is a resistance term, the losses are accounted for.

4.2 Motortypes

Brushless motor:

- Permanent magnets on the rotor
- Electrical commutation of the stator.

5 Electromechanic Systems

Biot-Savart:

$$F = \int_L Id\vec{L} \times \vec{B} = I \cdot (\vec{l} \times \vec{B}) = q \cdot (\vec{v} \times \vec{B})$$

Faradays Induction Law:

$$U = -v \cdot (\vec{l} \times \vec{B})$$

Usually we take the orthogonal case.

Motor Law:

$$F(t) = \kappa_{mot} \cdot I(t)$$
$$P_m = \omega_c \cdot T_m = \omega_c \cdot \kappa \cdot I$$

Generator Law:

$$U_{ind}(t) = \kappa_{el} \cdot \omega(t)$$

5.1 DC Motor

Assumption: Lossless

$$P_{el} = P_{mech} \quad \Leftrightarrow \quad U_{ind}I = T\omega$$
$$\kappa_{el}I\omega = \kappa_{mot}I\omega \quad \Rightarrow \quad \kappa_{el} = \kappa_{mot}$$

5.2 Motor Equations

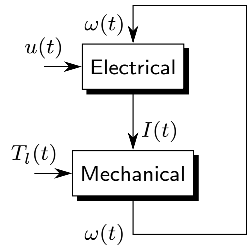
Two Reservoirs (Electircal and Kinetic Energy)
Mechanical:

$$\Theta\frac{d}{dt}\omega(t) = \overbrace{T_m(t)}^{\text{motor}}\overbrace{-T_l(t)}^{\text{load}}\overbrace{-d\omega(t)}^{\text{friction}} = \kappa I(t) - T_l(t) - d\omega(t)$$

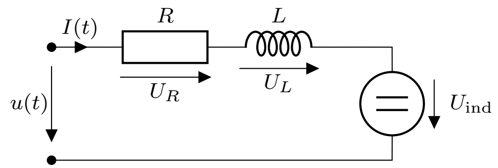
Electrical Motor:
Draw Circuit Diagramm & Use Kirchhoff

$$-u(t) + U_R(t) + U_L(t) + U_{ind}(t) = 0$$
$$\Rightarrow L\frac{d}{dt}I(t) = -RI(t) - \kappa\omega(t) + u(t)$$

Causality Diagram



Circuit:



Steady State:

$$L\frac{dI(t)}{dt} = 0 \Leftrightarrow I(t) = \frac{u(t) - \kappa\omega(t)}{R}$$

Nice to know

For $R_{new} < R_{old}$ the current increases, the speed at zero torque production $T_m = 0$ stays constant, the motor torque T_m at $\omega = 0$ increases and the slope of $T_m = f(\omega)$ changes.
electric losses are modeled with the resistance

Different Inductance: Voltage and Load Torque Jump

- High L means more extreme speed reaction but slower Motor Torque reaction
- Lower L means better speed reaction and faster Motor Torque reaction

6 Mechanical Systems

6.1 Mechanical Energy

6.1.1 Kintetic Energy

Translational:

$$T_t(t) = \frac{1}{2} m v^2(t)$$

Rotational:

$$T_r(t) = \frac{1}{2} \Theta \omega^2(t)$$

Complete Kinetic Energy:

$$T = \frac{1}{2} m \vec{v}_P^T \vec{v}_P + m \vec{v}_P^T (\vec{\omega} \times \vec{r}_{PS}) + \frac{1}{2} \vec{\omega}^T \Theta \vec{\omega}$$

- $\vec{v}_P$ velocity of the point P
- $\vec{r}_{PS}$ is the position vector from P to the center of gravity S
- $\vec{\omega}$ rotational speed of the body (same for each point)
- m mass of the body
- Θ_P Moment of Inertia of the body in point P.

If P is chosen to be equal to 0 or S the equation simplifies.

6.1.2 Potential Energy

Function of the Position: (Not velocity)

$$U(t) = U(\vec{r}(t))$$

Gravity	Linear Spring	Torsional Spring
$U = mgh$	$U = \frac{1}{2} k_{\text{lin}} x^2$	$U = \frac{1}{2} k_{\text{rot}} \varphi^2$

Conservative Force

A force is conservative if it can be written as the gradient of a potential.

$$F = -\frac{\partial U}{\partial \vec{q}}^T$$

6.1.3 Moment of Inertia

Definition:

$$\Theta = \iiint_B r(\vec{r}) dm$$

Steiners Theorem:

$$\Theta = \Theta_{CM} + m \cdot d^2$$

Rod:	$\Theta_{CM} = \frac{ML^2}{12}$
Cylinder:	$\Theta_{CM} = \frac{MR^2}{2}$
Hoop:	$\Theta_{CM} = MR^2$
Solid Ball:	$\Theta_{CM} = \frac{2MR^2}{5}$
m-t Ball:	$\Theta_{CM} = \frac{2MR^2}{3}$

Pointmass has zero moment of intertia with respect to its center of gravity.

6.2 Euler Method

Power:

$$P_F = \vec{F} \cdot \vec{v} \qquad P_T = \vec{T} \cdot \vec{\omega}$$

Total Energy:

$$E(t) = T(t) + U(t)$$

Energy Conservation:

$$\frac{d}{dt} E(t) = \sum_{i=i}^{k} P_i(t)$$

6.3 Newton

Translational:

$$\frac{d}{dt} m \cdot \vec{v}(t) = \sum F_i(t)$$

Rotational:

$$\frac{d}{dt} \Theta \cdot \vec{w}(t) = \sum T_i(t)$$

6.4 Drag Forces

Aerodynamic Force:

$$F_a = \frac{1}{2} \rho c_w A v_{rel}^2$$

Rolling Friction:

$$F_r = c_r F_N$$

Pendulum

Equation of motion of a pendulum:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin(\theta) = 0 \xrightarrow{\theta \approx 0} \frac{d^2\theta}{dt^2} + \frac{g}{l} \theta = 0$$

Solution:

$$\theta(t) = \theta_0 \cdot \cos\left(\sqrt{\frac{g}{l}} t\right), \quad T_0 = 2\pi \sqrt{\frac{g}{l}}$$

6.5 Lagrange Formalism

Degrees of Freedom:

2D DOF $= 3n - k$

3D DOF $= 6n - k$

$k =$ holonomic constraints, $n =$ number if bodies

6.5.1 Generalized Coordinates

Set of independent coordinates that describes the behaviour of the constrained system.

$$\vec{q}(t) = [q_1(t), \ldots, q_{DOF}(t)]^T$$

Generalzied Coordinates are not unique!!!
Minimum possible amount of generalized coordinates equalts the number of degrees of freedom (DOF).

6.5.2 Constraints

Holonomic Constraint:

Restriction of the reachable configuration. Reduce the number of variables used to describe the system. Independent of $\dot{\vec{q}}(t)$.

$$f(\vec{q}, t) = 0$$

Decrease the number of DOFs.

Non-Holonomic Constraint: (no change in DOFs)

Restriction of the trajectory. Dependent on $\dot{\vec{q}}(t)$.

$$f(\vec{q}(t), \dot{\vec{q}}(t), t) = 0$$

Do not decrease the number of DOFs.

Caution

If a Non-Holonomic constraint can be integrated over time it is Holonomic.

Non-Holonomic: $\dot{x} = 2xy$

Holonomic: $\dot{x} = \dot{\varphi} R \xrightarrow{\int dt} x = \varphi R - x_0$

6.5.3 Generalized Forces

Non-Conservativ Forces Acting in the System

Force Acting in A:

$$\vec{Q}_A = J_A^T \vec{F}$$

$$\vec{v}_A = J_A \cdot \dot{\vec{q}} + \xi_A$$

$\vec{v}_A$ velocity in A, ξ_A is the offset term.

Torque Acting in B:

$$\vec{Q}_B = J_B^T \vec{M}$$

$$\vec{\omega}_B = J_B \cdot \dot{\vec{q}} + \xi_B$$

$\vec{\omega}_B$ angular velocity in B, ξ_B is the offset term.

6.5.4 Procedure

1. Identify a set of generalized coordinates $\vec{q}(t)$
2. Is the System Holonomic or Non-Holonomic
3. Define the Lagrange Function

$$L(\vec{q}, \dot{\vec{q}}) = T(\vec{q}, \dot{\vec{q}}) - U(\vec{q}, \dot{\vec{q}})$$

4. Compute the generalized Forces $\vec{Q}_i$
5. Comput the Equation
   - Holonomic System:

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_k}\right\} - \frac{\partial L}{\partial q_k} = Q_k^{nc}$$

   - Non-Holonomic System: $n + \nu$ equations

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_k}\right\} - \frac{\partial L}{\partial q_k} - \sum_{j=1}^{\nu} \mu_j \alpha_{j,k} = Q_k^{nc}$$

$$\alpha_j^T \dot{\vec{q}}(t) = 0, \quad j = 1, \ldots, \nu$$

$$\alpha_j^T = [\alpha_{j,1}, \ldots, \alpha_{j,n}], \quad \alpha_{j,k} \in \mathbb{R}$$

Resulting Equation

$$M(q(t)) \cdot \ddot{q}(t) = f(q(t), \dot{q}(t), u(t))$$

M is allways a symmetric matrix.

7 Thermodynamic Systems

7.1 First Law

Open System:

$$\frac{dU}{dt} = \dot{Q} - \dot{W} + \sum \dot{H}_{in} - \sum \dot{H}_{out}$$

Internal Energy: $U(t) = c_v \cdot m \cdot \vartheta$
Enthalpy: $H(t) = c_p \cdot m \cdot \vartheta$
Enthalpy Flow: $\dot{H}(t) = c_p \cdot \dot{m} \cdot \vartheta$
$\dot{Q}(t)$ Heat Flow, $\dot{W}(t)$ Mechanical Power.
For incompressible Solids and Fluids we have $c_v = c_p$.

7.1.1 Heat Transfer

Conduction	$\dot{Q} = \frac{\kappa A}{l}(T_1 - T_2)$	Fourier
Convection	$\dot{Q} = kA(T_1 - T_2)$	Newton
Radiation	$\dot{Q} = \epsilon\sigma A(T_1^4 - T_2^4)$	Ste & Boltz.

- κ: thermal conductivity $[\mathrm{W/Km}]$
- k: heat transfer coeff $[\mathrm{W/Km^2}]$
- ϵ: emissivity < 1
- σ: Stefan-Boltzmann const. $4.670 \cdot 10^{-8}\mathrm{W/K^4m^2}$

7.2 Ideal Gases

Ideal Gas Law:

$$pV = n\bar{R}\vartheta = nMR\vartheta = mR\vartheta$$

- **Pressure:** p, **Volume:** V, **Temperature:** ϑ
- **# of Molecules:** n $[\mathrm{mol}]$, **Mass:** m $[\mathrm{kg}]$
- **Molar Mass:** $M = \frac{m}{n}$ $[\mathrm{kg\,mol^{-1}}]$

Gas Constant

- **Universal:** $\bar{R} = 8.314\mathrm{J\,mol^{-1}\,K^{-1}}$
- **Specific:** $R = \frac{\bar{R}}{M} = c_p - c_v$
- $\kappa = \frac{c_p}{c_v}$

7.2.1 Energy

Internal Energy:

$$U = mc_v(\vartheta - \vartheta_0) = mc_v\vartheta$$

Enthalpy:

$$H = U + pV = mc_v\vartheta + mR\vartheta = mc_p\vartheta$$

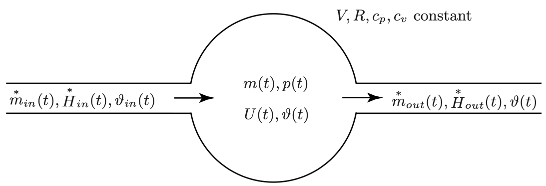
7.3 Lumped Parameters

For the lumped parameter assumption, the thermodynamic states are assumed to be constant inside the receiver. Therefor the outflow temeprature must be the same as the temperature in the receiver.

7.4 Pipe Temperatur

The pipe temperature of a insulated pipe, can only be described by a PDE.

7.5 Gas Receiver



Reservoirs:

Energy $U(t) : \vartheta(t)$, Mass $m(t) : p(t)$

Assumptions: V, R, c_p, c_v are Constant
Starting Equations:

$$\frac{dm}{dt} = \dot{m}_{in} - \dot{m}_{out}$$

$$\frac{dU}{dt} = \dot{Q} - \dot{W} + \sum \dot{H}_{in} - \sum \dot{H}_{out}$$

Don't forget the energy of fluid flowing in or flowing out.

7.5.1 Adiabatic Gas Receiver $\dot{Q} = 0$

Temperature (Energy):

$$\frac{d\vartheta}{dt} = \frac{R\vartheta}{c_V V p}\left[\dot{m}_{\mathrm{in}} c_p \vartheta_{\mathrm{in}} - \dot{m}_{\mathrm{out}} c_p \vartheta - (\dot{m}_{\mathrm{in}} - \dot{m}_{\mathrm{out}}) c_V \vartheta\right]$$

Pressure (Mass):

$$\frac{dp(t)}{dt} = \frac{\kappa \cdot R}{V}\left(\dot{m}_{in} \cdot \vartheta_{in} - \dot{m}_{out} \cdot \vartheta\right), \quad \kappa = \frac{c_p}{c_v}$$

7.5.2 Isothermal Gas Receiver $\vartheta = const.$

Temperature (Energy):

$$\frac{d\vartheta}{dt} = 0 \quad \Rightarrow \quad pV = mR\vartheta$$

Pressure (Mass): $(\vartheta = \vartheta_{in} = \vartheta_{out})$

$$\frac{dp(t)}{dt} = \frac{R\vartheta}{V}\left[\dot{m}_{\mathrm{in}}(t) - \dot{m}_{\mathrm{out}}(t)\right]$$

7.6 Eiswürfel in Wasser

Ein Eiswürfel in Wasser gibt einen Wärmestrom an das Wasser ab:

$$\overset{*}{Q} = L_f \overset{*}{m}_s$$

Wobei: L_f die spezifische Schmelenthalpie und $\overset{*}{m}_s$ der Schmelzwasserstrom ist.
Der Eiswürfel gibt über den Schmelzwasserstrom auch noch einen Enthalpiestrom an das Wasser ab:

$$\overset{*}{H} = \overset{*}{m}_s T_e c_w$$

Wobei: T_e die Eistemperatur (constant) ist und

7.7 Isentropic Relations

Temperature and Pressure:

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

Temperature and Volume:

$$\frac{T_2}{T_1} = \left(\frac{v_1}{v_2}\right)^{\gamma-1}$$

Pressure and Volume:

$$\frac{p_2}{p_1} = \left(\frac{v_1}{v_2}\right)^{\gamma}$$

Pressure:

$$\frac{\rho_2}{\rho_1} = \left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}}$$

Energy of a Fluid:

$$c_p \cdot T_1 + \frac{v_1^2}{2} = c_p \cdot T_2 + \frac{v_2^2}{2}$$

7.8 Increasing Engine Power

If $\dot{m}_{cyl}$ (into the cylinder) is model with a isenthalpic throttel and EGR throttel is closed at max engine power.

- Higher turbine efficiency = higher power
- Increasing heat removal by intercooler (Before the intake) = higher power
- Insulating exhaust manifold = higher power
- Reducing the intake volume = no effect
- Reducing the moment of inertai of the turbocharger = no effect

7.9 Turbocharger

Usually all massflows are algebraic equations and are therefore no state variables. Pressures and temperatures on the other hand can be state variables (different for every situation).

8 Fluiddynamic Systems

8.1 Valves

Incompressible: Bernoulli

$$\overset{*}{m}(t) = c_d A(t)\sqrt{2\rho}\sqrt{p_{\text{in}} - p_{\text{out}} + \frac{1}{2}\rho v_{in}^2}$$

We can often neglect $v_{in} \to v_{in} \approx 0$:

$$\overset{*}{m}(t) = c_d A(t)\sqrt{2\rho}\sqrt{p_{\text{in}} - p_{\text{out}}}$$

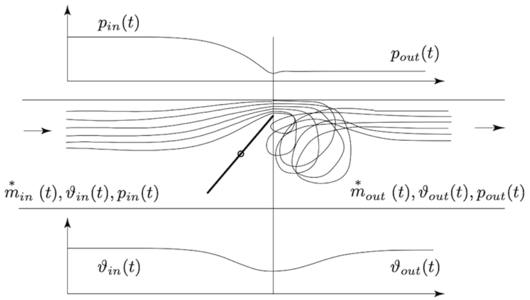
Compressible: Isenthalpic Throttle

$$\overset{*}{m}(t) = c_d A(t)\frac{p_{\text{in}}(t)}{\sqrt{R\vartheta_{\text{in}}(t)}}\Psi(p_{\text{in}}(t), p_{\text{out}}(t))$$

Caution for the correct sign!!!

$$\Psi(p_{\text{in}}, p_{\text{out}}) = \begin{cases} \sqrt{\kappa\left(\frac{2}{\kappa+1}\right)^{\frac{\kappa+1}{\kappa-1}}}, & ① \\ \frac{p_{\text{out}}}{p_{\text{in}}}^{\frac{1}{\kappa}} \cdot \sqrt{\frac{2\kappa}{\kappa-1}\left[1-\left(\frac{p_{\text{out}}}{p_{\text{in}}}\right)^{\frac{\kappa-1}{\kappa}}\right]}, & ② \end{cases}$$

$$p_{cr} = \left(\frac{2}{\kappa+1}\right)^{\frac{\kappa}{\kappa-1}} p_{in}, \qquad \begin{cases} ① : p_{out} < p_{cr} \\ ② : p_{out} \geq p_{cr} \end{cases}$$



Choked Flow

For $p_{out} < p_{cr}$ we reached the sonic speed at the outlet and the flow is choked (can't go faster).
For air $\gamma = 1.4$ sonic if

$$p_{out} < \frac{1}{2} \cdot p_{in}$$

When p_{out} reaches p_{cr}, the flow in the narrowest part reaches sonic conditions.
The flow is chocked at this velocity and no further speed increase can take place.

Approximation

We can also use an approximation for Ψ:

$$\Psi = \begin{cases} \frac{1}{\sqrt{2}}, & \text{for } p_{out} < 0.5p_{in} \\ \sqrt{\frac{2p_{out}}{p_{in}} \cdot \left[1 - \frac{p_{out}}{p_{in}}\right]}, & \text{for } p_{out} \geq 0.5p_{in} \end{cases}$$

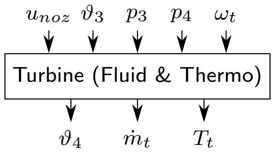
Both of the formulations have singularity at $p_{out} = p_{in}$

Opening Area

$$A_v = \pi R_v^2 - \left((1-x)R_v^2\right)\pi = A_{v0}\left(1-(1-x)^2\right)$$

8.2 Turbine

Caution: Algebraic Block not Dynamic.



Open System:

$$\frac{dE}{dt} = \dot{H}_{in} - \dot{H}_{out} - \dot{W}_t + \dot{Q}$$

Adiabatic and Static: $\dot{Q} = 0$, $\frac{dE}{dt} = 0$.

$$P_t = \dot{W}_t = \dot{H}_{in} - \dot{H}_{out} = \dot{m}_t \cdot c_p \cdot (\vartheta_3 - \vartheta_4)$$

Isentropic Relations:

$$\frac{\vartheta_3}{\vartheta_{4,is}} = \left(\frac{p_3}{p_4}\right)^{\frac{\kappa-1}{\kappa}} = \Pi_t^{\frac{\kappa-1}{\kappa}}, \quad \eta_t = \frac{\vartheta_3 - \vartheta_4}{\vartheta_3 - \vartheta_{4,is}}$$

Isentropic Velocity:

$$c_{us} = \sqrt{2 \cdot c_p \cdot \vartheta_3\left[1 - \Pi_t^{\frac{1-\kappa}{\kappa}}\right]}, \quad \tilde{c}_{us} = \frac{r_t \cdot \omega_t}{c_{us}}$$

Temperature: η_t from Maps or Charts

$$\vartheta_4 = \vartheta_3\left[1 - \eta_t\left(1 - \Pi_t^{\frac{1-\kappa}{\kappa}}\right)\right], \quad \Pi_t = \frac{p_3}{p_4}$$

Mass Flow: $\dot{\mu}_t$ form Maps or Charts

$$\dot{m}_t = \frac{p_3}{p_{ref,0}}\sqrt{\frac{\vartheta_{ref,0}}{\vartheta_3}} \cdot \dot{\mu}_t$$

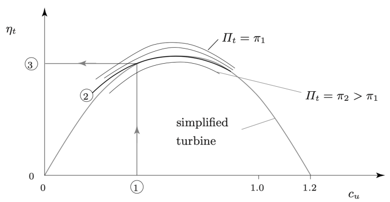
Torque:

$$T_t = \frac{P_t}{\omega_t} = \frac{\eta_t \cdot \dot{m}_t \cdot c_p \cdot \vartheta_3}{\omega_t}\left[1 - \Pi_t^{\frac{1-\kappa}{\kappa}}\right]$$

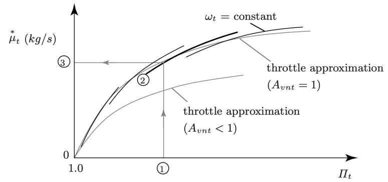
Power:

$$P_t = \dot{m}_t \cdot c_p \cdot \vartheta_3 \cdot \eta_t \cdot \left[1 - \Pi_t^{\frac{1-\kappa}{\kappa}}\right]$$

Efficiency Map Turbine

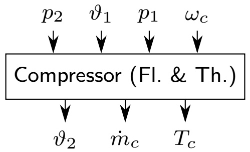


Massflow Map Turbine



8.3 Compressor

Caution: Algebraic Block not Dynamic.



Open System:

$$\frac{dE}{dt} = \dot{H}_{in} - \dot{H}_{out} - \dot{W}_c + \dot{Q}$$

Adiabatic and Static: $\dot{Q} = 0$, $\frac{dE}{dt} = 0$.

$$P_c = \dot{W}_c = \dot{H}_{in} - \dot{H}_{out} = \dot{m}_c \cdot c_p \cdot (\vartheta_2 - \vartheta_2)$$

Isentropic Relations:

$$\frac{\vartheta_{2,is}}{\vartheta_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\kappa-1}{\kappa}} = \Pi_c^{\frac{\kappa-1}{\kappa}}, \quad \eta_c = \frac{\vartheta_{2,is} - \vartheta_1}{\vartheta_2 - \vartheta_1}$$

Temperature: η_c from Maps or Charts

$$\vartheta_2 = \vartheta_1\left[1 + \frac{1}{\eta_c}\left(\Pi_c^{\frac{\kappa-1}{\kappa}} - 1\right)\right], \quad \Pi_c = \frac{p_2}{p_1}$$

Mass Flow: $\dot{\mu}_c$ form Maps or Charts

$$\dot{m}_t = \frac{p_1}{p_{ref,0}}\sqrt{\frac{\vartheta_{ref,0}}{\vartheta_1}} \cdot \dot{\mu}_c$$

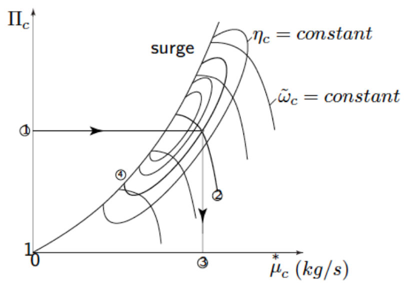
Torque:

$$T_t = \frac{P_c}{\omega_c} = \frac{\dot{m}_c \cdot c_p \cdot \vartheta_1}{\eta_c \cdot \omega_t}\left[\Pi_c^{\frac{\kappa-1}{\kappa}} - 1\right]$$

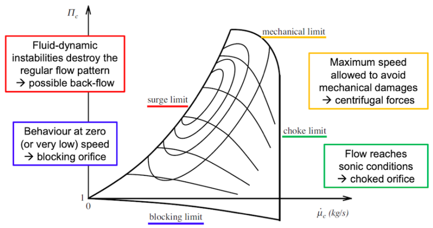
Power:

$$P_c = \frac{\dot{m}_c \cdot c_p \cdot \vartheta_1}{\eta_c}\left[\Pi_c^{\frac{\kappa-1}{\kappa}} - 1\right]$$

Compressor Efficiency & Massflow Map



Compressor Operational Limits



9 Chemical Systems

9.1 Mol Basics

One mol of A is $6.022 * 10^{23}$ molecules.
The Concentration is defined as the number of molecules per volume:

$$[A] = \frac{n_A}{V} = [\mathrm{mol\,m^{-3}}]$$

The molar mass of species A: $M_A[kg/mol]$ is defined the mass of 1 mol of A.

9.2 Stoichiometry

$$\alpha A + \beta B \rightleftharpoons \gamma C + \delta D$$

Forward Reaction:

$$\frac{d^-}{dt}[A] = -\alpha \cdot r^- [A]^\alpha [B]^\beta$$

Backwards Reaction:

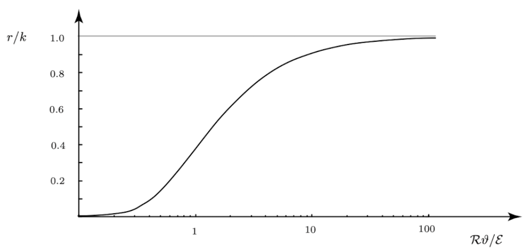
$$\frac{d^+}{dt}[A] = \alpha \cdot r^+ [C]^\gamma [D]^\delta$$

Whole Equation:

$$\frac{d}{dt}[A] = \alpha \cdot (\underbrace{r^-[C]^\gamma[D]^\delta}_{\text{forward reaction}} - \underbrace{r^+[A]^\alpha[B]^\beta}_{\text{inverse reaction}})$$

Arrhenius Model:

$$\underbrace{r^{+/-}}_{\text{rate coeff.}} = \underbrace{k^{+/-}(\vartheta, p, \dots)}_{\text{pre exp. factor}} \underbrace{e^{(-E^{+/-})/\overline{R}\vartheta}}_{\text{Boltzmann term}}$$



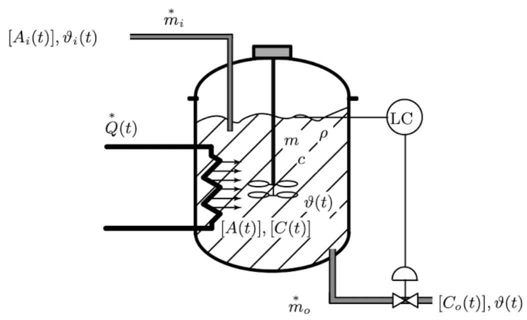
- With $\overline{R} = 8.314 \left[\frac{J}{molK}\right]$ being the universal gas constant
- E are the activation energies
- Boltzmann term: fraction of all collisions that have sufficient energy to start a reaction
- In most cases (k^+, k^-, E^+, E^-) must be determined experimentally

Adding Massflow:

$$\frac{d}{dt}[A]_{flow} = \frac{d}{dt}[A] + \underbrace{\frac{\dot{m}_{A,in} - \dot{m}_{A,out}}{VM_A}}_{\dot{n}_A/V \text{ (i.e flow)}}$$

9.3 Continuously Stirred Tank Reactor

Modeling



Reaction:

$$A + B \rightarrow C$$

Assumptions

- Molecule A is limiting species $\rightarrow [B] = const.$
- C is continuously removed $\rightarrow A + B \nleftarrow C$
- m, c and ρ are constant
- CSTR is adiabatic
- Lumped parameters $\rightarrow [C_o(t)] = [C(t)]$

Mass Balance:

$$\frac{d}{dt}m = 0 \rightarrow \overset{*}{m}_i = \overset{*}{m}_o = \overset{*}{m} \tag{1}$$

$$\overset{*}{V}_i = \overset{*}{V}_o = \overset{*}{V} = \frac{\overset{*}{m}}{\rho} \tag{2}$$

Three Reservoirs:

- n_A: level variable $[A]$
- n_C: level variable $[C]$
- U: level variable ϑ

Conservation of A:

$$\frac{d}{dt}n_A = \underbrace{\overset{*}{V} \cdot [A_i(t)]}_{\text{inflow}} - \underbrace{\overset{*}{V} \cdot [A(t)]}_{\text{outflow}} - \underbrace{Vr^-[A(t)][B]}_{\text{reaction}}$$

Conservation of C:

$$\frac{d}{dt}n_C = -\underbrace{\overset{*}{V} \cdot [A(t)]}_{\text{outflow}} + \underbrace{Vr^-[A(t)][B]}_{\text{reaction}}$$

Conservation of Energy:

$$\frac{d}{dt}U(\vartheta, n_A, n_C) = \overset{*}{H}_i(\vartheta_i(t)) - \overset{*}{H}_o(\vartheta(t)) + \overset{*}{Q}(t)$$

$$\overset{*}{H}_i(\vartheta_i(t)) = \overset{*}{m} \cdot c \cdot \vartheta_i(t) \quad \overset{*}{H}_o(\vartheta(t)) = \overset{*}{m} \cdot c \cdot \vartheta(t)$$

Chemical Reaction Energy:

$$dU = \frac{\partial U}{\partial \vartheta} \cdot d\vartheta + \frac{\partial U}{\partial n_A} \cdot dn_A + \frac{\partial U}{\partial n_B} \cdot dn_B + \frac{\partial U}{\partial n_C} \cdot dn_C$$
$$= \rho \cdot V \cdot c \cdot d\vartheta + H_A \cdot dn_A + H_B \cdot dn_B + H_C \cdot dn_C$$

With H_A, H_B, H_C being the enthalpies of formation. Stochiometry gives us: $-dn_A = -dn_B = dn_C$

Result

$$\tau \frac{d}{dt}[A(t)] = [A_i(t)] - (1 + \tau \cdot k \cdot e^{-\frac{E}{R\vartheta(t)}}) \cdot [A(t)]$$

$$\tau \frac{d}{dt}[C(t)] = -[C(t)] + \tau \cdot k \cdot e^{-\frac{E}{R\vartheta(t)}} \cdot [A(t)]$$

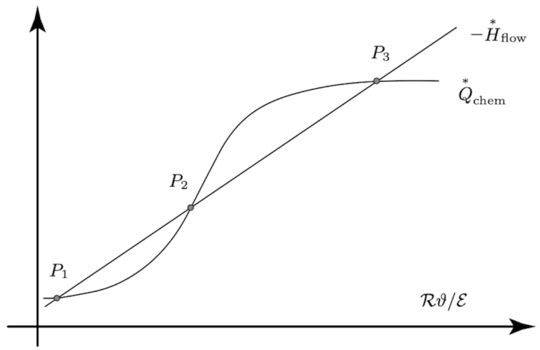
$$\tau \frac{d}{dt}\vartheta(t) = \vartheta_i(t) - \vartheta(t) + \underbrace{\frac{\overset{*}{Q}(t)}{\rho \cdot c \cdot \overset{*}{V}}}_{Controll}$$
$$+ \underbrace{\frac{\tau}{\rho \cdot c} \cdot H_0 \cdot k \cdot e^{-\frac{E}{R\vartheta(t)}}[A(t)]}_{Enthalpy}$$

$$\tau := \frac{V}{\overset{*}{V}}, \quad k := k^- \cdot [B], \quad H_0 = H_A + H_B - H_C$$

Chemical Equilibrium in a CSTR

Heat Removed by massflow: $\overset{*}{Q} = 0, \vartheta_{in} = c_1, [A_i] = c_2$

$$\overset{*}{Q}_{chem}(\vartheta) = H_0 \cdot \frac{V \cdot k \cdot e^{-\frac{E}{R\vartheta}}}{1 + \tau \cdot k \cdot e^{-\frac{E}{R\vartheta}}} \cdot [A_i]$$



- P_1: Stable but to slow
- P_2: Unstable but usefull $\rightarrow$ Controlling
- P_3: Stable but to hot

10 Optimization

10.1 Matrix Identities

Lemma 1:
$$\frac{d}{dx}c^T \cdot x = c, \quad c, x \in \mathbb{R}^n$$
Lemma 2:
$$\frac{d}{dx}x^T \cdot M \cdot x = 2M \cdot x, \quad M = M^T \in \mathbb{R}^{n\times n}, x \in \mathbb{R}^n$$
Lemma 3:
$$\frac{d^2}{dx^2}x^T \cdot M \cdot x = 2M, \quad M = M^T \in \mathbb{R}^{n\times n}, x \in \mathbb{R}^n$$

10.2 Parameter Optimization

Definitions:
$$\pi = [\pi_1, \dots, \pi_m]^T \in \mathbb{R}^m, \quad L : \mathbb{R}^m \to \mathbb{R}_+$$
Sufficient Condition for π_0 to be a local minimum:
$$\left.\frac{\partial L(\pi)}{\partial \pi}\right|_{\pi=\pi_0} = 0, \quad \left.\frac{\partial^2 L(\pi)}{\partial \pi^2}\right|_{\pi=\pi_0} > 0$$
Necessary Condition for π_0 to be a local minimum:
$$\left.\frac{\partial L(\pi)}{\partial \pi}\right|_{\pi=\pi_0} = 0, \quad \left.\frac{\partial^2 L(\pi)}{\partial \pi^2}\right|_{\pi=\pi_0} \geq 0$$

10.3 Optimization: Equality Constraint

Definitions:
$$\pi = [\pi_1, \dots, \pi_m]^T \in \mathbb{R}^m, \quad \text{Control Variable}$$
$$x = [x_1, \dots, x_n]^T \in \mathbb{R}^n, \quad \text{State Variables}$$
$$z = [\pi, x]^T, \quad \text{for convencience}$$
$$f : \mathbb{R}^{m+n} \to \mathbb{R}^n, \quad L : \mathbb{R}^{m+n} \to \mathbb{R}_+$$
Find π_0, x_0 which minimize L and satisfy $f(\pi_0, x_0) = 0$.
Solution: $n = 1$
$$\left.\frac{\partial L(z)}{\partial z}\right|_{z=z_0} + \lambda \cdot \left.\frac{\partial f(z)}{\partial z}\right|_{z=z_0} = 0$$

10.4 Numerical Algorithms

- **Semi-Analytical:** preformance index $L(\pi)$ and its gradient known (faster and less numerical error)
- **Fully Numerical:** only index is known and gradient is computed with a finite difference

10.4.1 First Order Methodes

- Guess an initial value $\pi(1)$
- Evaluate the Gradient $\left.\frac{\partial L(\pi)}{\partial \pi}\right|_{\pi=\pi(1)}$
- Determine the new iteration point:
$$\pi(i+1) = \pi(i) - h(i) \cdot \left.\frac{\partial L(\pi)}{\partial \pi}\right|_{\pi=\pi(1)}$$
- Check wether the difference $|L(\pi(i+1)) - L(\pi(i))|$ is smaller than a predetermined threshold ϵ.

Problems with small "ravines" and a good $h(i)$ is critical.
Nesterov's Algorithm: Solution for small ravines
$$\pi(i+1) = \rho(i) - h(i) \cdot \left.\frac{\partial L(\pi)}{\partial \pi}\right|_{\pi=\rho(1)}$$
$$\rho(i+1) = \pi(i+1) + \frac{i}{i+3} \cdot (\pi(i+1) - \pi(i))$$
Not every step satisfies $L(\pi(i+1)) < L(\pi(i))$ but will converge faster to the local minimum.

11 Model Parametrization

11.1 Planning Experiments

Determine the Parameters of a gray box Model or Validate the Systems. Very Important to not use the same data for these two tasks!!!

11.2 Linear Least Squares

Conditions:

- Coefficients have to enter linearly
- No singularities in the date.

Model: Algebraic
$$y(k) = h^T(u(k)) \cdot \pi + e(k)$$

- $k \in [1, r]$ = number of measurements
- $u(k) \in \mathbb{R}^m$ = k-th input vector
- $y(k) \in \mathbb{R}$ = k-th output
- $e(k) \in \mathbb{R}$ = k-th error
- $h() \in \mathbb{R}^q$ = Regressor (Non-Linear in $u(k)$ but algebraic and known exactly)
- $\pi \in \mathbb{R}^q$ = Vector with all Parameters

System of Equations:
$$\underbrace{\begin{bmatrix} y(1) \\ \vdots \\ y(r) \end{bmatrix}}_{\tilde{y} \in \mathbb{R}^r} = \underbrace{\begin{bmatrix} h^T(1) \\ \vdots \\ h^T(r) \end{bmatrix}}_{H \in \mathbb{R}^{r\times q}} \cdot \begin{bmatrix} \pi_0 \\ \pi_1 \\ \vdots \end{bmatrix} + \underbrace{\begin{bmatrix} e(1) \\ \vdots \\ e(r) \end{bmatrix}}_{\tilde{e} \in \mathbb{R}^r}$$
Error:
$$\tilde{e} = \tilde{y} - H \cdot \pi, \quad \tilde{e} \in \mathbb{R}^r$$
Goal: minimize e^2
$$\pi_{LS} = \arg\min \tilde{e}^T \tilde{e} = \arg\min(\tilde{y} - H\pi)^T(\tilde{y} - H\pi)$$
Solution:
$$\pi_{LS} = (H^T H)^{-1} H^T \tilde{y}$$
Use the Variable with the error as y!!!

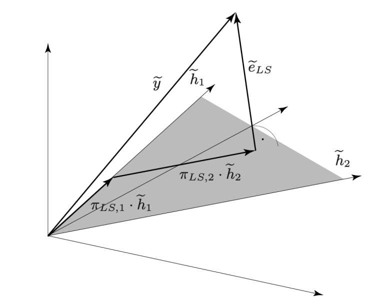
11.2.1 Weighted Least Squares

If not all measurements are equally good we can use a symmetric and positive definite weight matrix.
$$\varepsilon = \tilde{e}^T \cdot W \cdot \tilde{e}, \quad W \in \mathbb{R}^{r\times r}$$
Solution:
$$\pi_{LS} = \left[H^T \cdot W \cdot H\right]^{-1} H^T \cdot W \cdot \tilde{y}$$

11.2.2 Comments: Linear Least Squares

If $\text{rank}\, M \neq q$ we can use the Moore-Penrose Pseudo Inverse. If Model is precise and e is zero mean the parameters are excact. The full rank means that the parameters should be non-redundant.

Geometric Interpretation



11.3 Iterative Least Squares

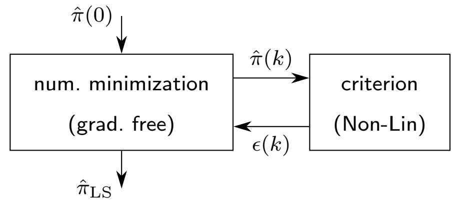
Main Idea:
Inversion is very time-consuming thus we use an iterative approach if a new measurement is avaliable.
$$\pi_{LS}(r+1) = f(\pi_{LS}(r), y(r+1)), \quad \pi_{LS}(0) = E\{\pi\}$$
Needed Lemma:
$M \in \mathbb{R}^{n\times n}, \det(M) \neq 0, v \in \mathbb{R}^n \; 1 + v^T \cdot M^{-1} \cdot v \neq 0$
$$[M + vv^\top]^{-1} = M^{-1} - \frac{M^{-1}vv^\top M^{-1}}{1 + v^\top M^{-1} v}$$
If M^{-1} is known, inversion of $M + K$ is easy.
Starting Point:
$$\pi_{\text{LS}}(r) = \underbrace{\left[\sum_{k=1}^{r} h(k)h^\top(k)\right]^{-1}}_{=:\Omega(r)} \sum_{k=1}^{r} h(k)y(k)$$
Definitions:
$$\Omega(r+1) = \Omega(r) - \frac{\Omega(r)h(r+1)h^\top(r+1)\Omega(r)}{1+c(r+1)}, \quad \Omega \in \mathbb{R}^{q\times q}$$
$$c(r+1) = h^\top(r+1)\Omega(r)h(r+1), \quad c \in \mathbb{R}$$
Solution:
$$\underbrace{\pi_{\text{LS}}(r+1)}_{\text{new}} = \underbrace{\pi_{\text{LS}}(r)}_{\text{old}} + \underbrace{\frac{\Omega(r)h(r+1)}{1+c(r+1)}}_{\text{direction}} \underbrace{[y(r+1) - h^\top(r+1)\pi_{\text{LS}}(r)]}_{\text{prediction error}}$$

11.3.1 Exponential Fogetting

If we now want to weight newer errors more we can use exponential forgetting.
$$\epsilon(r) = \textstyle\sum_{k=1}^{r} \lambda^{r-k} \cdot [y(k) - h^\top(k)\pi_{\text{LS}}(r)]^2, \; \lambda < 1$$
Solution:
$$\pi_{\text{LS}}(r+1) = \pi_{\text{LS}}(r) + \frac{\Omega(r)h(r+1)}{\lambda + c(r+1)} \frac{[y(r+1) - h^\top(r+1)\pi_{\text{LS}}(r)]}{1}$$
Update Equation:
$$\Omega(r+1) = \frac{1}{\lambda}\Omega(r)\left[\mathbb{I} - \frac{1}{\lambda + c(r+1)} h(r+1)h^\top(r+1)\Omega(r)\right]$$

11.4 Non-Linear Least Squares

Model: Dynamic
$$\frac{d}{dt}\hat{x}(t) = f(\hat{x}(t), u(t), \hat{\pi}), \quad \hat{x} \in \mathbb{R}^n, u \in \mathbb{R}^m$$
$$\hat{y}(t) = g(\hat{x}(t), u(t), \hat{\pi}), \quad \hat{y} \in \mathbb{R}^p, \hat{\pi} \in \mathbb{R}^q$$
Error Preformance:
$$\varepsilon = \sum_{i=1}^{r} \rho_i (y_i(\pi) - \hat{y}_i(\hat{\pi}))^2, \quad \rho_i \in \mathbb{R}_+$$
Finding the optimal π that minimizes ε we use non-linear programming.



11.5 Matlab

To calculate the least squares solution one can use the following commands:

- H\y
- mldivide(H,y)

12 Linear Systems

Obtained Model:

$$\frac{d}{dt}z(t) = f(z(t), v(t), t), \quad w(t) = g(z(t), v(t), t)$$

Problems:

- Non-normalized:
  - Numerical Problems (Different Magnitude)
  - Not the same units
- Non-Linear: There is no theory

12.1 Normalization

System Operates around a set point.

$$x_i(t) = \frac{z_i(t)}{z_{i0}} \quad u_i(t) = \frac{v_i(t)}{v_{i0}} \quad y_i(t) = \frac{w_i(t)}{w_{i0}}$$

Those normalized variables will have no units! (Derivatives will have unit $\frac{1}{s}$)
Such a transform does not change the systems characteristics. **Vector Notation:**

$$z = T \cdot x \qquad T = \text{diag}(z_{1,0} \dots z_{n,0})$$

Normalized System:

$$\frac{d}{dt}x(t) = f_0(x(t), u(t), t), \quad y(t) = g_0(x(t), u(t), t)$$

12.2 Linearization

Linearization around a small neighborhood of a chosen equilibrium point $\{x_e, u_e\}$

$$B_r := \{x \in \mathbb{R}^n \mid \|x - x_x\|^2 + \|u - u_e\|^2 \leq r\}$$

Equilibrium Point: $f_0(x_e, u_e, t) = 0$

$$\delta x = x - x_e \quad \delta u = u - u_e \quad \delta y = y - y_e$$

Taylor Expansion: $\mathcal{O}(\delta x^2, \delta u^2) \to 0$

$$\frac{d}{dt}\delta x(t) = \frac{\partial f_0}{\partial x}\Big|_{x_e,u_e}\delta x(t) + \frac{\partial f_0}{\partial u}\Big|_{x_e,u_e}\delta u(t)$$

$$\delta y(t) = \frac{\partial g_0}{\partial x}\Big|_{x_e,u_e}\delta x(t) + \frac{\partial g_0}{\partial u}\Big|_{x_e,u_e}\delta u(t)$$

Matrices

$$A = \begin{bmatrix} \frac{\partial f_{0,1}}{\partial x_1}\Big|_e & \dots & \frac{\partial f_{0,1}}{\partial x_n}\Big|_e \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{0,n}}{\partial x_1}\Big|_e & \dots & \frac{\partial f_{0,n}}{\partial x_n}\Big|_e \end{bmatrix}^{n \times n}$$

$$B = \begin{bmatrix} \frac{\partial f_{0,1}}{\partial u_1}\Big|_e & \dots & \frac{\partial f_{0,1}}{\partial u_m}\Big|_e \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{0,n}}{\partial u_1}\Big|_e & \dots & \frac{\partial f_{0,n}}{\partial u_m}\Big|_e \end{bmatrix}^{n \times m}$$

$$C = \begin{bmatrix} \frac{\partial g_{0,1}}{\partial x_1}\Big|_e & \dots & \frac{\partial g_{0,1}}{\partial x_n}\Big|_e \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{0,p}}{\partial x_1}\Big|_e & \dots & \frac{\partial g_{0,p}}{\partial x_n}\Big|_e \end{bmatrix}^{p \times n}$$

$$D = \begin{bmatrix} \frac{\partial g_{0,1}}{\partial u_1}\Big|_e & \dots & \frac{\partial g_{0,1}}{\partial u_m}\Big|_e \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{0,p}}{\partial u_1}\Big|_e & \dots & \frac{\partial g_{0,p}}{\partial u_m}\Big|_e \end{bmatrix}^{p \times m}$$

In General $\{A, B, C, D\}$ depend on time, but if the system is time invariant i.e. $f = f(x(t), u(t))$ they are constant.

Coordinate Transformation

A linearized system may be described in other coordinates. The change of coordinates is given by the similarity transform:

$$x = T\tilde{x}, \quad T \in \mathbb{R}^{n \times n}, \quad \det(T) \neq 0$$

In the new coordinates, the system is then described by:

$$\frac{d}{dt}\tilde{x}(t) = T^{-1}AT\tilde{x}(t) + T^{-1}Bu(t)$$

$$y(t) = CT\tilde{x}(t) + Du(t)$$

The fundamental system properties (IO-behavior, stability, controllability) are independent of the coordinates chosen.

12.3 Solution of Linear ODE

Linear ODE:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$
$$y = Cx + Du$$

Solution:

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Concolution:

$$\sigma(t) * u(t) = \int_0^t \sigma(t-\rho)u(\rho)d\rho$$

Impulse Response:

$$\sigma(t) = Ce^{At}B$$

Matrix Exponential

$$e^{At} = I + \frac{1}{1!}At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \dots$$

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$$

In general:

$$e^A \cdot e^B \neq e^{A+B}$$

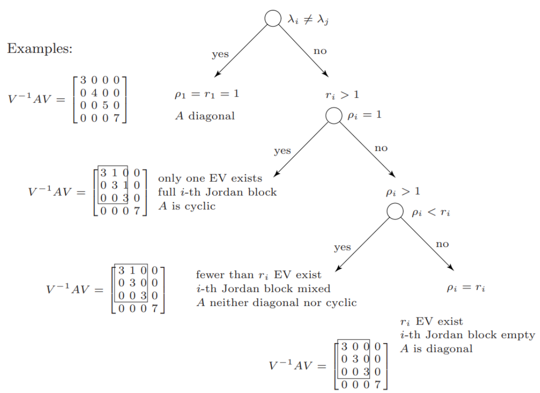
But if: $AB = BA$ then:

$$e^A \cdot e^B = e^{A+B}$$

And because At and $A\tau$ commute for arbitrary $t, \tau \in \mathbb{R}$:

$$(e^{At})^{-1} = e^{-At}$$

12.3.1 Jordan Forms



12.4 Jordan Form

We are looking at a matrix $A \in \mathbb{R}^{n \times n}$.

Eigenvectors: $v_i \in \mathbb{R}^n$

$$Av_i = \lambda v_i, \quad A \in \mathbb{R}^{n \times n}, \quad \lambda \in \mathbb{C}$$
$$\det(\lambda I - A) = 0$$

Even for real matrices $A \in \mathbb{R}^{n \times n}$ the eigenvalues λ_i and the eigenvectors v_i are in general complex entities. However they always arise in complex conjugate pairs.

If n *linearly independent* eigenvectors exist, then $T = [v_1, \dots, v_n]$ will diagonalize A:

$$AT = T\Lambda \quad \Rightarrow \quad T^{-1}AT = \Lambda$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ \dots & \ddots & \ddots & \dots \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}$$

If all λ_i are distinct then all v_i will always be linearly independent $\to$ A is diagonalizable.
This is equal to: $r_i = \rho_i = 1$ for all i.

Not all λ_i are distinct

- r_i: multiplicity of λ_i (Algebraic Multiplicity)
- ρ_i: rank loss of $\lambda_i I - A$ (Geometric Multiplicity) $\rho_i = n - \text{rank}(\lambda_i - A)$, n = systemrank.

Three Cases can occur:

- **Cyclic:** $\rho_i = 1 \Rightarrow$ Jordan Form
  1 indep. evec exists for r_i identical eval λ_i
- **Mixed:** $\rho_i < r_i \Rightarrow$ Jordan Form
  amount of indep. evec $< r_i$
- **Diagonalizable:** $\rho_i = r_i \Rightarrow$ Diagonalize
  sufficient indep evecs exist to diagonalize the part of A that belongs to λ_i

Jordan Form: Cyclic Case

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & \dots \\ 0 & \lambda_i & 1 & 0 & \dots \\ & & \ddots & \ddots & \\ 0 & \dots & \dots & 0 & \lambda_i \end{bmatrix}$$

For the mixed case the $r_i - \rho_i$ upper diagonal elements will be 1.

Systems with mixed or cyclic Jorad blocks associated to multiple eigenvalues on the imaginary axis will always have some states growing out of bound $\to$ unstable.

Generalized Eigenvectors

In the cyclic and mixed case A is not diagonalizable. Therefor we need generalized eigenvectors w_i
To get $\tilde{T}$ wich transforms A into the Jordan Form ($\tilde{T}^{-1}A\tilde{T} = J$) we need generalzed eigenvectors.

$$(\lambda_i I - A) \cdot w_i = v_i$$

Transformation Matrix:

$$\tilde{T} = [v_1, w_1, v_2, w_2, \dots]$$

12.5 Stability of Linear Systems

We are looking at the following system:

$$\frac{d}{dt}x(t) = A \cdot x(t), \quad x(0) = x_0, \quad 0 < \|x_0\| < \infty$$

Notice: for stability we set the input $u(t)$ to zero!

Definition of Stability:

$$\underbrace{\lim_{t \to \infty} \|x(t)\| = 0}_{\text{asympt. stable}}, \quad \underbrace{\|x(t)\| < \infty \ \forall t}_{\text{stable}}, \quad \underbrace{\lim_{t \to \infty} \|x(t)\| = \infty}_{\text{unstable}}$$

For Diagonalizable Matrices we have: $\sigma_i = \text{Re}(\lambda_i)$

Asympt. Stable	all	$\sigma_i < 0$
Stable	all	$\sigma_i \leq 0$
Unstable	any	$\sigma_i > 0$

For Cyclic or Mixed Matrices we have: $\sigma_i = \text{Re}(\lambda_i)$

Asympt. Stable	all	$\sigma_i < 0$
Unstable	any	$\sigma_i > 0$

If there are multiple $\sigma_i = 0$ then the system is only stable if the corresponding Jordan Blocks J_i are diagonal.
$\rho_{\sigma_i = 0} = r_{\sigma_i = 0} \, \forall \, i$

Systems with mixed or cyclic Jorad blocks associated to multiple eigenvalues on the imaginary axis will always have some states growing out of bound $\to$ unstable.

Here stability is a global concept: if the eq. point $x = 0$ is stable, then this is true for all finite initial conditions $x(0)$.

Check Stability of Linear System

1. Calculate all Eigenvalues $\lambda_i = \sigma_i + \omega_i j$ of A with:
   $$\det(\lambda I - A) = 0$$
2. If all σ_i are:
   (a) $\sigma_i > 0 \Rightarrow$ System is unstable
   (b) $\sigma_i < 0 \Rightarrow$ System is asymptotically stable
3. If one or more σ_i are zero, we need to further investigate the situation:
4. Calculate all r_i (algebraic multiplicity)
5. Calculate all ρ_i (geometric multiplicity):
   $$\rho_i = n - \text{rank}(\lambda_i I - A), \quad A \in \mathbb{R}^{n \times n}$$
6. For all r_i and ρ_i if one is cyclic or mixed, the system is unstable!
   (This also counts for the non linear system)

12.6 Reachability and Observability

Reachability: reach a state

Reachability Matrix:

$$\mathcal{R}_n = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

The system is fully reachable if $\text{rank}(\mathcal{R}_n) = n$.
For SISO Systems this means full rank.

Controllability: bring a state to the origin

The set of all states $x(0) \neq 0$, that can be forced to the origin in **finite time**, by a suitable controll signal $u(t)$.
For linear continuous-time systems the set of controllable and reachable states is identical.
If the system is completely reachable, it is also completely controllable.
A completely controllable systems can force any $x(0) \neq 0$ to the origin.

Observability: doesen't depend on x_0

Observability Matrix:

$$\mathcal{O}_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

The system is fully observable if $\text{rank}(\mathcal{O}_n) = n$
For SISO Systems this means full rank.

12.7 Balanced Realization

$\mathcal{R}, \mathcal{O}$ deliver only yes/no answer
$\Rightarrow$ we want quantitative information
System must be normalized!!!

12.7.1 Gramian Matrices

Controllability Gramian: symmetric & pos. definite

$$W_R = \int_0^\infty e^{A\sigma} B B^\top e^{A^\top \sigma} d\sigma$$

The closer W_R is to a singular matrix, the less controllable the corresponding system will be.
Observability Gramian: symmetric & pos. definite

$$W_O = \int_0^\infty e^{A^\top \sigma} C^\top C e^{A\sigma} d\sigma$$

The closer W_O is to a singular matrix, the less observable the corresponding system will be.

Computation of the Gramian

If System is Hurwitz (A asymptotically stable) we use two Lyapunov Equations.

$$A W_R + W_R A^\top = -B B^\top$$
$$A^\top W_O + W_O A = -C^\top C$$

Facts about Grammians

- Gramians only exist iff system: $\{A, B, C, D\}$ is asymptot. stable.
- Gramians are by construction symmetric and positive definite $\Rightarrow$ σ_i are all positive.

12.8 Order Reduction

We will Transform the System $T \cdot x_b = x$, such that

$$W_{R,b} = W_{O,b} = \text{diag}(\sigma_i), \quad i = 1, ..., n$$

Transformation:

$$T = T_R T_O, \quad W_R = V_R \Lambda_R^2 V_R^\top \to T_R = V_R \Lambda_R$$
$$\widetilde{W}_O = T_R^\top W_O T_R = V_O \Lambda_O^2 V_O^\top \ \to \ T_O = V_O \Lambda_O^{-1/2}$$

After the Transformation the Gramians of the transformed system: $T^{-1}AT, T^{-1}B, CT, D$ will have the following form:

$$W_{R,b} = W_{O,b} = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \sigma_n \end{bmatrix}, \sigma_1 \geq ... \geq \sigma_n \geq 0$$

The states that are nearest to 0 can be omitted as they are not good observable and not good controllable.

System Order Reduction Algorithm

After trasnforming the system in the order reduction form, one can partition the system: **System:**

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Du(t)$$

$x_1 \in \mathbb{R}^{n-\nu}$ are the important states.
$x_2 \in \mathbb{R}^{\nu}$ are the not important states.

We can now just omit x_2 and end up with the system:

$$\frac{d}{dt} x_1(t) = A_{1,1} x_1(t) + B_1 u(t)$$
$$y(t) = C_1 x_1(t) + Du(t)$$

Just omitting x_2 will change the DC-Gain.

If this is to be avoided, a singular pertubation approach is better, where the dynamics of states x_2 is neglected but not their DC contributions.

$$\frac{d}{dt} x_2(t) \approx 0 \ \Rightarrow \ x_2(t) \approx -A_{2,2}^{-1} \left[A_{2,1} x_1(t) + B_2 u(t)\right]$$

Resulting System:

$$\frac{d}{dt} x_1(t) = \left[A_{1,1} - A_{1,2} A_{2,2}^{-1} A_{2,1}\right] x_1(t) + \left[B_1 - A_{1,2} A_{2,2}^{-1} B_2\right] u(t)$$

$$y(t) = \left[C_1 - C_2 A_{2,2}^{-1} A_{2,1}\right] x_1(t) + \left[D - C_2 A_{2,2}^{-1} B_2\right] u(t)$$

DC-Gain:

$$\dot{x} = Ax + b = 0, \quad u(t) = 1$$

12.9 Zero Dynamics

System: SISO

$$P(s) = C\,[s\mathbb{I} - A]^{-1}\,B$$

Transferfunction:

$$P(s) = k \frac{s^m + b_{m-1}s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{m-1} + \dots + a_1 s + a_0}$$

- n: highest power denominator, # of integrators
- m: highest power numerator
- $r = n - m$: relative degree
- k: input gain

Canonial Coordinates with Gain k:

$$\frac{d}{dt} x(t) = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ k \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_0 & \dots & b_{m-1} & 1 & 0 & \dots \end{bmatrix} x(t)$$

This form has the minimum amount of parameters!
They have no physical meaning.

Alternative Definition of r:
Number of derivatives necessary before u appears in y

$$y(t) = Cx(t)$$
$$\dot{y}(t) = C\dot{x}(t) = CAx(t) + CBu(t) = CAx(t)$$
$$\dots$$
$$y^{(r)}(t) = CA^r x(t) + CA^{r-1}Bu(t) = CA^r x(t) + ku(t)$$

Zerodynamics from State Space

Solve $y(t) = 0$ to get the zerodynamics.
System:

$$\dot{x} = \begin{bmatrix} -2 & a \\ -1 & 0 \end{bmatrix} \cdot x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot u$$

$$y = [1 \ 0] \cdot x$$

We get:

$$y(t) = x_1(t) = 0$$

That x_1 stayes zero we also have $\dot{x}_1 = 0$.
As a result we get the zero dynamics.

$$\dot{x}_1 = ax_2 + u = 0 \to \dot{x}_2 = -u = ax_2$$

Zero Dynamics:

Special inputs $u^*(t)$ and IC x^* for which $y(t) = 0$
For $y(t) = 0 \ \forall t$ all derivatives of y must $= 0$

Coordinate Transform: $z = \Phi^{-1} x$

$$z_1 = y = Cx = [b_0 x_1 + \dots + b_{m-1} x_m + x_{m+1}]$$
$$z_2 = \dot{y} = CAx = [b_0 x_2 + \dots + b_{m-1} x_{m+1} + x_{m+2}]$$
$$\dots$$
$$z_r = y^{r-1} = CA^{r-1} x = [b_0 x_r + \dots + b_{m-1} x_{n-1} + x_n]$$

$$z_{r+1} = x_1 \quad \dots \quad z_n = x_{n-r} \qquad z = \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \xi = \begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix}, \eta = \begin{bmatrix} z_{r+1} \\ \vdots \\ z_n \end{bmatrix}$$

New Coordinates: $y = \xi_1$

$$\frac{d}{dt}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \left[\begin{array}{ccccc|ccccc} 0 & 1 & 0 & \dots & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & 1 & 0 & \dots & \dots & \dots & 0 \\ - & - & r^T & - & - & - & - & s^T & - & - \\ \hline 0 & \dots & \dots & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 & \dots & \dots & 0 & 1 \\ - & - & p^T & - & - & - & - & q^T & - & - \end{array}\right] \cdot \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \left[\begin{array}{c} 0 \\ \dots \\ 0 \\ k \\ \hline 0 \\ \dots \\ \dots \\ 0 \end{array}\right] \cdot u$$

$r^\top, s^\top$ not important here $(m = n - r)$

$$q^\top = [b_0, -b_1, \dots, -b_{n-r-1}] \quad p^\top = [1, 0, \dots, 0]$$

To have $y(t) = 0 \,\forall t$ we have to initialise the system with:

$$\xi^*(0) = 0, \quad u^*(t) = -\frac{1}{k} s^\top \eta^*(t), \quad \eta^*(0) \neq 0$$

Zero Dynamic States:

$$\frac{d}{dt}\eta(t) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ - & - & q^\top & - & - \end{bmatrix} \eta^*(t) = Q\eta^*(t)$$

$\Rightarrow$ Q asympt. stable $\Rightarrow$ **System is Minimum Phase**
(all zeros have negative Real parts)

Unstable Zero Dynamics zero with pos. real part if:

- system is non-min phase
- system's zero dynamics unstable
- internal states η can diverge without y affected

Consequences

- u may not be chosen such that y is (almost) 0 before states η associated with zero dynamics are (almost) zero
- **Feedback control more difficult**
- imposes constraint of bandwidth on CL-System
  $\Rightarrow$ slower (smaller) than slowest nmp zero
  System has to first get the nmp zeros fixed, before it can start to controll the output.

13 Nonlinear Systems

13.1 Equilibrium Sets

Linear Systems:

- 1 isolated equilibrium point
- entire equilibrium subspaces
- periodic orbits with the same frequency but arbitrary amplitude
- if linear system is asymp stable it is always exponentially asymp stable

Non-Linear Systems: Limit Sets

- can have **infinitely many** isolated equilibrium points
- equilibrium point can have **finite region of attraction**
- equilibrium point can be non-exponetially asymptotically stable.
- if an equilibrium point is unstable the state of the system can "escape to infinity" in finite time
- can have isolated periodic orbits; all trajectories that start close enough converge to this orbit.
- "Strange attractors" - bounded sets to which non-periodic trajectories converge if sufficiently close

13.2 Lyapunov Stability

Lyapunov stability is always connected to a constant equilibrium point x_e of a system.
System: Assume $x_e = 0$ w/o loss of generality

$$\frac{d}{dt}x(t) = f(x(t), t), \quad x(t_0) = x_0 \neq 0, \quad f(x_e, t) = 0$$

If $x_e \neq 0$ then use the transform:

$$\tilde{x} = x - x_e$$

Lyapunov Stable at $\mathbf{t = t_0}$:
If you can find some $r(R, t_0)$ for any $R > 0$ such that:
if: $\|x_0\| < r \leq R$ then $\|x(t)\| < R \, \forall t > t_0$

Uniformly Lyapunov Stable: if $r(R) \neq f(t_0)$

Asymptotically Stable:
Uniformly Lyapunov Stable and Attractive

$$\lim_{t\to\infty} x(t) = x_e = 0$$

Exponentially Asymptotically Stable:
if constant $a > 0, b > 0$ exist such that:

$$\|x(t)\| \leq ae^{-bt}\|x(0)\|$$

Facts

Usually only exponetially asymptotically stable systems are accepteable for technical applications.

Linear Systems:
If an equilibrium point of a linear system is asymptotically stable, then it is always expontially asymptotically stable.

Non-Linear Systems:
If an equilibrium point of a non-linear system is asymptotically stable, then it is **not** always expontially asymptotically stable.

Local attractiveness does not imply global stability.

13.2.1 From Linear to Non-Linear

If the linear system is

- **unstable** (Any $\mathrm{Re}(\lambda_i) > 0$) the non-linear system is also unstable.
- **asymptotically stable** (All $\mathrm{Re}(\lambda_i) < 0$) the non-linear system is also stable.
- **stable** (One or more $\mathrm{Re}(\lambda_i) = 0$) we need further knowledge of the system to decide.
  If however through further analysis the system is unstable we can conclude that the non-linear system will be unstable.

13.3 2nd-Order Systems

System:

$$\frac{d}{dt}x_1(t) = f_1(x_1, x_2), \quad x_1(0) = x_{1,0} \qquad (3)$$

$$\frac{d}{dt}x_2(t) = f_2(x_1, x_2), \quad x_2(0) = x_{2,0} \qquad (4)$$

Poincaré-Bendixson Theorem
CT diff'bar systems cannot exhibit deterministic chaos

Linearized system

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad a_{i,j} = \frac{\partial f_i}{\partial x_j} \quad \frac{d}{dt}\delta x(t) = A\delta x(t)$$

Lyapunov Principle:

The local behavior of the original nonlinear system and of the linearized system have the same characteristics.

If some Eigenvalues have $\mathrm{Re}(\lambda) = 0$, the principle doesn't hold and we need further analysis of the system.

Also applies to systems of higher order!

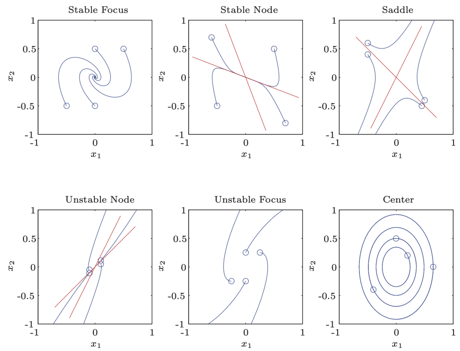
Eigenvalues	Linearized Sys.	Nonlin. Sys.
$\lambda_{1,2} \in \mathbb{C}_-$	Stable Focus	Stable Focus
$\lambda_{1,2} \in \mathbb{R}_-$	Stable Node	Stable Node
$\lambda_1 \in \mathbb{R}_+, \lambda_2 \in \mathbb{R}_-$	Saddle	Saddle
$\lambda_{1,2} \in \mathbb{R}_+$	Unstable Node	Unstable Node
$\lambda_{1,2} \in \mathbb{C}_+$	Unstable Focus	Unstable Focus
$\mathrm{Re}(\lambda_{1,2}) = 0$	Center	???

ACHTUNG
strictly local concept! regions of stability can be small
Lyapunov principle also holds for non-linear systems of higher order
$\Rightarrow$ the local stability properties of the isolated equilibrium point $x_e = 0$ of a time-invariant nonlinear system:

$$\frac{d}{dt}\vec{x}(t) = f(x(t)), \vec{x}(0) \neq 0$$

are fully described by the first-order approximation A of $f()$, provided A has no eigenvalues with zero real part.

Graphic Interpretation



Bottom right only valid for linear system.

13.4 Lyapunov Theory

If one is not interested in only local behaviour or $\mathrm{Re}\{\lambda_i\} = 0$ for some i than one can use the lyapunov theory for stability analytics.

Definitions

Nondecreasing function:

$$\alpha : \mathbb{R}_+ \to \mathbb{R}_+, \quad \alpha(0) = 0, \alpha(q) \geq \alpha(p) \, \forall p > q$$

Strictly Positive Functon:

$$V(x, t) > 0 \quad \forall x \neq 0, \forall t, \qquad V(\vec{0}, 0) = 0$$

Lyapunov Candidate Function: $V : \mathbb{R}^{n+1} \to \mathbb{R}_+$

- $V(x, t)$ is strictly positive
- two functions $\beta(x), \alpha(x)$ exist that satisfy:
  $\beta(\|x\|) \leq V(x, t) \leq \alpha(\|x\|)$

13.4.1 Global Stability

Uniformly Globally/Locally Lyapunov Stable:

$$\frac{d}{dt}V = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(x, t) \leq 0 \quad \forall x(t) \neq 0, \; \forall t$$

Uniformly Globally/Locally Asymptotically Stable:
$-\frac{d}{dt}V(x, t)$ has to be positive definite.

$$-\frac{d}{dt}V(x, t) > 0, \forall x \neq 0, \forall t \quad -\frac{d}{dt}V(\vec{0}, 0) = 0$$

Finding a Function is very difficult!!!
Lyapunov Theorem provides **sufficient** *but not* **necessary** *conditions*

Function for Linear Systems: $Q = Q^T > 0$ arbitrary

$$PA + A^\top P = -Q \;\Rightarrow\; V(x) = x^\top Px$$

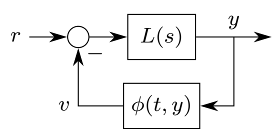
$$\frac{d}{dt}V(x) = -x^TQx$$

P is symmetric and positive definite ($P^T = P$).
Solution only exists if A is is Hurwitz.

This is no new information but it can help find a function for the non-linear case.

Achtung:
If $\frac{d}{dt}V$ does not fulfill criteria, Lyapunov theorem does not provide any conclusion on stability of the equilibrium.

13.5 Circle Criterion



- $L(s)$: LTI, SISO dynamic part
- $\phi(t, y)$: memoryless, time-varying nonlinearity

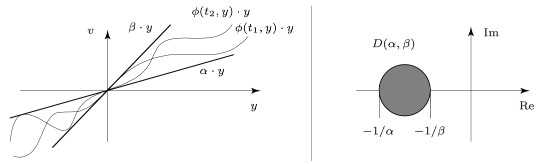
Nonlinearity assumed to be "sector bounded":

$$\alpha y < \phi(t, y)y < \beta y \quad \alpha, \beta \in \mathbb{R} \quad 0 < \alpha < \beta$$

Circle Criterion

Assume $L(s)$ is strictly proper transferfunction with n_+ unstable poles & n_0 purely Im. poles.
Assume $\phi(t, y)$ is sector bounded. CL system is asymptotically stable if:

1. Nyquist curve $L(j\omega)$ does not enter disk $D(\alpha, \beta)$
2. $L(j\omega)$ encircles $D(\alpha, \beta)$ $n_+ + n_0/2$ times



This result is sufficient and necessary!

13.6 Popov Criterion

Powerful Result for fewer Systems.
Additional Constraints (compared to circle criterion)

- $L(s)$ may not have unstable poles
- $\phi(.)$ must be time invariant

Popov Criterion *Assume $L(s), \phi(.)$ fulfill above conditions. CL system asymptotically stable if:*

$$\mathrm{Re}\left[(1 + rj\omega)L(j\omega)\right] + \tfrac{1}{\beta-\alpha} + \tfrac{\alpha}{\beta+\alpha}|L(j\omega)|^2 > 0 \quad \forall\omega$$

Yields global results if the constraints are met.

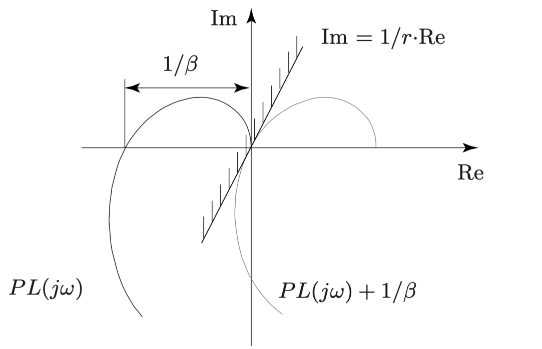
Special Case: $\alpha = 0$

Popov Plot:

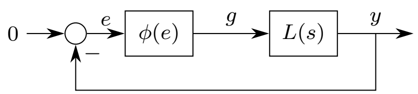
$$PL = \mathrm{Re}[L] + j\omega\mathrm{Im}[L]$$

Criterion:

$$\mathrm{Re}[L] - r\omega\mathrm{Im}[L] + \tfrac{1}{\beta} > 0, \quad \Rightarrow \mathrm{Im}[PL] < \tfrac{1}{r}\mathrm{Re}[PL + \tfrac{1}{\beta}]$$



13.7 Describing Functions



Describing Function Special class of NL, SISO systems

- $L(s)$: dynamic linear system, **low-pass**
- $L(s)$ has to be asymptotically stable
- $\phi(e)$: static nonlinear system, **odd** $\phi(-e) = -\phi(e)$
- $\phi(e)$ must be time invariant

Objecitve:
Predict the presence of limit cycles. Only Approximations.
Limit Cycle:
Sustained periodic oscillations of CL-system

Linear Systems

Linear case $\phi(e) = e \Rightarrow$ CL stability boundary:

$$e = a\sin(\omega t) \quad y = a\sin(\omega t - \pi)$$

Conditions

- amp. of e = amp. of y
- phase of y lags e by $-\pi$

$$|L(j\omega)| = 1 \quad \angle(L(j\omega)) = -\pi \text{ or}$$

$$L(j\omega) = e^{j\pi} = -1 \;\Rightarrow\; 1 + L(j\omega) = 0$$

Non Linear Case

Main Idea: if $e(t)$ periodic, g periodic as well

$$g(t) = \phi(a\sin(\omega t)) = \sum_{i=1}^{\infty} k_i(a)\sin(i\omega t + \varphi_i(a))$$

L is Low-Pass $\Rightarrow$ only first harmonic of g important

$$g(t) \approx k_1(a)\sin(\omega t + \varphi_1(a))$$

Describing Function:

$$\mathrm{DF}(a) = \frac{k_1(a)e^{j\varphi_1(a)}}{a}$$

Changes induced by $\phi(.)$ on amp & phase of $e(t)$.
Only dependent on the amplitude a and not ω.
Nyquist Diagram plot both $\mathrm{DF}(a)$ and $L(j\omega)$
$\Rightarrow$ marginally stable when:

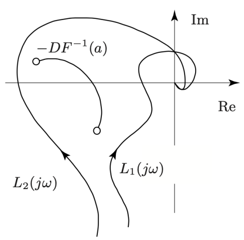
$$k_1(a)\cdot|L(j\omega)| = a|\mathrm{DF}(a)|\cdot|L(j\omega)| = a$$

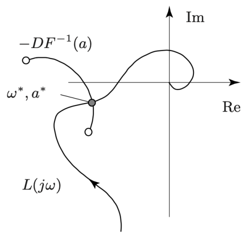
$$\varphi_1(a) + \angle(L(j\omega)) = \angle(\mathrm{DF}(a)) + \angle(L(j\omega)) = -\pi$$

$$\Rightarrow\; 1 + \mathrm{DF}(a)\cdot L(j\omega) = 0$$

3 Cases can occur:

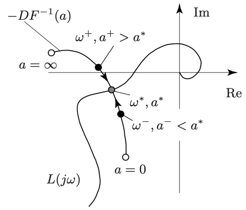
- $L(j\omega)$ neither intersects nor encircles $-\mathrm{DF}^{-1}(a)$
  $\Rightarrow$ CL-system probably **asymptotically stable w/o limit cycles**
- $L(j\omega)$ does not intersect $-\mathrm{DF}^{-1}(a)$, but encircles it
  $\Rightarrow$ CL-system probably **unstable**
- $L(j\omega)$ intersects $-\mathrm{DF}^{-1}(a)$
  $\Rightarrow$ CL-system **can produce limit cycle**



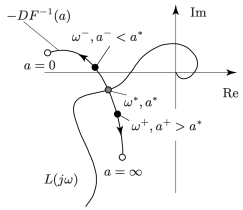


Stability of Limit Cycle

Stable



Unstable



Gedankenexperiment for stable

- system is on limit-cycle $\omega = \omega^*, a = a^*$
- at t_0 disturbance $\Rightarrow a \to a^+ > a^*$
- $L(j\omega)$ does not encircle $\mathrm{DF}^{-1}(a^+) \Rightarrow$ stable $a \to a^*$

The curve $-DF(a)$ is a generalization of the point -1 which, according to Nyquist, may not be part of $L(j\omega)$ to avoid sustained harmonic oscillations.

14 Chaos Theory

Key Ideas:

- Period doubling
- self similarity
- sensitivity to ICs
- strange attractors

Limit Set $x_\infty \in \mathbb{R}^n$ is limit point if there is a solution to

$$\frac{d}{dt}x(t) = f(x(t)), \quad x(0) \neq 0$$

that passes infintely many times arbitrarily close to x_∞.

The limit set of the point x_0 is the set of all limit points of the solutiion that start at $x(0) = 0$.

Closed, Bounded Region:
subset Ω of $\mathbb{R}^n$, Ω finite, $\partial\Omega \in \Omega$

14.1 Poincaré-Bendixson Theorem

System: time-invariant, 2nd order, CT, smooth, nonlinear

$$\frac{d}{dt}x_1(t) = f_1(x_1, x_2)$$

$$\frac{d}{dt}x_2(t) = f_2(x_1, x_2)$$

Theorem:
If L a limit set of system completely contained in Ω, L is either eq-point or periodic solution of system
$\Rightarrow$ Chaos is not possible.

Higher Order Systems:
Chaos is possible but not guaranteed.
(only necessary not sufficient)

Linear Systems

Linear Systems of any order can't have chaotic behaviour.

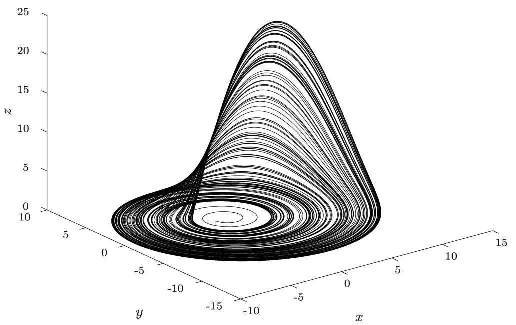
14.1.1 Rössler System

Simpelest Chaotic System.

$$\frac{d}{dt}x(t) = -y - z \tag{5}$$

$$\frac{d}{dt}y(t) = x + ay \tag{6}$$

$$\frac{d}{dt}z(t) = b + xz - cz \tag{7}$$



14.2 Time Variant Systems

A *time-variant* system with n states, can be extended by the state $\dot{t} = 1, t(0) = 0$ and will then become a *time-invariant* system of order $n+1$.
A second order time-variant system in particular can thus have chaotic solutions.

14.3 Limit Sets Extended

Chaotic Attractor (Strange Attractor):
The limit set is neither a equilibrium point nor a periodic solution. But they do not diverge to infinity. So the set is still a limit set.

Region of Attraction:
Onces this region is reached, it is never left.

Limit Cycle:
If a non linear system starts sufficiently close to the limit cycle, will orbit around that limit cycle.

14.4 Discrete Systems

Discrete Systems can have chaos at any order.

Logistics Equation:

$$x_{k+1} = f(x_k) = \mu \cdot x_k(1 - x_k), \quad \mu \in [1, 4]$$

Equilibria:

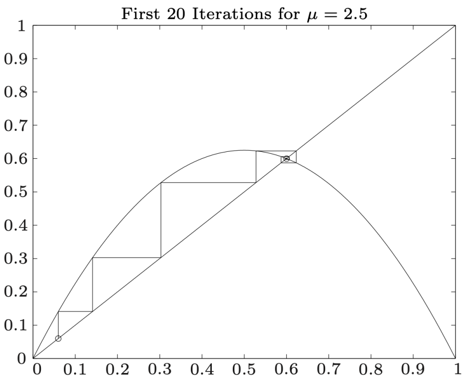
$$x_0 = \mu \cdot x_0(1 - x_0) \Rightarrow x_0 = \left\{0, 1 - \frac{1}{\mu}\right\}$$

Equilibrium Point is Asymptotically Stable if $|\frac{\partial f}{\partial x}(x_0)| < 1$. **Derivatives at Equilibria:**

$$\frac{d}{dx}f(x)|_{x_0} = \begin{cases} \mu & x_0 = 0 \\ 2 - \mu & x_0 = 1 - 1/\mu \end{cases}$$

- $x_0 = 0$ unstable for $\mu \in (1, r]$
- $x_0 = 1 - 1/\mu$ astable for $\mu \in (1, 3)$

Stability



Periodic Orbit

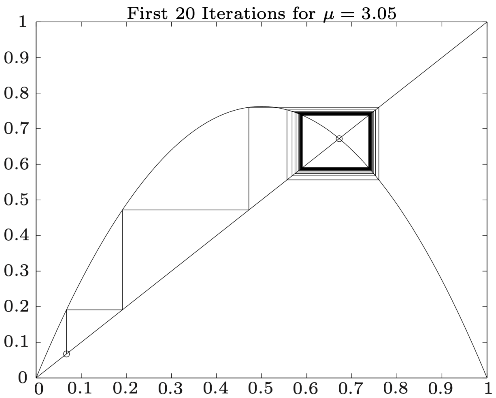
Jumps between 2 points

$$x_{0,3} = \left(1 + 1/\mu + \sqrt{1 - 2/\mu - 3/\mu^2}\right)/2$$

$$x_{0,4} = \left(1 + 1/\mu - \sqrt{1 - 2/\mu - 3/\mu^2}\right)/2$$

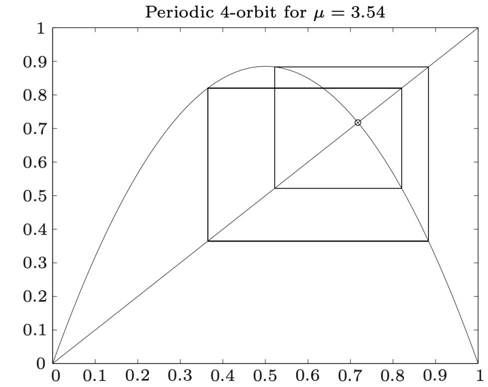
so long as

$$\mu < \mu_2 \approx 3.4494897\ldots$$



4 Point Orbit

$$\mu_2 < \mu < \mu_\infty \approx 3.5699456\ldots$$

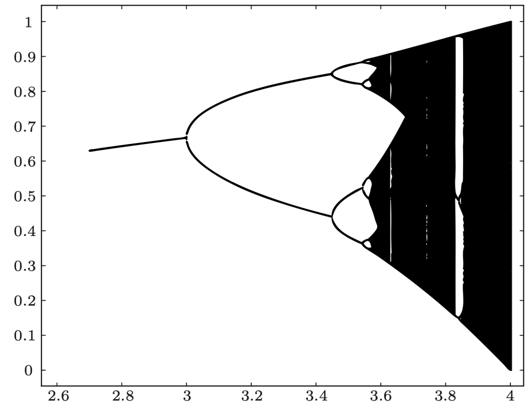


This can go on and on until we get a periodicity of ∞.

Infinite Period

$$\mu_\infty < \mu$$

Extremely complex behavior
There are values $\mu_\infty < \mu < 4$ with periodic orbits
Bifurcation Diagram



14.5 Phase Diagramm

The Phase diagramm of the follwing points are:

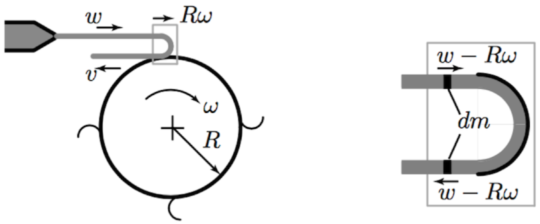
- Equilibrium: A single point
- Periodic Solution: Closed trajectory
- Quasi-Periodic Solution: Not a closed trajectory
- Chaotic Attractor: Fractall ftructure

15 Examples

15.1 Hydraulic Systems

15.1.1 Pelton Trubine

Conversion of Potential Energy (Pressure) to Kinetic Energy to Electric Energy via Momentum Exchange.



Change of Linear Momentum and Mass Element:

$$dB = 2(w - R\omega)dm, \qquad dm = \dot{V}\rho dt$$

Force Balance:

$$F_T = \dot{B} = 2(w - \omega R)\rho\dot{V}$$

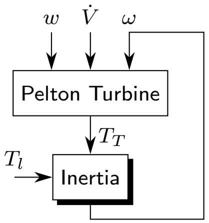
Torque:

$$T_T = 2\rho R\dot{V}(w - \omega R)$$

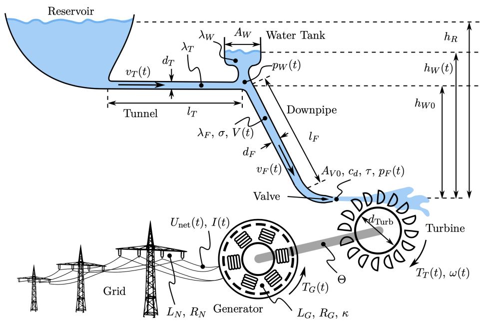
Power:

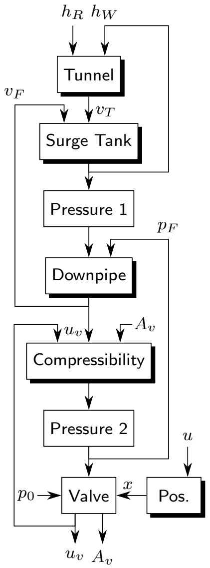
$$P_T = 2\rho R\dot{V}(w - \omega R)\omega$$

Quadratic: Max: $\omega = w/2R$, Min: $\omega = \{0, w/R\}$



15.1.2 Example: HEPP with Surge Tank





Tunnel

$$\frac{d}{dt}v_T = \frac{g(h_R - h_W)}{l_T} - \frac{\lambda_T}{2d_T}\text{sign}(v_T)v_T^2$$

Surge Tank

$$\frac{d}{dt}\bar{h}_W = \frac{v_T A_T - v_F A_F}{A_W}$$

$$h_W = \bar{h}_W + \lambda_W sign(\frac{d\bar{h}_W}{dt})(\frac{d\bar{h}_W}{dt})^2$$

Pressure 1

$$p_W = \rho g(h_W - h_{W0})$$

Downpipe

$$\frac{d}{dt}v_F = \left(\frac{p_W - p_F}{\rho l_F} + \frac{g h_{W0}}{l_F}\right) - \frac{\lambda_F}{2d_F}sign(v_F)v_F^2$$

Compressibility

$$\frac{dV}{dt} = v_F A_F - u_v A_v$$

$$p_F = \frac{V - V_0}{\sigma_0 V_0} + \rho g h_R$$

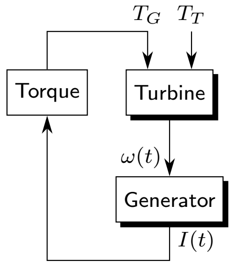
Valve

$$\frac{dx(t)}{dt} = u(t) \qquad A_v = A_{v,0}\left(1 - (1-x)^2\right)$$

$$u_v(t) = \begin{cases} c_d\sqrt{\frac{2(p_F - p_0)}{\rho}} & x > 0 \\ 0 & x = 0 \end{cases}$$

15.2 Electromechanic Systems

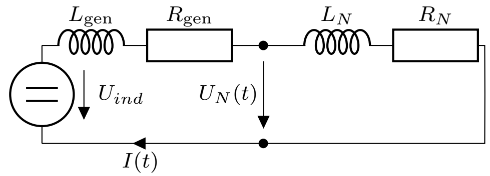
15.2.1 Example: Trubine Generator



Turbine (friction small)

$$\frac{d}{dt}\omega(t) = \frac{1}{\Theta}\left(T_T(t) - T_G(t)\right)$$

Grid Network

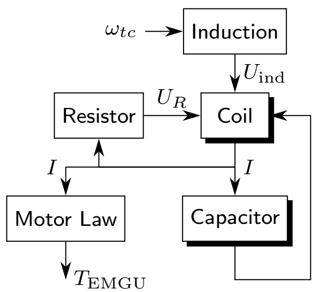


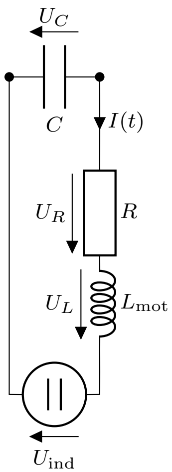
Generator & Grid

$$T_G(t) = \kappa I(t) \qquad \frac{d}{dt}I(t) = \frac{\kappa\omega(t) - R_{tot}I(t)}{L_{tot}}$$

$$\Rightarrow U_N(t) = L_N\frac{d}{dt}I(t) + R_N I(t)$$

15.2.2 Example: EMGU





$$\frac{d}{dt}I(t) = \frac{1}{L_{\text{mot}}}(U_C - U_{\text{ind}} - U_R)$$

$$\frac{d}{dt}U_C(t) = \frac{d}{dt}\frac{Q}{C} = \frac{-I(t)}{C} \qquad T_{\text{EMGU}} = \kappa_M I(t)$$

15.3 Rotational Gears:

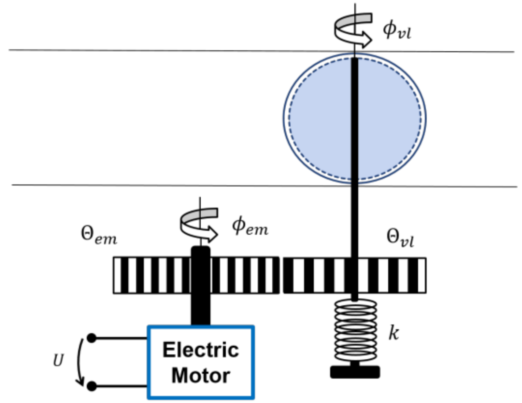
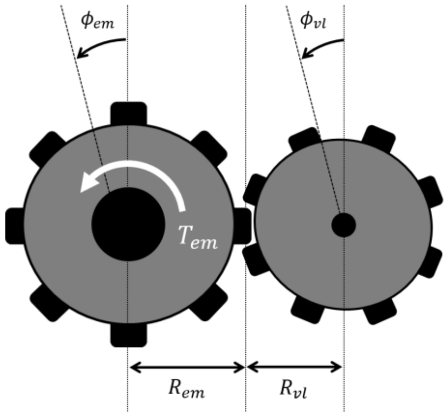


Figure 7: Blow-off valve system (lateral view).



Gears are connected: (same force and rotation)

$$R_{em}\cdot\omega_{em} = -R_{vl}\cdot\omega_{vl} \Rightarrow \omega_{em} = -\frac{R_{vl}}{R_{em}}\cdot\omega_{vl}$$

Equation left gear: (connection force F_{con})

$$\Theta_{em}\frac{d\omega_{em}}{dt} = T_{em} - R_{em}\cdot F_{con}$$

Equation right gear:

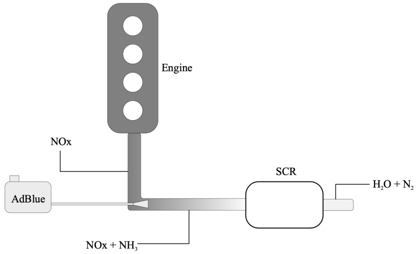
$$\Theta_{vl}\frac{d\omega_{vl}}{dt} = -k(\phi_0 - \phi_{vl}) - R_{vl}\cdot F_{con}$$

Together we get:

$$\frac{d\omega_{vl}}{dt} = \frac{-\frac{R_{vl}}{R_{em}}\cdot T_{em} - k(\phi_{vl} - \phi_0)}{\Theta_{vl} + \left(\frac{R_{vl}}{R_{cm}}\right)^2\Theta_{em}}$$

15.4 Chemical Systems

15.4.1 Example: SCR NOX System



Assumptions

- NO_x : only NO
- $[O_2] \gg [A_i] \Rightarrow$ rates independant of O_2
- Heat generated negligible

Chemical Reactions

Adsorption

$NH_3 \leftrightarrow NH_3^*$

Secondary Oxidization

$NH_3^* + \frac{3}{4}O_2 \rightarrow \frac{1}{2}N_2 + \frac{3}{2}H_2O$

Main SCR Reaction

$$NH_3^* + NO + \frac{1}{4}O_2 \rightarrow N_2 + \frac{3}{2}H_2O$$

Resevoir	Level Var.	Resevoir	Level Var.
Thermal	ϑ	NH_3	$[NH_3]$
NO	$[NO]$	NH_3^*	$[NH_3^*]$

Conservation Law

$$\frac{d}{dt}[NH_3^*] = r_{ads}[NH_3] - r_{des}[NH_3^*] - r_{SCR}[NO][NH_3^*] - r_{ox}[NH_3^*]$$

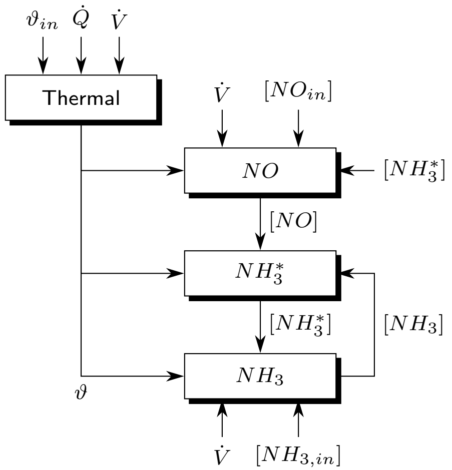
$$\frac{d}{dt}n_{NO} = \dot{V}[NO_{in}] - \dot{V}[NO] - V r_{SCR}[NO][NH_3^*]$$

$$\frac{d}{dt}[NO] = \frac{\dot{V}}{V}[NO_{in}] - \frac{\dot{V}}{V}[NO] - r_{SCR}[NO][NH_3^*]$$

Reaction Rates

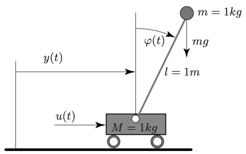
$$r_i = k_i e^{-E_i/R\vartheta}$$

Causality Diagram:



15.5 Mechanical Systems

15.5.1 Example: Inverted Pendulum



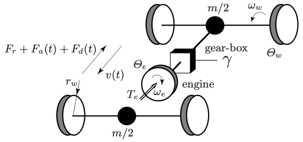
Assumptions

- no friction in pivot
- massless bar
- 2 rigid bodies $\Rightarrow$ 2 DOF

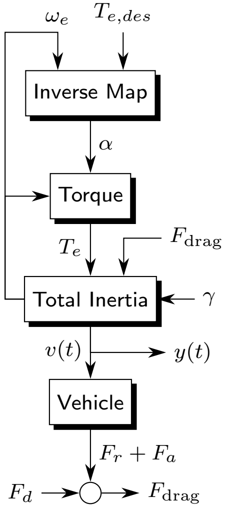
Lagrange Steps

1. Input: force on cart $u(t)$
   Output: angle of pendulum $\varphi(t)$
2. Coordinates:
   $$q_1 = y \quad q_2 = \varphi$$
3. Langrange Function (normalized):
   $$T_1(t) = \frac{\dot{y}^2}{2}, \quad U_1(t) = 0$$
   $$T_2(t) = \frac{\dot{y}^2}{2} + \cos(\varphi)\dot{\varphi}\dot{y} + \frac{\dot{\varphi}^2}{2}, \quad U_2(t) = \cos(\varphi)g$$
4. Dynamic Equations:
   $2\ddot{y} + \ddot{\varphi}\cos(\varphi) - \dot{\varphi}^2\sin(\varphi) = 0$
   $\ddot{\varphi} + \cos(\varphi)\ddot{y} - g\sin(\varphi) = 0$
   $$\ddot{y} = \frac{\dot{\varphi}^2\sin(\varphi) - g\cos(\varphi)\sin(\varphi) + u}{2 - \cos^2(\varphi)}$$
   $$\ddot{\varphi} = \frac{2g\sin(\varphi) - \dot{\varphi}^2\cos(\varphi)\sin(\varphi) - \cos(\varphi)u}{2 - \cos^2(\varphi)}$$

15.5.2 Example: Vehicle



- ω_e: Engine turn rate
- ω_w: Wheel turn rate
- r_w: Wheel radius
- γ: Gear-box ratio
- α: throttle position



Assumptions

- γ piecewise constant
- no wheel slip: $v_v = r_w\omega_w$
  no d-train slip: $\omega_w = \gamma\omega_e$
- must overcome F_{roll} & F_{aero}
- other forces lumped: F_d
- no potential energy

Step 1 Input: Engine torque T_e, **Output:** Signal $y \propto v_v(t)$
Step 2 Only **Reservoir** is kinetic energy
$\Rightarrow$ **Level Variable:** $v(t)$

$$E_{tot} = \frac{1}{2}mv^2 + 4\frac{1}{2}\Theta_w\omega_w^2 + \frac{1}{2}\Theta_e\omega_e^2 = \frac{1}{2}\left[m + \frac{4\Theta_w}{r_w^2} + \frac{\Theta_e}{\gamma^2 r_w^2}\right]v(t)^2$$

Step 3 $\frac{d}{dt}E_{tot} = P_+ - P_-$

$$P_+ = T_e(t)\omega_e(t) \qquad P_- = (F_r + F_a(t) + F_d(t))v(t)$$

Step 4 Rearrange step 3 to

$$M(\gamma, m)\frac{d}{dt}v(t) = \frac{T_e(t)}{r_w\gamma} - \left(c_r mg + \frac{1}{2}\rho c_w A v^2 + F_d(t)\right)$$

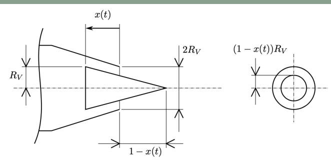
Gear-Ratio-Dependent Total Inertia:

$$M(\gamma, m) = \left[m + \frac{4\Theta_w}{r_w^2} + \frac{\Theta_e}{\gamma^2 r_w^2}\right]$$

Note T_e not technically input, rather $T_{e,des}$ mapped to throttle pos. α, (Nonlinear!). Dynamic effects small

15.6 Fluiddynmaic System

15.7 Example: Nozzle Valve



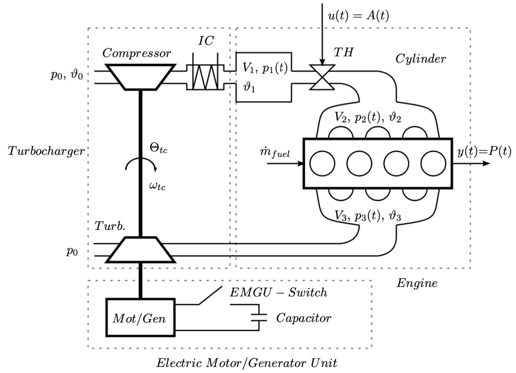
Input signal:
with stroke time $\tau = 1$

$$\frac{d}{dt}x = u$$

Flow Velocity Incomp!

$$u_v(t) = \begin{cases} c_d\sqrt{\frac{2(p_F - p_0)}{\rho}} & x > 0 \\ 0 & x = 0 \end{cases}$$

15.7.1 Example: TC-Engine



Engine

Modelled as 3 **Gas Receivers**

Resevoir	Level Var.
Volume IC to Throttle	p_1
Volume Throttle to Engine	p_2
Volume Engine to Turbine	p_3

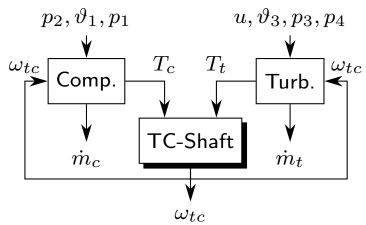
$$\frac{dp_1}{dt} = \frac{R\vartheta_1}{V_1}(\dot{m}_c - \dot{m}_{\text{thr}}), \quad \frac{dp_2}{dt} = \frac{R\vartheta_2}{V_2}(\dot{m}_{\text{thr}} - \dot{m}_{\text{cyl}})$$

$$\frac{dp_3}{dt} = \frac{R\vartheta_3}{V_3}(\dot{m}_{\text{cyl}} + \dot{m}_{\text{fuel}} - \dot{m}_{\text{t}})$$

Turbocharger

Only One Dynamic Element

Resevoir	Level Var.
Kinetic Energy	ω_{tc}



Kinetic Energy

$$\frac{d}{dt}\omega_{tc} = \frac{T_{\text{tot}}}{\Theta_{tc}} = \frac{(T_t - T_c)}{\Theta_{tc}}$$

16 Multiple Choice Questions

16.1 Modeling

16.1.1 Mechanical Systems

Allgemein

(1 point) A system can exhibit dynamic behaviour without having any reservoir.

- ☐ True.
- ☒ False.

Explanation: If a system does not have any reservoirs, no quantities (which we call "level variables") can be stored. Thus, all relationships are algebraic and no dynamics can occur.

Lagrange

(1 point) The choice of a minimal set of generalized coordinates (meaning that the number of generalized coordinates matches the number of degrees of freedom) is unique.

- ☐ True.
- ☒ False.

Explanation: A set of generalized coordinates is not unique. Its components only need to be independent, i.e., no component can be described by the others. The fact that the set is minimal still does not imply that the choice is unique.

(1 point) A constraint in the form of $f(\mathbf{q}, \dot{\mathbf{q}}) = 0$ is always non-holonomic.

- ☐ True.
- ☒ False.

Explanation: If the constraint in the form of $f(\mathbf{q}, \dot{\mathbf{q}}) = 0$ can be integrated such that $\tilde{f}(\mathbf{q}) = 0$, then the constraint is holonomic.

(1 point) In order to compute the kinetic energy of a body, the point chosen as reference must lie on that body.

- ☐ True
- ☒ False

Explanation: Although resulting in extremely cumbersome computations (coupling terms in the energies, time-dependent inertias, etc.) it is not forbidden to use points that are not part of body $\mathcal{B}$ as reference.

Statement	true	false
The Lagrange formalism can be used only for holonomic systems.		X
The minimum possible number of generalized coordinates equals the number of degrees of freedom.	X	
A point mass features a moment of inertia equal to zero (with respect to its center of gravity).	X	
x_1 and x_2, i.e. the x-coordinates of point masses m_1 and m_2 respectively, are a possible set of generalized coordinates for the excavator's arm (see Figure 2).		X

Pendel

(1 point) The amplitude of the small-angle oscillations is dependent on (answer each of the following statements)

Statement	true	false
the initial angle θ_0	X	
the length of the bar l		X
the mass m of the pendulum		X
gravity		X

(1 point) The period for small-angle oscillations is dependent on (answer each of the following statements)

Statement	true	false
the initial angle θ_0		X
the length of the bar l	X	
the mass m of the pendulum		X
gravity	X	

Explanations: The nonlinear system dynamics are: $\ddot{\theta}(t) = -\frac{g}{l}\sin\theta(t)$, with $\theta(t = 0) = \theta_0$. For small angles, the system can be linearized around $\theta = 0$, and the Laplace transform used to derive a dynamic equation in the Laplace domain in the form: $s^2\theta(s) + \frac{g}{l}\theta(s) = U(s)$. The corresponding transfer function is $\theta(s) = \frac{U(s)}{s^2+\frac{g}{l}}$, where the natural pulsation is $\omega_0 = \sqrt{\frac{g}{l}} = \frac{2\pi}{T_0}$. Thus the small-angle oscillations period is $T_0 = 2\pi\sqrt{\frac{l}{g}}$. Therefore the period is dependent on l, g, but not on initial conditions or mass.

16.1.2 Thermodynamical Systems

(1 point) You want to perfectly describe an insulated pipe through which water flows. What will be the nature of the resulting mathematical model for the input/output temperature behavior?

- ☐ It is a system of ordinary differential equations.
- ☒ It is a partial differential equation.
- ☐ It is a system of algebraic equations.

Explanation: The pure delay behavior resulting in this scenario can only be captured by a partial differential equation.

(1 point) Which of the following statements is **not** implied by the lumped parameter assumption?

- ☐ The thermodynamic states are assumed to be the same all over the receiver.
- ☒ $\overset{*}{m}_{\text{in}} = \overset{*}{m}_{\text{out}}$.
- ☐ $\vartheta_{\text{out}} = \vartheta(t)$, where $\vartheta(t)$ is the temperature in the receiver.

Explanation: In the lumped parameter assumption, the thermodynamic states (i.e., pressures, temperatures, composition, etc.) are assumed to be the same all over the receiver volume. It also requires that the outflowing gas has the same temperature as the gas inside the receiver. However, the assumption does not require that inflowing and outflowing mass flows are equal.

(1 point) Now assume that there is no outflow, i.e., $\overset{*}{m}_{\text{out}}(t) = 0\,\forall t$, and that the incoming mass flow $\overset{*}{m}_{\text{in}}$ and temperature ϑ_{in} are constant. The receiver could be modeled using either an adiabatic or an isothermal assumption. Mark the correct ending for the following sentence:

The rate at which pressure increases

- ☐ is identical for the isothermal and adiabatic assumptions.
- ☐ is faster for the isothermal assumption.
- ☒ is faster for the adiabatic assumption.

Explanation: One needs to compare the differential equations for the pressure $p(t)$. For the adiabatic assumption, the equation reads (remember, $\overset{*}{m}_{\text{out}} = 0$):

16.1.3 Electromechanical Systems

(1 points) The *brushless* DC electric motor has:

Statement	true	false
Permanent magnets on the rotor	X	
Mechanical commutation of the current in the rotor coil		X
Permanent magnets on the stator		X
Electrical commutation of the stator current	X	

16.1.4 Linear Systems

(1 point) Consider a linear, time-invariant, continuous-time and non-minimum phase system. Answer each of the following statements:

Statement	true	false
It has non-minimum phase zeros.	X	
The bandwidth of the closed-loop system with a stabilizing controller should be higher than the "fastest" non-minimum phase zero.		X
The bandwidth of the closed-loop system with a stabilizing controller should be smaller than the "slowest" non-minimum phase zero.	X	
The system's zero dynamics are unstable.	X	

(1 point) Consider a linear time-invariant continuous-time system. Answer each of the following statements:

Statement	true	false
The set of reachable and controllable states is identical.	X	
If it is completely reachable, it is also completely controllable.	X	
Whether a state is reachable or not depends on the initial conditions.		X
A completely controllable system can be brought to the origin only for some well chosen initial conditions.		X

(1 point) Answer each of the following statements:

Statement	true	false
If the equilibrium point of a linear time-invariant system is asymptotically stable, then it is always exponentially asymptotically stable.	X	
If the equilibrium point of a nonlinear system is asymptotically stable, then it is always exponentially asymptotically stable.		X

Statement	*True*	*False*
A normalization of the form $z_i = z_{i,0} \cdot x_i$ changes the stability characteristics of the system. (Here, z_i are the level variables, x_i their normalized counterparts, and $z_{i,0}$ denotes the normalization constants for $i = 1, \dots, n$, with n the order of the system.)		✗
If we want to carry out a balanced order reduction, it is important to normalize the system first, in order to be able to compare the magnitudes of the state variables among each other.	✗	
A linear time-invariant system of the form $\frac{d}{dt}x(t) = A \cdot x(t)$ can be Lyapunov unstable even though $\text{Re}(\lambda_i) \leq 0$ holds for all eigenvalues λ_i of the system matrix A.	✗	

Explanation: For the first two statements, please refer to the respective chapters in the lecture script. For the third statement: If there are eigenvalues with zero real part ($\text{Re}(\lambda_i) = 0$) and the matrix A is a cyclic or mixed matrix, then the system may be unstable even though no eigenvalues have a positive real part (see the discussion based on Jordan forms in the script).

16.1.5 Non Linear Systems

(1 point) Answer each of the following statements:

Statement	true	false
The specific type of stability of a certain equilibrium point x_e of a nonlinear system can always be deduced from the type of stability of the linearized system evaluated at x_e.		X
The stability of a nonlinear system around an equilibrium point can be studied using Lyapunov functions. If no Lyapunov function is found, you can conclude that the nonlinear system is unstable around x_e		X

Solution

1. See script page 142. If the linear system possesses eigenvalue(s) on the imaginary axis at a certain equilibrium point, then it is not possible to conclude about the type of stability of the corresponding nonlinear system at this equilibrium point.
2. See script page 146: "If no Lyapunov function is found, i.e., if a chosen Lyapunov function candidate turns out not to satisfy the conditions of the Lyapunov theorems, then no conclusion can be drawn. In this case, the system can still be stable or asymptotically stable.

Description: Consider a generic nonlinear system of the form $\dot{x} = f(x, t)$.

Q27 **(1 point)** Given $x \in \mathbb{R}^2$, the system can have chaotic solutions.

- ☒ True.
- ☐ False.

Explanation: A time-varying system can be represented in a time-invariant form by adding the state $\dot{t} = 1$ with $t(0) = 0$. Therefore, a 2-dimensional time-varying system would result in a 3-dimensional time-invariant system which, according to Poincaré-Benixson theorem, can indeed have chaotic solutions.

Q28 **(1 point)** Given $x \in \mathbb{R}^3$, the system has periodic solutions.

- ☐ True.
- ☒ False.

Explanation: Periodic solutions **can** occur in time-invariant systems with more than 2 dimensions. However, this condition is only necessary. Consider, e.g., $\dot{x} = [1,\ 1,\ 1]^\top$ with $x \in \mathbb{R}^3$, resulting in the non-periodic solution $x(t) = x(0) + [1,\ 1,\ 1]^\top \cdot t$.

Description: Consider a generic nonlinear time-invariant system of the form $\dot{x} = f(x)$.

Q29 **(1 point)** If an equilibrium of the system is locally attractive then it is also stable.

- ☐ True.
- ☒ False.

Explanation: For nonlinear systems, local attractiveness does not imply stability.

Q30 **(1 point)** We can assess whether an equilibrium of the system is unstable using Lyapunov theorem.

- ☐ True.
- ☒ False.

Explanation: Lyapunov theorem provides **sufficient**, but **not necessary**, conditions to assess the stability of equilibria. Therefore, it cannot be used to assess the instability of equilibria. As an example, if the chosen Lyapunov function $V(x)$ does not fulfill $\dot{V}(x) \leq 0$, Lyapunov theorem does not provide any conclusion on the stability of the equilibrium $x^* = 0$.

Description: Consider a nonlinear time-invariant system of the form $\dot{x} = f(x)$ with equilibrium x^* and linearization around it $\delta\dot{x} = A\delta x$. We exclusively refer to the equilibrium of the nonlinear system with x^*, while denoting the equilibrium of the linearized system with δx^*.

Q31 **(1 point)** If x^* is stable, then δx^* is stable.

- ☒ True.
- ☐ False.

Explanation: Any equilibrium is either stable or unstable. If x^* is stable, then it is not unstable. Since δx^* unstable implies x^* unstable, δx^* cannot be unstable and it is therefore stable.

Q32 **(1 point)** If δx^* is unstable, then x^* is unstable.

- ☒ True.
- ☐ False.

Explanation: δx^* unstable implies x^* unstable.

Description: All questions below address a time-invariant smooth (differentiable) nonlinear system of the form $\dot{x} = f(x)$ with $x \in \Re^n$ and $1 < n < \infty$.

Q31 **(1 point)** Every periodic solution is a limit set.

- ☒ True.
- ☐ False.

Explanation: As explained in Section 5.3.2 of the lecture script (version HS19), periodic trajectories form limit sets.

Q32 **(1 point)** Every chaotic attractor is a limit set.

- ☒ True.
- ☐ False.

Explanation: As explained in Section 5.3.3 of the lecture script (version HS19), a chaotic attractor trajectory does not reach an equilibrium or a periodic solution, but does not diverge to infinity either, making it a limit set.

Q33 **(1 point)** If the system has an asymptotically stable isolated equilibrium point x_e it cannot have other limit sets.

- ☐ True.
- ☒ False.

Explanation: An asymptotically stable isolated equilibrium point does not prevent the system to have other limit sets.

You are still analyzing the previous system. In addition you know that $n = 2$ and that the system dynamics have a limit cycle $\partial\Gamma$ and a region of attraction Γ.

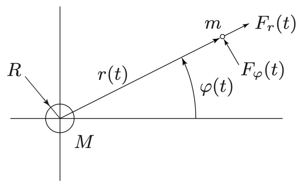
Q34 **(1 point)** Choose the correct statement.

- ☐ If the system starts at $x_0 \notin \Gamma$, $x(t)$ will never reach the limit cycle $\partial\Gamma$.
- ☐ There could be an equilibrium point x_e lying on the limit cycle $\partial\Gamma$.
- ☒ If the system starts at $x_0 \in \Gamma$, $x(t)$ never leaves the region Γ.

Explanation: If the system starts at $x_0 \notin \Gamma$ but sufficiently close to the limit cycle $\partial\Gamma$, the system can reach it, therefore the first option is wrong. To confute the second option we can think of what equilibrium point means: a point x_e that, once reached, is never left anymore. This clearly goes against the concept of limit cycle, which represents a set of points periodically reached ($n = 2$). Finally, the third option represents the definition of region of attraction, i.e., a region that once reached is never left anymore.

17 Geostationary Satellite

17.1 Problem Definition



- R: Radius Earth
- M: Mass Earth
- m: Mass Sat.
- r: Center of Earth → Sat.
- φ: orbit ∠ of Sat.

Assumptions

1. only **Earth-Sat.** System considered
2. $M \gg m \Rightarrow$ CG at center of Earth
3. Sat. always in Eq-Plane
4. Attitude controlled by other control system

17.2 Nonlinear Model

Lagrange Functions

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{r}}\right] - \frac{\partial L}{\partial r} = F_r \qquad \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\varphi}}\right] - \frac{\partial L}{\partial \varphi} = F_\varphi r$$

$$L = T - U$$

Kinetic Energy

$$T = \tfrac{1}{2} m \dot{r}^2 + \tfrac{1}{2} m (r\dot{\varphi})$$

Potential Energy

$$U = \int_R^r G\frac{Mm}{\rho^2} d\rho = GMm\left(\tfrac{1}{R} - \tfrac{1}{r}\right), \quad r > R$$

Minimum Velocities min speed to reach altitude r

$$v_0(r) = \sqrt{2GM\left(\tfrac{1}{R} - \tfrac{1}{r}\right)} \quad (E_{kin,0} = V(r))$$

escape velocity $v_\infty = \lim_{r\to\infty}$

$$v_\infty = \sqrt{\tfrac{2GM}{R}} \approx 1.12 \times 10^4 \text{m/s}$$

Lagrange System Dynamics

$$m\ddot{r} = mr\dot{\varphi}^2 - GMm\tfrac{1}{r^2} + F_r$$

$$mr^2\ddot{\varphi} = -2mr\dot{\varphi}\dot{r} + F_\varphi r$$

$$\Rightarrow \ddot{r} = r\dot{\varphi}^2 - GM\tfrac{1}{r^2} + u_r$$

$$\ddot{\varphi} = -2\dot{\varphi}\dot{r}\tfrac{1}{r} + \tfrac{1}{r}u_\varphi$$

Geostationary Orbit

$u_r = 0$	$\ddot{r} = 0$	$\dot{r} = 0$	$r = r_0$
$u_\varphi - 0$	$\ddot{\varphi} = 0$	$\dot{\varphi} = 0$	$\varphi = \omega_0 t$

Sidereal Angular Velocity

$$\omega_0 = \frac{2\pi \text{rad}}{86\,144\,\text{s}} \approx 7.29 \times 10^{-5} \text{rad/s}$$

Geostationary Radius

$$r_0 = \left(\frac{GM}{\omega_0^2}\right)^{1/3}$$

17.3 Systems Analysis

State-Space

$$x = \begin{bmatrix} r & \dot{r} & \varphi & \dot{\varphi}\end{bmatrix}^\top, \quad u = \begin{bmatrix} u_r & u_\varphi \end{bmatrix}^\top$$

$$\frac{d}{dt}x(t) = f(x(t), u(t)) = \begin{bmatrix} x_2 \\ x_1 x_4^2 - \frac{GM}{x_1^2} + u_1 \\ x_4 \\ -2x_2\frac{x_4}{x_1} + \frac{u_2}{x_1}\end{bmatrix}$$

$$y(t) = h(x(t)) = \begin{bmatrix} \frac{x_1}{r_0} \\ x_3 \end{bmatrix}$$

Nominal Orbit

$$x = \begin{bmatrix} r_0 & 0 & \omega_0 t & \omega_0 \end{bmatrix}^\top, \quad u = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$$

ACHTUNG Not eq-point! Periodic solution but same methodology works for linearization

Linearization

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega_0^2 & 0 & 0 & 2r_0\omega_0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-2\omega_0}{r_0} & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{r_0} \end{bmatrix}$$

$$C = \begin{bmatrix} \frac{1}{r_0} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \end{bmatrix}$$

Note In general the Matrices will be time dependent when linearized around eq-orbit ⇒ Special case here

Stability

$$\det(s\mathbb{I} - A) = \cdots = s^2(s^2 + \omega_0^2)$$

Roots: $\{0, 0, +j\omega_0, -j\omega_0\}$ ⇒ Oscillations with $f = \omega_0$
double root in origin ⇒ might be unstable
Rank: $\text{rank}(s\mathbb{I} - A)|_{s=0} = 3 \Rightarrow \rho = 1, r = 2 \Rightarrow A$ cyclic
⇒ **Linearized System Unstable!**

Controllability

$$\mathcal{R} = \begin{bmatrix} 0 & 0 & 1 & 0 & \dots \\ 1 & 0 & 0 & 2\omega_0 & \dots \\ 0 & 0 & 0 & \frac{1}{r_0} & \dots \\ 0 & \frac{1}{r_0} & \frac{-2\omega_0}{r_0} & 0 & \dots \end{bmatrix}$$

Completely Controllable
first 4 cols lin. ind.
$\det(\mathcal{R}) = -\frac{1}{r_0^2} \neq 0$

Radial Thruster Failure $B_2 = [0, 0, 0, 1/r_0]^\top$
⇒ Still Completely Controllable

Tangential Thruster Failure $B_1 = [0, 1, 0, 0]^\top$
⇒ **NOT** Completely Controllable

Observability

$$\mathcal{O} = \begin{bmatrix} \frac{1}{r_0} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{r_0} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Completely Observable
first 4 rows lin. ind.

Radial Sensor Failure $C_2 = [0, 0, 1, 0]^\top$
⇒ Still Completely Observable
Tangential Sensor Failure $C_1 = [\frac{1}{r_0}, 0, 0, 0]^\top$
⇒ **NOT** Completely Observable
Transfer Function

$$P(s) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = C(s\mathbb{I} - A)^{-1}B = \frac{C\text{Adj}(s\mathbb{I} - A)B}{\det(s\mathbb{I} - A)}$$

$$P(s) = \begin{bmatrix} \frac{1}{r_0(s^2+\omega_0^2)} & \frac{2\omega_0}{r_0 s(s^2+\omega_0^2)} \\ \frac{-2\omega_0}{r_0(s^2+\omega_0^2)} & \frac{s^2 - 3\omega_0^2}{r_0 s^2(s^2+\omega_0^2)} \end{bmatrix}$$

- TF shows linearized system unstable
- Completely controllable & observable with tangential thruster & sensor working
  ⇒ P_{22} is only one that has pole/zero cancellations
- even if system is stabilizable with only tangential thruster & sensor, it is difficult. Corresponding SISO TF P_{22} has NMP-zero (opposite direction at start) at $\sqrt{3}\omega_0$ (which limits attainable crossover freq.)