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Analysis 3 Summary

Based on the lecture "Analysis 3" held by Prof. M. Iacobelli HS21

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Basics

Let $u = u(x,y,z)$

Partial Derivative: $\partial_x u = \frac{\partial}{\partial x} u = u_x$ (same for y, z).

Gradient: $\nabla u = (u_x \ u_y \ u_z)^T$

Laplacian: $\Delta u = \nabla^2 u = u_{xx} + u_{yy} + u_{zz}$

PDE: Partial Differential Equation

Important equations:

Burger's equation: $u_t - uu_x = 0$ **Transport equation**: $u_y + \overset{\in \mathbb{C}}{c} u_x = 0$

Waveequation: $u_{tt} - c^2 \Delta u = 0$

Heat equation: $\Delta u - K u_t = 0$

Laplace equation: $\Delta u = 0$ **Poisson equation**: $\Delta u = f(u) \neq 0$

Condition Types:

Initial Condition: $(x_0, \overset{t=0}{0})$ **Boundary Condition**: $(\overset{x=0}{0}, t)$

Well posed problem: (otherwise: ill-posed)
1. Existence: Problem has a solution
2. Uniqueness: Problem has only one solution
3. Stability: Small change in eq. or data ⇒ small change in the solution.

Strong solutions: if all the derivatives of the solution that are in the PDE exist and are continuous. Bem: Strong sol. ⇒ weak sol.

Weak solutions: Are only valid in some domain of the problem.

Superposition principle: if a PDE is linear and homogeneous, then any lin. combination of solutions is also a solution.

Classification of PDEs

Order: highest order partial derivative. $u_{xyz} + u_y + u_{xx} = 0 \rightarrow 3$

Linearity: If u and all its partial derivatives only appear linearly

Quasi-linearity: linear in highest order derivative term

Bem: $\overset{\alpha}{\widetilde{u_{xyz}}} \overset{\beta}{\widetilde{u_{xxx}}} + u_{xx} = 0$ is quasilinear!

Semi-linearity: all derivatives are linear (but not u itself) in PDE

Homogeneity: $f(x) = 0$ (no term not depending on u)

ODE solving Methods

linear ODE w. constant coefficients: $\boxed{a_n x^{(n)}(t) + \ldots + a_0 x(t) = f(t)}$

→ general solution: $x(t) = x_p(t) + x_h(t)$

$x_h(t)$: Ansatz: $x_h(t) = e^{\lambda t} \Rightarrow \text{Chp}(\lambda): x^{(n)} \rightarrow \lambda^n \Rightarrow \lambda_1, \lambda_2, \ldots$

$\Rightarrow x_h(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} + \ldots$

$x_p(t)$: Ansatz: $x_p(t) \sim$ similar to $f(x)$ → plug in ODE & find coeff.

separable ODE: $\boxed{\frac{d}{dx} y(x) = f(x) \cdot g(y)} \Rightarrow \frac{1}{g(y)} dy = f(x) dx$

$\Rightarrow G(y) = \int \frac{1}{g(y)} dy = \int f(x) dx = F(x) \Rightarrow$ solve for y

Variation of constant: $\boxed{\frac{d}{dx} y(x) = a(x) \cdot y + b(x)}$

$A(x) = \int a(x) dx \qquad P(x) = \int b(x) e^{-A(x)} dx \Rightarrow y(x) = F \cdot e^{A(x)} + P(x) e^{A(x)}$

Most important ODEs:

$$\begin{cases} x'(t) - \lambda x(t) = 0 \Rightarrow x(t) = Ce^{\lambda t} \\ x''(t) + \lambda x(t) = 0 \Rightarrow x(t) = \alpha \sin(\sqrt{\lambda} t) + \beta \cos(\sqrt{\lambda} t) \\ x''(t) - \lambda x(t) = 0 \Rightarrow x(t) = \alpha \sinh(\sqrt{\lambda} t) + \beta \cosh(\sqrt{\lambda} t) \end{cases}$$

1. Order linear & quasilinear PDEs

↳ how to solve: Method of Characteristics (M.o.C)

general idea: PDE: $F(x, y, u, u_x, u_y) = 0$ → establish a relation between the solution u and the tangent plane to the graph u.

General form:

$$\begin{cases} a(x,y) u_x(x,y) + b(x,y) u_y(x,y) = c(x,y,u) & \text{PDE} \\ u(x_0(s), y_0(s)) = u_0(s) & \text{initial condition} \end{cases}$$

Cooking recipe:

1. Find the initial curve from the initial condition:
   $\Gamma(s) = (x_0(s), y_0(s), \tilde{u}_0(s)) \equiv (x(s,0), y(s,0), \tilde{u}(s,0))$
   $x(s,t)$, $y(s,t)$ are the characteristics. $(\tilde{u}(s,0) = u(s,0))$
2. Solve the following system of ODEs:
   $$\begin{cases} \partial_t x(t,s) = a(x(t,s), y(t,s)) & \text{w. } x(0) = x_0(s) \\ \partial_t y(t,s) = b(x(t,s), y(t,s)) & \text{w. } y(0) = y_0(s) \\ \partial_t u(t,s) = c(x(t,s), y(t,s), u(t,s)) & \text{w. } u(0) = u_0(s) = \tilde{u}_0(s) \end{cases}$$
3. Express s and t depending on x and y
4. Rücktrafo: Insert $s(x,y)$ & $t(x,y)$ in $u(t,s)$ and find $u(x,y)$
   $\tilde{u}(t,s) = u(x(t,s), y(t,s)) \Leftrightarrow \underset{\text{gesucht}}{u(x,y)} = \tilde{u}(t(x,y), s(x,y))$
5. Verify the solution (plug it in PDE and see if it matches).

Transversality condition: must be fulfilled for the problem to have a sol.

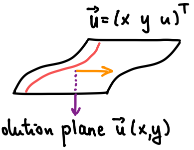
$$\boxed{J = \det \begin{bmatrix} x_t(0,s) & y_t(0,s) \\ x_s(0,s) & y_s(0,s) \end{bmatrix} = \det \begin{bmatrix} a_0(s) & b_0(s) \\ \partial_s x_0 & \partial_s y_0 \end{bmatrix} \neq 0}$$

Where the transversality condition is fulfilled, there exists a local solution, because the mapping $x = x(t,s)$ and $y = y(t,s)$ is invertible. if $J = 0$, then there is 0 or ∞-many solutions :(.

Obstacles to global existence: ↯
- i) solutions can blow up in finite time
- ii) If the characteristics cross the initial curve $\Gamma(s)$ more than once
- iii) If derivatives intersect with each other

Graphical interpretation of M.o.C:

We can rewrite the problem as: $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} u_x \\ u_y \\ -1 \end{pmatrix} = 0$,



normal to solution plane $\vec{u}(x,y)$

so $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is $\perp$ to the normal → so tangent to $\vec{u}$.

By integrating the tangent over t, we get the solution (since tangent = derivative to $\vec{u}$). So we get the following Differential equation:

$\frac{d}{dt} \begin{pmatrix} x \\ y \\ u \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ With this we get the characteristics which are tangent to $\vec{u}$. We can then knit the surface of $\vec{u}$ w. the help of I.C.

Conservation laws:

Conservation laws are PDEs describing the evolution of conserved quantities. Here x is the spacial variable and y the time variable. (d.h. $y \sim t$, $y > 0$)

General formulation: $u = u(x,y)$

$$\boxed{\left.\begin{matrix} u_y + (F(u))_x = 0 \\ u_y + c(u) u_x = 0 \end{matrix}\right\} \text{equivalent}}$$

With initial data $u(x,0) = h(x)$

$u: \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$

$F: \mathbb{R} \rightarrow \mathbb{R}$

$c(u) = \partial_u F(u(x))$

$F(u) = \int c(u) du$

$\boxed{F(u) = \text{flux}}$

Transport equation: $\boxed{u_y + c u_x = 0}$ (special case of conservation law w. $c \in \mathbb{R}$)

How to solve? → nach wie vor M.o.C :)

What's new though? → Now Characteristics are straight lines.

And: The Transversality condition will guarantee local existence and uniqueness of a solution up until the critical time.

Critical time: $\boxed{y_c = \inf_{s \in \mathbb{R}: (c(u_0(s)))_s < 0} \left\{ \frac{-1}{(c(u_0(s)))_s} \right\}}$ Inf = greatest lower bound

$u_0(s)$ einsetzen, dann ableiten.

or $\boxed{y_c = -\left(\inf_{s \in \mathbb{R}} \frac{d}{ds} \left(\frac{df}{du} (u(s,0)) \right) \right)^{-1}}$

and $\boxed{y_c = +\infty}$ if $c(u_0(s))_s \geq 0$. then $u(x,y) = u_0(x - c(u(x,y))y)$

y_c is the time after which there is no smooth solution to the problem.

Bem: If the initial condition $u(x,0)$ is never decreasing, we will not have a critical time y_c :)

Also: if $y_c < 0$, then no critical time (since $y \geq 0$)

But what happens after y_c? → We must introduce weak solutions that satisfy the PDE in each Region D_i ($D = \bigcup_{i=1} D_i$) and the integral form of the PDE in the whole domain D:

$$\boxed{\int_{x_1}^{x_2} [u(x,y)]_{y_1}^{y_2} dx = -\int_{y_1}^{y_2} [f(u(x,y))]_{x=a}^{x=b} dy}$$

u^+: Solution on the right side
u^-: Solution on the left side.

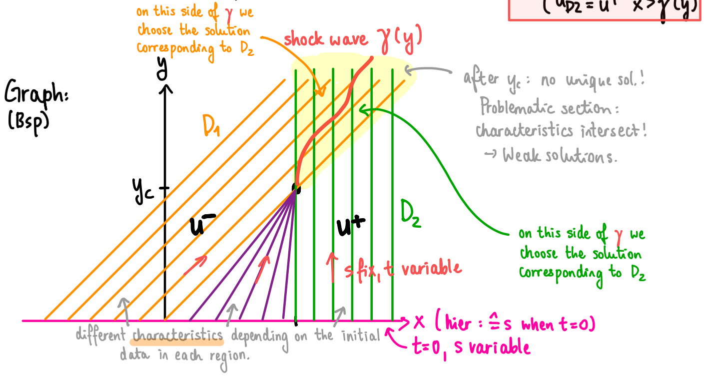
Boundaries between D_i are called shock Waves.

Rankine Hugoniot: $\boxed{\partial_y \gamma(y) = \frac{F(u^+) - F(u^-)}{u^+ - u^-}} \ \hat{=}$ speed of shockwave

Shock Waves have to satisfy the RH-condition!

The weak solution on the problematic section is then: $\boxed{u = \begin{cases} u_{D1} = u^- & x < \gamma(y) \\ u_{D2} = u^+ & x > \gamma(y) \end{cases}}$

Graph: (Bsp)



Entropy condition: $\boxed{c(u^+) < \partial_y \gamma(y) < c(u^-)}$ bzw. $\boxed{F'(u^+) < \partial_y \gamma(y) < F'(u^-)}$

A weak solution satisfies the entropy condition, if characteristics only enter shock waves but do not emanate from them.

⇒ use this to select the best weak solution.

(weak sol. that satisfies entropy condition is better than one that doesn't)

cooking recipe finding weak solutions:

1. Calculate y_c.
   If $\exists y_c \neq \infty$ → classical sol. break down somewhere → we need weak solutions for domain after y_c!
2. Before y_c: solve for classical solutions
3. After y_c: solve for weak solutions:
   - 3.1 Calculate the shock Wave $\gamma(y)$:
     - slope $\gamma'(y)$ (w.R.t. y!) w. Rankine Hugoniot
     - shock wave goes through critical point (x_c, y_c)
       (x_c: find out either graphically or calculate where the characteristics intersect.)
   - 3.2 Check if entropy condition is satisfied.
   - 3.3 On the "left side" of shock wave: $u = u^-$ on the "right side": $u = u^+$
4. Write down the whole solution (depending on domain & time section!)

Important property: Sol. of conservation laws are constant along their characteristics $(x(t,s), y(t,s), \tilde{u}(t,s))$, which are straight lines.
$\forall s \in \mathbb{R}$, the characteristic through point $(x,y) = (s,0)$ is the line in the (x,y)-plane through $(s,0)$ with slope $\frac{1}{c(u_0(s))}$ and on this line, u takes the constant value $u_0(s)$.

→ fix s and vary t. constant means: $\partial_t(\text{characteristics}) = 0$

Also: The transport eq. does not smooth out w. time. Dh: if there are any singularities, these stay & are transported along the characteristics!

2. Order PDEs

General formulation: $\boxed{L\{u\} = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g}$ principal part

where a, b, c, d, e, f, g can depend on x and y.

Bem: We can describe a 2nd order lin. PDE as an Operator $L\{u\}$.

Classification: Discriminant: $\boxed{\delta(L) = b^2 - ac}$

↳ Hyperbolic: $\delta(L) > 0$ → d'Alemberts formula
↳ Parabolic: $\delta(L) = 0$
↳ Elliptic: $\delta(L) < 0$
} Separation of variables

Hyperbolic PDEs: The Wave equation in ℝ

Homogeneous Wave equation: $\boxed{u_{tt} - c^2 u_{xx} = 0}$

$u = u(x,t)$
$(x,t) \in \mathbb{R} \times (0, +\infty)$
$c \in \mathbb{R}$: Wave speed

To solve this equation, we introduce new variables:

$\boxed{\xi(x,t) = x + ct}$ and $\boxed{\eta(x,t) = x - ct}$

$w(\xi, \eta) = u(x(\xi,\eta), t(\xi,\eta))$ Recall: $w(\xi(x,t), \eta(x,t))_t = w_\xi \frac{\partial \xi}{\partial t} + w_\eta \frac{\partial \eta}{\partial t}$

$$\left.\begin{aligned} u_{tt} &= c^2(w_{\xi\xi} - 2w_{\xi\eta} + w_{\eta\eta}) \\ u_{xx} &= w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta} \end{aligned}\right\} \text{eq: } u_{tt} - c^2 u_{xx} = -4c^2 w_{\xi\eta} = 0$$

⇒ $\boxed{w_{\xi\eta} = 0}$ is the canonical form of the wave equation.

This equation implies Separation of Variables (SoV):

$w(\xi, \eta) = F(\xi) + G(\eta)$ so $u(x,t) = \underbrace{F(x+ct)}_{\text{backward travelling}} + \underbrace{G(x-ct)}_{\text{forward travelling wave}}$

F, G are characteristics and: $\begin{cases} F \text{ is constant along } x+ct \\ G \text{ is constant along } x-ct \end{cases}$

Classification of PDEs in these variables:

$u(x,y) \mapsto w(\xi,\eta) = w(\xi(x,y), \eta(x,y))$

$$\begin{cases} \text{Hyperbolic: } & w_{\xi\eta} + \tilde{d}w_\xi + \tilde{e}w_\eta + \tilde{f}w = \tilde{g} \\ \text{Parabolic: } & w_{\xi\xi} + \tilde{d}w_\xi + \tilde{e}w_\eta + \tilde{f}w = \tilde{g} \\ \text{Elliptic: } & w_{\xi\xi} + w_{\eta\eta} + \tilde{d}w_\xi + \tilde{e}w_\eta + \tilde{f}w = \tilde{g} \end{cases}$$

Cauchy Problem w. homogeneous waveequation: Without B.C.!

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & (x,t) \in \mathbb{R} \times (0, \infty) \\ u(x,0) = f(x) & x \in \mathbb{R} \quad \text{initial position} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \quad \text{initial velocity} \end{cases}$$

} initial conditions

The solution is given by: **d'Alembert's formula for homogeneous Waveequations:**

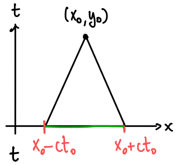
$$\boxed{u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\, dy}$$

Properties:

- Singularities propagate along the characteristics
- If g and f are even/odd/periodic then u is also even/odd/periodic
  ↳ So: if a problem is well posed only for some x, we can even- or odd-extend the problem and solve with d'Alembert's formula.
  ↳ such that we can eliminate the B.C.

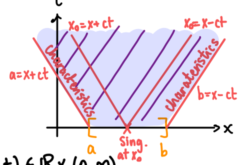
Domain of dependence:

The solution in (x_0, y_0) depends on $f(x_0 + ct_0)$, $f(x_0 - ct_0)$ and g in the interval $[x_0 - ct_0, x_0 + ct_0]$



Region of influence:

All points satisfying $x - ct \leq b$, $x + ct \geq a$ are dependent on the initial condition on the interval $[a,b]$



Inhomogeneous Wave equation:

Cauchy Problem: $\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) \neq 0 & (x,t) \in \mathbb{R} \times (0, \infty) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$

F(x,t) = external force acting on the Wave

Bem: no B.C.

There are 2 ways to solve this Problem:

1) d'Alembert for inhomogeneous Waveequation:

zuerst nach ξ integrieren, danach nach τ

$$\boxed{u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds + \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi, \tau)\, d\xi\, d\tau}$$

2) Superposition principle:

- Find/guess a $v(x,t)$ such that $v_{tt} - c^2 v_{xx} = F(x,t)$
- Use superposition for linear PDE: Since u solves the C.P. and v is as defined above → define $w = u - v$:

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0 \\ w(x,0) = u(x,0) - v(x,0) \\ w_t(x,0) = u_t(x,0) - v_t(x,0) \end{cases}$$

⇒ We now have a homogeneous Cauchy Problem! :)

- Solve for w (using d'Alembert for homogeneous Waveequation
- finally: $u = w + v$

Bem: If we want to solve a waveequation on some domain, we extend the problem and make the initial conditions coincide w. the initial problem.

Wave equation with Boundary Conditions: 2 Cases:

Case 1: If we can use symmetry to "eliminate" the B.C: Extend the Problem w. suitable f(x), g(x), then use homogeneous/inhomogeneous d'Alembert depending on PDE.

Case 2: Use Separation of Variables (genauer in section w. Heat eq.)

Parabolic PDEs - The Heat equation

General form of the heat equation: $\boxed{u_t - Ku_{xx} = 0}$

Homogeneous 2nd order linear PDE

Cauchy problem:

$$\begin{cases} u_t - Ku_{xx} = 0 & (x,t) \in [0,L] \times [0, \infty) \quad \text{PDE} \\ u(x,0) = f(x) & \text{Initial Condition} \\ \text{one of: } \begin{cases} u(0,t) = u(L,t) = 0 & \text{Dirichlet} \\ u_x(0,t) = u_x(L,t) = 0 & \text{von Neumann} \\ \text{mixed} \end{cases} \end{cases}$$

Boundary condition:

How to solve these Problems: (used to solve linear PDEs in general).

Separation of Variables: $u(x,t) = X(x)T(t)$

For the heat equation: We follow the following procedure:

1) Identify the problem:
   - 1.1) PDE
   - 1.2) Boundary condition
   - 1.3) Initial condition
2) Apply separation of variables to PDE and extract ODEs. Assume $u(x,t) = X(x)T(t)$

   $XT' - c^2 X''T = 0 \Leftrightarrow \frac{T'}{c^2 T} = \frac{X''}{X} \overset{!}{=} -\lambda$

   - 2.1) ODE for X (w. B.C.)
   - 2.2) ODE for T (w. I.C.)

   $\begin{cases} \lambda > 0 \\ \lambda = 0 \\ \lambda < 0 \end{cases}$
3) Find general solution for X → make case distinction for λ & find out λ
4) Find general solution for T (use λ we found out in 3))
5) Formulate general solution for $u(x,t) = X(x)T(t)$

   In general: We do not want the trivial sol. $u(x,y) = 0$.

   Don't forget: general solution = linear combination of all possible solutions!
6) Use the initial conditions to determine the coefficients
7) Enjoy and write down the full solution. But don't panic! There are Shortcuts :)

① Homogeneous Wave & Heat equations w. homogeneous Boundary Conditions

Heat: $u_t - c^2 u_{xx} = 0$

Initial conditions: $u(x,0) = f(x)$

Wave: $u_{tt} - c^2 u_{xx} = 0$

$u(x,0) = f(x)$, $u_t(x,0) = g(x)$

1. General solution for T:

   Heat: $\boxed{T_n = e^{-c^2 \left(\frac{n\pi}{L}\right)^2 t}}$

   Wave: $\boxed{T_n = A_n \cos\left(\frac{n\pi}{L} ct\right) + B_n \sin\left(\frac{n\pi}{L} ct\right)}$

2. General solution for X: (depending on boundary conditions):

   Dirichlet: $u(0,t) = u(L,t) = 0$ ⇒ $\boxed{X_n = \alpha_n \sin\left(\frac{n\pi}{L} x\right)}$ $n = 1, 2, 3, \dots$

   von Neumann: $\begin{cases} u_x(0,t) = u_x(L,t) = 0 \\ \text{or} \\ u_x(0,t) = u_x(L,t) \end{cases}$ ⇒ $\boxed{X_n = \alpha_n \cos\left(\frac{n\pi}{L} x\right)}$ $n = 0, 1, 2, \dots$

   } $\lambda_n = \left(\frac{n\pi}{L}\right)^2$

3. Finally, combine both: → at exam: show that these indeed satisfy the B.C. !

⇒ **Heat equation** $u_t - Ku_{xx} = 0$ homogeneous !!

Dirichlet B.C: $\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) e^{-K\left(\frac{n\pi}{L}\right)^2 t}}$

w. $A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$ ↙ Fourier decomposition :) or take directy from I.C.

von Neumann B.C: $\boxed{u(x,t) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi}{L}x\right) e^{-K\left(\frac{n\pi}{L}\right)^2 t}}$

w. $B_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$ or take directly from I.C.

⇒ **Wave equation:** $u_{tt} - c^2u_{xx} = 0$ homogeneous !!

Dirichlet B.C: $\boxed{u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[A_n\cos\left(\frac{n\pi}{L}ct\right) + B_n\sin\left(\frac{n\pi}{L}ct\right)\right]}$

w. $A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$ or take directly from I.C.

$B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$

von Neumann : $\boxed{u(x,t) = \frac{A_0 + B_0 t}{2} + \sum_{n=1}^{+\infty}\cos\left(\frac{n\pi}{L}x\right)\left[A_n\cos\left(\frac{cn\pi}{L}t\right) + B_n\sin\left(\frac{cn\pi}{L}t\right)\right]}$

w. $A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$ or take directly from I.C.

$B_0 = \frac{2}{L}\int_0^L g(x)\,dx$, $B_n = \frac{2}{cn\pi}\int_0^L g(x)\cos\left(\frac{n\pi}{L}x\right)dx$

But what do I do for the inhomogeneous Heat-/Waveequation?

② Homogeneous Boundary Conditions, inhomogeneous PDE:

1. Use the solution for the homogeneous case without solving for T(t):

$$u(x,t) = \begin{cases} \sum_{n=1}^{\infty} T_n(t)\underbrace{\sin\left(\frac{n\pi}{L}x\right)}_{X(x)} & \text{Dirichlet B.C.} \\ \sum_{n=0}^{\infty} T_n(t)\underbrace{\cos\left(\frac{n\pi}{L}x\right)} & \text{von Neumann B.C.} \end{cases}$$

⚠ If mixed: do the whole derivation. Find X by solving the corresponding homogeneous Problem.

2. Then express the inhomogenity in the corresponding basis, And insert :

$$\sum_n T_n'' X_n(x) - c^2\sum_n T_n X_n''(x) = h(x,t) = \sum_n c_n(t)X_n(x)$$

$$\left.\begin{array}{l}\sum_n T_n(0)X_n(x) = f(x) = \sum_n c_nX_n(x) \\ \sum_n T_n'(0)X_n(x) = g(x) = \sum_n c_nX_n(x)\end{array}\right\}\text{initial conditions}$$

We thus get: $\begin{cases} T_n''(t) + c^2\left(\frac{n\pi}{L}\right)^2 T_n(t) = c_n(t) \\ T_n(0) = \frac{2}{L}\int_0^L X_n(x)f(x)\,dx \\ T_n'(0) = \frac{2}{L}\int_0^L X_n(x)g(x)\,dx \end{cases}$ ⇒ solve these ODE for each n. Then use I.C. to determine the coefficients.

③ If the Boundary Conditions are inhomogeneous:

1. find a $w(x,t)$ that satisfies the Boundary Conditions & define $\vartheta = u - w$.
2. Then solve for ϑ like ① or ② depending on Homogeneity of PDE.
3. Finally, $u = \vartheta + w$

Elliptic PDEs – The Laplace & Poisson equation

Laplace equation: $\Delta u = u_{xx} + u_{yy} = 0$

Poisson equation: $\Delta u = u_{xx} + u_{yy} = \rho(x,y)$ (inhomogeneous Laplace)

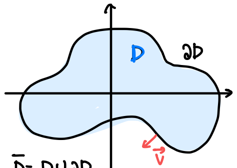
$u(x,y)$ is a harmonic function if it solves the Laplace equation.

For elliptic equations: x and y are both spacial variables (so no time variable)

Dirichlet Problem:

$$\begin{cases} \Delta u = \rho(x,y) & (x,y)\in D \\ u = g(x,y) & (x,y)\in\partial D \end{cases}$$

D is a domain in $\mathbb{R}^2$:



$\bar{D} = D \cup \partial D$

$\vec{v}$ = Normal vector on ∂D

∂_v = Richtungsableitung in Richtung von $\vec{v}$.

Neumann Problem:

$$\begin{cases} \Delta u = \rho(x,y) & (x,y)\in D \\ \partial_v u = \vec{v}\cdot\vec{\nabla u} = g(x,y) & (x,y)\in\partial D \end{cases}$$

Problem of the 3rd kind:

$$\begin{cases} \Delta u = f(x,y) & (x,y)\in D \\ u + \partial_v u = g(x,y) & (x,y)\in\partial D \end{cases}$$

Maximum / Minimum:

Conditions for a Maximum in (x_0,y_0): $\begin{cases} \nabla u(x_0,y_0) = 0 \\ \Delta u(x_0,y_0) \le 0 \\ u_{xx}(x_0,y_0) \le 0 \\ u_{yy}(x_0,y_0) \le 0 \end{cases}$

Conditions for a Minimum in (x_0,y_0): $\begin{cases} \nabla u(x_0,y_0) = 0 \\ \Delta u(x_0,y_0) \ge 0 \\ u_{xx}(x_0,y_0) \ge 0 \\ u_{yy}(x_0,y_0) \ge 0 \end{cases}$

Existence of solution to the Neumann Problem: A necessary condition for the existence of a solution to the Neumann Problem is:

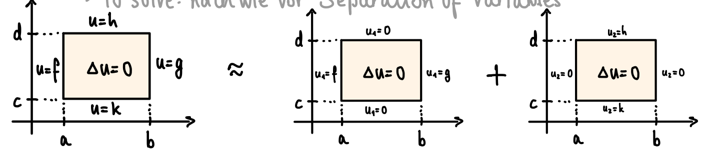
$$\boxed{\int_{\partial D} \overset{\text{Wegintegral}}{g(x(s),y(s))\,ds} = \int_D \overset{\text{Flächenintegral}}{\rho(x,y)}\,dx\,dy_{\text{über SF}}}$$

Inhomogeneity of the Laplace equation

Proof: Gauss Theorem

Laplace's equation in rectangular & circular domains:

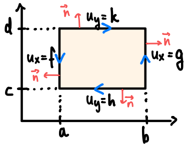
↳ To solve: nach wie vor Separation of Variables



General Procedure:

1. check the necessary conditions for the existance of a solution:

Neumann: $\boxed{\oint_{\partial D}\partial_n u(s)\,ds = \int_c^d g\,dy - \int_c^d f\,dy + \int_a^b k\,dx - \int_a^b h\,dx \overset{!}{=} 0}$



Remember: $\partial_n = \nabla\cdot\vec{n}$, $\vec{n}$ shows away from domain.

Dirichlet: the Boundary conditions must be continuous:

$$k(a,c) \overset{!}{=} f(a,c) \wedge g(b,c) = k(b,c) \wedge g(b,d) = h(b,d) \wedge h(a,d) = f(a,d)$$

Bem: Sometimes, these conditions are not met. In these cases, try to modify the problem slightly by adding some function to u.

2. If needed: split the problem in 2 sub-problems such that boundaries are zero on opposite sides. Then verify the conditions (Step 1.) again.

↱ when B.C. condition is not satisfied

If needed: introduce $\tilde{u} = u + \alpha(x^2 + y^2)$ and find α such that the condition is fulfilled.

If $\Delta u \neq 0$ (Poisson) → find f such that $\vartheta := u + f$ is $\Delta\vartheta = 0$

Or for D.B.C: Add a harmonic polynomial $p_H(x,y) = a_0 + a_1x + a_2y + a_3xy$ → $\tilde{u} = u + p_H$. Then scale a_i to make the boundaries of $\tilde{u}$ continuous.

3. Solve problems for u_1 and u_2 (or $\tilde{u}_1$ and $\tilde{u}_2$, or simply u) w. separation of variables: $u = X(x)Y(y)$ → use B.C. to find coefficients
   - In the homogeneous direction (here: x for u_2, y for u_1) ↖ solve for this direction first → "find out" λ_n

D.B.C: $X = \sum_{n=1}^{\infty} A_n \sin\left(\sqrt{\lambda_n}\,(x - \underline{a})\right)$

$Y = \sum_{n=1}^{\infty} A_n \sin\left(\sqrt{\lambda_n}\,(y - \underline{c})\right)$ ↖ offsets!

N.B.C: $X = \alpha_0 x + \beta_0 + \sum_{n=1}^{\infty} A_n\cos\left(\sqrt{\lambda_n}\,(x-a)\right)$ $\lambda_n = \left(\frac{\pi n}{b-a}\right)^2$ if solving for X_n

$Y = \alpha_0 x + \beta_0 + \sum_{n=1}^{\infty} A_n\cos\left(\sqrt{\lambda_n}\,(y-c)\right)$ $\lambda_n = \left(\frac{\pi n}{d-c}\right)^2$ if solving for Y_n

   - In the other direction:

D.B.C: $X = \sum_{n=1}^{\infty} C_n\sinh\left(\sqrt{\lambda_n}\,(x-a)\right) + D_n\sinh\left(\sqrt{\lambda_n}\,(x-b)\right)$

$Y = \sum_{n=1}^{\infty} C_n\sinh\left(\sqrt{\lambda_n}\,(y-c)\right) + D_n\sinh\left(\sqrt{\lambda_n}\,(y-d)\right)$

N.B.C: $X = \alpha_0 x + \beta_0 + \sum_{n=1}^{\infty} C_n\cosh\left(\sqrt{\lambda_n}\,(x-a)\right) + D_n\cosh\left(\sqrt{\lambda_n}\,(x-b)\right)$

$Y = \alpha_0 x + \beta_0 + \sum_{n=1}^{\infty} C_n\cosh\left(\sqrt{\lambda_n}\,(y-c)\right) + D_n\cosh\left(\sqrt{\lambda_n}\,(y-d)\right)$

4. Use B.C. to find the coefficients
5. If needed: add both solutions: $u = u_1 + u_2$
6. Subtract $\alpha(x^2-y^2)$ (p_H) if used: $u = \tilde{u} - \alpha(x^2-y^2)$ $(u = \tilde{u} - p_H)$

Laplace equation in Polar coordinates:

Laplace Operator: $\boxed{\Delta w = w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\theta\theta}}$

Problem: $\boxed{\Delta w = 0 \text{ inside a given domain + B.C.}}$

Method: nach wie vor Separation of Variables: $w(r,\theta) = R(r)\Theta(\theta)$

General solutions:

$\Theta_n(\theta) = \begin{cases} A_0 & n = 0 \\ A_n\sin(n\theta) + B_n\cos(n\theta) & n \neq 0 \end{cases}$ $n = \sqrt{\lambda_n}$

$R_n(r) = \begin{cases} C_0 + \underline{D_0\log(r)} & n = 0 \\ C_n\cdot r^n + \underline{D_n r^{-n}} & n \neq 0 \end{cases}$ Discard if $r = 0 \in D$

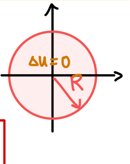
General final solution:

$$\boxed{w(r,\theta) = E + D_0\log(r) + \sum_{n=1}^{+\infty}\left[A_n\sin(n\theta) + B_n\cos(n\theta)\right]r^n + \left[C_n\sin(n\theta) + D_n\cos(n\theta)\right]\cdot r^{-n}}$$

Boundary Types: Bem: ∀Types: insert B.C. in general sol & find coefficients.

Type I: Circle: $\bar{D} = \{0 \le r \le R,\ 0 \le \theta < 2\pi\}$

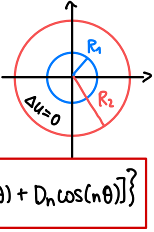
Boundary Conditions: $\begin{cases} w(R,\theta) = f(\theta) & \text{given} \\ \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases}$ } periodicity conditions



General solution: $\boxed{w(r,\theta) = C_0 + \sum_{n=1}^{+\infty} r^n\left(A_n\sin(n\theta) + B_n\cos(n\theta)\right)}$

Type II: Ring: $\bar{D} = \{R_1 \le r \le R_2,\ 0 \le \theta < 2\pi\}$

Boundary Conditions: given: $\begin{cases} w(R_1,\theta) = f(\theta) \\ w(R_2\theta) = g(\theta) \end{cases}$ periodicity: $\begin{cases} \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta(2\pi) \end{cases}$

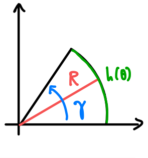


General solution:

$$\boxed{w(r,\theta) = E + F\log(r) + \sum_{n=1}^{\infty}\left\{r^n\left[A_n\sin(n\theta) + B_n\cos(n\theta)\right] + r^{-n}\left[C_n\sin(n\theta) + D_n\cos(n\theta)\right]\right\}}$$

Type III: Circle Section: $\bar{D}=\{0\leq r\leq R\ ,\ 0\leq\theta\leq\gamma\}$

Boundary conditions: $w(R,\theta)=h(\theta)$



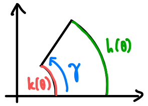
In this case, we only look at N.B.C & D.B.C:

D.B.C: $\begin{cases}\Theta(0)=0\\ \Theta(\gamma)=0\end{cases}$ N.B.C: $\begin{cases}\Theta'(0)=0\\ \Theta'(\gamma)=0\end{cases}$

$$\boxed{w(r,\theta)=\sum_{n=1}^{+\infty}A_n\sin\left(\frac{n\pi}{\gamma}\theta\right)r^{\frac{n\pi}{\gamma}}}\qquad\boxed{w(r,\theta)=A_0+\sum_{n=1}^{+\infty}A_n\cos\left(\frac{n\pi}{\gamma}\theta\right)r^{\frac{n\pi}{\gamma}}}$$

Type IV: Ring section: $\bar{D}=\{R_1\leq r\leq R_2\ ,0\leq\theta\leq\gamma\}$

Boundary conditions: $\begin{cases}w(R_1,\theta)=k(\theta)\\ w(R_2,\theta)=h(\theta)\end{cases}$



In this case, we only look at D.B.C and N.B.C:

D.B.C: $\begin{cases}\Theta(0)=0\\ \Theta(\gamma)=0\end{cases}\Rightarrow$ $\boxed{w(r,\theta)=\sum_{n=1}^{+\infty}A_n\sin\left(\frac{n\pi}{\gamma}\theta\right)r^{\frac{n\pi}{\gamma}}+B_n\sin\left(\frac{n\pi}{\gamma}\theta\right)r^{-\frac{n\pi}{\gamma}}}$

N.B.C: $\begin{cases}\Theta'(0)=0\\ \Theta'(\gamma)=0\end{cases}\Rightarrow$ $\boxed{w(r,\theta)=A_0+B_0\log(r)+\sum_{n=1}^{+\infty}A_n\cos\left(\frac{n\pi}{\gamma}\theta\right)r^{\frac{n\pi}{\gamma}}+B_n\cos\left(\frac{n\pi}{\gamma}\theta\right)r^{-\frac{n\gamma}{\gamma}}}$

Theorems and Principles

For 1st order PDEs:

Existence and uniqueness Theorem for 1st order PDEs:

Assume $\exists s_0\in\mathbb{R}$ so that the transversality condition holds. The $\exists!$ (there exists a unique) solution u of the Cauchy problem defined in a neighborhood of $(x(0,s_0),y(0,s_0))$.

For hyperbolic equations:

Uniqueness of the Solution of the Waveequation:

The solution to $\begin{cases}u_{tt}-c^2u_{xx}=F(x,t) & ,x\in\mathbb{R}\ ,t\in(0,+\infty)\\ u(x,0)=f(x) & ,x\in\mathbb{R}\\ u_t(x,0)=g(x) & ,x\in\mathbb{R}\end{cases}$ is unique.

For elliptic equations: remember: solutions are harmonic functions

Uniqueness for the Dirichlet Problem for the Poisson equation

$\begin{cases}\Delta u=f & \text{in } D\\ u=g & \text{on } \partial D\end{cases}$ then the Problem has at most one solution $u\in C^2(D)\cap C(\bar{D})$ ⚠ Sol. for Laplace is not unique

Weak Maximum / Minimum principle:

Let D be a bounded (→ darf auch offen sein) Domain & $u(x,y)\in C^2(D)\cap C(\bar{D})$ a harmonic function. Then u will take its maximum/minimum on ∂D.

$$\Rightarrow\ \boxed{\max_{\bar{D}}u=\max_{\partial D}u}\quad\&\quad\boxed{\min_{\bar{D}}u=\min_{\partial D}u}$$

Proof: let $u_\varepsilon(x,y)=u(x,y)+\varepsilon(x^2+y^2)$ w. $\varepsilon>0$, u harmonic.
If $\max u_\varepsilon(x,y)=u(x_0,y_0)\in D$ then $\Delta u_\varepsilon(x_0,y_0)\overset{!}{\leq}0$. But on the other hand, $\Delta u_\varepsilon(x_0,y_0)=\Delta u+\Delta\varepsilon(x^2+y^2)\big|_{(x_0,y_0)}=\varepsilon(2+2)=4\varepsilon>0$ ↯
So u_ε must obtain its maximum on ∂D.

$$\Rightarrow\max_{\bar{D}}u\leq\max_{\bar{D}}v=\max_{\partial D}v=\max_{\partial D}u+\max_{\partial D}\varepsilon(x^2+y^2)$$

And for $\varepsilon\to0$ we have $\max_{\bar{D}}u\leq\max_{\partial D}u$ ■

Mean value Theorem:

Let $u(x,y)$ be harmonic in D and let $B_R(x_0,y_0)\subseteq D$ be a Ball of radius R centered in (x_0,y_0). Then:

$$\boxed{u(x_0,y_0)=\frac{1}{2\pi R}\int_{\partial B_R(x_0,y_0)}u(x(s),y(s))\,ds=\frac{1}{2\pi}\int_0^{2\pi}u(x_0+R\cos\theta,y_0+R\sin\theta)\,d\theta}$$

in Words: "the central value of u ($u(x_0,y_0)$) is equal to the average value of u along the boundary of the Ball centered at (x_0,y_0)"

Strong maximum principle:

Let $u(x,y)$ be harmonic in D. If u reaches its maximum inside D, then u is constant on all D. analogeously for minimum.

Proof: Use the mean value principle and the fact that in a circle around (x_0,y_0), the values must average (x_0,y_0), so they must all be equal to $u(x_0,y_0)$ since $u(x_0,y_0)$ is the maximum. Then the function is ¢ on a circle that we can expand on all D.

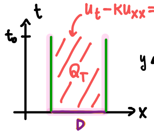
For Parabolic equations:

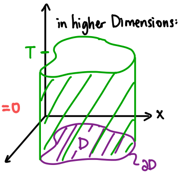
Weak Maximum / Minimum Principle for the heat equation:

Let $u_t=k\Delta u$ on the Domain $Q_T=[0,T]\times D$. The parabolic boundary is defined as: $\partial_pQ_T=\{\underline{\{0\}\times D}\cup\underline{[0,T]\times\partial D}\}$.

Now, let u be a sol. of $u_t=k\Delta u$ on Q_T and let D be a bounded domain. Then u reaches its maximum / minimum on ∂_pQ_T.

$$\boxed{\max_{Q_T}u=\max_{\partial_pQ_T}u}$$





↳ Very useful to prove the uniqueness of the solution to the Poisson eq.

Uniqueness of the Dirichlet Problem for the Heat equation:

$\begin{cases}u_t-k\Delta u=f & \text{in } Q_T\\ u(0,x)=g & \text{on } D\\ u(t,x)=h & \text{on } [0,T]\times\partial D\end{cases}$ has a unique solution.

Extra

Tipps for Proofs:

- When you have to prove that a solution is unique: Assume that u_1 and u_2 solve the problem. then set $v=u_1-u_2$ and prove that $v\equiv0$ (by argumenting w. Max/Min) → then $u_1=u_2$.
- When you have to prove the weak maximum principle:
  Try to show that $\max_{\partial D}u\geq\max_{\bar{D}}u$ $\min_{\partial D}u\leq\min_{\bar{D}}u$
  The other way round is obvious since $\partial D\subset D$ ⇒ you thus get "="

Show that u is a weak solution:

1. u has to satisfy the PDE in each region
2. (If any) discontinuities: Across discontinuities, the shock wave has to satisfy the Rankine-Hugoniot condition.

Wichtige Ungleichungen: $\max(a-b)\leq\max a-\min b\ \wedge\ \min(a-b)\geq\min a-\max b$

$$\min a+\min b\leq\min(a+b)\leq\begin{Bmatrix}\max a+\min b\\ \min a+\max b\end{Bmatrix}\leq\max(a+b)\leq\max a+\max b$$

Max/Min auf Rand finden:

Sei $\gamma:[a,b]\to\mathbb{R}^n$, $t\mapsto\gamma(t)$ die Parametrisierung des Randes. Dann sind die Kandidaten für Max/ Min auf den Rand die Punkte, die $\boxed{\frac{d}{dt}(f\circ\gamma)(t)=0}$ erfüllen.

Richtungsableitung:

$$\boxed{\partial_\nu f(\vec{x})=\lim_{h\to0}\frac{f(\vec{x}+h\vec{\nu})-f(\vec{x})}{h}}$$

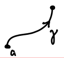
$\vec{x}$ und $\vec{\nu}$ in denselben Koordinaten!

Wegintegral über Skalarfeld:

falls γ geschlossen: Gebiet auf meiner linken Seite!

$$\boxed{\int_\gamma f(x)\,dx=\int_a^b f(\gamma(t))\,\|\dot{\gamma}(t)\|\,dt}$$

w. $\gamma:[a,b]\longrightarrow\mathbb{R}^n$, $t\mapsto\gamma(t)$



Transformationsregel bei Integralen:

$$\boxed{\int_{\Phi(\Omega)}f(\vec{y})\,d\vec{y}=\int_\Omega f(\Phi(\vec{x}))\,|\det d\Phi(x)|\,dx}$$

Polar: $\iint f(x,y)\,dx\,dy=\int_0^{2\pi}\int_0^R f(r\cos\theta,r\sin\theta)\,r\,dr\,d\theta$ Kugel: $r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\oint_{B_R}f(s)\,ds=\int_0^{2\pi}f(x(r,\theta),y(r,\theta))\,R\,d\theta$$

Kettenregel:

1D: $\big(f(u(s))\big)_s=\partial_uf(u(s))\,\partial_su(s)$ $\quad\frac{d}{ds}f(u(s))=\frac{df}{du}\cdot\frac{du}{ds}$

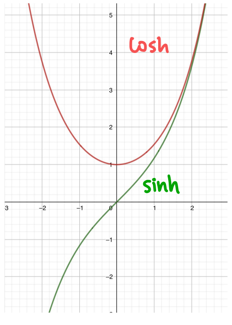
2D: $\frac{\partial}{\partial s}f(x(s,t),y(s,t))=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$

$$\frac{\partial^2}{\partial s^2}f(x(s,t),y(s,t))=\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}\right)=\frac{\partial f}{\partial x}\frac{\partial^2x}{\partial s^2}+\frac{\partial f}{\partial y}\cdot\frac{\partial^2y}{\partial s^2}$$

⚠ The chain rule is rigorous only for smooth functions. It is false for functions with jumps!

Trigrules:

- $\cos(x)=\frac{1}{2}(e^{ix}+e^{-ix})$
- $\sin(x)=\frac{1}{2i}(e^{ix}-e^{-ix})$
- $\cos^2(x)+\sin^2(x)=1$
  $\begin{cases}\cos(-x)=\cos(x)\\ \sin(-x)=-\sin(x)\\ \tan(-x)=-\tan(x)\end{cases}$
- $\cosh(x)=\frac{1}{2}(e^x+e^{-x})$
- $\sinh(x)=\frac{1}{2}(e^x-e^{-x})$
- $\tan(x)=\frac{\sin(x)}{\cos(x)}$
- $\sin'(x)=\cos(x)$
- $\sin^3x=\frac{1}{4}(3\sin x-\sin(3x))$
- $\sin(x\pm y)=\sin(x)\cos(y)\pm\sin(y)\cos(x)$
- $\cos^3x=\frac{1}{4}(3\cos x+\cos(3x))$
- $\cos(x\pm y)=\cos(x)\cos(y)\mp\sin(x)\sin(y)$
- $\arctan\left(\frac{1}{x}\right)=\frac{\pi}{2}-\arctan(x)$
- $\sin^2(x)=\frac{1}{2}-\frac{1}{2}\cos(2x)$
- $\cos^2(x)=\frac{1}{2}+\frac{1}{2}\cos(2x)$
- $\sin\alpha\sin\beta=\frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$
- $\cos\alpha\cos\beta=\frac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta))$
- $\sin\alpha\cos\beta=\frac{1}{2}(\sin(\alpha-\beta)+\sin(\alpha+\beta))$
- $\sin\alpha\pm\sin\beta=2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$
- $\cos\alpha+\cos\beta=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
- $\cos\alpha-\cos\beta=-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)$



- $\int_0^{2\pi}\sin^{(2n+1)}(x)\,dx=\int_0^{2\pi}\cos^{(2n+1)}(x)\,dx=0\quad n\in\mathbb{N}$
- $\int_0^{2\pi}\sin^2(x)\,dx=\int_0^{2\pi}\cos^2(x)\,dx=\pi$