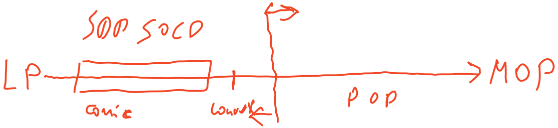


Definition 2.0.1 — Mathematical optimization problem. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in [m]$. A *mathematical optimization problem* in n variables has the form.

$$\begin{aligned} \min\ & f(x) \\ & g_i(x) \le 0, \qquad \text{for } i \in [m], \\ & x \in \mathbb{R}^n. \end{aligned}$$

The function f is called the *objective function* and functions g_i are called *(inequality) constraint functions*.

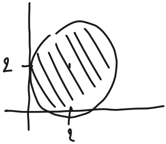
$f(x) = c^T x$ $\qquad$ $g_i(x) = a_i^T x - b_i$

Let $f, g_1, \ldots, g_m \in \boxed{\mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_n]}$. We start with the following definition.

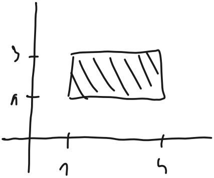
Definition 2.0.2 — Basic closed semialgebraic set. A set $\mathcal{G}_+ \subset \boxed{\mathbb{R}^n}$ is a *basic closed semialgebraic* set if there exists a set of polynomials $\mathcal{G} = \{\boxed{g_0 := 1}, g_1, \ldots, g_m\} \subseteq \mathbb{R}[x]$ such that

$= \mathbb{R}[x_1, \ldots x_n]$

$$\mathcal{G}_+ := \{x \in \mathbb{R}^n \mid g(x) \ge 0, \text{ for all } g \in \mathcal{G}\}$$



$g_1(x,y) = 4 - (x-2)^2 - (y-2)^2$



$g_1(x,y) = (x-1)(4-x)$

$g_2(x,y) = (y-1)(3-y)$

Remark

$\exists\ \mathcal{G}^1 \neq \mathcal{G}^2 \subseteq \mathbb{R}[x_1 \ldots x_n]$ s.t. $\mathcal{G}^1_+ = \mathcal{G}^2_+$

Exercise 2.1 Consider the following two sets of polynomials $\mathscr{G}^1, \mathscr{G}^2 \subset \mathbb{R}[x,y]$:

$$\mathscr{G}^1 = \{4-(x-2)^2-(y-2)^2,\ -(x-1)(4-x),\ -(y-1)(3-y)\}$$

and

$$\mathscr{G}^2 = \mathscr{G}^1 \cup \{x^4y^2+x^2y^4-3x^2y^2+1\}$$

Prove that $\mathscr{G}^1_+ = \mathscr{G}^2_+$.

$M(x,y) \geq 0 \quad \forall x,y \in \mathbb{R}$

$\mathcal{G}_+ = \{x \in \mathbb{R}^n : g(x) \geq 0 \ \forall g \in \mathcal{G}\}$

$$x^4y^2 + x^2y^4 + 1 \geq 3x^2y^2 \quad \forall x,y \in \mathbb{R}$$

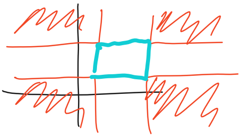
$$\frac{1}{3}(\underbrace{x^4y^2}_{a_1 \geq 0} + \underbrace{x^2y^4}_{a_2 \geq 0} + \underbrace{1}_{a_3 \geq 0}) \geq \sqrt[3]{x^6y^6} \quad \forall x,y \in \mathbb{R}$$

AGM

Not convex

$$g_1(x,y) = (x-1)(4-x) \quad | (-1)$$

$$g_2(x,y) = (y-1)(3-y) \quad | (-1)$$



Definition 2.0.3 — Closed semialgebraic set. A finite union of basic closed semialgebraic sets in $\mathbb{R}^n$ is called a *closed semialgebraic set*.

R Semialgebraic sets are closed under finite unions, intersections, and complementation.

Theorem 2.0.1 — Tarski—Seidelberg theorem. Let X be a semialgebraic set in $\mathbb{R}^{n+1}$ and $\pi_n : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection defined as $\pi_n(x_1,\ldots,x_n,x_{n+1}) = (x_1,\ldots,x_n)$. Then $\pi_n(X)$ is a semialgebraic set in $\mathbb{R}^n$.

- does not work for basic semialgebraic set. SDP spectrahedra
- Cylindrical Algebraic Decomposition double exponential

$=, \leq \quad \wedge, \vee, \cup, \cap, \quad \forall, \exists$

Definition 2.0.4 — Polynomial optimization problem. Let $f, g_1, \ldots, g_m \in \mathbb{R}[x_1,\ldots,x_n]$. A *polynomial optimization problem* in general form takes the form

$$\begin{aligned} &\min f(x) \\ &\quad g_i(x) \geq 0, \qquad \text{for } i \in [m], \\ &\quad x \in \mathbb{R}^n. \end{aligned}$$

Proposition Let $f, g_1, \ldots g_m \in \mathbb{R}[x_1, \ldots x_n]$

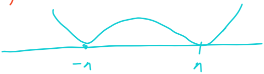
Let $\mathcal{G} = \{g_1, \ldots g_m\}$

A POP can be written as

$$\min \boxed{f(x)}$$
$$\text{s.t. } x \in \boxed{\mathcal{G}_+}$$

Polynomial Optimization problems are not necessarily convex!

example $f(x) = x^4 - 2x^2 + 1 = (x^2-1)^2$

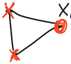


Ex 2.2 solving POP is NP-hard.

Max Independent Set

Input $G(V,E)$ $|V| = n$

Output: $S \subseteq V$: Vertices in S do not share an edge



$x_i \in \{0,1\}$

$\sum x_i$

$x_i + x_j \leq 1 \quad \forall (i,j) \in E$

$x_i \in \{0,1\} \quad \forall i \in [n]$

$=$

$x_i^2 - x_i \geq 0$

$x_i^2 - x_i \leq 0$ $\Leftrightarrow$ $x_i^2 - x_i = 0$

Definition 2.1.1 — Convex optimization problem. Let $f, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ be convex functions. A *convex optimization problem* in general form is the following program:

$$\min f(x) \quad \text{convex}$$
$$g_i(x) \le 0, \qquad \text{for } i \in [m].$$

$h_j(x) = 0$ for $j \in [k]$

affine

$\boxed{h_j(x)} \le 0$

$\boxed{-h_j(x)} \le 0$

Abstract convex optimization problem

min $f(x)$ – convex

s.t. $x \in C$ $\quad$ C – convex

Goal:

Find inequality representation of C

$C := \{x \in \mathbb{R}^n : \underbrace{g(x)}_{\text{convex}} \le 0, \underbrace{h(x)}_{\text{affine}} = 0\}$

Example $\min f(x) := x_1^2 + x_2^2$

$g(x) := \cancel{x_1 / (1 + x_2^2)^2} \le 0$

$h(x) := (x_1 + x_2)^2 = 0$

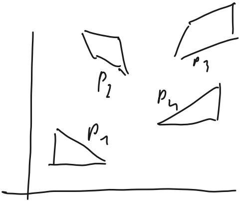
$$\boxed{\begin{aligned} &\min f(x) \\ &\tilde{g}(x) := x_1 \le 0 \\ &\tilde{h}(x) := x_1 + x_2 = 0 \end{aligned}}$$

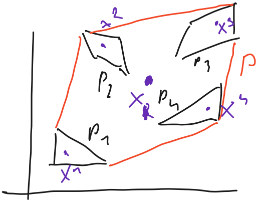
Convex optimization problem

■ **Example 2.6** Consider a finite set of k polytopes $P_i \subseteq \mathbb{R}^n$, for $i \in [k]$. We are interested in optimizing, possibly an affine function, over the convex hull of the union of polytopes $P_1, \dots, P_k$, namely over a set P defined as:

$$\boxed{P = \operatorname{conv}\left(\bigcup_{i=1}^{k} P_i\right).}$$

Since the convex hull of the union of $P_1, \dots, P_k$ is a polytope, we are interested in its inequality representation.





$\bar{P} := \{ (\underbrace{x, x^1, \dots, x^k}_{\in \mathbb{R}^n}, \underbrace{\lambda}_{\mathbb{R}^k}) \in \mathbb{R}^{n(k+1)+k} \mid$

$x = \sum_{i=1}^{k} \lambda_i x^i$ $\qquad$ $x - \sum_{i=1}^{n} \lambda_i x^i = 0$

$\sum_{i=1}^{k} \lambda_i = 1$

$x^i \in P_i \quad \forall i \in [k] \}$

$P_i = \{ x \in \mathbb{R}^n : A_i x \leq b \}$

$x_i \in P_i$

$A_i x_i \leq b$

$y = \lambda_i x_i$

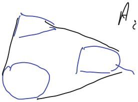
$A y \leq \lambda_i b$

$\tilde{P} := \{ (y, y^1, \dots, y^k, \lambda) \in \mathbb{R}^{n(k+1)+k} \mid$

$y = \sum_{i=1}^{k} y^i$

$y^i \in \lambda_i P_i$ $\qquad$ $\lambda_i P_i := \{ y \in \mathbb{R}^n :$

$\sum_{i=1}^{k} \lambda_i = 1 \}$ $\qquad$ $A_i y \leq \lambda_i b_i \}$



$C_i = \{ x \in \mathbb{R}^n :$

$f_{ij} \leq 0 \quad \forall j \in [m]$

convex $\}$

Convex problems cover a broad class of problems, including linear programs and many other classes that we will study in the next chapters. Special classes of convex problems include:

	f	$g_1 \dots g_m$
LP	affine	affine
QP	convex quadratic	affine $a_i^T x - b_i$
QCQP	convex quadratic	convex quadratic
SOCP	affine	$g_i = \|A_i x - b_i\|_2 + c_i^T x + d_i$ convex
SDP	affine	$g_i(x) = \lambda_{max}(A_i^0 + x_1 A_i^1 + \dots + x_n A_i^n)$, $\in \mathbb{R}^{n \times n}$ symmetric

=/

Proposition 2.1.1 Let f, h, g_i, for $i \in [m]$, be convex functions. The program

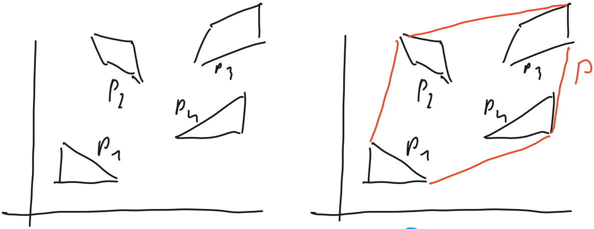
$$\min\{f(x) \mid g_i \leq 0,\ h(x) \leq 0,\ \text{for } i \in [m]\} \tag{2.2}$$

is equivalent to the convex problem

$$\min\{f(x) \mid g_i \leq 0,\ h(x) = 0,\ \text{for } i \in [m]\} \tag{2.3}$$

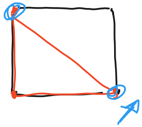
if at any optimal solution x^* of the convex problem we have $h(x^*) = 0$.

Additional page:
Motivation for Disjunctive Programming



Consider the following optimization problem

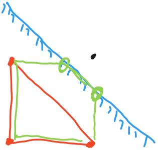
$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t} \quad & x_2 \leq -x_1 + \tfrac{3}{2} \\ & x_1, x_2 \in \{0,1\} \end{aligned}$$



In red, we have convex hull of feasible solutions. Red arrow shows direction of optimization. Blue vertices are optimal solutions, that give opt=1.

Since we cannot solve 0/1 optimization problems in general, usually we relax the problem to the following form.

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_2 \leq -x_1 + \tfrac{3}{2} \\ & x_1, x_2 \in [0,1] \end{aligned}$$



Since we relaxed the problem, the feasibility region got bigger (the green region). The optimal value of the relaxed problem (the green points) is 3/2. Which is bigger than the integral optimal solution that was 1.

One of the successful ways to provide tighter relaxation is to use Cutting Plane methods. In this method we generate additional linear constraints to make our feasibility region as tight as possible. An example of cutting plane method are Chvatal-Gomory cuts.

Example of Chvátal-Gomory cut.

Let the following two constraints be among the ones that describe the feasibility region

$$21x_1 + 4x_2 \geq 21$$

$$6x_1 + 5x_2 \geq 13$$

This implies that any solution must satisfy

$$27x_1 + 9x_2 \geq 34$$

and thus

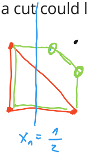
$$3x_1 + x_2 \geq 3 + \frac{7}{9}$$

and in particular any integer solution $(x_1, x_2 \in \{0,1\})$ must satisfy

$$3x_1 + x_2 \geq 4$$

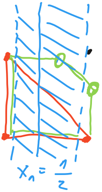
since integer variables multiplied by integer coefficients cannot give fractional value.

Even more general type of cuts is the class of Split cuts.
In a split cut we "cut" our feasible region in two pieces by cutting it with a hyperplane, and then delete a strip "around" the hyperplane in a way to be sure that none of the integral solutions was deleted. The we take a DISJUNTION of these two parts, that is a convex hull of union of these sets. Let's see how such a cut could look like in our case.



we take the following hyperplane $x_1 = \frac{1}{2}$

Now we will try to construct a "strip" around the hyperplane such that we do not delete any of the integral solutions. Such a "strip" is shown below.

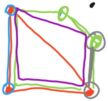


After deleting the strip we get two polytops P1 and P2 and we optimize over convex hull of its union. The polytopes are defined in the following way.

$$P_1 := \{(x_1, x_2) : \; x_2 \leqslant -x_1 + \tfrac{3}{2}, \; x_1 \leqslant \lfloor \tfrac{1}{2} \rfloor = 0, \; x_1, x_2 \in [0,1]\}$$

$$P_2 := \{(x_1, x_2) : \; x_2 \leqslant -x_1 + \tfrac{3}{2}, \; x_1 \geqslant \lceil \tfrac{1}{2} \rceil = 1, \; x_1, x_2 \in [0,1]\}$$

Now we can draw P1 (in blue), P2 (in gray) and convex hull of its union (in purple).



One can see that purple region (obtained using split cut and DISJUNCTIVE PROGRAMMING) is tighter that the green relaxation.

Remark: Can you see why Chvátal-Gomory cut is a split cut?