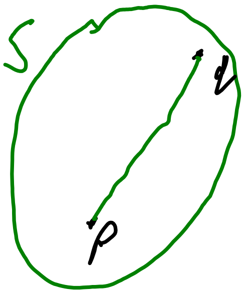
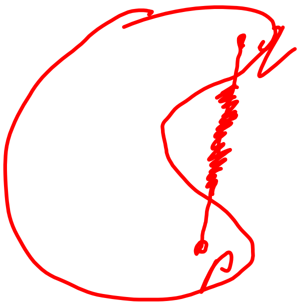
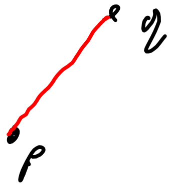
Definition 1.1.2 — Convex set. A set $S \subseteq \mathbb{R}^n$ is *convex* if $[p,q] \in S$ for all $p, q \in S$.





$\{\lambda p + (1-\lambda) q \mid 0 \leq \lambda \leq 1\}$



Proposition:

Let $S, T \subseteq \mathbb{R}^n$ be convex sets. Then

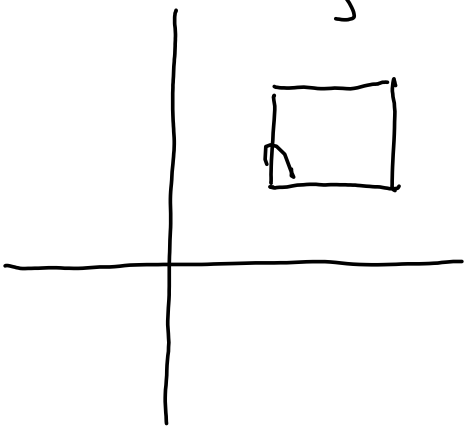
- $S \cap T$ is convex
- $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$:

  $\{Ax + b \mid x \in S\}$ is convex

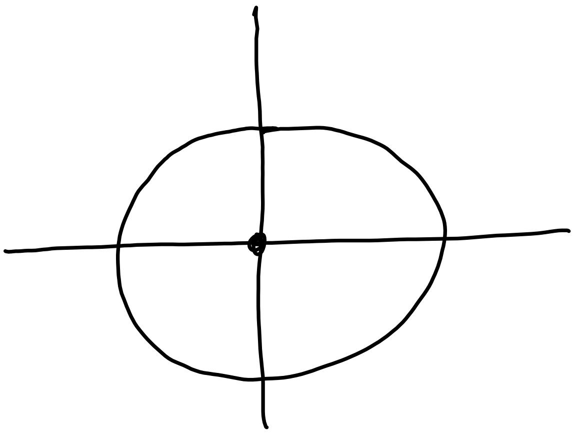
- the Minkowski sum of S and T is convex

$S \oplus T = S + T = \{x + y \mid x \in S, y \in T\}$

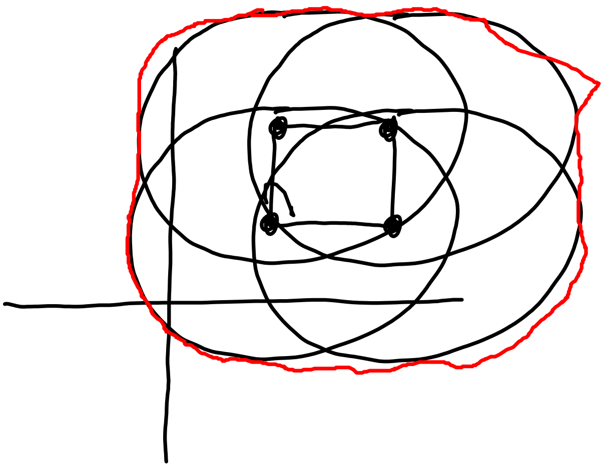
S



T

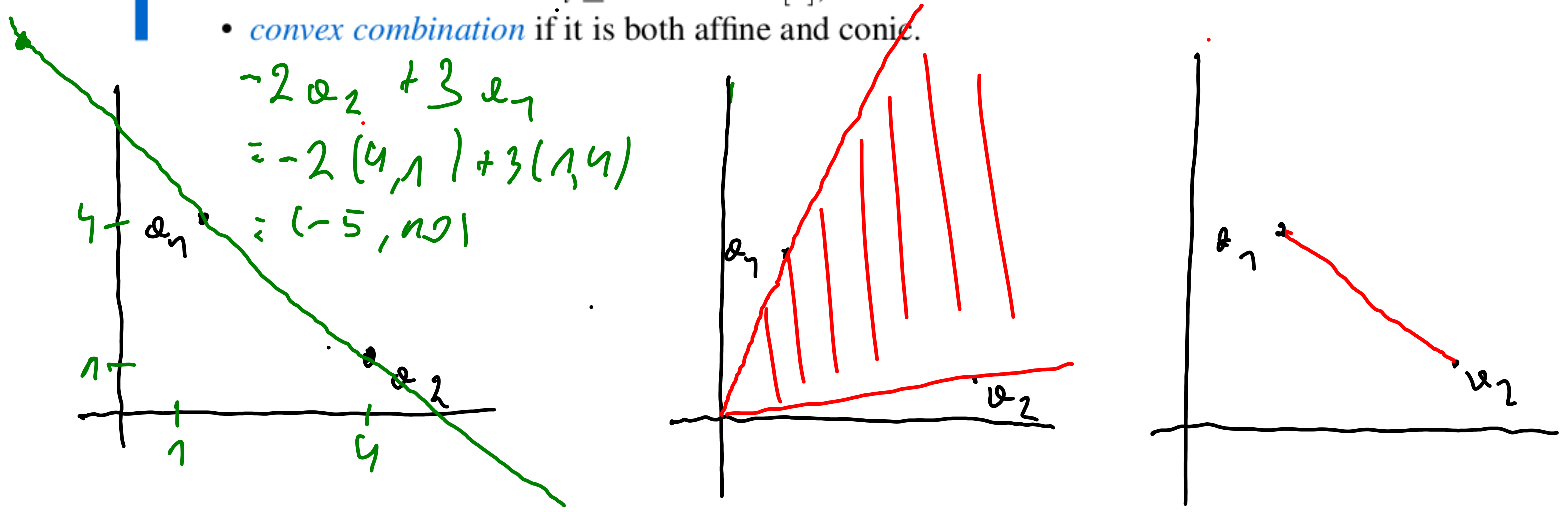


$S \oplus T$



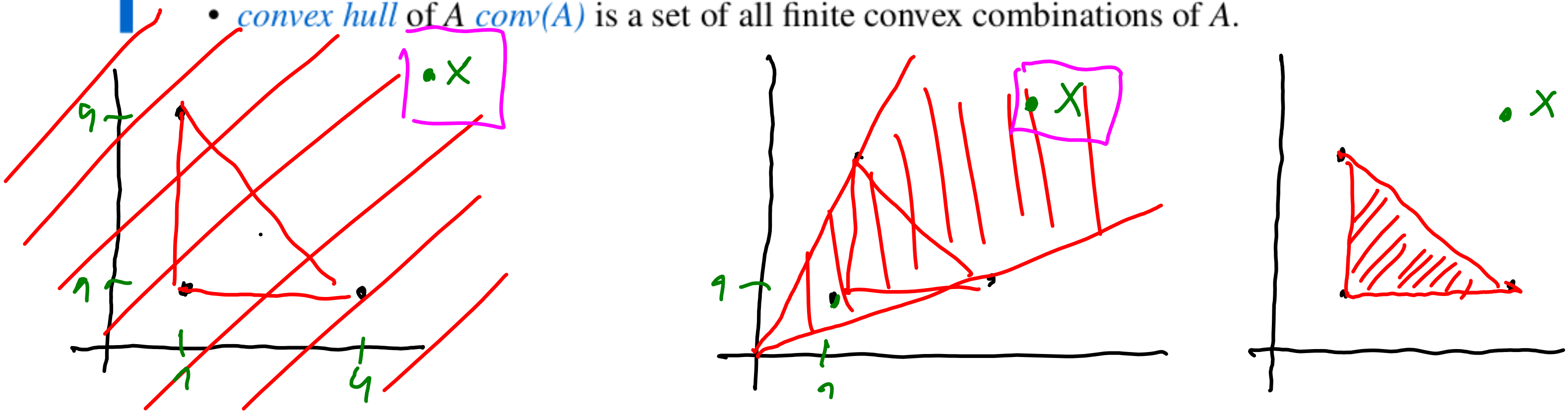
Definition 1.1.3 — Affine, conic and convex combination. Let $a_1, \ldots, a_k \in \mathbb{R}^n$. For $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, a linear combination $\boxed{\sum_{i=1}^k \lambda_i a_i}$ is called:

- *affine combination* combination if $\sum_{i=1}^k \lambda_i = 1$,
- *conic combination* if $\lambda_i \geq 0$ for all $i \in [k]$,
- *convex combination* if it is both affine and conic.



Definition 1.1.4 — Affine, conic, convex hull. Let $A \subseteq \mathbb{R}^n$. Then:

- *affine hull* of A *aff(A)* is a set of all finite affine combinations of A,
- *conic hull* of A *cone(A)* is a set of all finite conic combinations of A,
- *convex hull* of A *conv(A)* is a set of all finite convex combinations of A.



$x = \boxed{-5(1,1) + 3(1,4) + 3(4,1)}$
$= (10,10)$

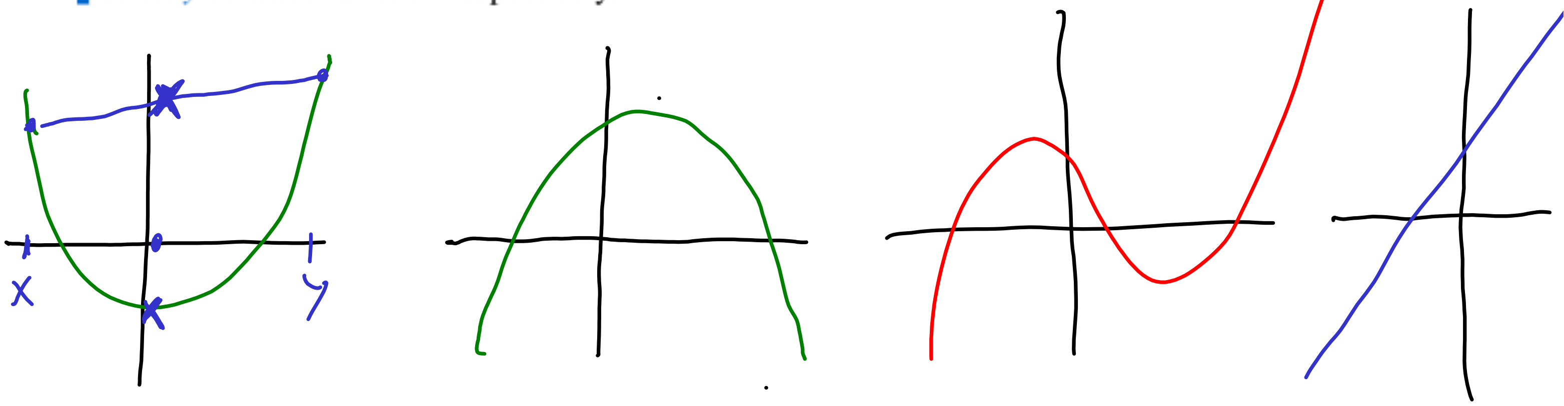
$x = \boxed{10(1,1)}$

Definition 1.1.5 — Convex function. Let $X \subseteq \mathbb{R}^n$ be a convex set. A function $f : X \to \mathbb{R}$ is *convex* if for all $\boxed{x, y \in X}$ and $\lambda \in [0,1]$ it holds that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

If the above inequality is strict, the function is called *strictly convex*.

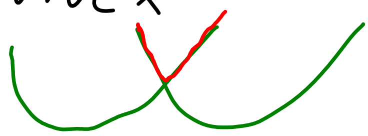
Moreover, if a function f is such that $-f$ is convex, strictly convex we call it *concave*, *strictly concave* function respectively.



Proposition For $X \subseteq \mathbb{R}^n$ convex set

Let $f_1, \dots f_s : X \to \mathbb{R}$ be convex functions

- $\sum_{i=1}^{s} \lambda_i f_i$ for $\lambda_i \in \mathbb{R}_{\geq 0}$ is convex
- $\max\{f_1 \dots f_s\}$ is convex



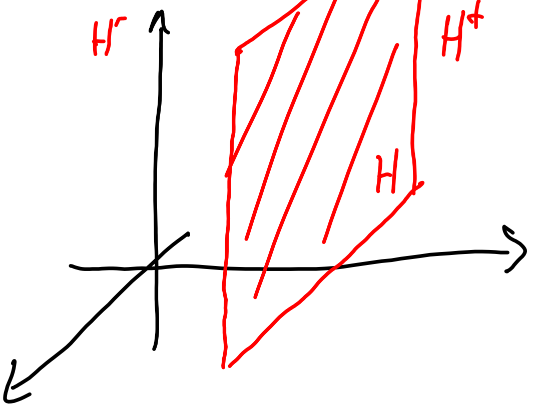
- $\underbrace{f_1(f_2(x))}_{f}$ is convex if f_1 non decreasing

$$f''(x) = \underbrace{f_1''(f_2(x))}_{\geq 0} \cdot \underbrace{f_2'(x)^2}_{\geq 0} + \underbrace{f_1'(f_2(x))}_{\geq 0} \underbrace{f_2''(x)}_{\geq 0} \geq 0$$

Polyhedra

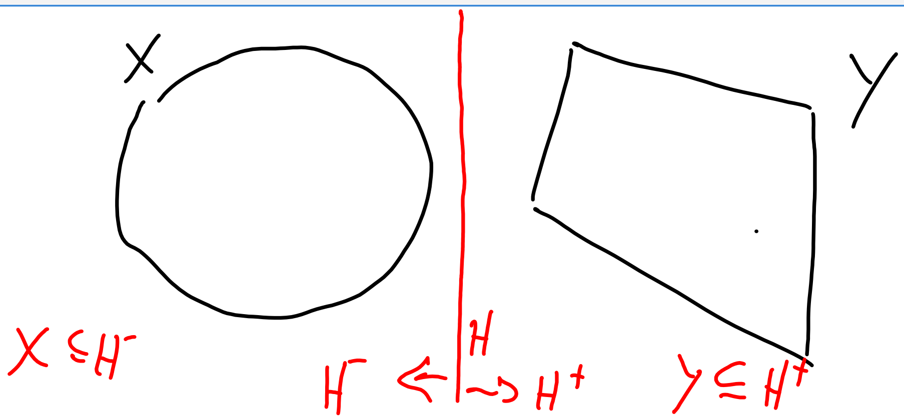
Definition 1.2.1 — Hyperplane, half-space. An $(n-1)$-dimensional subspace $H = \{x \in \mathbb{R}^n : a^\top x = \beta\}$ for $a \in \mathbb{R}^n \setminus \{0\}$ and $\beta \in \mathbb{R}$ is called a *hyperplane*.

Moreover a hyperplane H defines a positive and negative *half-space* $H^+ = \{x \in \mathbb{R}^n : a^\top x \geq \beta\}$ and $H^- = \{x \in \mathbb{R}^n : a^\top x \leq \beta\}$, respectively.

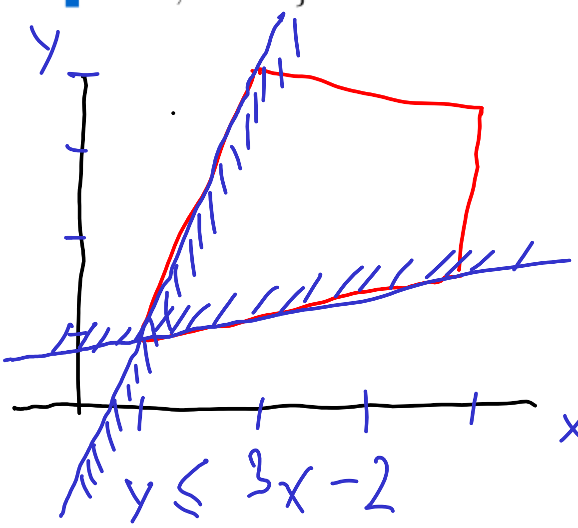


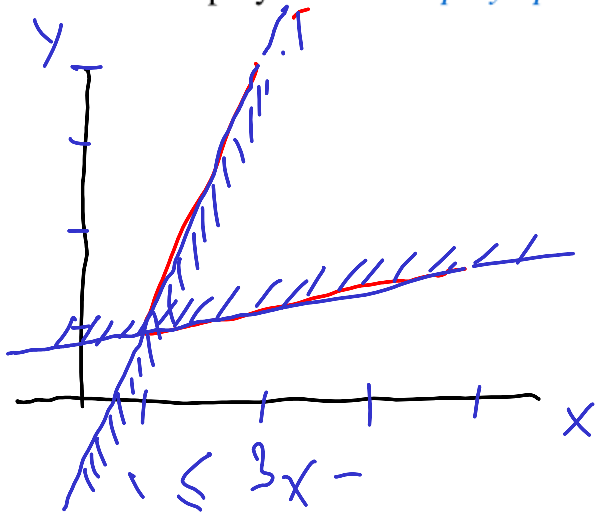
Theorem 1.2.1 — Separating hyperplane theorem. Let $X, Y \in \mathbb{R}^n$ be two nonempty, disjoint convex sets. Then there exists $a \in \mathbb{R}^n$, $a \neq 0$ and $b \in \mathbb{R}$ such that:

- $a^\top x \leq b$ for all $x \in X$,
- $a^\top y \geq b$ for all $y \in Y$.

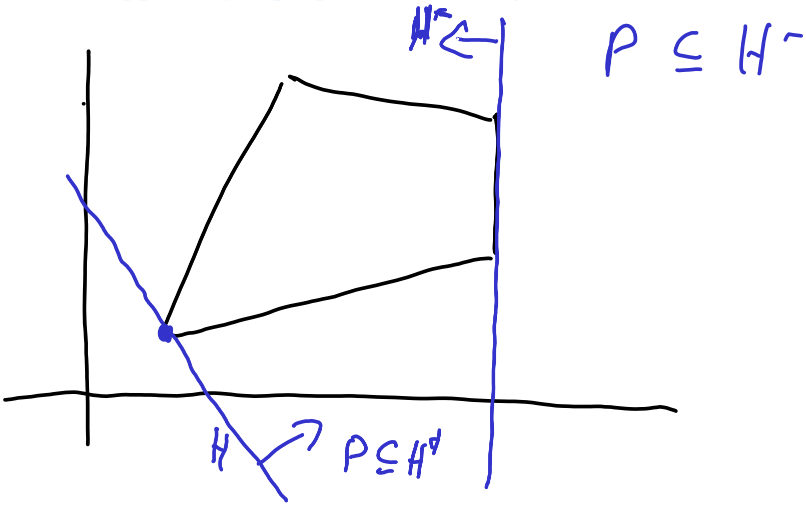


Definition 1.2.2 — Polyhedron, polytope. A *polyhedron* $P = \{x \in \mathbb{R}^n : Ax \leq b, \text{ for } A \in R^{m \times n}, b \in R^n\}$ is a finite intersection of half-spaces. A bounded polyhedron is a *polytope*.



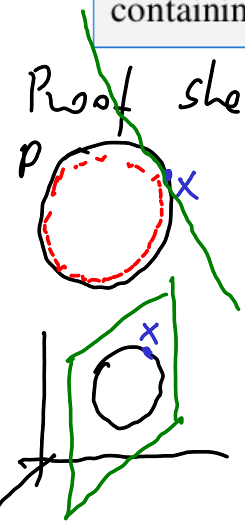


Definition 1.2.3 — Supporting hyperplane. For a given set $P \subseteq \mathbb{R}^n$ a hyperplane H is called a *supporting hyperplane* if $P \cap H \neq \varnothing$ and either $P \subseteq H^+$ or $P \subseteq H^-$.



Theorem 1.2.2 — Supporting hyperplane theorem. For any nonempty convex set $P \subseteq \mathbb{R}^n$ and a point x belonging to the boundary of the set P, there exists a supporting hyperplane containing x.

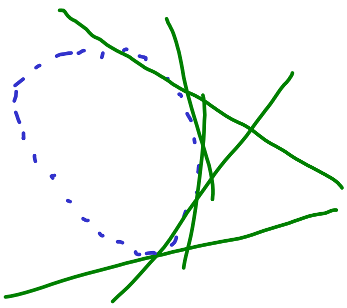
Proof sketch:



Case 1: Interior is non empty, we apply separating hyperplane theorem to set $\{x\}$ and interior of P

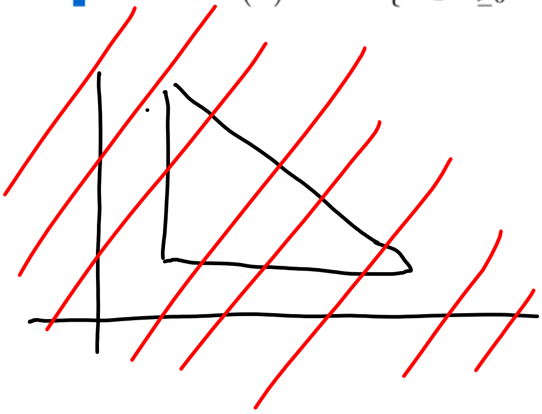
Case 2: Interior is empty $\Rightarrow$ it must lie in the affine set of dimension $n-1$ or less $\Rightarrow$ any hyperplane containing that affine set;

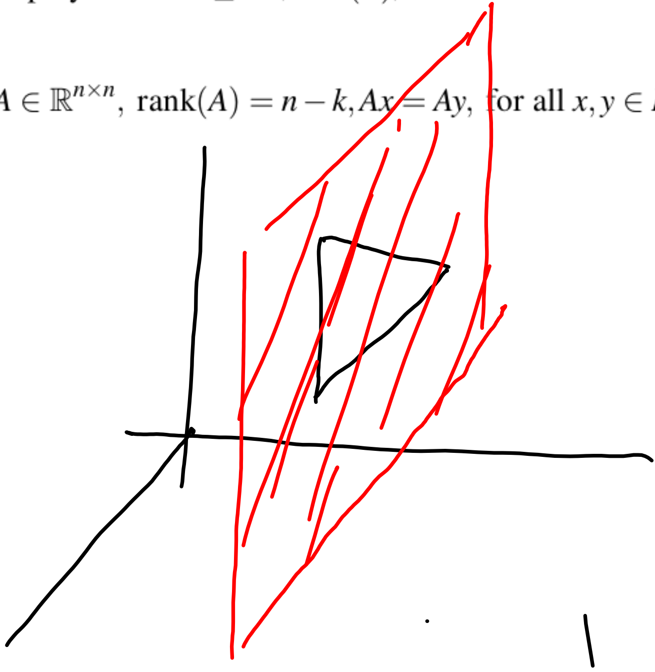
R There exists a partial converse of the Supporting hyperplane theorem, Theorem 1.2.2: If a closed, non-empty set $P \subseteq \mathbb{R}^n$ has a supporting hyperplane at every point x on its boundary, then P is convex.



Definition 1.2.4 The *dimension* of a polyhedron $P \subseteq \mathbb{R}^n$, $dim(P)$, is the dimension of its affine hull $\text{aff}(P)$, i.e.,:

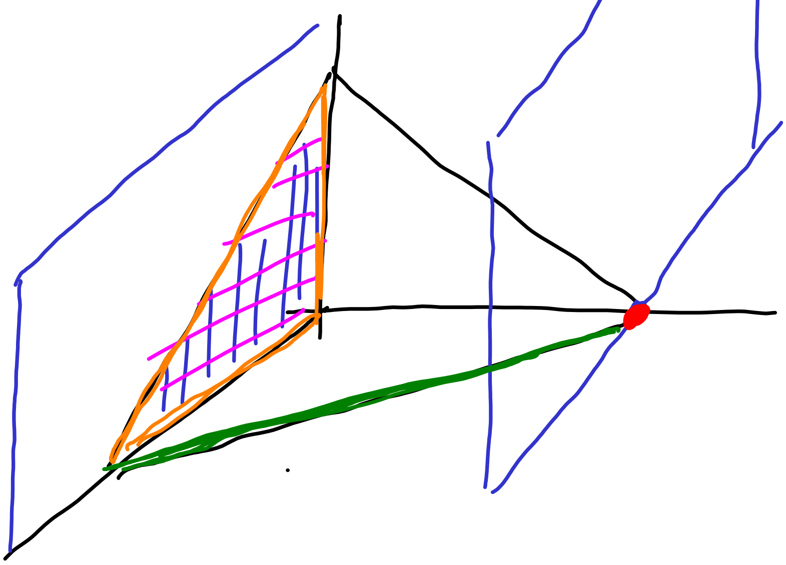
$$dim(P) := \min\{k \in \mathbb{Z}_{\geq 0} : \exists A \in \mathbb{R}^{n \times n},\ \text{rank}(A) = n - k, Ax = Ay,\ \text{for all } x, y \in P\}$$





Definition 1.2.5 — Face, vertex, edge, facet. Let $P \subseteq \mathbb{R}^n$ be a non-empty polyhedron, then:

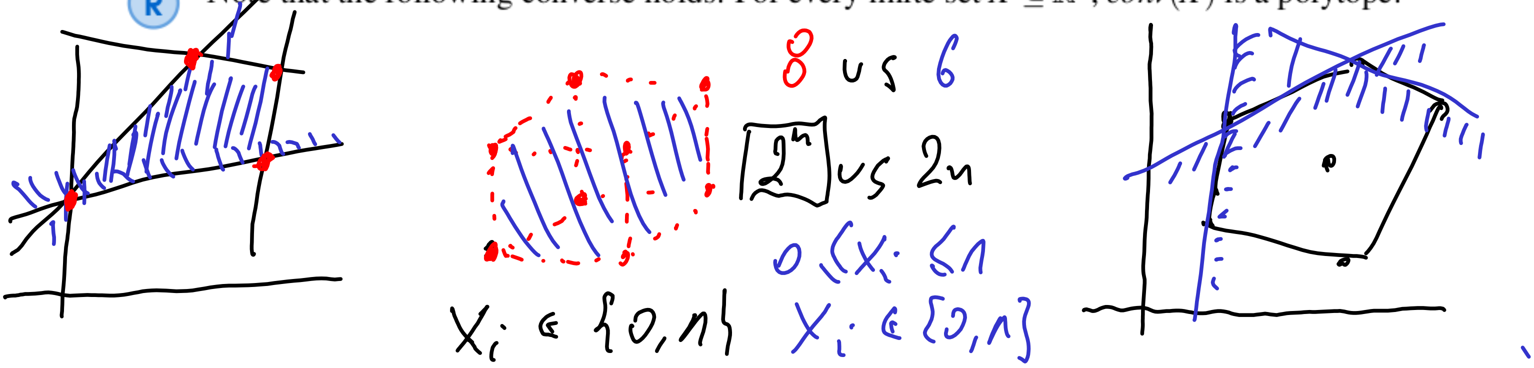
1. a *face* of P is either P itself or the intersection of P with a supporting hyperplane,
2. a *vertex* of P is a 0-dimensional face of P,
3. an *edge* of P is a 1-dimensional face of P,
4. a *facet* of P is a $(dim(P) - 1)$-dimensional face of P.



R Every face of a polyhedron P is also a polyhedron of dimension $dim(P)$ or less.

Proposition 1.2.3 — Vertex representation. Every polytope is the convex hull of its vertices.

> **R** Note that the following converse holds: For every finite set $X \subseteq \mathbb{R}^n$, $conv(X)$ is a polytope.

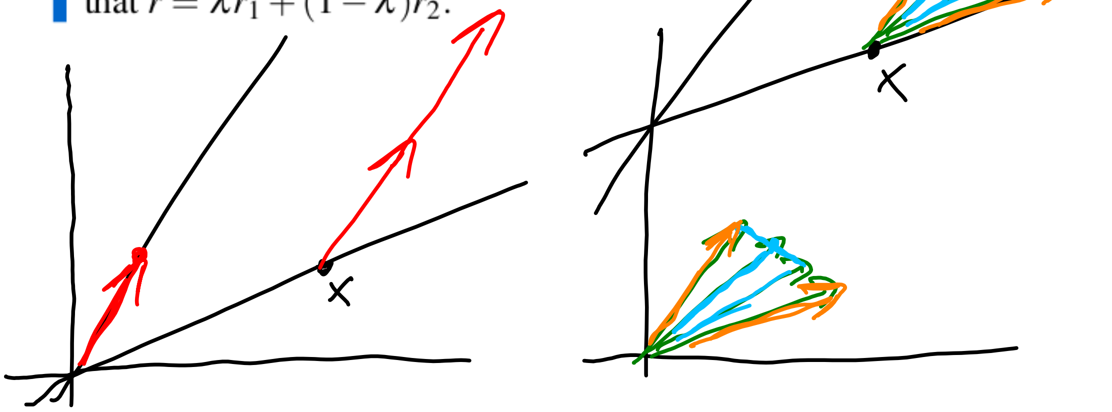


Definition 1.2.6 Let $P \subseteq \mathbb{R}^n$ be a polyhedron. A point $r \in \mathbb{R}^n$ is a $\boxed{\textit{ray}}$ of P if and only if

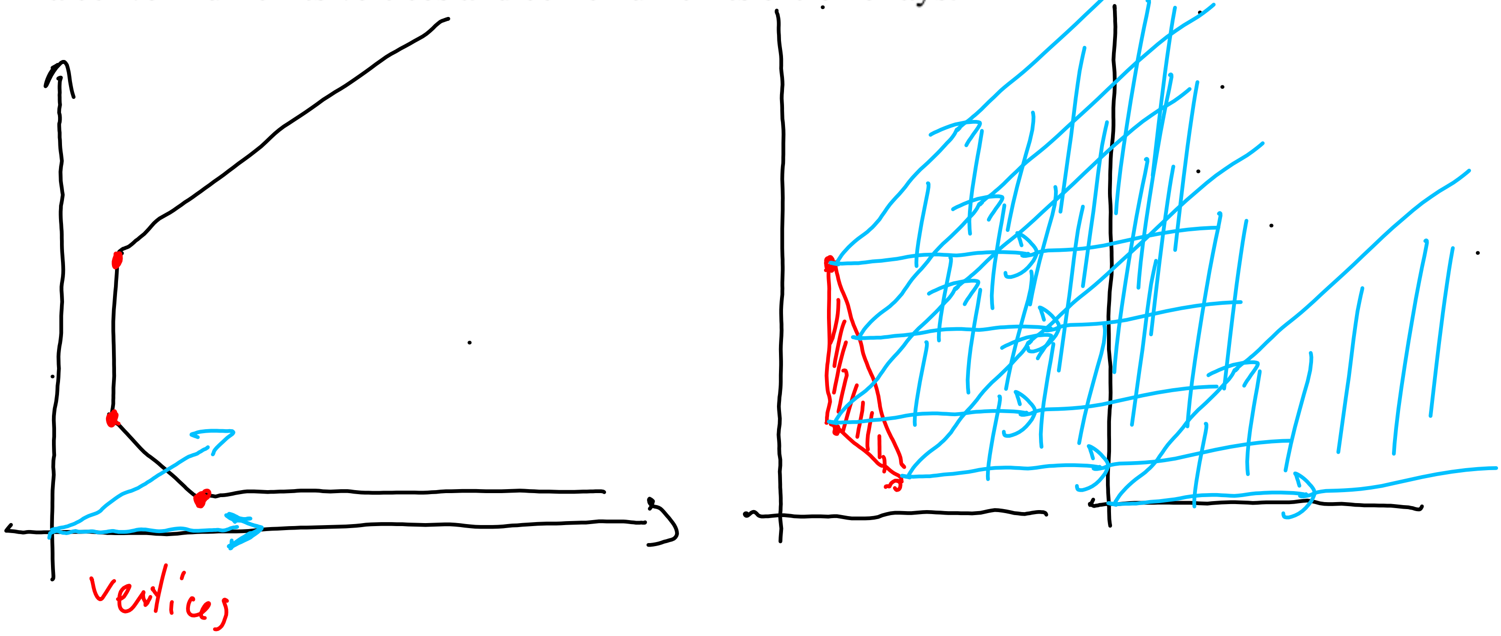
$$\{x+\lambda r \mid \lambda \geq 0\} \subseteq P$$

for any point $x \in P$.

Moreover r is an *extreme ray* of P if it is not in a line segment between two distinct rays of P i.e., there are no distinct rays r_1, r_2 (i.e., $r_1 \neq \mu r_2$ for any $\mu > 0$) of P and $0 < \lambda < 1$ such that $r = \lambda r_1 + (1-\lambda) r_2$.

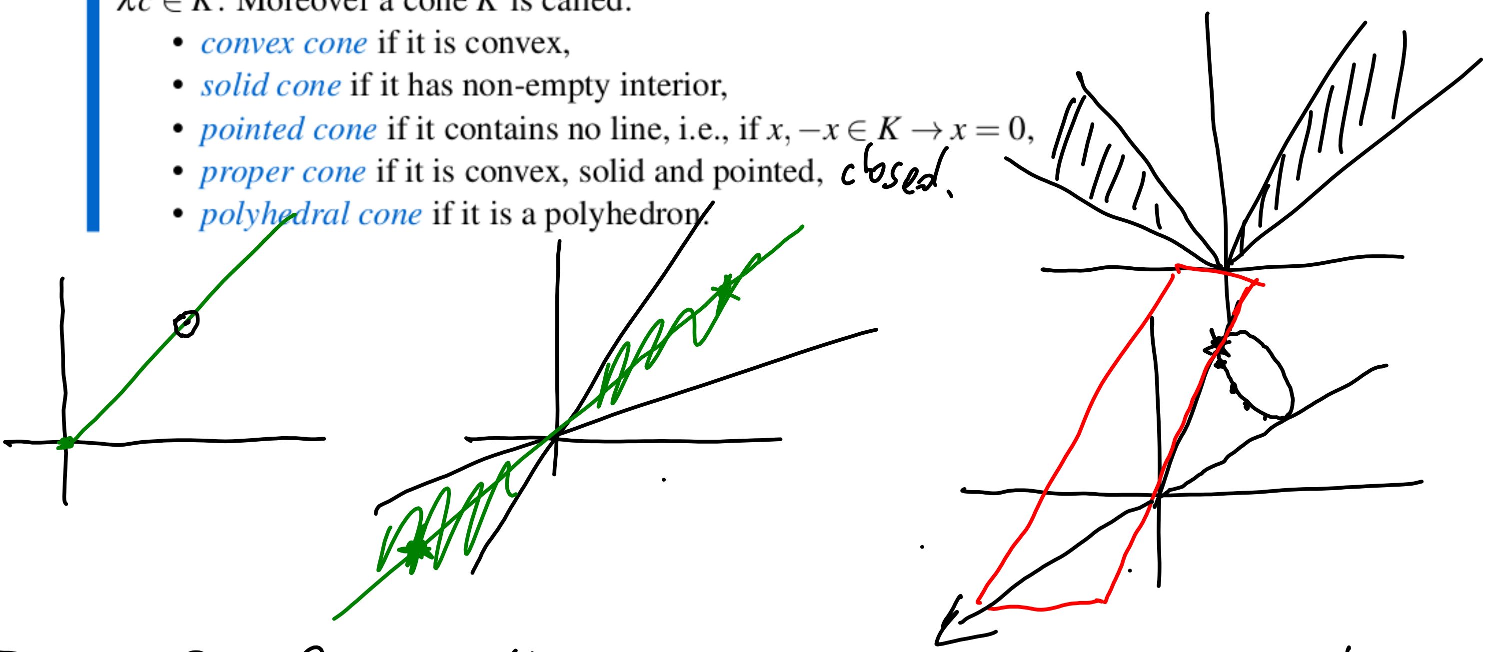


Proposition 1.2.4 — Minkowski Resolution Theorem. Every polyhedron is a Minkowski sum of a convex hull of its vertices and conic hull of its extreme rays.



Definition 1.3.1 — Cone. A set $K \subseteq \mathbb{R}^n$ is called a *cone* if for any $c \in K$ and $\lambda \in \mathbb{R}_{\geq 0}$ implies $\lambda c \in K$. Moreover a cone K is called:

- *convex cone* if it is convex,
- *solid cone* if it has non-empty interior,
- *pointed cone* if it contains no line, i.e., if $x, -x \in K \to x = 0$,
- *proper cone* if it is convex, solid and pointed, closed.
- *polyhedral cone* if it is a polyhedron.



Ex 1.7 Prove that nonnegative orthant $\mathbb{R}^n_{\geq 0}$ is a proper cone.

- $\mathbb{R}^n_{\geq 0}$ is a cone $\forall x \in \mathbb{R}^n_{\geq 0}$ $\forall \lambda \geq 0$ $\lambda x \in \mathbb{R}^n_{\geq 0}$
- $\mathbb{R}^n_{\geq 0}$ is solid

take $\bar{x} \in (1, 1, \dots 1)$ $r = \frac{1}{2}$

$B(\bar{x}, r) \overset{?}{\subseteq} \mathbb{R}^n_{\geq 0}$

$B(\bar{x}, r) \subseteq \{ x = (x_1, \dots x_n) :$

$\underbrace{\bar{x}_i - r}_{\frac{1}{2}} \leq x_i \leq \underbrace{\bar{x}_i + r}_{\frac{3}{2}} \} \subseteq \mathbb{R}^n_{\geq 0}$

- $\mathbb{R}^n_{\geq 0}$ is pointed

$\forall x \in \mathbb{R}^n_{\geq 0}$ $x_i \geq 0$ $\forall i$

$-x$ $\cancel{x_i} \leq 0$

- $\mathbb{R}^n_{\geq 0} = \{ x \in \mathbb{R}^n : x_i \geq 0$ $\forall i \in [n] \}$

R Any polyhedral cone K admits an inequality representation of the following form $K = \{x \in \mathbb{R}^n : Ax \leq 0\}$ for some matrix $A \in \mathbb{R}^{m \times n}$. Moreover the converse is true. That is, for every matrix $A \in \mathbb{R}^{m \times n}$ the set $K = \{x \in \mathbb{R}^n : Ax \leq 0\}$ is a polyhdral cone.

Definition 1.3.2 — Dual cone. Let $K \subseteq \mathbb{R}^n$ be a cone. The *dual cone* of K is the set defined as

$$K^* := \{l \in \mathbb{R}^n : \langle l, x \rangle \geq 0, \text{ for all } x \in K\}.$$

If $K = K^*$, the cone K^* is called *self dual*.

Ex 1.8 $\quad K = \mathbb{R}^n_{\geq 0}$ is self dual

$K \subseteq K^*$ $\quad \forall l \in K$ we prove that $\forall x \in K$

$$\langle l, x \rangle \geq 0$$

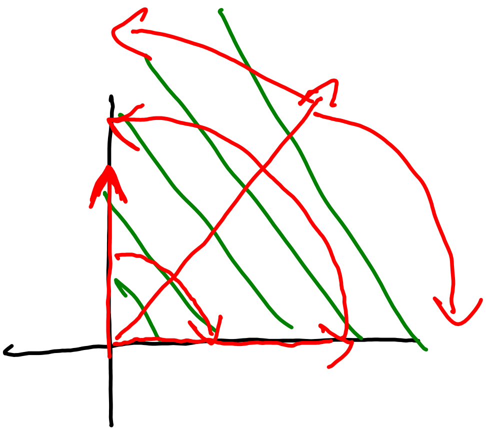
$$\sum_{i=1}^{n} \underbrace{l_i}_{\geq 0} \cdot \underbrace{x_i}_{\geq 0} \geq 0$$

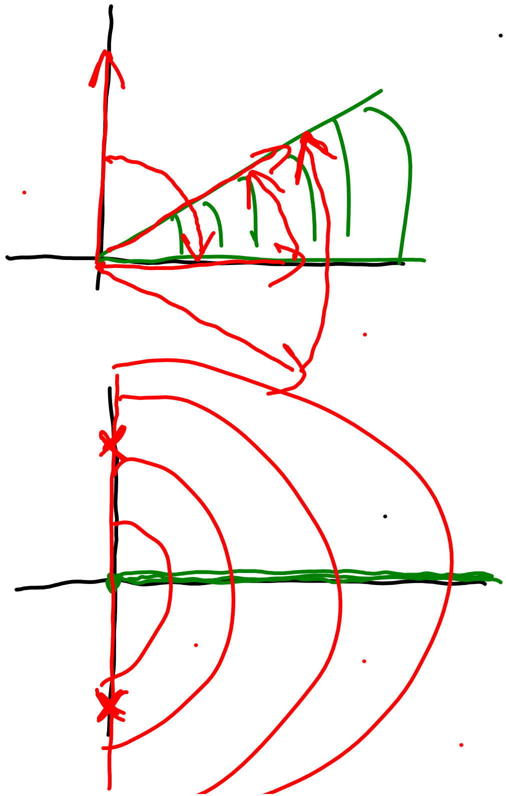
$K = K^*$

$K^* \subseteq K$ $\quad$ For the sake of contradiction assume $\exists \bar{l} \in K^*$ and $\exists r \in \mathbb{N}$ $\bar{l}_r < 0$.

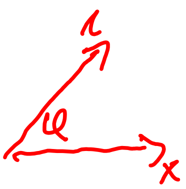
Take $x = (0, \dots, 0, \overset{r\text{'th}}{1}, 0 \dots 0) \in K$

$$\langle \bar{l}, x \rangle = \bar{l}_r < 0 \ \lightning$$

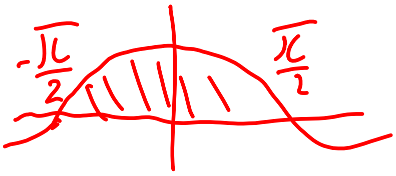




$K^* = \{l : \langle l, x \rangle \geq 0 \ \forall x \in K\}$



$$\cos(\varphi) = \frac{\langle l, x \rangle}{|x|\,|l|}$$



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Proposition 1.3.2 Let $K_1, K_2 \subseteq \mathbb{R}^n$ be cones. Then

1. K_1^* is closed and convex (even if K_1 is not convex),
2. $K_1 \subseteq K_2 \Rightarrow K_2^* \subseteq K_1^*$,
3. K_1 is solid $\Rightarrow K_1^*$ is pointed. If $\overline{K_1}$ is pointed and convex $\Rightarrow K_1^*$ is solid,
4. $K_1^{**} = \overline{conv(K)}$, if K_1 is closed and convex, $K_1 = K_1^{**}$,
5. If K_1 is proper, then K_1^* is proper.

Proof 5. K_1 - convex, closed, pointed, solid

K_1^* - convex, closed } P1

- pointed P3a

- solid P3b

Definition: Cone $K \subseteq \mathbb{R}^n$ is full dimensional if $\dim(K) = n$

Proposition: If K is solid $\Rightarrow$ K is full dimensional

K is solid $\Rightarrow \exists x \in K \; r > 0 \quad B(x,r) \subseteq K$

$\dim(\operatorname{aff}(B(x,r))) = n$

1) K_1^* is closed and convex

convexity: $p, q \in K_1^*$ and $\lambda \in [0,1]$

goal: $\lambda p + (1-\lambda) q \in K_1^*$

let $x \in K_1$ $\quad \langle p, x \rangle \geq 0 \qquad \langle q, x \rangle \geq 0$

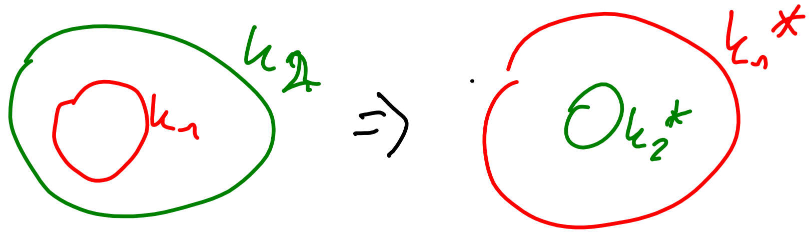
$$\langle \lambda p + (1-\lambda) q, x \rangle = \underbrace{\lambda}_{\geq 0} \underbrace{\langle p, x \rangle}_{\geq 0} + \underbrace{(1-\lambda)}_{\geq 0} \underbrace{\langle q, x \rangle}_{\geq 0} \geq 0$$

$x \in K_1$ and let $(y_{(k)})_{k\in\mathbb{N}}$ be a Cauchy sequence in K_1 with limit $y = \lim_{k\to\infty} y_{(k)}$

$$\langle y_{(k)}, x\rangle \geq 0 \quad \forall k \in \mathbb{N}$$

$$\langle y, x\rangle = \langle \lim_{k\to\infty} y_k, x\rangle = \lim_{k\to\infty} \underbrace{\langle y_k, x\rangle}_{\geq 0} \geq 0$$

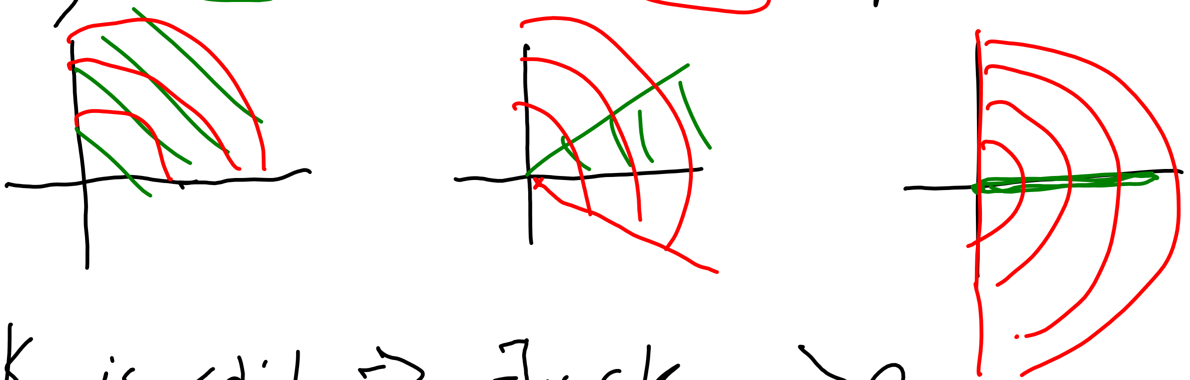
2. $\underline{K_1} \subseteq \underline{K_2} \Rightarrow \underline{K_2^*} \subseteq \underline{K_1^*}$



$K_1^* = \{y \mid \langle y, x\rangle \geq 0 \ \forall x \in K_1\}$

$K_2^* = \{y \mid \langle y, x\rangle \geq 0 \ \forall x \in K_2\} \subseteq \{y \mid \langle y, x\rangle \geq 0 \ \forall x \in K_1\} = K_1^*$

3a) $\boxed{K_1}$ is solid $\Rightarrow$ $\boxed{K_1^*}$ is pointed



K_1 is solid $\Rightarrow \exists x \in K_1 \ r > 0$

$$B(x, r) \subseteq K_1$$

$$\boxed{\text{Goal: if } (y \in K_1^* \wedge -y_1 \in K_1^*) \Rightarrow y = 0}$$

Let $z \in B(x, r)$

if $(y \in K_1^* \wedge -y \in K_1^*) \Rightarrow \underbrace{\langle y,z\rangle \geq 0 \quad \langle -y,z\rangle \geq 0}_{\langle y,z\rangle = 0}$

Let $i \in [n]$ $\quad \boxed{y_i = 0}$

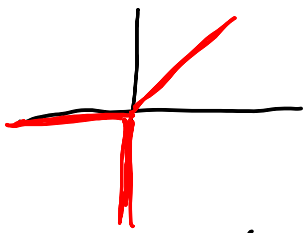
$$z = x + r\underbrace{e^i}_{\substack{= \\ (0,0,\dots 0,\underset{i'th}{1},0\dots 0)}} \in B(x,r)$$

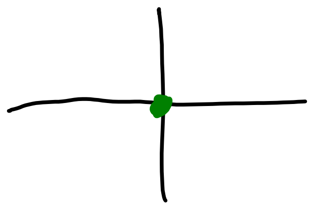
$$\langle y,z\rangle = \underbrace{\langle y,x\rangle}_{=0} + \langle y, re^i\rangle = r y_i = 0 \quad \Rightarrow \quad y_i = 0$$

$x \in B(x,r)$

3b) K_1 is convex

$\overline{K_1}$ is pointed $\Rightarrow K_1^*$ is solid





Note: • K_1^*, $(\overline{K_1})^*$ - closed and convex P1

• $K_1^* = (\overline{K_1})^*$

$K_1 \subseteq \overline{K_1} \Rightarrow (\overline{K_1})^* \subseteq K_1^*$ by P2

Show $K_1^* \subseteq (\overline{K_1})^*$ ...

$$(\overline{K_1})^* = K_1^*$$

$\{y \mid \langle x,y\rangle \geq 0 \ \forall x \in \overline{K_1}\}$ $\qquad \{y \mid \langle y,x\rangle \geq 0 \ \forall x \in K_1$

$K_n^* = \{y \mid \langle y, x\rangle \geq 0 \quad \forall x \in \bar{K}_n\}$

- $\operatorname{int} K_n^* = \{y \mid \langle y, x\rangle > 0 \quad \forall x \in \bar{K}_n \setminus \{0\}\}$

For the sake of contradiction assume

$$\exists\, y \in \operatorname{int} K_n^* \quad \text{and} \quad x \in \bar{K}_n \setminus \{0\} :$$
$$\langle y, x\rangle = 0$$

$\exists\, r > 0 \qquad B(y, r) \in \operatorname{int} K_n^*$

Note $y \neq 0$ otherwise $B(0, r) \in \operatorname{int} K_n^*$

$$\Downarrow$$
$$K_n^* = \mathbb{R}^n$$
$$\Downarrow$$
$$K_n = \{0\} \text{ not pointed}$$

Let $z \in B(y, r)$ Let $i \in [n] : x_i \neq 0$

$$z = y + r\,\underbrace{e^i}_{\substack{\{0,0,\dots 0\,1\,0 \dots 0\} \\ i\text{'th}}} \cdot (-\operatorname{sgn}(x_i))$$

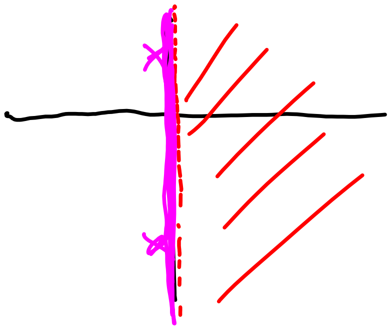
$$\langle z, x\rangle = \underbrace{\langle y, x\rangle}_{=0} + \langle r e^i \cdot (-\operatorname{sgn}(x_i)), x\rangle = x_i(-\operatorname{sgn}(x_i)) < 0 \quad \lightning$$

Since $\bar{K}_n$ is closed and pointed $\exists\, H = \{x : a^T x = 0\}$

$\bar{K}_n \subseteq H^+ \qquad \bar{K}_n \cap H = \{0\} \Rightarrow \forall x \in \bar{K}_n \setminus \{0\} \quad \langle a, x\rangle > 0$

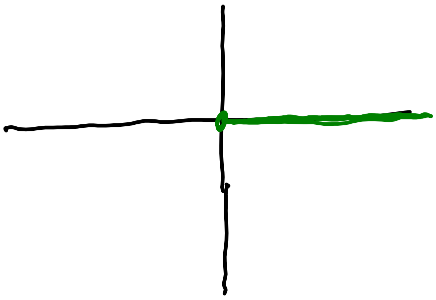
$$\Downarrow$$
$$a \in \operatorname{int} K^*$$

K_1 pointed



$\{(x_1, x_2) : x_1 > 0\}$

K_1^*



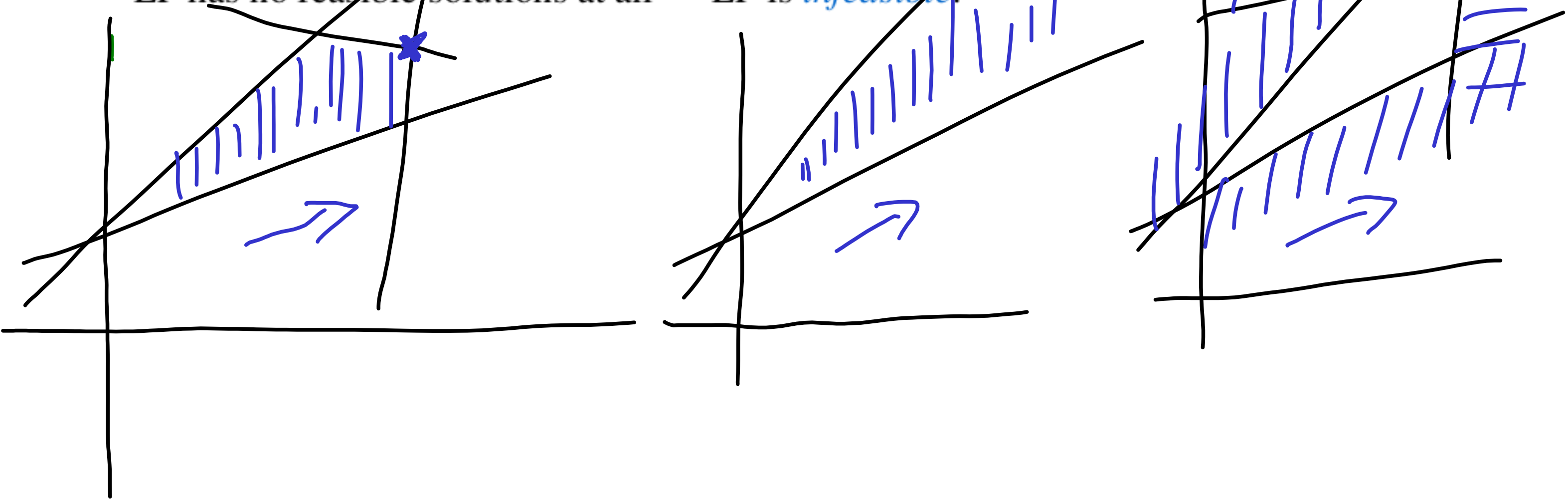
$\{(y_1, y_2) :$
$y_2 = 0$
$y_1 \geq 0\}$

Definition 1.4.1 — Linear Program (LP). Let $c \in \mathbb{R}^n$, $A, C, E \in \mathbb{R}^{m\times n}$ and $b, d, f \in \mathbb{R}^m$. An optimization problem is called a *linear program* in a *general/standard/canonical form* if it is of the following form:

General :	*Standard* :	*Canonical* :
$\min c^\top x$	$\min c^\top x$	$\max c^\top x$
$Ax \geq b$	$Ax = b$	$Ax \leq b$
$Cx \leq d$	$x \geq 0$	$x \geq 0$
$Ex = f$		

$x \in \mathbb{R}^{+}_{\geq 0}$

For a linear program one of the following three cases can happen.
- LP attains a finite optimum — LP is *feasible*,
- optimal solution of the LP is unbounded — LP is *unbounded*,
- LP has no feasible solutions at all — LP is *infeasible*.



$$\begin{aligned}
\max\ & 2x_1 + 7x_2 \\
& 2x_1 + 7x_2 \le 4 \\
& x_1 + 5x_2 \le 7 \\
& x_1, x_2 \ge 0
\end{aligned}$$

Definition 1.4.2 — Dual of an LP. Let $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. A *primal* and *dual* formulation of a linear program in a standard/canonical form is the following:

$$\begin{array}{llll}
\textit{Standard}: & & \textit{Canonical}: & \\
\textit{Primal}: \min c^\top x & \textit{Dual}: \max b^\top y & \textit{Primal}: \max c^\top x & \textit{Dual}: \min b^\top y \\
\quad Ax = b & \quad c - A^\top y \ge 0 & \quad Ax \le b & \quad A^\top y \ge c \\
\quad x \ge 0 & & \quad x \ge 0 & \quad y \ge 0
\end{array}$$

Theorem 1.4.1 — Weak duality for linear programs. Let x, y be a feasible solution to the primal, dual formulation, respectively, of some linear program in a standard form. Then

$$c^\top x \geq b^\top y.$$

Theorem 1.4.2 — Strong duality for linear programs. [**PapadimitriouS82**] If both primal and dual are feasible, then at optimality their costs are equal. That is, there exist feasible solutions x^*, y^* for primal and dual respectively such that $c^\top x^* = b^\top y^*$.

Proposition 1.4.3 For a primal and a dual formulations of a linear program the following holds:

- Primal finite $\Rightarrow$ dual finite,
- Primal unbounded $\Rightarrow$ dual infeasible,
- Primal infeasible $\Rightarrow$ dual unbounded or infeasible,

and, vice versa, by the solution of Exercise 1.10.

Theorem 1.4.4 — Complementary slackness. Let x, y be feasible primal and dual solutions for linear program in the canonical form, respectively. Then x and y are optimal solutions if and only if

$$(b - Ax)^\top y = 0 \quad \text{and} \quad (A^\top y - c)^\top x = 0.$$