Definition 1.1.2 — Convex set. A set $S \subseteq \mathbb{R}^n$ is *convex* if $[p,q] \in S$ for all $p,\ q \in S$.

Definition 1.1.3 — Affine, conic and convex combination. Let $a_1, \ldots, a_k \in \mathbb{R}^n$. For $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, a linear combination $\sum_{i=1}^k \lambda_i a_i$ is called:

- *affine combination* combination if $\sum_{i=1}^k \lambda_i = 1$,
- *conic combination* if $\lambda_i \geq 0$ for all $i \in [k]$,
- *convex combination* if it is both affine and conic.

.

Definition 1.1.4 — Affine, conic, convex hull. Let $A \subseteq \mathbb{R}^n$. Then:

- *affine hull* of A *aff(A)* is a set of all finite affine combinations of A,
- *conic hull* of A *cone(A)* is a set of all finite conic combinations of A,
- *convex hull* of A *conv(A)* is a set of all finite convex combinations of A.

.

Definition 1.1.5 — Convex function. Let $X \subseteq \mathbb{R}^n$ be a convex set. A function $f : X \to \mathbb{R}$ is *convex* if for all $x, y \in X$ and $\lambda \in [0,1]$ it holds that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

If the above inequality is strict, the function is called *strictly convex*.

Moreover, if a function f is such that $-f$ is convex, strictly convex we call it *concave*, *strictly concave* function respectively.

Definition 1.2.1 — Hyperplane, half-space. An $(n-1)$-dimensional subspace $H = \{x \in \mathbb{R}^n : a^\top x = \beta\}$ for $a \in \mathbb{R}^n \setminus \{0\}$ and $\beta \in \mathbb{R}$ is called a *hyperplane*.

Moreover a hyperplane H defines a positive and negative *half-space* $H^+ = \{x \in \mathbb{R}^n : a^\top x \geq \beta\}$ and $H^- = \{x \in \mathbb{R}^n : a^\top x \leq \beta\}$, respectively.

Theorem 1.2.1 — Separating hyperplane theorem. Let $X,\ Y \in \mathbb{R}^n$ be two nonempty, disjoint convex sets. Then there exists $a \in \mathbb{R}^n$, $a \neq 0$ and $b \in \mathbb{R}$ such that:

- $a^\top x \leq b$ for all $x \in X$,
- $a^\top y \geq b$ for all $y \in Y$.

Definition 1.2.2 — Polyhedron, polytope. A *polyhedron* $P = \{x \in \mathbb{R}^n : \ Ax \leq b,$ for $A \in R^{m \times n},\ b \in R^n\}$ is a finite intersection of half-spaces. A bounded polyhedron is a *polytope*.

.

Definition 1.2.3 — Supporting hyperplane. For a given set $P \subseteq \mathbb{R}^n$ a hyperplane H is called a *supporting hyperplane* if $P \cap H \neq \varnothing$ and either $P \subseteq H^+$ or $P \subseteq H^-$.

.

Theorem 1.2.2 — Supporting hyperplane theorem. For any nonempty convex set $P \subseteq \mathbb{R}^n$ and a point x belonging to the boundary of the set P, there exists a supporting hyperplane containing x.

R There exists a partial converse of the Supporting hyperplane theorem, Theorem 1.2.2: If a closed, non-empty set $P \subseteq \mathbb{R}^n$ has a supporting hyperplane at every point x on its boundary, then P is convex.

Definition 1.2.4 The *dimension* of a polyhedron $P \subseteq \mathbb{R}^n$, $dim(P)$, is the dimension of its affine hull aff(P), i.e.,:

$$dim(P) := \min\{k \in \mathbb{Z}_{\geq 0} : \exists A \in \mathbb{R}^{n\times n},\ \text{rank}(A) = n-k, Ax = Ay,\ \text{for all } x, y \in P\}$$

.

Definition 1.2.5 — Face, vertex, edge, facet. Let $P \subseteq \mathbb{R}^n$ be a non-empty polyhedron, then:

1. a *face* of P is either P itself or the intersection of P with a supporting hyperplane,
2. a *vertex* of P is a 0-dimensional face of P,
3. an *edge* of P is a 1-dimensional face of P.
4. a *facet* of P is a $(dim(P)-1)$-dimensional face of P.

.

(R) Every face of a polyhedron P is also a polyhedron of dimension $dim(P)$ or less.

Proposition 1.2.3 — Vertex representation. Every polytope is the convex hull of its vertices.

(R) Note that the following converse holds: For every finite set $X \subseteq \mathbb{R}^n$, $conv(X)$ is a polytope.

Definition 1.2.6 Let $P \subseteq \mathbb{R}^n$ be a polyhedron. A point $r \in \mathbb{R}^n$ is a *ray* of P if and only if

$$\{x + \lambda r | \lambda \geq 0\} \subseteq P$$

for any point $x \in P$.

Moreover r is an *extreme ray* of P if it is not in a line segment between two distinct rays of P i.e., there are no distinct rays r_1, r_2 (i.e., $r_1 \neq \mu r_2$ for any $\mu > 0$) of P and $0 < \lambda < 1$ such that $r = \lambda r_1 + (1 - \lambda) r_2$.

Proposition 1.2.4 — Minkowski Resolution Theorem. Every polyhedron is a Minkowski sum of a convex hull of its vertices and conic hull of its extreme rays.

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Definition 1.3.1 — Cone. A set $K \subseteq \mathbb{R}^n$ is called a *cone* if for any $c \in K$ and $\lambda \in \mathbb{R}_{\geq 0}$ implies $\lambda c \in K$. Moreover a cone K is called:

- *convex cone* if it is convex,
- *solid cone* if it has non-empty interior,
- *pointed cone* if it contains no line, i.e., if $x, -x \in K \to x = 0$,
- *proper cone* if it is convex, solid and pointed,
- *polyhedral cone* if it is a polyhedron.

.

R Any polyhedral cone K admits an inequality representation of the following form $K = \{x \in \mathbb{R}^n : Ax \leq 0\}$ for some matrix $A \in \mathbb{R}^{m \times n}$. Moreover the converse is true. That is, for every matrix $A \in \mathbb{R}^{m \times n}$ the set $K = \{x \in \mathbb{R}^n : Ax \leq 0\}$ is a polyhderal cone.

Definition 1.3.2 — Dual cone. Let $K \subseteq \mathbb{R}^n$ be a cone. The *dual cone* of K is the set defined as

$$K^* := \{l \in \mathbb{R}^n : \langle l, x \rangle \geq 0, \text{ for all } x \in K\}.$$

If $K = K^*$, the cone K^* is called *self dual*.

.

Proposition 1.3.2 Let $K_1, K_2 \subseteq \mathbb{R}^n$ be cones. Then

1. K_1^* is closed and convex (even if K_1 is not convex),
2. $K_1 \subseteq K_2 \Rightarrow K_2^* \subseteq K_1^*$,
3. K_1 is solid $\Rightarrow K_1^*$ is pointed. If $\overline{K_1}$ is pointed and convex $\Rightarrow K_1^*$ is solid,
4. $K_1^{**} = \overline{conv(K)}$, if K_1 is closed and convex, $K_1 = K_1^{**}$,
5. If K_1 is proper, then K_1^* is proper.

Definition 1.4.1 — Linear Program (LP). Let $c \in \mathbb{R}^n$, $A, C, E \in \mathbb{R}^{m\times n}$ and $b, d, f \in \mathbb{R}^m$. An optimization problem is called a *linear program* in a *general/standard/canonical form* if it is of the following form:

$$\begin{array}{lll} \textit{General}: & \textit{Standard}: & \textit{Canonical}: \\ \min c^\top x & \min c^\top x & \max c^\top x \\ \quad Ax \ge b & \quad Ax = b & \quad Ax \le b \\ \quad Cx \le d & \quad x \ge 0 & \quad x \ge 0 \\ \quad Ex = f & & \end{array}$$

For a linear program one of the following three cases can happen.

- LP attains a finite optimum — LP is *feasible*,
- optimal solution of the LP is unbounded — LP is *unbounded*,
- LP has no feasible solutions at all — LP is *infeasible*.

$$\begin{aligned} \max \ & 2x_1 + 7x_2 \\ & 2x_1 + 7x_2 \leq 4 \\ & x_1 + 5x_2 \leq 7 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Definition 1.4.2 — Dual of an LP. Let $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. A *primal* and *dual* formulation of a linear program in a standard/canonical form is the following:

$$\begin{aligned} & Standard: & & & & Canonical: \\ & Primal: \min c^\top x & & Dual: \max b^\top y & & Primal: \max c^\top x & & Dual: \min b^\top y \\ & Ax = b & & c - A^\top y \geq 0 & & Ax \leq b & & A^\top y \geq c \\ & x \geq 0 & & & & x \geq 0 & & y \geq 0 \end{aligned}$$

Theorem 1.4.1 — Weak duality for linear programs. Let x, y be a feasible solution to the primal, dual formulation, respectively, of some linear program in a standard form. Then

$$c^\top x \geq b^\top y.$$

Theorem 1.4.2 — Strong duality for linear programs. [**PapadimitriouS82**] If both primal and dual are feasible, then at optimality their costs are equal. That is, there exist feasible solutions x^*, y^* for primal and dual respectively such that $c^\top x^* = b^\top y^*$.

Proposition 1.4.3 For a primal and a dual formulations of a linear program the following holds:

- Primal finite $\Rightarrow$ dual finite,
- Primal unbounded $\Rightarrow$ dual infeasible,
- Primal infeasible $\Rightarrow$ dual unbounded or infeasible,

and, vice versa, by the solution of Exercise 1.10.

Theorem 1.4.4 — Complementary slackness. Let x, y be feasible primal and dual solutions for linear program in the canonical form, respectively. Then x and y are optimal solutions if and only if

$$(b - Ax)^\top y = 0 \quad \text{and} \quad (A^\top y - c)^\top x = 0.$$