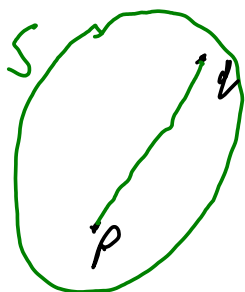
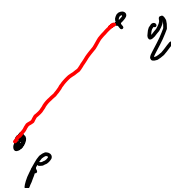


Definition 1.1.2 — Convex set. A set $S \subseteq \mathbb{R}^n$ is **convex** if $[p, q] \in S$ for all $p, q \in S$.



$$\{\lambda p + (1-\lambda)q \mid 0 \leq \lambda \leq 1\}$$



Proposition:

Let $S, T \subseteq \mathbb{R}^n$ be convex sets. Then

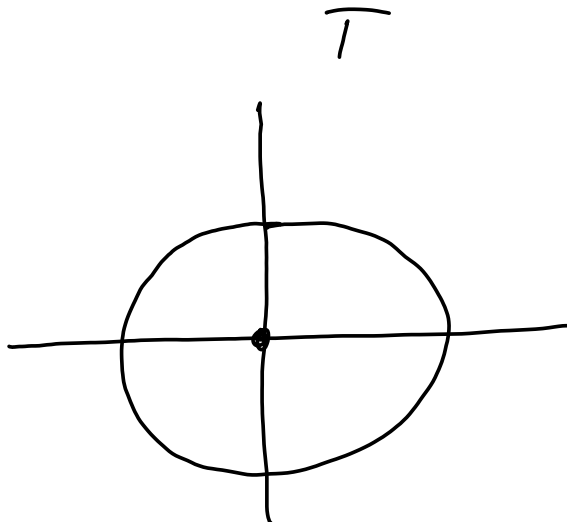
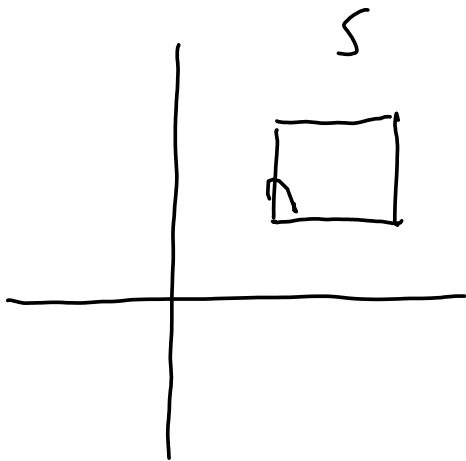
- $S \cap T$ is convex

- $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^m$

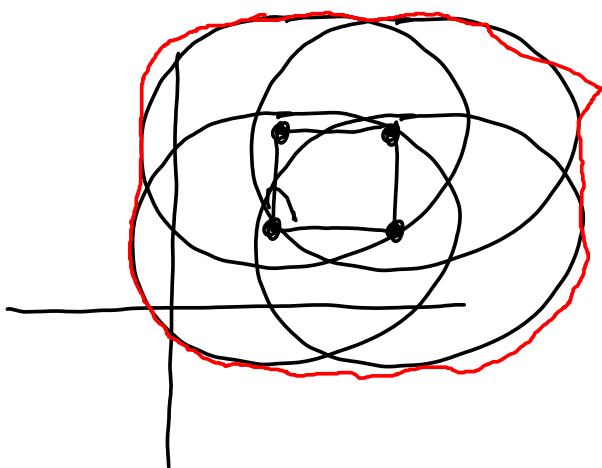
$\{Ax + b \mid x \in S\}$ is convex

- the **Minkowski sum** of S and T is convex

$$S \oplus T = S + T = \{x + y \mid x \in S, y \in T\}$$

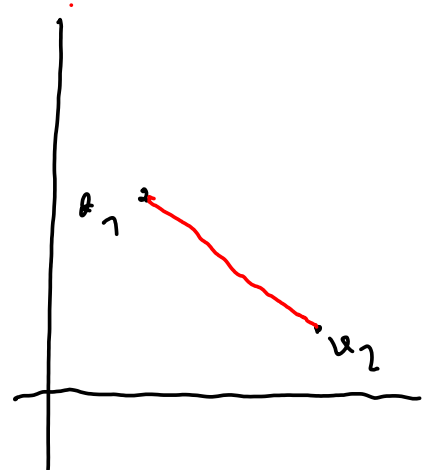
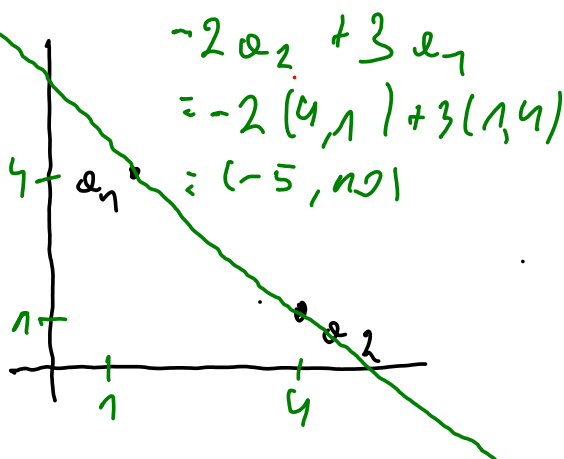


S $\otimes$ T



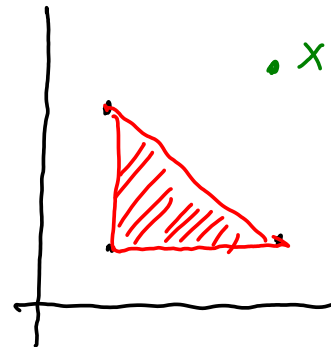
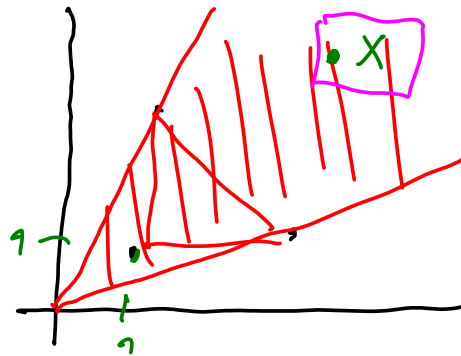
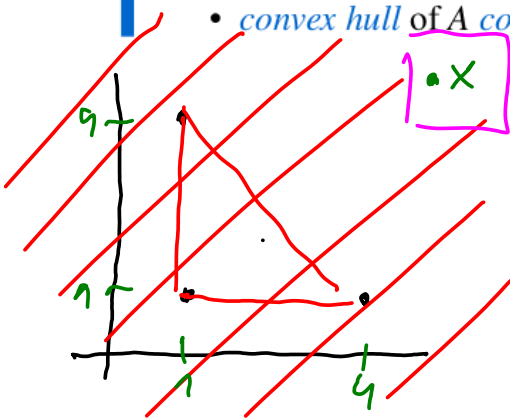
Definition 1.1.3 — Affine, conic and convex combination. Let $a_1, \dots, a_k \in \mathbb{R}^n$. For $\lambda_1, \dots, \lambda_k \in \mathbb{R}$, a linear combination $\sum_{i=1}^k \lambda_i a_i$ is called:

- *affine combination* if $\sum_{i=1}^k \lambda_i = 1$,
- *conic combination* if $\lambda_i \geq 0$ for all $i \in [k]$,
- *convex combination* if it is both affine and conic.



Definition 1.1.4 — Affine, conic, convex hull. Let $A \subseteq \mathbb{R}^n$. Then:

- *affine hull* of A $\text{aff}(A)$ is a set of all finite affine combinations of A ,
- *conic hull* of A $\text{cone}(A)$ is a set of all finite conic combinations of A ,
- *convex hull* of A $\text{conv}(A)$ is a set of all finite convex combinations of A .



$$X = -5(1, 1) + 3(1, 4) + 3(4, 1) = (10, 10)$$

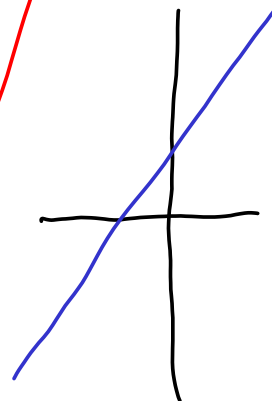
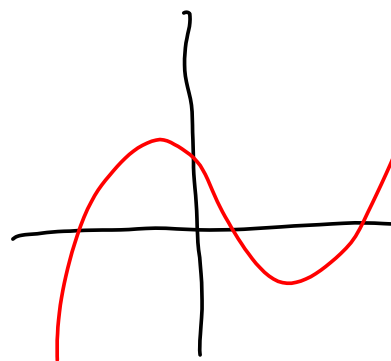
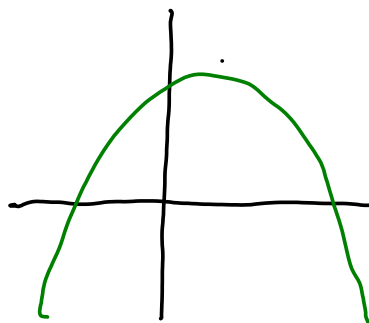
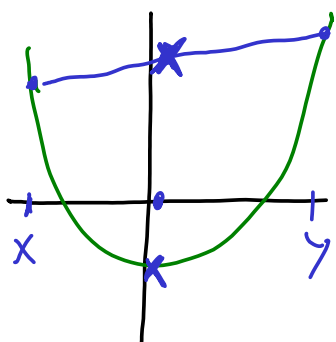
$$X = 10(1, 1)$$

Definition 1.1.5 — Convex function. Let $X \subseteq \mathbb{R}^n$ be a convex set. A function $f : X \rightarrow \mathbb{R}$ is *convex* if for all $x, y \in X$ and $\lambda \in [0, 1]$ it holds that

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

If the above inequality is strict, the function is called *strictly convex*.

Moreover, if a function f is such that $-f$ is convex, strictly convex we call it *concave*, *strictly concave* function respectively.



Proposition For $X \subseteq \mathbb{R}^n$ convex set

Let $f_1, \dots, f_s : X \rightarrow \mathbb{R}$ be convex functions

• $\sum_{i=1}^s \lambda_i f_i$ for $\lambda_i \in \mathbb{R}_{\geq 0}$ is convex

• $\max \{f_1, \dots, f_s\}$ is convex

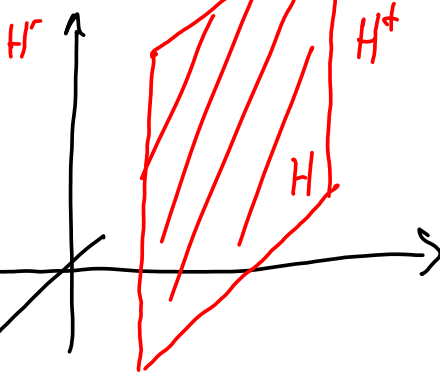
• $f_1(f_2(x))$ is convex if f_1 non decreasing

$$f''(x) = \underbrace{f_1''(f_2(x))}_{\geq 0} \underbrace{f_2'(x)^2}_{\geq 0} + \underbrace{f_1'(f_2(x))}_{\geq 0} \underbrace{f_2''(x)}_{\geq 0} \geq 0$$

Polyhedra

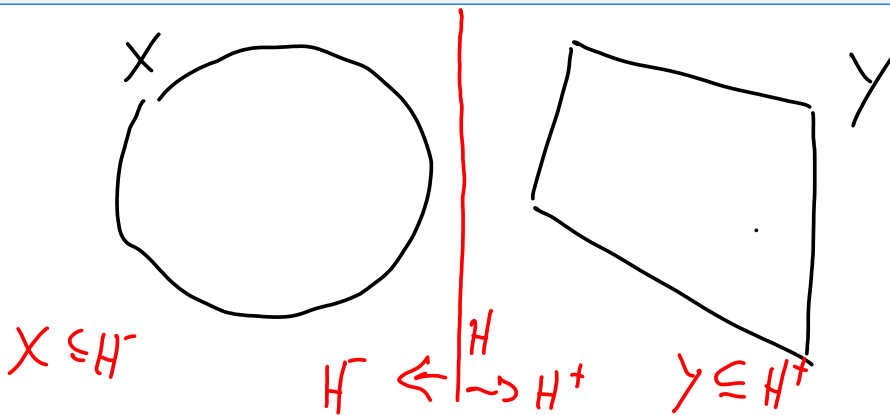
Definition 1.2.1 — Hyperplane, half-space. An $(n-1)$ -dimensional subspace $H = \{x \in \mathbb{R}^n : a^\top x = \beta\}$ for $a \in \mathbb{R}^n \setminus \{0\}$ and $\beta \in \mathbb{R}$ is called a *hyperplane*.

Moreover a hyperplane H defines a positive and negative *half-space* $H^+ = \{x \in \mathbb{R}^n : a^\top x \geq \beta\}$ and $H^- = \{x \in \mathbb{R}^n : a^\top x \leq \beta\}$, respectively.

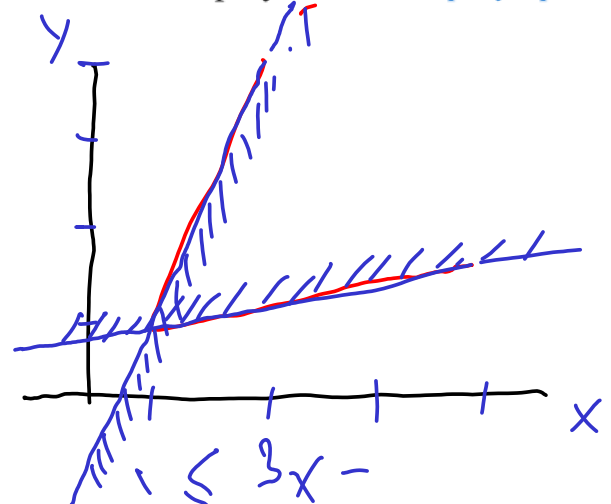
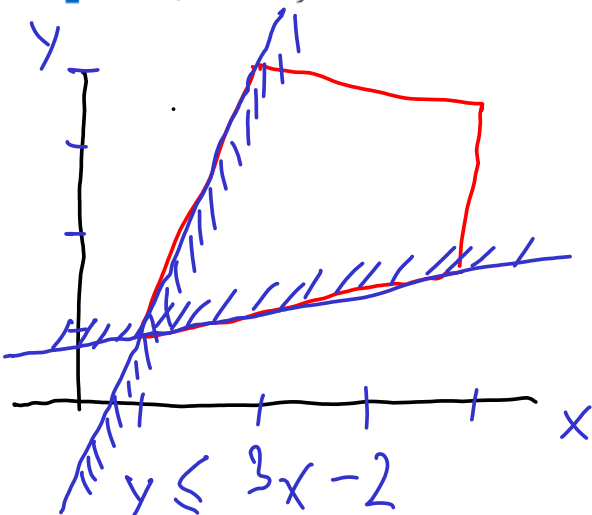


Theorem 1.2.1 — Separating hyperplane theorem. Let $X, Y \in \mathbb{R}^n$ be two nonempty, disjoint convex sets. Then there exists $a \in \mathbb{R}^n$, $a \neq 0$ and $b \in \mathbb{R}$ such that:

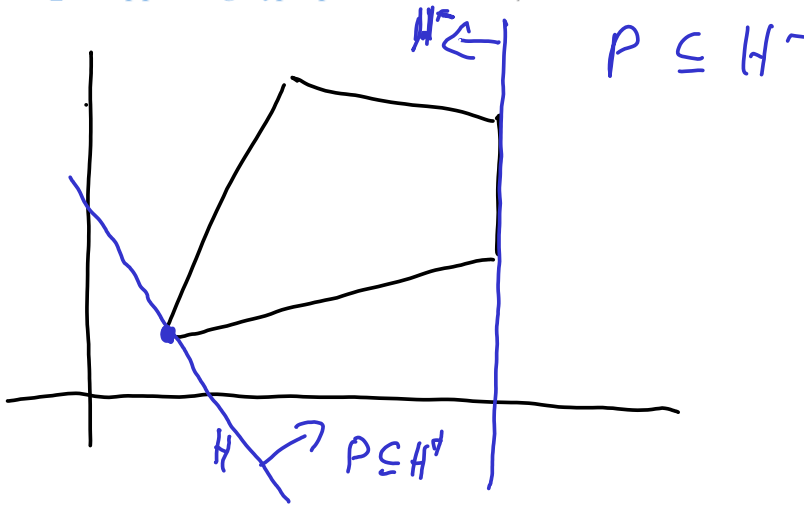
- $a^\top x \leq b$ for all $x \in X$,
- $a^\top y \geq b$ for all $y \in Y$.



Definition 1.2.2 — Polyhedron, polytope. A *polyhedron* $P = \{x \in \mathbb{R}^n : Ax \leq b, \text{ for } A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m\}$ is a finite intersection of half-spaces. A bounded polyhedron is a *polytope*.

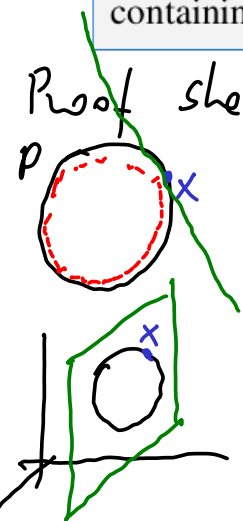


Definition 1.2.3 — Supporting hyperplane. For a given set $P \subseteq \mathbb{R}^n$ a hyperplane H is called a *supporting hyperplane* if $P \cap H \neq \emptyset$ and either $P \subseteq H^+$ or $P \subseteq H^-$.



Theorem 1.2.2 — Supporting hyperplane theorem. For any nonempty convex set $P \subseteq \mathbb{R}^n$ and a point x belonging to the boundary of the set P , there exists a supporting hyperplane containing x .

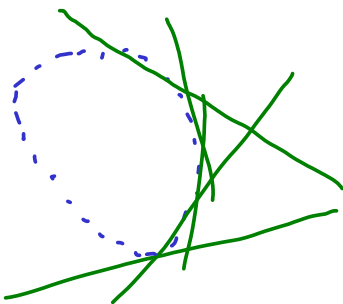
Proof sketch:



Case 1: Interior is non empty, We apply separating hyperplane theorem to set $\{x\}$ and interior of P

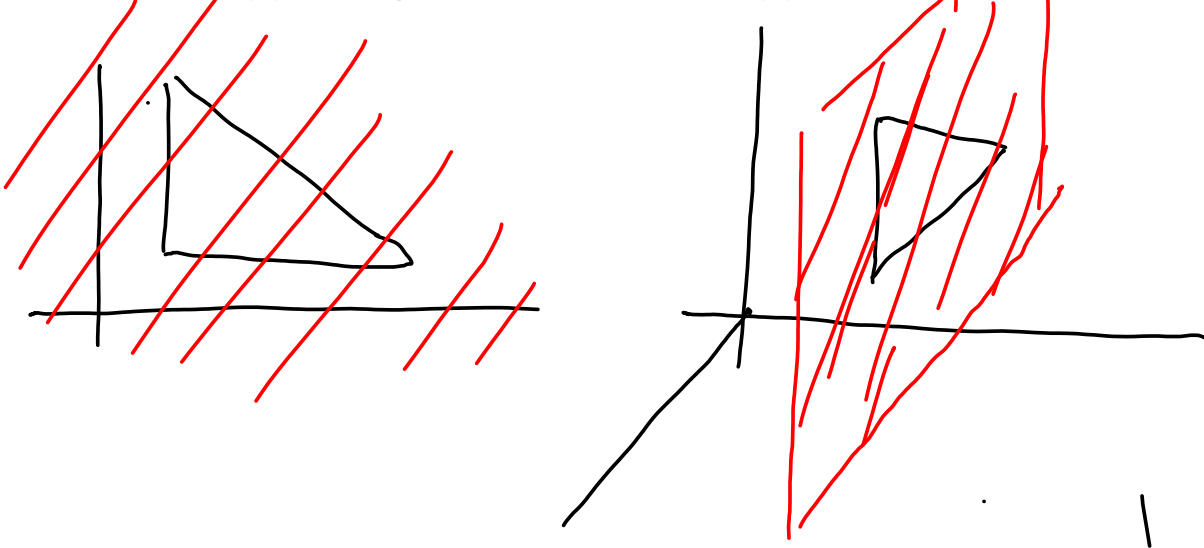
Case 2: Interior is empty $\Rightarrow$ it must lie in the affine set of dimension $n-1$ or less $\Rightarrow$ any hyperplane containing that affine set;

R There exists a partial converse of the Supporting hyperplane theorem, Theorem 1.2.2: If a closed, non-empty set $P \subseteq \mathbb{R}^n$ has a supporting hyperplane at every point x on its boundary, then P is convex.



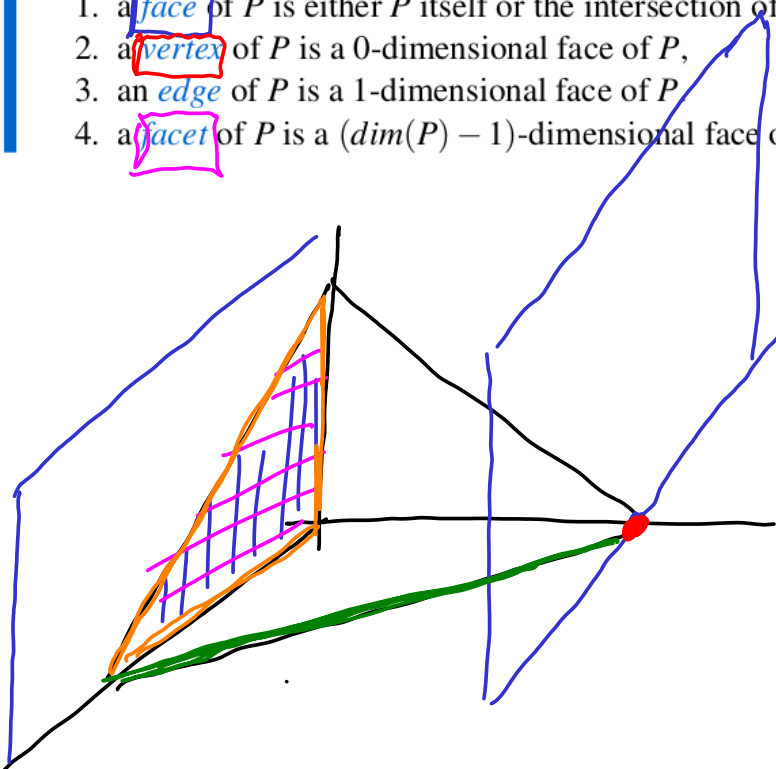
Definition 1.2.4 The *dimension* of a polyhedron $P \subseteq \mathbb{R}^n$, $\dim(P)$, is the dimension of its affine hull $\text{aff}(P)$, i.e.,:

$$\dim(P) := \min\{k \in \mathbb{Z}_{\geq 0} : \exists A \in \mathbb{R}^{n \times n}, \text{rank}(A) = n - k, Ax = Ay, \text{ for all } x, y \in P\}$$



Definition 1.2.5 — Face, vertex, edge, facet. Let $P \subseteq \mathbb{R}^n$ be a non-empty polyhedron, then:

1. a *face* of P is either P itself or the intersection of P with a supporting hyperplane,
2. a *vertex* of P is a 0-dimensional face of P ,
3. an *edge* of P is a 1-dimensional face of P ,
4. a *facet* of P is a $(\dim(P) - 1)$ -dimensional face of P .



Every face of a polyhedron P is also a polyhedron of dimension $\dim(P)$ or less.

Proposition 1.2.3 — Vertex representation. Every polytope is the convex hull of its vertices.

R Note that the following converse holds: For every finite set $X \subseteq \mathbb{R}^n$, $\text{conv}(X)$ is a polytope.

Definition 1.2.6 Let $P \subseteq \mathbb{R}^n$ be a polyhedron. A point $r \in \mathbb{R}^n$ is a **ray** of P if and only if

$$\{x + \lambda r \mid \lambda \geq 0\} \subseteq P$$

for any point $x \in P$.

Moreover r is an **extreme ray** of P if it is not in a line segment between two distinct rays of P i.e., there are no distinct rays r_1, r_2 (i.e., $r_1 \neq \mu r_2$ for any $\mu > 0$) of P and $0 < \lambda < 1$ such that $r = \lambda r_1 + (1 - \lambda)r_2$.

Proposition 1.2.4 — Minkowski Resolution Theorem. Every polyhedron is a Minkowski sum of a convex hull of its vertices and conic hull of its extreme rays.

Definition 1.3.1 — Cone. A set $K \subseteq \mathbb{R}^n$ is called a *cone* if for any $c \in K$ and $\lambda \in \mathbb{R}_{\geq 0}$ implies $\lambda c \in K$. Moreover a cone K is called:

- *convex cone* if it is convex,
- *solid cone* if it has non-empty interior,
- *pointed cone* if it contains no line, i.e., if $x, -x \in K \rightarrow x = 0$,
- *proper cone* if it is convex, solid and pointed,
- *polyhedral cone* if it is a polyhedron.

R Any polyhedral cone K admits an inequality representation of the following form $K = \{x \in \mathbb{R}^n : Ax \leq 0\}$ for some matrix $A \in \mathbb{R}^{m \times n}$. Moreover the converse is true. That is, for every matrix $A \in \mathbb{R}^{m \times n}$ the set $K = \{x \in \mathbb{R}^n : Ax \leq 0\}$ is a polyhedral cone.

Definition 1.3.2 — Dual cone. Let $K \subseteq \mathbb{R}^n$ be a cone. The *dual cone* of K is the set defined as

$$K^* := \{l \in \mathbb{R}^n : \langle l, x \rangle \geq 0, \text{ for all } x \in K\}.$$

If $K = K^*$, the cone K^* is called *self dual*.

Proposition 1.3.2 Let $K_1, K_2 \subseteq \mathbb{R}^n$ be cones. Then

1. K_1^* is closed and convex (even if K_1 is not convex),
2. $K_1 \subseteq K_2 \Rightarrow K_2^* \subseteq K_1^*$,
3. K_1 is solid $\Rightarrow K_1^*$ is pointed. If $\overline{K_1}$ is pointed and convex $\Rightarrow K_1^*$ is solid,
4. $K_1^{**} = \overline{\text{conv}(K)}$, if K_1 is closed and convex, $K_1 = K_1^{**}$,
5. If K_1 is proper, then K_1^* is proper.

Definition 1.4.1 — Linear Program (LP). Let $c \in \mathbb{R}^n$, $A, C, E \in \mathbb{R}^{m \times n}$ and $b, d, f \in \mathbb{R}^m$. An optimization problem is called a *linear program* in a *general/standard/canonical form* if it is of the following form:

General :

$$\begin{aligned} \min \quad & c^\top x \\ & Ax \geq b \\ & Cx \leq d \\ & Ex = f \end{aligned}$$

Standard :

$$\begin{aligned} \min \quad & c^\top x \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

Canonical :

$$\begin{aligned} \max \quad & c^\top x \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

For a linear program one of the following three cases can happen.

- LP attains a finite optimum — LP is *feasible*,
- optimal solution of the LP is unbounded — LP is *unbounded*,
- LP has no feasible solutions at all — LP is *infeasible*.

$$\begin{aligned}
\max \quad & 2x_1 + 7x_2 \\
\text{s.t.} \quad & 2x_1 + 7x_2 \leq 4 \\
& x_1 + 5x_2 \leq 7 \\
& x_1, x_2 \geq 0
\end{aligned}$$

Definition 1.4.2 — Dual of an LP. Let $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. A *primal* and *dual* formulation of a linear program in a standard/canonical form is the following:

Standard :

$$\begin{aligned}
\text{Primal : } \min c^\top x \quad & \text{Dual : } \max b^\top y \\
Ax = b \quad & c - A^\top y \geq 0 \\
x \geq 0 \quad &
\end{aligned}$$

Canonical :

$$\begin{aligned}
\text{Primal : } \max c^\top x \quad & \text{Dual : } \min b^\top y \\
Ax \leq b \quad & A^\top y \geq c \\
x \geq 0 \quad & y \geq 0
\end{aligned}$$

Theorem 1.4.1 — Weak duality for linear programs. Let x, y be a feasible solution to the primal, dual formulation, respectively, of some linear program in a standard form. Then

$$c^\top x \geq b^\top y.$$

Theorem 1.4.2 — Strong duality for linear programs. [PapadimitriouS82] If both primal and dual are feasible, then at optimality their costs are equal. That is, there exist feasible solutions x^*, y^* for primal and dual respectively such that $c^\top x^* = b^\top y^*$.

Proposition 1.4.3 For a primal and a dual formulations of a linear program the following holds:

- Primal finite $\Rightarrow$ dual finite,
- Primal unbounded $\Rightarrow$ dual infeasible,
- Primal infeasible $\Rightarrow$ dual unbounded or infeasible,

and, vice versa, by the solution of Exercise 1.10.

Theorem 1.4.4 — Complementary slackness. Let x, y be feasible primal and dual solutions for linear program in the canonical form, respectively. Then x and y are optimal solutions if and only if

$$(b - Ax)^{\top} y = 0 \quad \text{and} \quad (A^{\top} y - c)^{\top} x = 0.$$