ETH Eidgenössische Technische Hochschule Zürich Swiss Federal Institute of Technology Zurich

Physics 1 - Waves and Thermodynamics Solution for Problem Set 0

Spring Semester 2018
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Problem 0.1 Guitarists

a) We start with two in-phase waves:

$$\begin{cases} y_A(r,t) = C_A(r)\cos(k(r-r_A)-\omega t) \\ y_B(r,t) = C_B(r)\cos(k(r-r_B)-\omega t) = C_B(r)\cos(k(r-r_A)-\omega t + k\Delta r) \end{cases} \tag{1}$$

Where we defined $\Delta r = r_A - r_B$, and the amplitudes at the listener position (r_L) are equal $C_A(r_L) = C_B(r_L) = C$. Adding the two waves together we get:

$$y_{int}(r_L,t) = 2C\cos\Big(\frac{k\Delta r}{2}\Big)\cos\Big(k(r_L-r_A)-\omega t+\frac{k\Delta r}{2}\Big) \tag{2}$$

The amplitude for the oscillating wave is $|2C\cos\Big(\frac{k\Delta r}{2}\Big)|$ and it have maximum when $k\Delta r = 2\pi n$, with $n = 1,2,3...$. Finally, we can relate k to f by $k = \frac{2\pi f}{v_{sound}}$ and get:

$$f_n = n\cdot\frac{v_{sound}}{\Delta r} = n\cdot\frac{300\,\frac{\mathrm{m}}{\mathrm{sec}}}{1\,\mathrm{m}} = \begin{cases} 300\,\mathrm{Hz} & n=1 \\ 600\,\mathrm{Hz} & n=2 \\ 900\,\mathrm{Hz} & n=3 \end{cases} \tag{3}$$

b) The oscillation frequency is given by $f_n = n\cdot\frac{v}{2L}$. The velocity of a wave in a string is $v = \sqrt{\frac{F_T}{\mu}}$. We are interested in the fundamental harmonic - $f_1 = \sqrt{\frac{F_T}{\mu}}\frac{1}{2L}$, so

$$F_T = \mu f_1^2(2L)^2 = 0.002\,\frac{\mathrm{kg}}{\mathrm{m}}\cdot 450^2\,\frac{1}{\mathrm{sec}^2}\cdot 1\,\mathrm{m}^2 = 405\,\mathrm{N} \tag{4}$$

c) The new frequency of the wave at the listener is

$$f_r = f_s\cdot\frac{v_{sound}}{v_{sound}-u} = 440\,\mathrm{Hz}\cdot\frac{300}{300-3} = 440\,\mathrm{Hz}\cdot\frac{100}{99}\approx 444.44\,\mathrm{Hz} \tag{5}$$

That means that the beating is $f_{beat} = 444.44\,\mathrm{Hz} - 440\,\mathrm{Hz} = 4.44\,\mathrm{Hz}$.

d) The intensity is inverse proportional to the square of the distance between the guitarist and the listener. Assuming that the sound wave propagate in half sphere we have $I_B(r) = \frac{P_{B,0}}{2\pi r^2}$, where $P_{B,0}$ is the power of the source (guitarist B). The distance between the guitarist to the listener was change from $r_1 = 12\,\mathrm{m}$ to $r_2 = 6\,\mathrm{m}$, therefor, the intensity increased by factor of 4 $\left(\frac{I_{B,r_2}}{I_{B,r_1}} = \frac{\frac{P_{B,0}}{2\pi\cdot 6^2}}{\frac{P_{B,0}}{2\pi\cdot 12^2}} = \frac{12^2}{6^2} = 4\right)$.

Problem 0.2 Standing waves

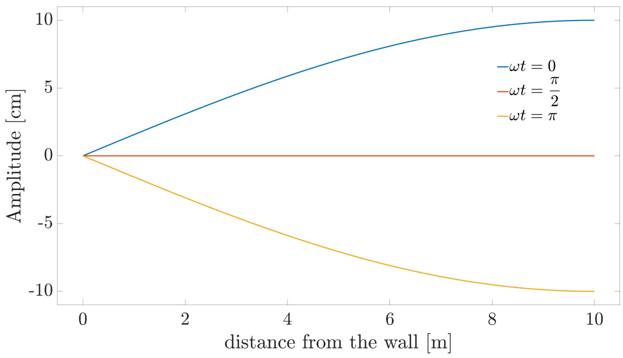
a) Because the bead is free to move on the pole there is anti-node in the pole position (and a node on the wall). This boundary conditions determine that all the length of the string is equal to odd multiplication of quarter wavelength $L = n\frac{\lambda_n}{4}$, with $n = 1, 3, 5, \ldots$. The frequencies are $f_n = \frac{v}{\lambda_n} = n\frac{v}{4L}$, with $v = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{10\,\mathrm{N}}{0.1\,\frac{\mathrm{kg}}{\mathrm{m}}}} = 10\,\frac{\mathrm{m}}{\mathrm{s}}$.
The first three frequencies are:

$$f_n = n \cdot \frac{10\,\frac{\mathrm{m}}{\mathrm{sec}}}{4 \cdot 10\,\mathrm{m}} = \begin{cases} 0.25\,\mathrm{Hz} & n = 1 \\ 0.75\,\mathrm{Hz} & n = 3 \\ 1.25\,\mathrm{Hz} & n = 5 \end{cases} \tag{6}$$

b) For the fundamental harmonic wave function we have $k = \frac{2\pi f}{v} = \frac{2\pi \cdot 0.25\,\mathrm{Hz}}{10\,\frac{\mathrm{m}}{\mathrm{sec}}} = \frac{2\pi}{40\,\mathrm{m}}$ and $\omega = 2\pi \cdot 0.25\,\mathrm{Hz}$. Therefor, in the range $0 < x < L$ we get:

$$y(x,t) = A\sin(kx)\cos(\omega t) = (10\,\mathrm{cm}) \cdot \sin\left(\frac{2\pi \cdot x}{40\,\mathrm{m}}\right)\cos\left(\frac{2\pi \cdot t}{4\,\mathrm{sec}}\right). \tag{7}$$

Sketches of the displacement in the $\hat{y}$-direction as function of the distance from the wall:



c) For calculating the wave energy we consider the kinetic and potential energy in a length segment dx and integrate over the entire string length.
The kinetic energy is given by $dE_k = \frac{1}{2}\mu\dot{y}^2 dx$, where μdx is the mass of a length segment dx.
The potential energy can be calculated by the work needed to stretch a string segment that have a displacement $y(x)$ in the $\hat{y}$ direction. The overall change in the length of the string due to this displacement is $dl = \sqrt{dx^2 + dy^2} - dx = dx\left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} - 1\right) \approx dx\left(1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 - 1\right) = \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx$. The potential energy is thus $dE_p = \frac{1}{2}F_T\left(\frac{dy}{dx}\right)^2 dx$.
Now, we will calculate the derivatives:

$$\dot{y} = -\omega A \sin(kx)\sin(\omega t)$$
$$\Longrightarrow \dot{y}^2 = \omega^2 A^2 \sin^2(kx)\sin^2(\omega t)$$

similarly:

$$\frac{\partial y}{\partial x} = kA\cos(kx)\cos(\omega t)$$
$$\implies \left(\frac{\partial y}{\partial x}\right)^2 = k^2A^2\cos^2(kx)\cos^2(\omega t)$$

The total mechanical energy of the segment is:

$$\begin{aligned} dE &= dE_k + dE_p \\ &= \frac{1}{2}\mu\omega^2A^2\sin^2(kx)\sin^2(\omega t)dx + \frac{1}{2}F_Tk^2A^2\cos^2(kx)\cos^2(\omega t)dx \end{aligned}$$

Using $k^2 = \frac{\mu\omega^2}{F_T}$ we get:

$$\begin{aligned} dE &= \frac{1}{2}\mu\omega^2A^2\Big(\sin^2(kx)\sin^2(\omega t) + \cos^2(kx)\cos^2(\omega t)\Big)dx \\ &= \frac{1}{4}\mu\omega^2A^2\Big(1 + \cos(2kx)\cos(2\omega t)\Big)dx \end{aligned}$$

Finally, performing the integration (where $k = \frac{2\pi}{4L}$) we get:

$$\begin{aligned} E &= \int_0^L \frac{1}{4}\mu\omega^2A^2\Big(1 + \cos(2kx)\cos(2\omega t)\Big)dx \\ &= \frac{1}{4}\mu\omega^2A^2L + \frac{1}{4}\mu\omega^2A^2 \cdot \underbrace{\frac{1}{2k}\sin(2\frac{2\pi}{4L}x)\cos(2\omega t)\Big|_0^L}_{=0} \\ &= \frac{1}{4}\mu\omega^2A^2L = \frac{1}{4}\cdot 0.1\,\frac{\mathrm{kg}}{\mathrm{m}}\cdot(2\pi\cdot 0.25\,\mathrm{Hz})^2\cdot(0.1\,\mathrm{m})^2\cdot 10\,\mathrm{m} \\ &= \frac{\pi^2}{16}\cdot 10^{-2}\,\mathrm{J} \approx 6.17\cdot 10^{-3}\,\mathrm{J} \end{aligned} \tag{8}$$

Due to energy conservation, we can also calculate the energy when we have maximal kinetic energy and zero potential energy. The kinetic energy per unit length of the string is:

$$dE_k(x) = \frac{1}{2}\underbrace{\mu dx}_{m}\dot{y}^2 = \frac{1}{2}\mu dx\omega^2A^2\sin^2(kx)\sin^2(\omega t) \tag{9}$$

The maximum potential energy is when $\omega t = \frac{\pi}{2}$:

$$dE_{k,max}(x) = dE(x) = \frac{1}{2}\mu dx\omega^2A^2\sin^2(kx) \tag{10}$$

$$E = \int_0^L \frac{1}{2}\mu dx\omega^2A^2\sin^2(kx) = \frac{1}{4}\mu\omega^2A^2L = \frac{\pi^2}{16}\cdot 10^{-2}\,\mathrm{J} \approx 6.17\cdot 10^{-3}\,\mathrm{J} \tag{11}$$

d) In a standing wave, without dissipation, the energy is conserved and is spatial bounded by the boundary (i.e., by the two ends of the string), therefor, no energy is being transmitted by the wave.
Another way to look on a standing wave is interference between counter propagating waves with the same amplitude, where each wave transmit the same amount of power but in the opposite direction from the other wave, such that the overall power transmitted by the wave is zero.

Problem 0.3 Heat capacity

a) The angular frequency is:

$$\omega' = \sqrt{\omega_0^2 - \left(\frac{1}{2\tau}\right)^2},$$

where $\omega_0^2 = \frac{k}{M_W} = \frac{40\,\frac{\text{kg}}{\text{sec}^2}}{0.1\,\text{kg}} = 400\,\frac{1}{\text{sec}^2}$ and $\left(\frac{1}{2\tau}\right)^2 = \frac{1}{360000}\,\frac{1}{\text{sec}^2}$. Therefor, $\omega' \approx \omega_0 = \sqrt{400}\,\frac{1}{\text{sec}} = 20\,\frac{1}{\text{sec}}$.

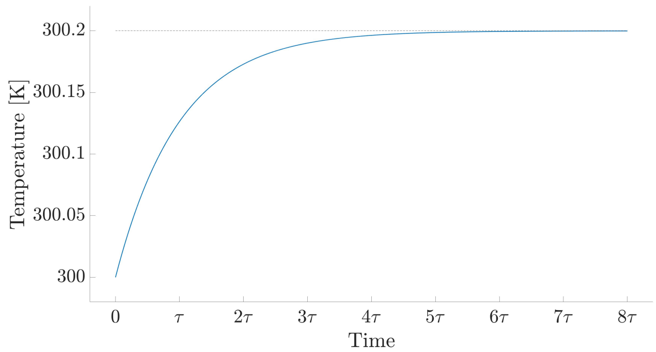
b) The entire mechanical energy of the spring was converted during the oscillation to internal energy of the mass and the Tungsten tube. The initial potential energy of the spring was:

$$E_p = \frac{1}{2}k\Delta L^2 = \frac{1}{2}\cdot 40\,\frac{\text{N}}{\text{m}}\cdot 0.5^2\,\text{m}^2 = 5\,\text{J}$$

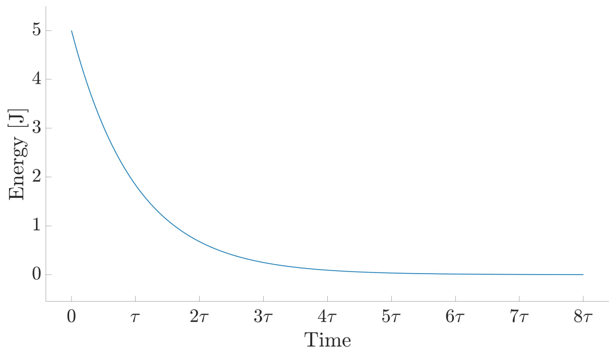
The change in the internal energy is thus:

$$\begin{aligned}\Delta U_{int} &= (M_W \cdot c_W + M\cdot c)\Delta T = E_p\\ \Longrightarrow c &= \frac{1}{M}\left(\frac{E_p}{\Delta T} - M_W\cdot c_W\right)\\ &= \frac{1}{0.1\,\text{kg}}\left(\frac{5\,\text{J}}{0.2\,\text{K}} - 0.1\,\text{kg}\cdot 0.135\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)\\ &= 115\,\frac{\text{J}}{\text{kg}\cdot\text{K}} = 0.115\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}\end{aligned} \tag{12}$$

c) Sketch of the temperature as function of time:



Sketch of the energy of the spring as function of time:



d) In the case of solids, the change in volume due to change in temperature is negligible, therefor we can estimate $\frac{dV}{dT} = 0$. That means that the change in the solid internal energy is due to heat $dU_{int} = dQ$ $(dW = 0)$.

$$
\begin{aligned}
dS &= \int_{T_i}^{T_f} \frac{dQ}{T} = \int_{T_i}^{T_f} \frac{M_W c_W dT}{T} \\
\Longrightarrow \Delta S &= M_W c_W \ln\left(\frac{T_f}{T_i}\right) = 0.1\,\mathrm{kg} \cdot 0.135\,\frac{\mathrm{kJ}}{\mathrm{kg}\cdot\mathrm{K}} \ln\left(\frac{300.2\,\mathrm{K}}{300\,\mathrm{K}}\right) \quad = 8.997 \cdot 10^{-3}\,\frac{\mathrm{J}}{\mathrm{K}}
\end{aligned} \tag{13}
$$

Because the temperature is changing very slowly in the integration we can also approximate it to be constant and the integral becomes:

$$
dS = \int_{T_i}^{T_f} \frac{M_W c_W dT}{T} \approx M_W c_W \frac{\Delta T}{T} \tag{14}
$$

if using the initial temperature we get:

$$
\Delta S \approx M_W c_W \frac{\Delta T}{T_i} = 0.1\,\mathrm{kg} \cdot 0.135\,\frac{\mathrm{kJ}}{\mathrm{kg}\cdot\mathrm{K}} \cdot \frac{0.2\,\mathrm{K}}{300\,\mathrm{K}} = 9 \cdot 10^{-3}\,\frac{\mathrm{J}}{\mathrm{K}}, \tag{15}
$$

while if using the final temperature we get:

$$
\Delta S \approx M_W c_W \frac{\Delta T}{T_i} = 0.1\,\mathrm{kg} \cdot 0.135\,\frac{\mathrm{kJ}}{\mathrm{kg}\cdot\mathrm{K}} \cdot \frac{0.2\,\mathrm{K}}{300.2\,\mathrm{K}} = 8.994 \cdot 10^{-3}\,\frac{\mathrm{J}}{\mathrm{K}}. \tag{16}
$$

Problem 0.4 Brayton cycle

a) We start with calculating the volume and temperature at the beginning of each step. We will use the given molar heat capacity at constant volume, $c_v = R$ (the molar heat capacity at constant pressure is $c_p = c_V + R = 2R$), as well as the Poisson constant $\gamma = \frac{c_p}{c_V} = 2$.

Step 1: Here we just have P_1, V_1 and T_1

Step 2: To calculate the temperature and volume in 2 we can use the $P - V$ relation in adiabatic process, $PV^\gamma = const.$ In addition, we can use the equation of state for ideal gas $PV = \tilde{n}RT$:

The volume:

$$\begin{aligned} P_1 V_1^\gamma &= P_2 V_2^\gamma \\ \Longrightarrow V_2^2 &= \frac{P_1}{P_2} \cdot V_1^2 \\ \Longrightarrow V_2 &= V_1 \Big(\frac{P_1}{4P_1}\Big)^{\frac{1}{2}} = \frac{1}{2} V_1 \end{aligned}$$

The temperature:

$$\begin{aligned} \frac{T_2}{T_1} &= \frac{P_2 V_2}{P_1 V_1} = \frac{4P_1 \cdot \frac{1}{2} V_1}{P_1 V_1} = 2 \\ \Longrightarrow T_2 &= 2T_1 \end{aligned}$$

Step 3: Here we have $P_2 = P_3$ (we now that $T_3 = 8T_1$ - it is the maximum temperature in the cycle), therefor we can use the equation of state to calculate the volume.

$$V_3 = \frac{\tilde{n}RT_3}{P_3} = \frac{8\tilde{n}RT_1}{4P_1} = 2V_1$$

Step 4: This step is similar to step 2 (or 3):

The volume:

$$\begin{aligned} P_4 V_4^\gamma &= P_3 V_3^\gamma \\ \Longrightarrow V_4^2 &= \frac{P_3}{P_4} \cdot V_3^2 \\ \Longrightarrow V_4 &= 2V_1 \Big(\frac{4P_1}{P_1}\Big)^{\frac{1}{2}} = 4V_1 \end{aligned}$$

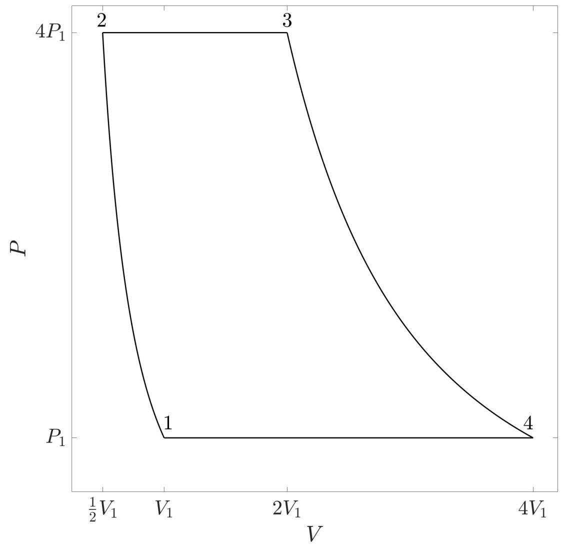
The temperature:

$$T_4 = T_1 \frac{V_4}{V_1} = 4T_1$$

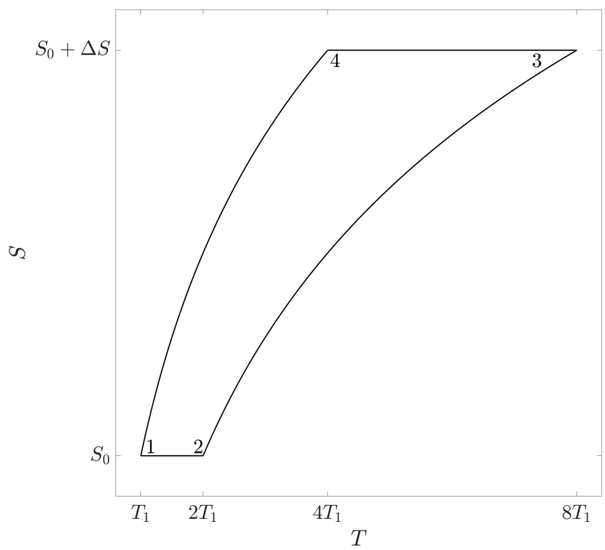
Overall we have:

$$\begin{cases} \text{Step 1: } P_1, & V_1, & T_1 \\ \text{Step 2: } P_2 = 4P_1, & V_2 = \frac{1}{2} V_1, & T_2 = 2T_1 \\ \text{Step 3: } P_3 = 4P_1, & V_3 = 2V_1\,, & T_3 = 8T_1 \\ \text{Step 4: } P_4 = P_1, & V_4 = 4V_1\,, & T_4 = 4T_1 \end{cases} \tag{17}$$

Sketch of the P-V diagram:



Sketch of the S-T diagram:



b) We will calculate the work in each step:

Step 1: in adiabatic process there is no heat transfer $\Delta Q = 0$, therefor:

$$\Delta U_{int} = \Delta W_{on\,gas} = \frac{f}{2}\tilde{n}R\Delta T = \tilde{n}R(T_2 - T_1) = \tilde{n}RT_1 = P_1V_1$$

where f=2 is the number of degrees of freedom of the gas.

Step 2: The work done on the gas in constant pressure is:

$$\Delta W_{on\,gas} = -P\Delta V = -4P_1(2V_1 - \frac{1}{2}V_1) = -6P_1V_1,$$

with $P = P_2$ and $\Delta V = V_3 - V_2$.
The change in the internal energy is:

$$\Delta U_{int} = \frac{f}{2}\tilde{n}R\Delta T = 6\tilde{n}RT_1 = 6P_1V_1$$

The heat that was transferred to the gas is:

$$\Delta Q = \Delta U_{int} - \Delta W = 12P_1V_1$$

Step 3: (Similar to step 1)

$$\Delta U_{int} = \Delta W_{on\,gas} = \frac{f}{2}\tilde{n}R\Delta T = -4\tilde{n}RT_1 = -4P_1V_1$$
$$\Delta Q = 0$$

Step 4: (Similar to step 2)

$$\Delta W_{on\,gas} = -P\Delta V = 3P_1V_1,$$

with $P = P_1$ and $\Delta V = V_1 - V_4$. The change in the internal energy is:

$$\Delta U_{int} = \frac{f}{2}\tilde{n}R\Delta T = -3\tilde{n}RT_1 = -3P_1V_1$$

The heat that was transferred to the gas is:

$$\Delta Q = \Delta U_{int} - \Delta W = -6P_1V_1$$

The total amount of work done by the engine in one cycle is:

$$W_{cyc,\,by} = -W_{cyc,\,on} = 6P_1V_1$$

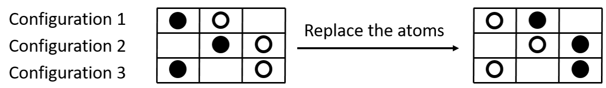
c) The efficiency of the engine is:

$$\eta = \frac{|W|}{Q_h} = \frac{W_{cyc,\,by}}{Q_h} = \frac{6P_1V_1}{12P_1V_1} = \frac{1}{2},$$

where Q_h is the heat that was transferred from the hot reservoir (during step 2).

Problem 0.5 Entropy and heat capacity

a) The entropy relates to the number of possible microstates of the system. We need to count the number of possible microstates of the system, or, in how many arrangement we can position the two atoms. We have $N = 3$ positions and $k = 2$ atoms. The number of possible configuration is $\Omega = \frac{N!}{(N-k)!} = \frac{3!}{(3-2)!} = 6$.



The entropy is $S = k_B \ln(\Omega) = k_B \ln(6) \approx 1.79 k_B$.

b) We have $N = 3$ atoms and each of them have $k = 3$ different possible spin configuration. The number of possible configuration is therefor $\Omega = k^N = 3^3 = 27$. The entropy is $S = k_B \ln(\Omega) = k_B \ln(27) \approx 3.3 k_B$.

c) The molar heat capacity at constant volume, c_v, relates to the change in internal energy by $\Delta U_{int} = \tilde{n} c_v \Delta T$.

From the equipartition theorem we know that energy of $\frac{1}{2}RT$ per mole is associated with every degree of freedom. The number of degrees of freedom for a molecule in the system is $f = 4$ (2 for translation, 2 for rotation). Therefor,

$$U_{int} = f \cdot \tilde{n} \frac{1}{2} RT$$
$$\Delta U_{int} = f \cdot \tilde{n} \frac{1}{2} R \Delta T = \tilde{n} c_v \Delta T$$
$$\Longrightarrow c_v = \frac{f}{2} R = 2R$$

d) When E is very large the molecules cannot rotate any more, therefor the number of available degree of freedom is reduced to $f = 2$ and the molar heat capacity is $c_v = \frac{f}{2} R = R$.

Qualitatively sketch of the molar heat capacity: