Disclaimer: Diese Zusammenfassung wurde im Rahmen der Vorlesung "Signal und Systemtheorie II" von Prof. Lygeros im FS22 geschrieben.

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Signal and Systems Theory II

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0. LinAlg stuff

Inverse: $\exists !$ if $\det(A)\neq 0$. if $A=\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in\mathbb{R}^{2\times 2}$: $A^{-1}=\frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a\end{bmatrix}$
↳ bzw $\lambda_i\neq 0\ \forall i$

Eigenvalues (EW) and Eigenvectors (EV): $Av=\lambda v$, $\lambda\in\mathbb{C}$
1) EW: λ s.t. $\det(A-\lambda I)=0$
2) EV zum EW: $(A-\lambda I)x=0$ → find x

Orthogonal matrix: $A^TA=AA^T=I$ d.h. $A^T=A^{-1}$

Cayley Hamilton Theorem: Every Matrix $A\in\mathbb{R}^{n\times n}$ satisfies its Chp:

$$\boxed{A^n+a_1A^{n-1}+a_2A^{n-2}+\ldots+a_nI=0}$$

Diagonilazability: if AM=GM $\forall\lambda_i$, Matrix is diagonalizable:
$A=TDT^{-1}$ with $D=\text{diag}(\lambda_i)$, T= EV zum entsprechenden EW.

Symmetric Matrices: $A=A^T$, $A\in\mathbb{R}^{n\times n}$
↳ EW $\in\mathbb{R}$ and EV orthogonal
↳ positive definite: $x^TAx>0\ \forall x\neq 0\Rightarrow\lambda_i\in\mathbb{R}_{>0}\ \forall i$ Notation: $A>0$
↳ positive semi-definite: $x^TAx\geq 0\ \forall x\neq 0\Rightarrow\lambda_i\in\mathbb{R}_{\geq 0}\ \forall i$ Notation: $A\geq 0$

Rank of a Matrix: dimension of the vector space generated / spanned by its columns. ≙ max. number of linearly independent cols of matrix. ≙ dim of vector space spanned by its rows (≙ # Pivots)

Full Rank: All Submatrices (and matrix itself) have det ≠ 0

If Matrix has not full rank: $\exists$ a nullspace. (aufgespannt von EV zu EW 0)
det ≠ 0 ⇒ $\nexists$ nullspace

Range (A): Gauss → span{ursprüngliche Spalten mit Pivots}

1. Modelling

Dynamics:

$$\boxed{\begin{aligned}\dot{x}(t)&=f(x(t),u(t),t)\overset{\text{LTI}}{=}Ax(t)+Bu(t)\\ y(t)&=h(x(t),u(t),t)=Cx(t)+Du(t)\end{aligned}}$$

$x(t)\in\mathbb{R}^n$ state, $u(t)\in\mathbb{R}^m$ input, $y(t)\in\mathbb{R}^p$ output
$A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{n\times n}$, $D\in\mathbb{R}^{n\times p}$

Examples: Pendulum: $ml\ddot{\theta}(t)=-dl\dot{\theta}(t)-mg\sin(\theta(t))$ nonlinear!
RLC-Circuit: $\ddot{u}_c(t)+\frac{R}{L}\dot{u}_c(t)+\frac{1}{LC}u_c(t)=\frac{1}{LC}u_q(t)$ linear!
→ w. $x(t)=\begin{pmatrix}u_c(t)\\ i_L(t)\end{pmatrix}$ → $\dot{x}(t)=\begin{bmatrix}0 & 1/C\\ -1/L & -R/L\end{bmatrix}x(t)+\begin{bmatrix}0\\ 1/L\end{bmatrix}u(t)$

Change of coordinates: $\begin{aligned}\dot{x}&=Ax+Bu\\ y&=Cx+Du\end{aligned}\ \xRightarrow{\hat{x}=Tx}\ \begin{aligned}\dot{\hat{x}}&=\hat{A}\hat{x}+\hat{B}u=TAT^{-1}\hat{x}+TBu\\ \hat{y}&=\hat{C}\hat{x}+\hat{D}u=CT^{-1}\hat{x}+Du\end{aligned}$
usually T = EV, $\hat{A}=\text{Diag}(\lambda_i)$

Time-Invariant: Dynamics do not depend explicitly on time t
↳ eliminate explicit time dependence by introducing time as an additional state.

Autonomous: Time invariant + no input variables u
↳ autonomisieren: t als state einführen. $z=(x,t)^T$, $\dot{z}=(\dot{x},1)$

Lipschitz: $\boxed{f:\mathbb{R}^n\to\mathbb{R}^n,\ \exists\lambda>0\text{ s.t. }\forall x,\hat{x}\in\mathbb{R}^n:\|f(x)-f(\hat{x})\|\leq\lambda\|x-\hat{x}\|}$
↳ Differentiable with bounded derivatives ⇒ Lipschitz
↳ Linear functions are Lipschitz.

Existence & Uniqueness of solutions: If f is Lipschitz, then $\dot{x}(t)=f(x(t),u(t))$, $x(0)=x_0\in\mathbb{R}^n$ has a unique solution.
if system is non-autonomous: $f(x,u,t)$ has to be Lipschitz in x, continuous in u and t & u(t) has to be continuous for "almost all" t for existence of sol.

2. Energy, Controllability, Observability

Energy: $\boxed{E(t)=\frac{1}{2}x^TQx}$ w. Q symmetrical and positive definite. $Q=Q^T>0$

Power: $P(t)=\frac{dE(t)}{dt}=\frac{x^T(A^TQ+QA)x}{2}=-\frac{x^TRx}{2}$ RLC: $Q=\begin{bmatrix}C & 0\\ 0 & L\end{bmatrix}$ (if $x=[u_c,i_L]^T$)

Lyapunov equation: $\boxed{A^TQ+QA=-R}$ w. $Q=Q^T>0$, $R=R^T>0$
↳ The EW of A have negative real part (d.h. the system is asympt. stable) if and only if for any $R=R^T>0$, $\exists !\ Q=Q^T>0$ s.t. $A^TQ+QA=-R$.
↳ The Lyapunov function is then $V(x)=\frac{1}{2}x^TQx$

Controllability: When we can steer the system from any initial cond. $\hat{x}_0\in\mathbb{R}$ to any final cond. $\hat{x}_N\in\mathbb{R}^n$ using appropriate inputs u_k, $k=0,1,\ldots,N-1$
↳ Controllability Matrix: $\boxed{P=[B\ \ AB\ \cdots A^{n-1}B]}\in\mathbb{R}^{n\times nm}$
Rank(P) $\overset{!}{=}$ n for controllability
↳ Controllability Gramian: $\boxed{W_c(t)=\int_0^t e^{A\tau}BB^Te^{A^T\tau}d\tau}\in\mathbb{R}^{n\times n}$ $W_c=W_c^T\geq 0$
System if controllable over (0,t) ⇔ $W_c(t)$ is invertible
↳ Set of controllable states: Range (P)
If System is not controllable and I.C. $x(0)=x_0=0$: only states that are ∈ Range (P) can be reached!

Observability: System is observable over [0,t] if given $u(\cdot):[0,t]\to\mathbb{R}^m$ and $y(\cdot):[0,t]\to\mathbb{R}^p$ we can uniquely determine $x(\cdot):[0,t]\to\mathbb{R}^n$.
↳ Observability Matrix: $\boxed{Q=\begin{bmatrix}C\\ CA\\ \vdots\\ CA^{n-1}\end{bmatrix}}\in\mathbb{R}^{np\times n}$ Rank(Q) $\overset{!}{=}$ n for observability
↳ Observability Gramian: $\boxed{W_o(t)=\int_0^t e^{A^T\tau}C^TCe^{A\tau}d\tau}\in\mathbb{R}^{n\times n}$, $W_o=W_o^T\geq 0$
W_o is invertible ⇔ System is observable over [0,t) ∀t.
↳ Set of unobservable states: Null(Q) ∨ $x\in\mathbb{R}^n$ s.t. $Ce^{A\tau}=0\ \forall\tau\in[0,t]$
⚠ A system is observable ⇔ x=0 is the only unobservable state.

Kalman decomposition: ∃ a change of coordinates $T\in\mathbb{R}^{n\times n}$ invertible s.t.

$$\hat{A}=TAT^{-1}=\begin{bmatrix}\hat{A}_{11} & 0 & \hat{A}_{13} & 0\\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} & \hat{A}_{24}\\ 0 & 0 & \hat{A}_{33} & 0\\ 0 & 0 & \hat{A}_{34} & \hat{A}_{44}\end{bmatrix}\quad \hat{x}(t)=Tx(t)=\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\\ \hat{x}_3(t)\\ \hat{x}_4(t)\end{bmatrix}$$

$\hat{x}_1(t)$ ← controllable & observable
$\hat{x}_2(t)$ ← controllable & unobservable
$\hat{x}_3(t)$ ← uncontrollable & observable
$\hat{x}_4(t)$ ← uncontrollable & unobservable

$\hat{C}=CT^{-1}=[\hat{C}_1\ \ 0\ \ \hat{C}_3\ \ 0]$, $\hat{B}=TB=\begin{bmatrix}\hat{B}_1\\ \hat{B}_2\\ 0\\ 0\end{bmatrix}$ controllable observable

↳ Stabilizability & detectability:
System is detectable ⇔ all EW of $\hat{A}_{22}$ and $\hat{A}_{44}$ have Re{EW} < 0
System is stabilizable ⇔ all EW of $\hat{A}_{33}$ and $\hat{A}_{44}$ have Re{EW} < 0

2. Continuous LTI Systems in time domain

System Solution: $\boxed{x(t)=\underbrace{\Phi(t)x_0}_{}+\underbrace{\int_0^t\Phi(t-\tau)Bu(\tau)d\tau}_{}}$
Total transition = Zero Input Transition (ZIT) + Zero state Transition (ZST)

$\boxed{y(t)=C\Phi(t)x_0+\int_0^tC\Phi(t-\tau)Bu(\tau)d\tau+Du(t)}$
Total response = Zero Input Response (ZIR) + Zero State Response (ZSR)

with Transition Matrix $\boxed{\Phi(t)=e^{At}=\sum_{n=0}^{\infty}\frac{I^nt^n}{n!}\in\mathbb{R}^{n\times n}}$ Properties: $\Phi(0)=I$, $\frac{d}{dt}\Phi(t)=A\Phi(t)$, $\Phi(-t)=[\Phi(t)]^{-1}$, $\Phi(t_1+t_2)=\Phi(t_1)\cdot\Phi(t_2)$

↳ for diagonalizable Matrices: $\Phi(t)=e^{At}=Te^{Dt}T^{-1}$
↳ for Nilpotent Matrices: $e^{Nt}=I+Nt+\frac{N^2t}{2}+\frac{N^3t}{3!}+\ldots\to 0$ ab k s.t. $N^k=0$
↳ if $A=N+D$, $N\cdot D=D\cdot N$: $e^{At}=e^{Nt}\cdot e^{Dt}$

Stability: System is stable iff
$\forall\varepsilon>0,\exists\delta>0$ s.t. if $\|x_0\|<\delta$ then $\|x(t)\|\leq\varepsilon\ \forall t\geq 0$
↳ if System not stable, System is unstable.
↳ Asymptotically stable if stable and $\lim_{t\to\infty}\|x(t)\|=0$
↳ For a System with diagonalisable Matrix A w. EW λ_i
① ↳ Stable ⇔ $\text{Re}\{\lambda_i\}\leq 0\ \forall i$
② ↳ Asymptotically stable ⇔ $\text{Re}\{\lambda_i\}<0\ \forall i$
③ ↳ Unstable ⇔ $\exists i:\text{Re}\{\lambda_i\}>0$
↳ for nondiagonalizable Matrix A: ①,② same, keine Aussage if $\exists i:\text{Re}\{\lambda_i\}=0$

Hurwitz Criterion: for 2. Order Polynomials: $\alpha\lambda^2+\beta\lambda+\gamma=0$
↳ α,β,γ same sign ⇔ $\text{Re}\{\lambda_i\}<0\ \forall i$ ⇔ Asym. stable
↳ α,β,γ not same sign ⇔ $\exists i:\text{Re}\{\lambda_i\}>0$ ⇔ Unstable.

Phase plane plots: on EV zu stable EW, stable. If EW ∈ ℂ, Circles.

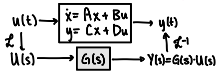
Impulse transition: ZST for $u(t)=\delta(t)$:
$h(t)=\int_0^t\Phi(t-\tau)B\delta(\tau)d\tau=e^{At}B=\Phi(t)B\Rightarrow x(t)=(H*u)(t)$

Unit step Response: ZST for $H(t)=\begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases}$

Output impulse Response: $K(t)=C\Phi(t)B+D\delta(t)\in\mathbb{R}^{p\times m}\Rightarrow y(t)=(K*u)(t)$

Stability with inputs: If $\text{Re}\{\lambda_i\}<0\ \forall i$, $\exists\alpha\geq 0$ s.t. ZST x(t) satisfies:
$\|u(t)\|\leq M\ \forall t\geq 0\Rightarrow\|x(t)\|\leq\alpha M\ \forall t\geq 0$
If in addition, $\lim_{t\to\infty}u(t)=0$ then $\lim_{t\to\infty}x(t)=0$.

3. Continuous LTI in frequency domain



⚠ s=jω

Transfer function $\boxed{G(s)=C(sI-A)^{-1}B+D}$
↳ in SS2: always proper rational!
$G(s)=\frac{\text{pol}(s)}{\text{pol}(s)}$ w. deg(den) ≥ deg(nom)

Laplace Transform, properties (in SS2: always assume f(t)=0 ∀t<0)
1) Linearity: $\mathcal{L}\{\alpha f(t)+\beta g(t)\}=\alpha\mathcal{L}\{f(t)\}+\beta\mathcal{L}(g(t))=\alpha F(s)+\beta G(s)$
2) s-shift: $\mathcal{L}\{e^{-at}f(t)\}=F(s+a)$
3) Time derivative: $\mathcal{L}\{\frac{d}{dt}f(t)\}=sF(s)-f(0^-)$, $\mathcal{L}\{\frac{d^2}{dt^2}f(t)\}=s^2F(s)-sf'(0^-)-f(0^-)$
4) Convolution: $\mathcal{L}\{(f*g)(t)\}=F(s)\cdot G(s)$ and $\mathcal{L}\{f(t)\cdot g(t)\}=(F*G)(s)$

Some important Laplace Transforms:
$\delta(t)$ ∘—● 1 $\sin(\omega t)$ ∘—● $\frac{\omega}{s^2+\omega^2}$
$H(s)$ ∘—● $1/s$
e^{-at} ∘—● $\frac{1}{s+a}$ $\cos(\omega t)$ ∘—● $\frac{s}{s^2+\omega^2}$

Laplace Transform of LTI-Systems:
$\begin{aligned}x&=Ax+Bu\\ y&=Cx+Du\end{aligned}\ \overset{\mathcal{L}}{\rightsquigarrow}\ \begin{aligned}X(s)&=(sI-A)^{-1}x_0+(sI-A)^{-1}BU(s)\\ Y(s)&=CX(s)+DU(s)\end{aligned}$

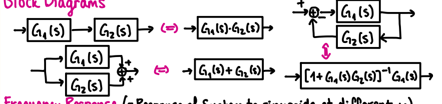
$(sI-A)^{-1}=\mathcal{L}\{e^{At}\}=\mathcal{L}\{\Phi(t)\}$ ⇔ $\Phi(t)=\mathcal{L}^{-1}\{(sI-A)^{-1}\}$

Sinusoidal inputs: for asymp. stable systems w. sinusoidal input, the steady state solution is also sinusoidal with:
↳ Same frequency as input
↳ Amplitude & Phase determined by system matrices.

Transfer function $G(s)=C(sI-A)^{-1}B+D$
↳ $Y(s)=G(s)\cdot U(s)$
↳ $\mathcal{L}^{-1}\{G(s)\}=K$ is the output impulse response, i.e. $y(t)=(K*u)(t)$
↳ SISO: $G(s)=\frac{(s-z_1)\cdot(s-z_2)\cdot\ldots\cdot(s-z_k)}{(s-p_1)\cdot(s-p_2)\cdot\ldots\cdot(s-p_n)}$, mostly strictly proper, i.e. k<n and D=0
↳ if no pole-zero cancellations (d.h. System is controllable & observable): denominator is Chp(A), i.e. poles p_i = EW of A.

Transfer function and Stability (⚠ provided: no pole zero cancellations)
↳ Distinct Poles: System is…
… asymptotically stable ⇔ $\text{Re}\{p_i\}<0\ \forall i$
… stable ⇔ $\text{Re}\{p_i\}\leq 0\ \forall i$
… unstable ⇔ $\exists i:\text{Re}\{p_i\}>0$
↳ Repeated Poles: System is…
… asymptotically stable ⇔ $\text{Re}\{p_i\}<0\ \forall i$
… unstable ⇔ $\exists i:\text{Re}\{p_i\}>0$
… no conclusion: if $\text{Re}\{p_i\}\leq 0\ \forall i$ and $\exists i:\text{Re}\{p_i\}=0$ ⇒ System might be stable or unstable depending on EV of A.

Block Diagrams

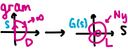


Frequency Response (= Response of System to sinusoids at different ω).
Consider a proper, asymp. stable SISO-System w. transfer fct. G(s). We apply u(t)=sin(ωt). Then Output settles to Sinusoid $y(t)=K\sin(\omega t+\phi)$ with:
↳ same frequency ω as u(t)
↳ Amplitude $K=|G(j\omega)|=\sqrt{\text{Re}[G(j\omega)]^2+\text{Im}[G(j\omega)]^2}$
↳ Phase $\phi=\angle G(j\omega)=\arctan(\text{Im}(G(j\omega))/\text{Re}[G(j\omega)])$
$G(s)=|G(s)|e^{i\angle G(s)}$
$\angle G(s)=\angle(s-z_1)+\angle(s-z_2)+\ldots-\angle(s-p_1)-\angle(s-p_2)-\ldots$

Principle of the argument

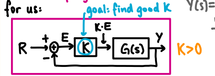
As s ∈ ℂ moves around, G(s) ∈ ℂ also moves.
If s travels around a closed curve D ⊆ ℂ, G(s) ∈ ℂ will travel around another closed curve L ⊆ ℂ ⇒ Nyquist plot!
Assume that the curve D is traversed in the clockwise direction and does not pass through any poles or zeros of G(s). Let:
N := # of times L encircles (0,0) in the (anti (−1)-) clockwise (+1) direction
Z := # of zeros of G(s) encircled by D
P := # of poles of G(s) encircled by D
Then $\boxed{N=Z-P}$ holds.

Nyquist Diagram
D-curve:

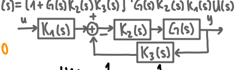


Spiegelsymmetrisch entlang Im-Achse.

Closed Loop Systems (CLS)
for us:



allgemein:
$Y(s)=[1+G(s)K_2(s)K_3(s)]^{-1}G(s)K_2(s)K_1(s)U(s)$



$\boxed{E(s)=\frac{1}{1+KG(s)}R(s)=H(s)R(s)=\frac{1}{F(s)}R(s)}$ $H(s):=\frac{1}{1+KG(s)}=:\frac{1}{F(s)}$
Open loop transfer function: G(s)
Closed loop transfer function: H(s)

Nyquist Stability Criterion
We want the error dynamics to be asym. stable (Y(s) tracks R(s)).
For this, we require: no zeros of F(s) w. positive real part!
Consider "D-Curve", map it under F(s) = 1+KG(s) / KG(s) / G(s) → "L-Curve"
N := Number of times resulting L curve encircles (0,0) / (−1,0) / (−1/K, 0) in the clockwise direction
Z := Number of closed loop poles (poles of H(s), zeros of F(s)) w. real part > 0
P := Number of open loop poles (poles of G(s), zeros of H(s)) w. real part > 0.
Closed loop system is asym. stable if $\boxed{Z=0}$. Hence, $\boxed{N\overset{!}{=}-P}$ (this is the requirement for N. → find K s.t. this is fulfilled!)

Bode Stability Criterion
Assume the open loop transfer function G(s) is asymptotically stable and its magnitude and phase Bode plots are monotone decreasing.
Then the closed loop system is asymptotically stable if and only if
$\boxed{|KG(j\omega)|<1}$ at the frequency where $\boxed{\angle G(j\omega)=-180°}$
↳ Small gain Theorem: Under the above conditions, if $|G(j\omega)|<1\ \forall\omega$, the closed loop system is asymptotically stable.

Gain and Phase Margins
Assume that the open loop system G(s) is asympt. stable.
Gain Margin = $1/G(j\omega_u)$ where $\angle G(j\omega_u)=-180°$
Phase Margin = Phase diff. $\angle G(j\omega_c)+180°>0$ where $|K\cdot G(j\omega_c)|=1$
⚠ System is asy. stable for K < gain margin
⚠ Overall Asymptotically stable: ∄K s.t. system is unstable
⚠ Allgemein: if open loop system G(s) has pole w. $\text{Re}\{p_i\}>0$ ⇒ unstable

Uncontrollable / Unobservable Systems: Pole-zero cancellation: EWs get lost

Pole-zero cancellation of the transfer function, Tests:
Detectability $\boxed{\text{rank}\begin{pmatrix}C\\ \lambda I-A\end{pmatrix}}$ and Stabilizability $\boxed{\text{rank}\,[B\ \ \lambda I-A]}$
↳ The λ's that reduce the rank of these tests are the ones which are being pole-zero cancelled in the transfer function
⇒ These λ are the EW ("modes") which are unobservable or uncontrollable correct?
Every other EW will appear in the transfer function!
if λ unstable and system uncontrollable, but rank [B λI−A] = n ⇒ λ is stabilizable.
if λ unstable and system unobservable, but rank $\begin{bmatrix}C\\ \lambda I-A\end{bmatrix}$ = n, ⇒ λ is detectable. (for λ stable, we don't care so much because already stable).

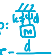
Transfer function Realization
Not unique. state space contains more information than Transfer function!

Initial value Theorem: $\lim_{t\to 0}f(t)=\lim_{s\to\infty}sF(s)$
Final value Theorem: $\lim_{t\to\infty}f(t)=\lim_{s\to 0}sF(s)$
whenever all limits exist.

Resonance: (may appear in Systems with two or more poles).

$\boxed{G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_ns+\omega_n^2}}$
ω_n .. Natural frequency
ζ .. damping ratio
K .. gain

Bsp:



$\omega_n=\sqrt{\frac{k}{m}}$, $\zeta=\frac{d}{2\sqrt{km}}$, $K=-\frac{1}{k}$

↳ frequency response: $G(j\omega)=\frac{K\omega_n^2}{(\omega_n^2-\omega^2)+j(2\zeta\omega_n\omega)}$
⇒ $\boxed{|G(j\omega)|=\frac{K\omega_n^2}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}}}$, $\boxed{\angle G(j\omega)=-\arctan\left(\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}\right)}$

1. For Stability, need ζ ≥ 0
2. For ζ ≥ 1 poles real (overdamped system)
3. For ζ = 1 poles real & equal (critical damping)
4. For 0 < ζ < 1 poles complex (under-damped system)
5. For ζ = 0 poles imaginary (undamped system)
6. For ζ ≥ 1/√2 magnitude bode plot decreasing in ω
7. For 0 ≤ ζ < 1/√2 magnitude bode plot has a max. at $\omega=\omega_n\sqrt{1-2\zeta^2}$ and $|G(j\omega)|=\frac{K}{2\zeta\sqrt{1-\zeta^2}}$

Bode Plot Kochrezept: $G(j\omega)=K_0(j\omega)^r\underbrace{G_1(j\omega)G_2(j\omega)\cdot\ldots\cdot G_n(j\omega)}_{G^*(j\omega)}$ $j\omega\hat{=}s$
Teilsysteme G_i:
$G_i(j\omega)=1+j\omega T_{n,i}$ Steigung +20dB/dec Phase +90°
$G_i(j\omega)=\frac{1}{1+j\omega T_{p,i}}$ Steigung −20dB/dec Phase −90°
1) Teilsystem nach aufsteigenden Eckfrequenzen $\omega_i=1/T_{n,i}$ sortieren
2) Startpunkt $(\omega_1, F_{dB}(\omega_1)=20\text{dB}\cdot\log_{10}(|K_0F^*_{ges}(0)|\cdot\omega_1^r)$ nach links: Gerade mit Steigung r·20dB/dec. nach rechts: jetzige St. + St. von ω_i
3) Phase: Startfrequenz ω_1 nach links: $\varphi(0)=\begin{cases}r\cdot 90° & \text{falls } K_0F^*_{ges}(0)>0\\ -180°+r\cdot 90° & \text{falls } K_0F^*_{ges}(0)<0\end{cases}$
4) Glieder 1. Ordnung: Phasenverlauf ±45°/dec zw. 0.1ω_i und 10ω_i

4. Discrete time LTI Systems

LTI: $\boxed{\begin{aligned}x_{k+1}&=Ax_k+Bu_k\\ y_k&=Cx_k+Du_k\end{aligned}}$ $x_k=x(kT)\in\mathbb{R}^n$, $u_k=u(kT)\in\mathbb{R}^m$, $y_k=y(kT)\in\mathbb{R}^p$

Solution: $\boxed{x_k=A^k\hat{x}_0+\underbrace{\sum_{i=0}^{k-1}A^{k-i-1}Bu_i}_{⊛}}$ ⊛ Discrete time convolution of input u_i and state impulse response $h_k=A^{k-1}B$

Stability:
System is stable if $\forall\varepsilon>0,\exists\delta>0$ s.t. $\|x_0\|<\delta\Rightarrow\|x_k\|\leq\varepsilon\ \forall k=0,1,\ldots$
Symstem is asymptotically stable if in addition, $\lim_{k\to\infty}\|x_k\|=0$.
↳ for diagonalizable Matrix A:
① Stable System ⇔ $\forall i\ |\lambda_i|\leq 1$
② Asymptotically stable system ⇔ $\forall i\ |\lambda_i|<1$
③ Unstable System ⇔ $\exists i:|\lambda_i|>1$
↳ for nondiagonalizable Matrix A: ②,③ same.
①: if $\exists\lambda_i=0$, then System might be stable or unstable.

Coordinate Change: Assume $\hat{x}_k=Tx_k$ for some $T\in\mathbb{R}^{n\times n}$, det(T) ≠ 0
System in the new coordinates is:
$\boxed{\begin{aligned}\hat{x}_{k+1}&=\hat{A}\hat{x}_k+\hat{B}\hat{u}_k\\ \hat{y}_k&=\hat{C}\hat{x}_k+\hat{D}\hat{u}_k\end{aligned}}$ with $\hat{A}=TAT^{-1}$, $\hat{B}=TB$, $\hat{C}=CT^{-1}$, $\hat{D}=D$

Energy & Power, Controllability, Observability: same as time continuous LTI Systems!

z-Transform (≙ Laplace for time-discrete systems).
$\boxed{F(z)=\mathcal{Z}\{f_k\}=\sum_{k=0}^{\infty}f_kz^{-k}}$
↳ Properties:
↳ Linearity: $\mathcal{Z}\{\alpha f_1(t)+\beta f_2(t)\}=\alpha F_1(z)+\beta F_2(z)$
↳ Time shift: $\mathcal{Z}\{f_{k-k_0}\}=z^{-k_0}F(z)$
↳ Convolution: $\mathcal{Z}\{(f*g)_k\}=\mathcal{Z}\{\sum_{i=0}^kf_ig_{k-i}\}=F(z)\cdot G(z)$
↳ Some common functions:
$\delta_k=\begin{cases}1, & k=0\\ 0, & k\neq 0\end{cases}$ ∘—● 1 $1_k=\begin{cases}1, & k\geq 0\\ 0, & k<0\end{cases}$ ●—∘ $\frac{z}{z-1}$ a^k ∘—● $\frac{z}{z-a}$ if |a|<1

Transfer function: $Y(z)=\underbrace{[C(zI-A)^{-1}B+D]}_{G(z)}U(z)$
↳ System is asymptotically stable ⇔ Poles of G(z) have $|p_i|<1$
↳ Pole/zero cancellations ⇔ System is uncontrollable/unobservable

Simulation: $x((k+1)\delta)=e^{A\delta}x(k\delta)\approx(I+A\delta)x_k$ if $\boxed{\delta<\frac{2}{\max_{i=1,\ldots,n}|\lambda_i|}}$ EW of A

5. Nonlinear Systems

$\boxed{\dot{x}(t)=f(x(t))}$ $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^m$, $y(t)\in\mathbb{R}^p$, $f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$
↳ in SS2: autonomous, time-invariant, f lipschitz.

Invariance: A set of states $S\subseteq\mathbb{R}^n$ is invariant if:
$\forall x_0\in S,\forall t\geq 0, x(t)\in S$ → Check Vektorfeld @ boundaries!
(for anywhere you start in the set, you remain in the set forever.)
x(t): sol. to $\dot{x}=f(x(t))$ starting at x_0

Equilibrium: A state $\hat{x}\in\mathbb{R}^n$ is called an equilibrium if: $\boxed{\dot{\hat{x}}=f(\hat{x})=0}$
↳ equilibrium points are an important class of invariant sets!
↳ if you start at an equilibrium, you stay at the equilibrium forever.
↳ For linear systems: Equilibria = Nullspace. D.h. if det(A)=0 ⇒ ∃∞ many eq.

Shifting equilibria to the origin
Change of Coordinates: $w(t)=x(t)-\hat{x}\in\mathbb{R}^n$
System in the new coordinates:
$\dot{w}(t)=\dot{x}(t)=f(x(t))=f(w(t)+\hat{x})=f(\hat{w}(t))$ has an equilibrium at $\hat{w}=0$

Periodic Orbit: A solution x(t) is a periodic orbit if:
$\boxed{\exists T>0,\forall t\geq 0, x(t+T)=x(t)}$ ↳ Equilibria are periodic orbits

Van der Pol Oscillator: $\ddot{\theta}(t)-\varepsilon(1-\theta(t)^2)\dot{\theta}(t)+\theta(t)=0$
↳ Model for dynamics of vacuum tube (transistor) circuits

Stability for nonlinear systems: property of invariant sets.
Trajectories stay close or converge into invariant sets.
↳ Stable equilibrium: An eq. $\hat{x}$ is called stable if: if not stable: unstable
$\forall\varepsilon>0,\exists\delta>0$ s.t. $\|x_0-\hat{x}\|<\delta\Rightarrow\|x(t)-\hat{x}\|<\varepsilon\ \forall t\geq 0$
↳ Asymptotic stability: An eq. $\hat{x}$ is locally asymptotically stable if it is stable and $\exists M>0$ s.t. $\|x_0-\hat{x}\|<M\Rightarrow\lim_{t\to\infty}x(t)=\hat{x}$
↳ Globally asymptotically stable: if it holds for any M>0.
⚠ if system has more than 1 eq ⇒ ∄ globally asymp. stable eq!

How to check Stability:
↳ Lyapunov first / Lyapunov indirect method: use Linearization
↳ Linearization: Let $\delta x(t)=x(t)-\hat{x}\in\mathbb{R}^n$ small.
Then: $\frac{d\delta x(t)}{dt}\approx A\delta x(t)$, $A=\begin{bmatrix}\frac{\partial f_1}{\partial x_1}(\hat{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\hat{x})\\ \vdots & \ddots & \vdots\\ \frac{\partial f_n}{\partial x_1}(\hat{x}) & \cdots & \frac{\partial f_n}{\partial x_n}(\hat{x})\end{bmatrix}\in\mathbb{R}^{n\times n}$ Jacobi Matrix
↳ if EW of A have $\text{Re}\{\lambda_i\}<0\ \forall i$ ⇔ eq. is locally asymp. stable.
↳ if ∃ at least one EW has $\text{Re}\{\lambda_i\}>0$ ⇔ eq. is unstable
↳ if ∃ at least one EW $\lambda_i=0$ ⇔ no conclusion possible.
↳ Lyapunov second / Lyapunov direct method
Assume ∃ an open set (d.h. without boundary) $S\subseteq\mathbb{R}^n$ with $\hat{x}\in S$ and a differentiable function $V(\cdot):\mathbb{R}^n\to\mathbb{R}$ s.t.:
1) $V(\hat{x})=0$ 2) $V(x)>0$, $\forall x\in S$ with $x\neq\hat{x}$ 3) $\frac{d}{dt}V(x(t))\leq 0\ \forall x\in S$
Then the equilibrium $\hat{x}$ is stable V(x): Lyapunov function
↳ if 3) $\frac{d}{dt}V(x(t))<0$, the eq. $\hat{x}$ is locally asymptotically stable.
↳ $\frac{d}{dt}V(x(t))=\sum_{i=1}^n\frac{\partial V}{\partial x_i}(x(t))\frac{d}{dt}x_i(t)=\sum_{i=1}^n\frac{\partial V}{\partial x_i}(x(t))f_i(x(t))=\nabla V(x(t))f(x(t))$
↳ if in addition, $\|x\|\to\infty\Rightarrow V(x)\to\infty$ then $\hat{x}$ is globally asymptotically stable.

La Salle's Theorem:
Assume ∃ compact invariant set $S\subseteq\mathbb{R}^n$ & a differentiable function $V(\cdot):S\to\mathbb{R}$ s.t. $\frac{d}{dt}V(x(t))=\nabla V(x)f(x)\leq 0\ \forall x\in S$
Let M be the largest invariant set contained in the set
$\bar{S}=\{x\in S|\nabla V(x)f(x)=0\}\subseteq\mathbb{R}^n$.
Then all trajectories starting in S tend to M as $t\to\infty$
Compact = bounded & closed ↳ region of attraction

Invariant Set: Trajectories can not escape the set.
Check vector field at the boundaries!
If somewhere, Richtungsableitung < 0 → then stuff will flow out from the boundary ⇒ not invariant ↯

Also with Lyapunov function: $\frac{d}{dt}V(x(t))\overset{!}{\leq}0\ \forall x\in$ Invariant set.

Wichtige Ableitungen:
↳ $(\ln(x))'=\frac{1}{x}$ ↳ $(\tan(x))'=1/\cos^2(x)$

Examples of Lyapunov functions:
Pendulum: $V(x)=\frac{1}{2}ml^2x_2^2+mgl(1-\cos(x_1))$
$V(x)=1-\cos(x_1)+\frac{x_2^2}{2}$