Definition 2.0.1 — Mathematical optimization problem. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in [m]$. A *mathematical optimization problem* in n variables has the form.

$$\begin{aligned} \min\ & f(x) \\ & g_i(x) \leq 0, \qquad \text{for } i \in [m], \\ & x \in \mathbb{R}^n. \end{aligned}$$

The function f is called the *objective function* and functions g_i are called *(inequality) constraint functions*.

Let $f, g_1, \ldots, g_m \in \mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_n]$. We start with the following definition.

Definition 2.0.2 — Basic closed semialgebraic set. A set $\mathscr{G}_+ \subset \mathbb{R}^n$ is a *basic closed semialgebraic* set if there exists a set of polynomials $\mathscr{G} = \{g_0 := 1, g_1, \ldots, g_m\} \subseteq \mathbb{R}[x]$ such that

$$\mathscr{G}_+ := \{x \in \mathbb{R}^n |\ g(x) \geq 0, \text{ for all } g \in \mathscr{G}\}$$

Exercise 2.1 Consider the following two sets of polynomials $\mathscr{G}^1, \mathscr{G}^2 \subset \mathbb{R}[x,y]$:

$$\mathscr{G}^1 = \{4-(x-2)^2-(y-2)^2,\ -(x-1)(4-x),\ -(y-1)(3-y)\}$$

and

$$\mathscr{G}^2 = \mathscr{G}^1 \cup \{x^4y^2+x^2y^4-3x^2y^2+1\}$$

Prove that $\mathscr{G}^1_+ = \mathscr{G}^2_+$. ∎

Definition 2.0.3 — Closed semialgebraic set. A finite union of basic closed semialgebraic sets in $\mathbb{R}^n$ is called a *closed semialgebraic set*.

R Semialgebraic sets are closed under finite unions, intersections, and complementation.

Theorem 2.0.1 — Tarski—Seidelberg theorem. Let X be a semialgebraic set in $\mathbb{R}^{n+1}$ and $\pi_n : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection defined as $\pi_n(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$. Then $\pi_n(X)$ is a semialgebraic set in $\mathbb{R}^n$.

Definition 2.0.4 — Polynomial optimization problem. Let $f, g_1, \ldots, g_m \in \mathbb{R}[x_1, \ldots, x_n]$. A *polynomial optimization problem* in general form takes the form

$$\begin{aligned} \min\ & f(x) \\ & g_i(x) \geq 0, \qquad \text{for } i \in [m], \\ & x \in \mathbb{R}^n. \end{aligned}$$

Definition 2.1.1 — Convex optimization problem. Let $f, g_1, \dots, g_m : \mathbb{R}^n \to \mathbb{R}$ be convex functions. A *convex optimization problem* in general form is the following program:

$$\begin{aligned} \min\ & f(x) \\ & g_i(x) \leq 0, \qquad \text{for } i \in [m]. \end{aligned}$$

■ **Example 2.6** Consider a finite set of k polytopes $P_i \subseteq \mathbb{R}^n$, for $i \in [k]$. We are interested in optimizing, possibly an affine function, over the convex hull of the union of polytopes $P_1, \ldots, P_k$, namely over a set P defined as:

$$P = \operatorname{conv}\left(\bigcup_{i=1}^{k} P_i\right).$$

Since the convex hull of the union of $P_1, \ldots, P_k$ is a polytope, we are interested in its inequality representation.

Convex problems cover a broad class of problems, including linear programs and many other classes that we will study in the next chapters. Special classes of convex problems include:

Proposition 2.1.1 Let f, h, g_i, for $i \in [m]$, be convex functions. The program

$$\min\{f(x)|\ g_i \leq 0,\ h(x) \leq 0,\ \text{for } i \in [m]\} \tag{2.2}$$

is equivalent to the convex problem

$$\min\{f(x)|\ g_i \leq 0,\ h(x) = 0,\ \text{for } i \in [m]\} \tag{2.3}$$

if at any optimal solution x^* of the convex problem we have $h(x^*) = 0$.

Definition 2.2.1 — conic program. Let $K \subseteq \mathbb{R}^n$ be a proper cone. Moreover, let $A, F \in \mathbb{R}^{m\times n}$, $b, g \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. A *conic program* has the following form:

$$\begin{aligned} &General: && Standard: \\ &\quad \min c^\top x && \quad \min c^\top x \\ &\qquad -Fx - g \in K && \qquad Ax = b \\ & && \qquad x \in K \end{aligned}$$

Definition 2.2.2 — dual vector space. Given a vector space V over a field F, the *dual vector space* V^* is defined as the set of all linear maps $\phi : V \to F$.

A pair of an element in the dual vector space and an element in the vector space form a *pairing* $\langle \cdot, \cdot \rangle : V^* \times V \to F$ defined as $\langle \phi, x \rangle := \phi(x)$.

Definition 2.2.3 — conic program duality in standard form. Given two real vector spaces S, T, a linear map $\mathscr{A} : S \to T$ and a proper cone $K \subseteq S$, for $b \in T$ and $c \in S^*$ the *primal* and the *dual* formulation of a conic program in standard form are the following.

$$\begin{aligned} &\textit{Primal}: \\ &\quad \min \langle c, x \rangle_S \\ &\qquad \mathscr{A}x = b \\ &\qquad x \in K \end{aligned} \qquad\qquad \begin{aligned} &\textit{Dual}: \\ &\quad \max \langle y, b \rangle_T \\ &\qquad c - \mathscr{A}^* y \in K^* \end{aligned}$$

Theorem 2.2.1 — Weak duality for conic programs. Let x, y be a feasible solution to the primal, dual formulations respectively, of some conic program in standard form. Then

$$\langle c, x \rangle_S \geq \langle y, b \rangle_T .$$