Exercise 8: Viscoelasticity

Review:

Simple models to describe viscoelastic[1] materials (this is a material that exhibits both *viscous* and *elastic* behaviour) are the **Maxwell** model and the **Kelvin–Voigt** model. Here, we make use of elements called *springs* and *dashpots*: the spring (think of an ordinary metal spring) stretches instantly under stress and will hold the load – when the load is removed, the spring will recover. The dashpot (also sometimes called a damper), is a plunger-like system that is submerged in a pot with a *Newtonian* fluid and moves at a rate that is proportional to the stress. When you remove the stress however, it does not recover.

In the **Maxwell** model, we conider a spring (with modulus E) and a dashpot (of viscocity η), which are arranged in **series**. The stress and strain in the Maxwell element is related by a first-order differential equation: both elements are subjected to the same stress ($\sigma_{spring} = \sigma_{dashpot}$), but are allowed an independent strain! We can write the total strain as

$$\varepsilon_{total} = \varepsilon_{spring} + \varepsilon_{dashpot} \ .$$

For both stress and strain, we can solve differential equation using appropriate initial conditions. We can describe the behavior of the *Maxwell* model with

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} \ .$$

On the other hand, the **Kelvin–Voigt** model has a spring (with modulus E) which is arranged in **parallel** to a dashpot (viscocity η). The stresses and strains in this model are also related by a first-order differential equation. We assume there is no bending between the elements, so both must be subjected to the same strain (unlike in the *Maxwell* model). We can also describe the beviour of the *Kelvin-Voigt* model with

$$\sigma = E \cdot \varepsilon + \eta \cdot \dot{\varepsilon} \ .$$

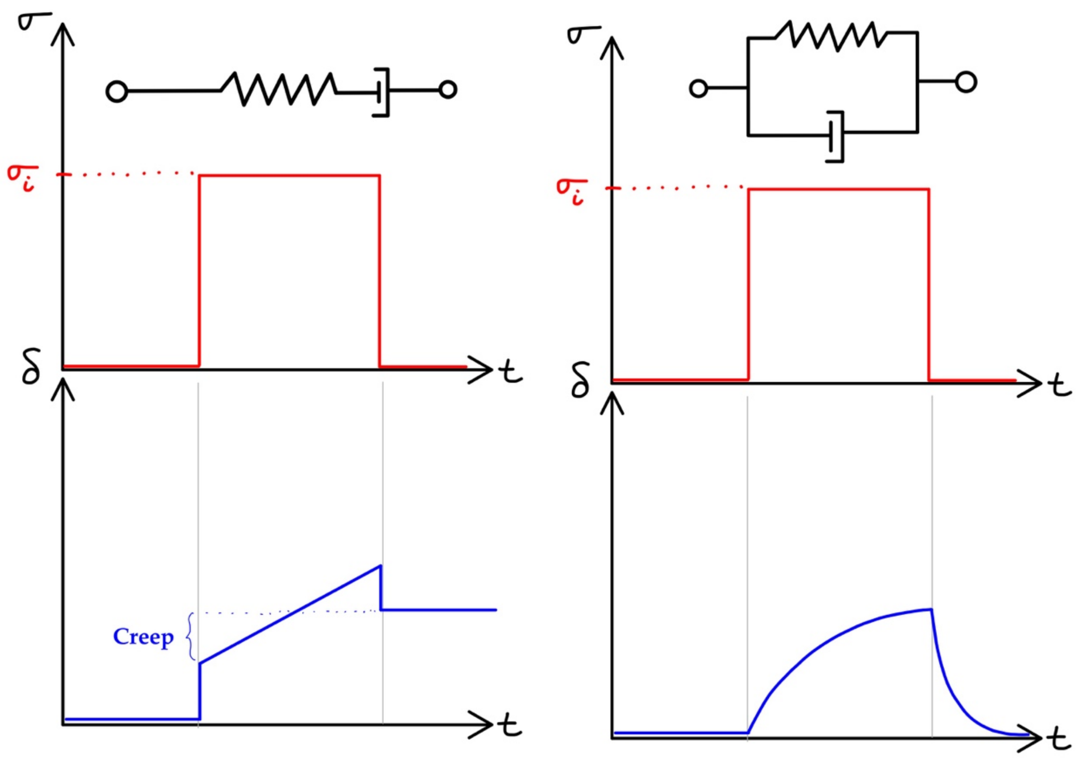
As we can see, both configurations (those of Maxwell and Kelvin-Voigt) will behave differently in creep and relaxation experiments. Often, both types of models are used together in a sort of serial and parallel configuration to model various types of viscoelastic behaviors.

[1] These materials respond with stress relaxation, creep and hysteresis.

1. **For an applied *step load* (instantaneously applied stress), sketch the deformation vs. time curves for the two material models below (*fig. 1*). What type of material behavior does this test characterize?**

Solution:
For step excitations of stress in the Maxwell model (left, the response represents the creeping behavior. Let's have a closer look: one component of strain will be instantaneous (this part is from the spring) and will relax when the stress is released later. Another component of the strain comes from the viscous element (and here it's the dashpot). This second, dash-pot related component, will grow with time as long as the stress is applied. Moreover, when the stress is removed, the dashpot will not return to its original position.

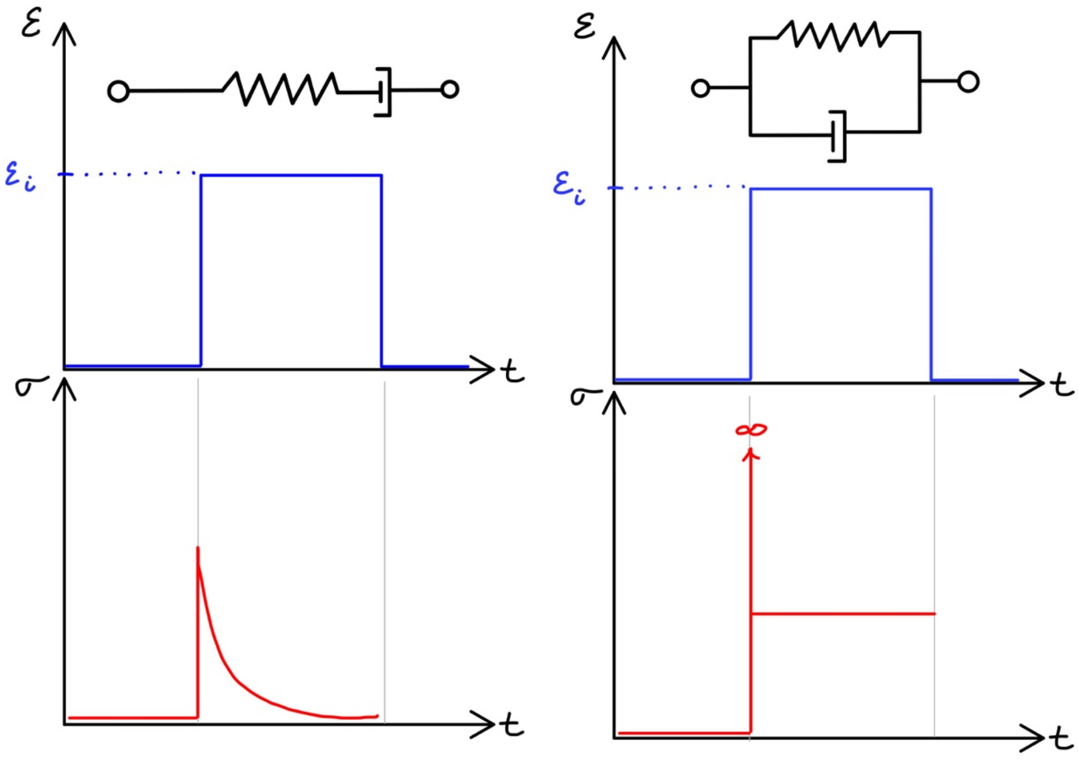
In the Voigt-Kelvin model, the strain response to a step excitation of stress reveals that the model describes viscoelastic solid behavior. Specifically, the material will deform at a decreasing rate and will asymptotically approach a steady-state strain. When released, the material will relax to its undeformed state. If a load σ is applied suddenly (like in our step excitation), the spring will want to deform right-away, but the system is held back by the dash-pot, which cannot react immediately due to its viscous "nature". This behavior can be observed in the plots below.



2. **For an applied *step deformation* (instantaneously applied displacement) sketch the stress vs. time curves for the two material models below (fig. 1). What type of material behavior does this test characterize?**

   Solution:
   In the Maxwell model (left), a step excitation of strain describes relaxation. Intuitively, this means that the stress required to hold the viscoelastic material at the constant strain will be found to decrease over time.

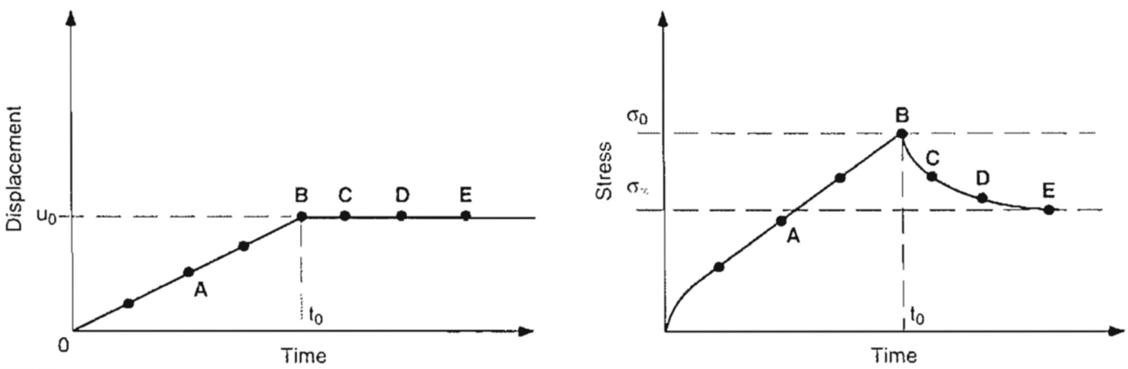
   In the Voigt-Kelvin model, a step-wise strain excitation leads to infinite stress. How does this all fit together? Well, if we want to have a constant, step strain excitation, we will need to apply an *infinite* (impulse, *dirac-delta-like*) stress to the system, since the dash-pot will not respond right away to a *finite* stress (the fluid-like character of the dashpot dominates the response for a step strain!).

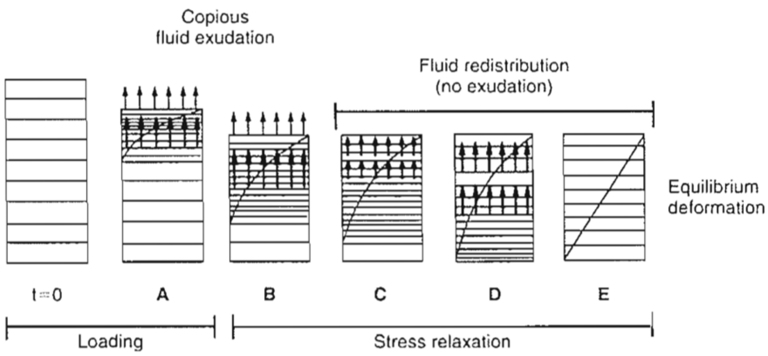


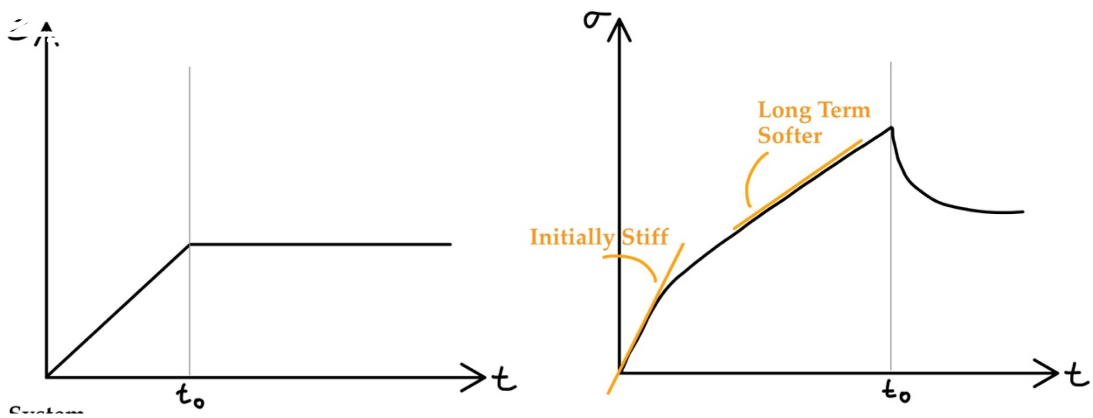
3. **Propose a spring-damper model (a configuration of springs and dampers) that can capture the effects of the tests on articular cartilage. Explain your choice of elements, and their contribution to the overall system response.**

Solution:

Given is the biphasic viscoelastic stress-relaxation response of articular cartilage in a one-dimensional compression experiment. A constant compression rate is applied to the tissue until $\varepsilon(t_0) = \varepsilon_{constant}$ is reached. Beyond t_0, the deformation is maintained. The stress response is shown on the right (notice how during the stress relaxation phase, the stress continuously decays along the curve B-C-D-E).







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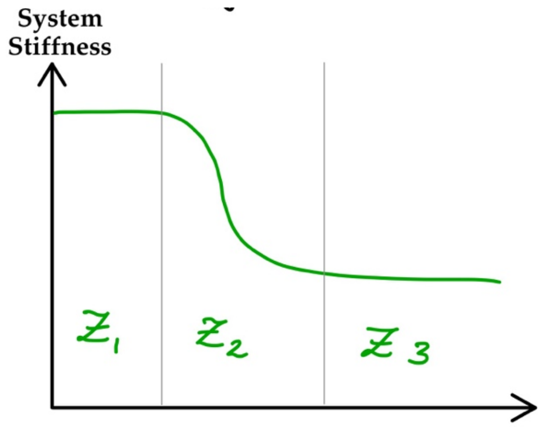
We can draw the system stiffness as being of the following form and divide it into three zones:



$\boldsymbol{Z_1}$: Zone 1 is the *Initially-Stiff Zone*: Here, there is low viscosity (high fluid flux) and a high elastic modulus (very stiff response as shown by the steep curve in the stress-time plot).
$\boldsymbol{Z_2}$: Zone 2 is the *Transition Zone*: Here, the viscosity is higher (shown by the slower flux) and a moderate elastic modulus (seen by the gentle shift from steep to *less*-steep curve).
$\boldsymbol{Z_2}$: Zone 3 is the *Long-Term Zone*: Here, there is no flux (or flow) left and a very low elastic modulus (long term softer).

We can model this type of behavior by introducing a system of *springs* and *dashpots* in the following way:

- For zone Z_1, we need to introduce a spring with a high elastic modulus (for the strong stiffness in the beginning) and a low-viscosity damper (so the deformation can take place quickly and is not delayed and does not lead to infinite stresses when displaced).
- For zone Z_2, we need to tweak these properties: for one, we need to have a spring with lower elastic modulus (to slowly lower the stiffness in a gradual way) as well as a damper with now higher viscosity (the fluid in the cartilage sample flows slower out).
- For zone Z_3 and beyond, we need to have a very low elastic modulus of the spring. Moreover, we will have no dashpot system, because as we reach constant displacement, $\varepsilon(t_0) = \varepsilon_{const}$, there is no copious fluid flux (the fluid in the tissue will merely redistribute). Intuitively, as we reach point B and *hold* the displacement, the dashpots from the previous two zones will shorten and reducing the stress. The steady-state stress for the compressed cartilage sample is then sustained by the spring of very low elastic modulus.