Quantum solid state magnetism

Thermodynamics of magnetic matter

→ insulating materials with 3d transition metal magnetic ions have simple magnetic Hamiltonian, magnetically active electrons are in the partially filled 3d shell.

→ no spherical symmetry, #levels remain $2L+1$, lowest is singlet
no freedom in orbital motion → orbital momentum quenching

→ Coulomb + Pauli → exchange interaction (12.12) Zheludev

Falls of exponentially with $|\vec{R}|$ ext. magnetic field

$$\hat{\mathcal{H}} = \sum_{\vec{r},\vec{R}} \sum_{\alpha,\beta} J^{\alpha\beta}_{\vec{R}} \hat{S}^{\alpha}_{\vec{r}} \hat{S}^{\beta}_{\vec{r}+\vec{R}} + \sum_{\vec{r}} \sum_{\alpha\beta} g^{\alpha\beta}_{\vec{r}} \hat{S}^{\alpha}_{\vec{r}} H^{\beta}_{\vec{r}}$$

location distance $\in \{x,y,z\}$ accounts for different ions t

includes spin-orbit and dipole-dipole

$S^2 \perp$ easy axis?

$R=0 \rightsquigarrow$ single-ion anisotropy

Linear response theory in 3d magnets

p.51 Lacroix
p.57 Lacroix

assume isotropic

$$\hat{M}(\vec{r}) = \sum_i g_i \mu_B \hat{\vec{S}}_i \delta(\vec{r}-\vec{r}_i) \qquad \text{magnetization}$$

immediate position of i-th e^-

sum over ion spin of ion at $\vec{r}$

$$\hat{M}(\vec{q}) \overset{\text{F.T.}}{=} \int d\vec{r}\, \hat{\vec{M}}(\vec{r}) e^{i\vec{q}\cdot\vec{r}} = g\mu_B F(\vec{q}) \sum_{\vec{r}} \hat{\vec{S}}_r e^{i\vec{q}\cdot\vec{r}} = g\mu_B F(\vec{q}) \hat{\vec{S}}(\vec{q})$$

magnetic form factor F.T. of ion

is measurable

$$G^S_{\alpha\beta}(\vec r,t) = \int \langle\langle \hat S^\alpha(\vec r+\vec r',0)\,\hat S^\beta(\vec r',t)\rangle\rangle \, d\vec r'$$ (spin correlation function)

statistical average

$$S^S_{\alpha\beta}(\vec q,\omega) \overset{F.T}{=} \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dt\int d\vec r\; G^S_{\alpha\beta}(\vec r,t)\, e^{i(\vec q\cdot\vec r-\omega t)}$$ (spin dynamic structure factor)

energy conservation

$$= \sum_{\lambda,\lambda'} p_\lambda \langle\lambda|\hat S^\alpha(\vec q)|\lambda'\rangle\langle\lambda'|\hat S^\beta(-\vec q)|\lambda\rangle\, \delta(E_{\lambda'}-E_\lambda-\hbar\omega)$$

describes the possible transitions in the system between the available spin states $|\lambda\rangle$.

Kramers-Kronig

reactive $\in \mathbb{C}$ dissipative $\in \mathbb{C}$

$$\chi_{\alpha\beta}(\vec q,\omega) = \chi'_{\alpha\beta}(\vec q,\omega) + i\chi''_{\alpha\beta}(\vec q,\omega)$$

Sokhotski

$$= \lim_{\varepsilon\to 0^+} \frac{(g\mu_B)^2}{V}\sum_{\lambda,\lambda'}(p_\lambda-p_{\lambda'})|F(\vec q)|^2\,\frac{\langle\lambda|\hat S^\alpha(\vec q)|\lambda'\rangle\langle\lambda'|\hat S^\beta(-\vec q)|\lambda\rangle}{E_{\lambda'}-E_\lambda-\hbar(\omega+i\varepsilon)}$$

fluctuation-dissipation $\Longrightarrow$ theorem

$$\chi''_{\alpha\beta}(\vec q,\omega) = \frac{\pi(g\mu_B)^2}{V}|F(\vec q)|^2\left(1-e^{-\hbar\omega/T}\right) S^S_{\alpha\beta}(\vec q,\omega)$$

$\Rightarrow$ $S^S_{\alpha\beta}(\vec q,\omega)$ is measurable (ESR, NMR, neutron scattering)

$q=0$ at any $\vec q$ and ω

Phase transitions

A phase transition is an abrupt change of system's properties as a function of smoothly varying parameter.

$\Rightarrow$ at some temperature/pressure/... the free energy or its various derivatives experience some kind of discontinuous behaviour.

$\rightsquigarrow$ phase transition only in thermodynamic limit (macroscopic systems)

continuous	discontinuous
· coherence length infinite	· coherence length finite
· system remains uniform	· phase coexistence
$\rightsquigarrow$ onset magnetic order usually	$\rightsquigarrow$ boiling water
· associated with symmetry breaking	
$\rightsquigarrow$ for $M \neq 0$ $\rightsquigarrow$ $t \leftrightarrow -t$ not invariant	
OP no jumps	OP jumps

Mean field description

A.wills chapter 5

unit cell, position in cell

$$\vec{M}(\vec{R},\vec{r}) = \vec{m}_{\vec{r}}\, e^{i\vec{Q}\cdot\vec{R}} + \vec{m}_{\vec{r}}^{\,*}\, e^{-i\vec{Q}\cdot\vec{R}} \in \mathbb{R}^3 \qquad (22)\ \text{A.wills}$$

N = # ions per unit cell, we need Q and $\underbrace{\vec{m} = (\vec{m}_{\vec{r}_1}, \dots, \vec{m}_{\vec{r}_N})}_{3N\text{-vector}}$

dimension P

basis of irrep

$$\vec{m}_{\Gamma} = \sum_{\gamma=1}^{n_P} L_{\Gamma,\gamma}\, \vec{X}_{\Gamma,\gamma} \qquad (49)\ \text{A.wills}$$

coefficient, amplitude

$\vec{L}_P := (L_{\Gamma,1}, \dots, L_{\Gamma,n_P})$ is a quantification of how much a symmetry is violated $\Rightarrow$ order parameter

$$F(T) = \frac{1}{2}\sum_{\Gamma}\alpha_\Gamma(T-T_\Gamma)|\vec{L}_\Gamma|^2 + O(|\vec{L}_\Gamma|^4)$$

all different and possibly negative

Different magnetic order types violate different symmetries $\Rightarrow$ "independant"

$$\Rightarrow \langle\langle S_{\vec{Q}}\rangle\rangle \propto |\vec{L}_\Gamma| \propto \sqrt{1-\frac{T}{T_\Gamma}}$$

Criticality and scaling

$$C_V \propto |T_c - T|^{-\alpha}$$

$$|\vec{L}| =: L \propto (T_c - T)^{\beta}$$

$$\chi_L \propto |T_c - T|^{-\gamma}$$

$$L \propto |H_L|^{1/\delta}$$

$$\xi \propto |T_c - T|^{-\nu}$$

$$\langle \vec{L}(0)\vec{L}(\vec{r})\rangle - \langle L\rangle^2 \overset{T=T_c}{\propto} r^{d-2+2}$$

	α	β	γ	ν
Mean field	0 (jump)	1/2	1	1/2
3D Ising	0,1	0,3	1,2	0,6
3D XY	−0,02	0,3	1,3	0,7
3D Heisenberg	−0,1	0,4	1,4	0,7
2D Ising	0 (log)	1/8	7/4	1

Universality classes

The dimensionality of the system and the particular symmetry being broken dictate mathematically identical description.

Scaling hypothesis

Assumption: $F(T,H_L) = \lambda f(\lambda^x |T-T_c|, \lambda^y H_L)$

$$
\left.\begin{array}{l}
C_p \sim |T-T_c|^{2-1/x} \\
\chi_L \sim |T-T_c|^{-\frac{2y}{x}-\frac{1}{x}} \\
L \sim (T_c-T)^{-y/x-1/x} \\
L \sim H_L^{-1-\frac{1}{y}}
\end{array}\right\} \text{reduction from } \alpha,\beta,\gamma,\delta \text{ to } x,y
$$

$\alpha + 2\beta + \gamma = 2$ (Rushbrooke)

$\gamma = \beta(\delta - 1)$ (Widom)

Hyperscaling

- include correlation length

Assumption: $\xi(T,H_L) = \lambda \, \varepsilon(\lambda^{1/\nu}|T-T_c|, \lambda^{\Delta/\nu} H_L)$

$$\overset{\lambda=|T-T_c|^{-\nu}}{\Longrightarrow} \quad \xi(T,H_L) = |T-T_c|^{-\nu} \, \varepsilon\left(\frac{|T-T_c|^{\Delta}}{H_L}\right)$$

$\rightsquigarrow \Delta = \beta + \gamma$

$d\nu = 2-\alpha$ (Josephson)

$\gamma = \nu(2-\eta)$ (Fisher)

$$\Longrightarrow \quad F(T,H_L) = \lambda^{-d} f_{therm}(\lambda^{1/\nu}|T-T_c|, \lambda^{(\beta+\gamma)/\nu} H_L)$$

not valid in quantum critical region

This form only holds if hyperscaling can be applied.

Magnetic systems are short ranged $\rightsquigarrow$ hyperscaling.

- Hyperscaling is not fulfilled in mean field treatment.

$$F \propto T V \xi^{-d} \propto |T-T_c|^{d\nu}$$

one cluster has energy T

Critical dynamics

The characteristic time at which the correlated cluster of size ξ disappears is given by

dynamical exponent

$$\tau \propto \xi^{z}$$

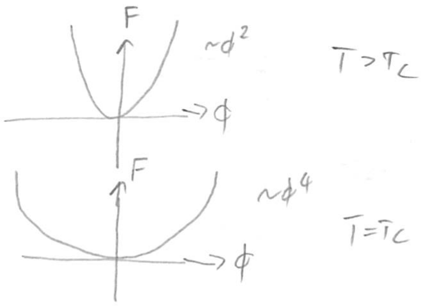
the larger the domain, the more time it takes to disappear

$$\Longleftrightarrow \quad \omega \propto q^{z}$$

$\omega_0 \sim \xi^{-z}$

$\rightsquigarrow$ critical slowing down

close to q_c the dynamics slow down dramatically and it takes a lot more time to get back to equilibrium.



Hohenberg - Mermin - Wagner

Continuous symmetry can not be spontaneously broken at finite temperature in 2D and 1D in a system with the interaction strength falling off fast (faster than a certain power law) with the distance.

exp decrease fulfills. The theorem only deals with thermal fluctuations. No statement regarding T=0 state. Beware: realistic materials have discreet symmetry, but ordering is suppressed.

The integrals diverge at low q (for $d \leq 2$). This is the manifestation of the magnetic order being destroyed by the low-energy fluctuations. These Goldstone modes are responsible for the destruction of the ordered phase.

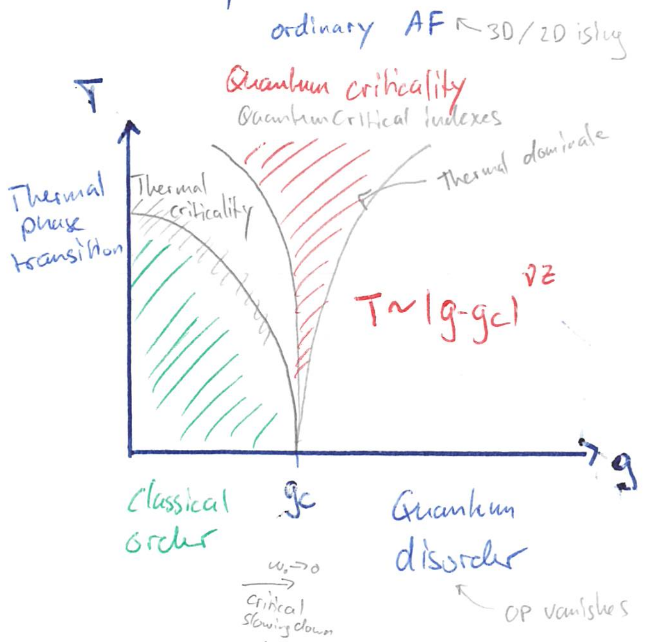
	Ferromagnet	Antiferromagnet
3D	$T_c > 0$	$T_N > 0$, order
2D Ising	$T_c > 0$	$T_N > 0$ order
2D XY	$T_c = 0$, but $T_{BKT} > 0$	$T_N = 0$, but $T_{BKT} > 0$
2D Heisenberg	$T_c = 0$	$T_N = 0$
1D Ising	order at $T=0$	order at $T=0$
1D XY	order at $T=0$	quantum critical at $T=0$
1D Heisenberg	order at $T=0$	quantum critical at $T=0$

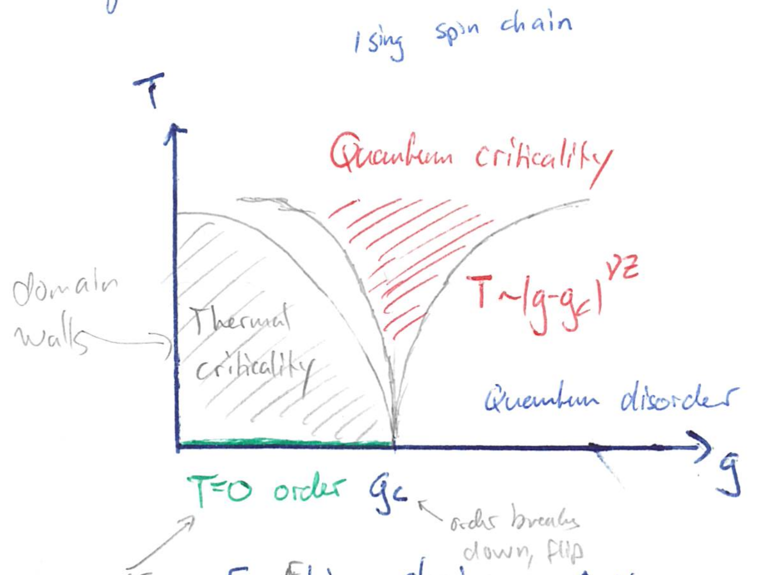
Quantum phase transitions

- Absence of thermal fluctuation not sufficient for order!
- At zero temperature the quantum fluctuations will play the leading role and some parameter other than T can tune their strength.
- The transitions between the different phases can occur as a function of these parameter at $T=0$ ⇝ quantum phase transitions
  the associated critical points are the quantum critical points

Even though qCP at $T=0$, the critical behaviour is at $T\gtrsim 0$

Generic quantum critical phase diagram





Eg. AF Ising chain, $g \mathrel{\hat{=}} H_{\text{transverse}}$

eg. $C_V \sim T^{d/z}$

in quantum critical region

g tunes the strength of the Quantum fluctuations

Quantum vs. thermal fluctuations

$\hbar\omega \overset{?}{\ll} k_B T$ but $\omega_0 \to 0 \Rightarrow$ always satisfied
but not for $T=0 \Rightarrow$ QPT

Approaching $T \to 0$, the role of zero-point fluctuations is increasing.

$\omega_0 \to 0$ for $g \to g_c$

$[\beta f] = L^{-d} = \xi^{-d}$

$\hbar\omega_0 \sim A\xi^{-z}$ (large domains fluctuate slower)

$$F \propto \xi^{-d}(T + \hbar\omega_0) \propto \xi^{-d}(T + A\xi^{-z})$$

$T=0$

$$\Rightarrow \quad F \propto \xi^{-d-z} \quad , \quad d_{\text{eff}} = d+z$$

dynamic

instead of $F \propto \xi^{-d}$ (thermal)

$\xi \sim |g-g_c|^{-\nu}$ OP

$$\hbar\omega_0 \propto \xi^{-z} \propto |g-g_c|^{z\nu} \xrightarrow[g \to g_c]{} 0$$

$\varphi = \nu z$

$\xi \xrightarrow[g\to g_c]{} \infty$

$\Rightarrow$ relative contribution from quantum fluctuations is $\dfrac{|g-g_c|^{\varphi}}{T}$

Cone: critical thermal fluctuations are dominant but there is no phase transition. For $g=g_c$, unable to find any other characteristic energy apart from temperature T.

The cone is $T \gtrsim |g-g_c|^{\varphi}$

$|g-g_c|$ small

Unlike thermal, in QPT thermodynamics can not be seperated from dynamics.

Static critical properties at quantum critical point

include g in F:

$$F(g,T,H_L) = \lambda^{-d-z} f_{quant}(\lambda^z T, \lambda^{1/\nu}|g-g_c|, \lambda^{(\beta+\gamma)/\nu} H_L)$$

$$\Rightarrow \quad C_p \propto T^{d/z} \quad , \quad F(g,T) = T^{d/z+1}\, \Phi\left(\frac{|g-g_c|^{\nu z}}{T}\right)$$

φ

$\Rightarrow$ The free energy weighted with the appropriate power law of temperature, will not vary along the $T = |g-g_c|^{\varphi}$ line

$d \longleftrightarrow d+z$ dynamic and static become entangled

Phase boundary shape

The semiclassical phase boundary is determined by the shift exponent ψ:

$$T_c \propto (g_c - g)^{\psi}$$

In Landau MF theory:

$$\psi = \frac{z}{d+z-2} \neq \varphi = z\nu$$

If different, then hyperscaling is violated.

Dynamic critical properties at quantum critical point.

Extension of
Scaling properties to the dynamics may not be possible for quantum phase transitions. In this case often: $d_{eff} = d+z \geq 4 \rightsquigarrow$ mean field critical behaviour. → hyperscaling violated

For $d_{eff} < 4 \Rightarrow$ hyperscaling holds, propose

$$S(\vec{q}, \omega, g, T) = \lambda^{\gamma + z\nu} \mathcal{F}(\lambda^{\nu}\vec{q}, \lambda^{z\nu}\omega, \lambda^{z\nu}T, \lambda|g-g_c|)$$

Along $g = g_c$ and $\lambda = T^{-1/z\nu}$

$\frac{q^z}{T}$ ↙   ↙ $\omega \sim q^z$

$$S(\vec{q}, \omega, T) = T^{-1-\gamma/z\nu}\, \sigma\left(\frac{\vec{q}}{T^{1/z}}, \frac{\omega}{T}\right)$$

Only energy of thermal fluctuations matter.
Absence of intrinsic energy scales, $g = g_c \Rightarrow$ contributions compensated

p. 406 Huang

Goldstone bosons

Assume spontaneously broken symmetry which is continuous. The Hamiltonian predicts identical energy, therefore a rotation of all the spins would cost no energy. Such an infinitesimal in-plane oscillation corresponds to a gapless bosonic particle known as Nambu-Goldstone boson. Spontaneous breaking of a continuous symmetry inevitably leads to the appearance of gapless bosonic excitations. This is the point of Goldstone's Theorem.

periodic modulation $\Rightarrow \varphi \Rightarrow$ particle

Mexican hat potential:

$$\Psi = |\Psi| e^{i\varphi}\,, \quad \text{complex order parameter}$$

$\varphi \overset{?}{=}$ direction of magnetization

$$E(\Psi) = -a(g-g_c)|\Psi|^2 + b|\Psi|^4 \,, \quad a,b>0$$

For $g<g_c \Rightarrow$ parabolic, $\Psi=0$ minimum, disordered, gapped

For $g>g_c \Rightarrow$ mexican hat, $|\Psi|=$ const minimum, ordered, gapless

Typically, the Goldstone boson dispersion law is given by

$$\omega \propto k$$

and #gapless modes $\overset{\text{normally}}{=}$ # broken symmetries

third is still present, $\hat{S}_z$ is OP, ↑↑↑ ↑↑↑ invariant, broken

Ferromagnet: two symmetries broken but only one goldstone mode wi

$$\omega \propto k^2$$

"anomalous"

reason: modes are not independent, $[\hat{S}^x, \hat{S}^y] = i\hbar \hat{S}^z$

Antiferromagnet: $\hat{S}^z$ is not the order parameter but the difference

Higgs boson

The quasiparticle associated with the longitudinal fluctuations of the order parameter would always require some energy to be created.

$\rightsquigarrow$ Gapped excitation

The mass scales with the order parameter.

ginzburg-landau $\Rightarrow$

$$\begin{cases} -2K|\delta\ddot{\psi}(t)| = 2a(g-g_c)|\delta\psi|, & \text{amplitude motion, } \omega = \sqrt{\frac{a(g-g_c)}{K}} = |\psi_0|\sqrt{\frac{b}{K}} \\ K\frac{a(g-g_c)}{b}\delta\ddot{\varphi}(t) = 0 & \omega = 0 \text{ does not cost energy} \end{cases}$$

at $g=g_c$ the Higgs and Goldstone modes are both gapless, degenerate and indistinguishable.

as gap $\sim$ O.P. $\overset{g=g_c}{\Rightarrow}$ gapless

Changes the amplitude of the O.P.

Weakly anisotropic case

Assume: $E(\psi) = a(g-g_c)|\psi|^2 + b|\psi|^4 - c|\psi|^2\cos^2\varphi$, $a,b,c>0$, $c \ll a$

$\Rightarrow$ discreet symmetry $\Rightarrow$ goldstone breaks down $\Rightarrow$ excitations massive

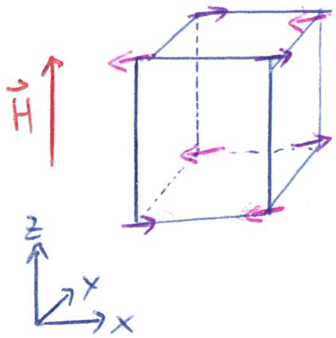
longitudinal: gap $\sim |\psi|$

For c small, the Goldstone bosons have a small gap $\rightsquigarrow$ pseudogoldstone modes

Antiferromagnet basics

Néel order

Assume AF on cubic lattice. ⇒ Groundstate is two-sublattice state



$$\langle\langle \hat{S}_{\vec{r}_A} \rangle\rangle = \begin{pmatrix} S \\ 0 \\ 0 \end{pmatrix}$$

$$\langle\langle S_{\vec{r}_B} \rangle\rangle = \begin{pmatrix} -S \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ H \end{pmatrix}$$

$$\hat{\mathcal{H}} = \sum_{\vec{r}} \left[\sum_{\vec{R}} J (\hat{\vec{S}}_{\vec{r}} \cdot \hat{\vec{S}}_{\vec{r}+\vec{R}}) - g\mu_B (\hat{\vec{S}}_{\vec{r}} \cdot \vec{H}) \right]$$

$$\Rightarrow |\langle\langle \hat{S}^x \rangle\rangle| = S \sqrt{1 - \left(\frac{H}{H_{sat}}\right)^2}$$

staggered magnetization along x is the order parameter.

Magnetization along z is <u>not</u> an order parameter: no spontaneous symmetry breaking.

part was skipped in the lecture

Spin wave theory

magnons ← interactions neglected   p. 186 White

Assumptions: dilute approximation, weakly deviate from equ., fully ordered $S \gg 1$

Linear spin wave theory can handle the spectrum of excitations in a structure described by a single propagation vector $\vec{Q}$. The spins are assumed to be fully ordered: $|\langle\langle \hat{\vec{S}} \rangle\rangle| = S$, and $S \gg 1$, and the deviations from the fully ordered ground state are small.

$$\hat{\mathcal{H}} = \sum_{\vec{R}} \sum_{\alpha = x,y,z} J_{\vec{R}} \hat{S}^{\alpha}_{\vec{r}} \hat{S}^{\alpha}_{\vec{r}+\vec{R}}$$

Assume that the spin spiral is the groundstate

$$\langle\langle \hat{\vec{S}}_{\vec{r}} \rangle\rangle = \begin{pmatrix} 0 \\ S \sin \vec{Q}\cdot\vec{r} \\ S \cos \vec{Q}\cdot\vec{r} \end{pmatrix}.$$

By introducing the pseudospin operators $\hat{\vec{S}}'_{\vec{r}}$ we map the spin spiral structure onto a fake ferromagnetic structure.

$$\hat{S}^x_{\vec{r}} = \hat{S}'^x_{\vec{r}}$$

$$\hat{S}^y_{\vec{r}} = \hat{S}'^y_{\vec{r}} \sin \vec{Q}\cdot\vec{r} + \underline{\hat{S}'^z_{\vec{r}} \cos \vec{Q}\cdot\vec{r}}$$

$$\hat{S}^z_{\vec{r}} = \underline{\hat{S}'^z_{\vec{r}} \cos \vec{Q}\cdot\vec{r}} - \hat{S}'^x_{\vec{r}} \sin \vec{Q}\cdot\vec{r}$$

$\Rightarrow$ Ground state of each pseudospin $\hat{\vec{S}}'$ is now $|S,S\rangle$

$$\Rightarrow \quad \hat{\mathcal{H}} = \sum_{\vec{R}} \sum_{\alpha,\beta = x,y,z} \tilde{J}^{\alpha\beta}_{\vec{R}} S'^{\alpha}_{\vec{r}} S'^{\beta}_{\vec{r}+\vec{R}}$$

Define the operators $\hat{a}^\dagger$, $\hat{a}$ that create and destroy the minimal possible on-site deviations: Is this the linear in linear spin wave theory

$$\boxed{\hat{S}'^{-} = \sqrt{2S}\,\hat{a}^\dagger \;,\quad \hat{S}'^{+} = \sqrt{2S}\,\hat{a} \;,\quad \hat{S}'^{z} = S - \hat{a}^\dagger \hat{a} \qquad \left(\begin{array}{c}\text{Holstein-}\\ \text{Primakoff}\\ \text{transformation}\end{array}\right)}$$

Precision is $\frac{1}{S}$. inserting the operators in the Hamiltonian and taking the Fourier transformation $\hat{a}_{\vec{r}} = \frac{1}{\sqrt{N}} \sum_{\vec{q}} \hat{a}_{\vec{q}}\, e^{-i\vec{q}\cdot\vec{r}}$

$$\hat{\mathcal{H}} = \overset{\sim S^2}{E_{GS}} + \sum_{\vec{q}} \left[2A_{\vec{q}}\, \hat{a}^\dagger_{\vec{q}} \hat{a}_{\vec{q}} + B_{\vec{q}} \left(\hat{a}^\dagger_{\vec{q}} \hat{a}^\dagger_{-\vec{q}} + \hat{a}_{-q} \hat{a}_{q} \right) \right]^{\sim S}$$

This Hamiltonian can be diagonalized by a Bogoliubov transformation

$$\hat{b}^\dagger_{q} = u_{\vec{q}}\, \hat{a}^\dagger_{\vec{q}} - v_{\vec{q}}\, \hat{a}_{-q}$$

$$\hat{b}_{\vec{q}} = u_{\vec{q}}\, \hat{a}_{\vec{q}} - v_{\vec{q}}\, \hat{a}^\dagger_{-q}$$

$$\Rightarrow \quad \hat{\mathcal{H}} = E_{GS} + \sum_{\vec{q}} \hbar\omega_{\vec{q}} \left(\hat{b}^\dagger_{\vec{q}} \hat{b}_{\vec{q}} + \frac{1}{2} \right)$$

- harmonic oscillator
- diagonal

with the dispersion: $\hbar\omega_q = \sqrt{A^2_{\vec{q}} - B^2_{\vec{q}}}$ and the structure factor

$$\left.\begin{aligned}
S'^{xx}(\vec{q},\omega) &= \frac{S}{2} |u_{\vec{q}} - v_{\vec{q}}|^2\, \delta(\omega - \omega_q) \\
S'^{yy}(\vec{q},\omega) &= \frac{S}{2} |u_{\vec{q}} + v_{\vec{q}}|^2\, \delta(\omega - \omega_q) \\
S'^{zz}(\vec{q},\omega) &= S^2\, \delta(\vec{q})\, \delta(\omega)
\end{aligned}\right\}$$

can be related to the actual dynamic structure factor

Large S approximation and well ordered-magnetic structure we have full account of all the correlation functions.

Spin wave decays

phonons: dof is atomic position $\longleftrightarrow$ spin waves: dof is direction of magnetic moments

This analogy only valid in semiclassical treatment

$$[\hat{p}^{\alpha}, \hat{p}^{\beta}] = 0 \quad \longleftrightarrow \quad [\hat{S}^{\alpha}, \hat{S}^{\beta}] = i\,\varepsilon_{\alpha\beta\gamma}\,\hat{S}^{\gamma}$$

xyz small for $S\to\infty$

$\Rightarrow$ no interaction $\qquad$ $\Rightarrow$ interaction

Not dramatic as long as validity limits ~~are~~ not outstepped.

Example: 2-dim square lattice $S=1/2$ AF

HMW $\Rightarrow$ LRO absent for $T>0$.

At $T=0$ $\Rightarrow$ reduced ordered momentum, 40% still fluctuating, #magnons large

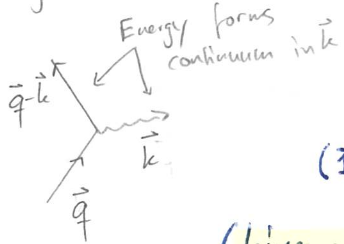
However, collinearity saves magnons as #magnons is conserved.

Magnon can not spontaneously decay into two - does not conserve the wavefunction parity under π rotation around the collinearity axis.

$\Rightarrow$ no magnon-magnon interaction in leading order p.223 Chernyshev

Two-magnon decays

if surface disappears $\Rightarrow$ magnon stable



external parameter

defines a surface for variable $\vec{k}$ value

Energy conserv. not always satisfied!!!

$$\hbar\omega(\vec{q}) = \hbar\omega(\vec{k}) + \hbar\omega(\vec{q}-\vec{k}) \qquad \text{(I)}$$

(kinematic condition)

$$E_2^{min}(\vec{q}) \leq \underbrace{\hbar\omega(\vec{k}) + \hbar\omega(\vec{q}-\vec{k})}_{\hbar\omega(\vec{q})} \leq E_2^{max}(\vec{q}) \qquad \text{(II)}$$

defined via $\vec{v}_{\vec{q}} = \vec{v}_{\vec{q}-\vec{k}}$

If $\hbar\omega(\vec{q}) < E_2^{min}(\vec{q}) \Rightarrow$ magnon is safe in whole Brillouin zone and higher-order processes also forbidden

is minimal at $\vec{v}_{\vec{q}} = \vec{v}_{\vec{q}-\vec{k}}$

Decays of acoustic mode

Heisenberg AF., for low energy ("acoustic"): $\hbar\omega(\vec{q}) = cq \rightsquigarrow$ not interesting

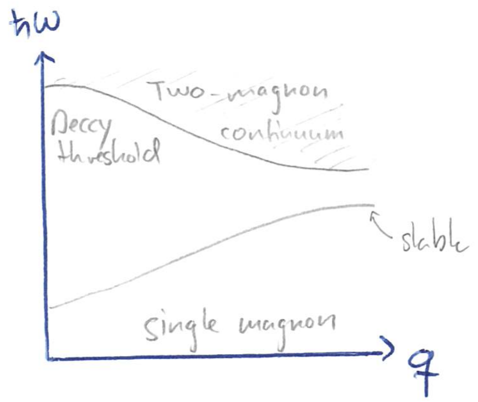
(I) always satisfied

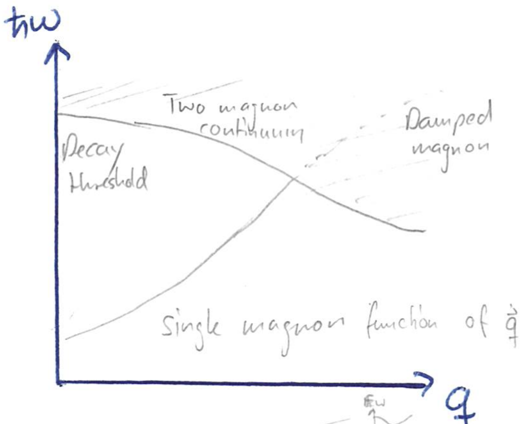
$\Rightarrow$ assume: $\hbar\omega(\vec{q}) \simeq cq + \alpha q^3$

small for $q \to 0$

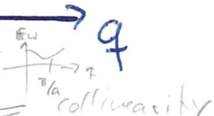
$$\Rightarrow \alpha \simeq \frac{c\varphi^2}{6(q-k)^2} > 0 \Rightarrow$$ decays allowed for $\alpha > 0$

convex





For $\vec{H}=0$, $\alpha < 0 \Rightarrow$ magnons stable in simple A.F.



Field-induced decays

$\omega \sim k^2$ | decay possible

A.F. $H \geq H_{sat}$ spectrum changes to ferromagnetic cosine-type (convex)

⇝ Spin waves only stable because spins collinear

but not zero | turns on decay channel | kinematic condition

For $H < H_{sat}$: no collinearity and convex ⇝ decay possible

↖ just before full polarization

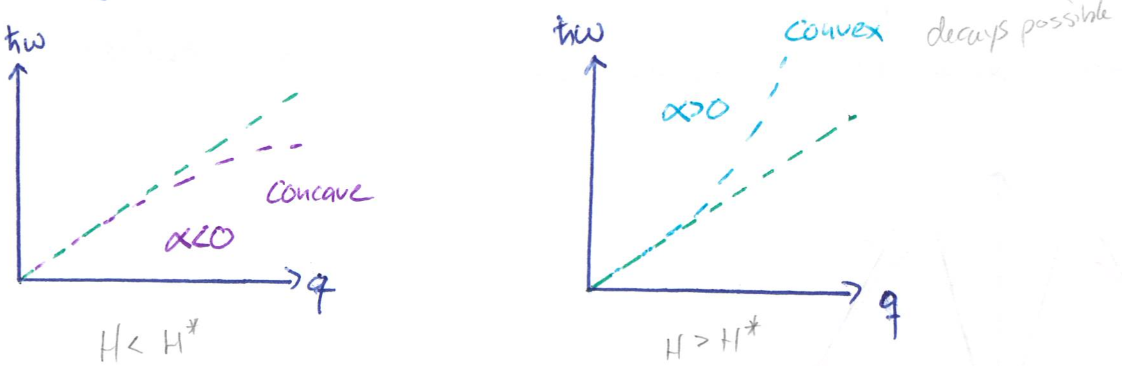
⟹ magnons heavily damped.

⇝ Spin waves are long lived for $H \to 0$ and $H \to H_{sat}$ but not inbetween.

square/cubic

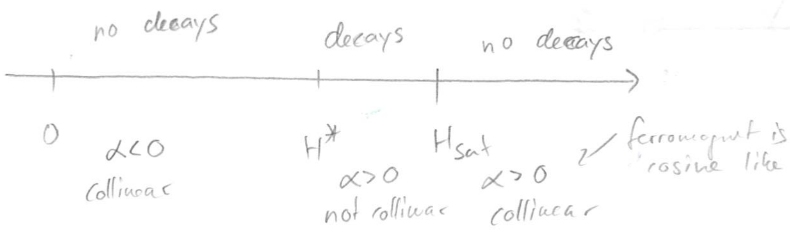
There is a threshold field $H^* \sim 0{,}76\, H_{sat}$ where α changes its sign.

Reminder:



but for $H > H_{sat}$ ⇝ no decays (collinearity)

AF:



Decays at zero field

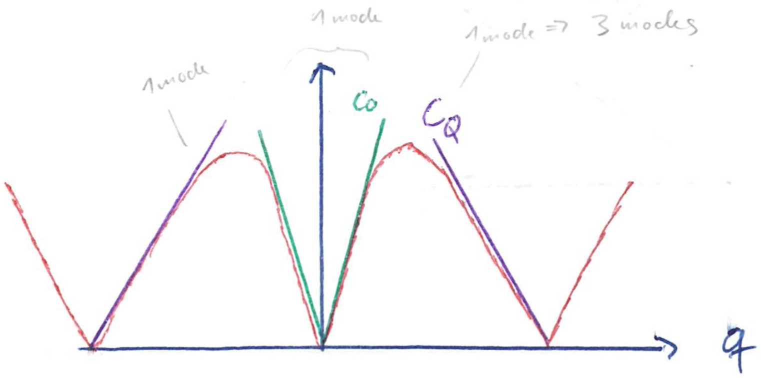
The condition

$$\alpha \simeq \frac{cq^2}{6(q-k)^2}$$ is not valid for multiple branches (complex structures).

For a spin structure with propagation vector $\vec{Q}$, there are three Goldstone modes: at $\vec{q}=0$ and $\vec{q}=\pm\vec{Q}$ with velocity c_0 and c_Q. If $c_0 > c_Q$

then $c_0 q = c_Q k + c_Q |q-k| \Rightarrow$ Fast magnons decay into slow even within linear approximation.

For non-collinear spin arrangements with several branches, linear spin wave theory produces very approximate results.

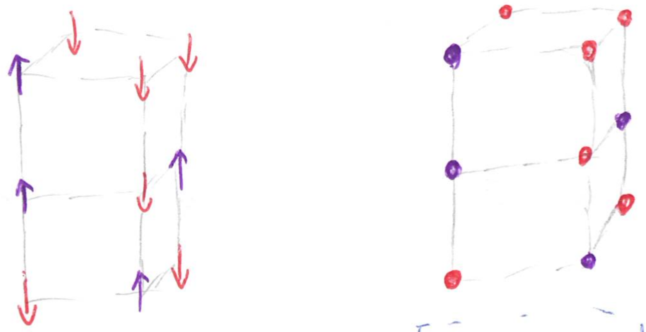


Batyev-Braginskii approach

Antiferromagnet

$$\hat{\mathcal{H}} = \sum_{\vec{r},\vec{R}} J\, \hat{\vec{S}}_{\vec{r}}\, \hat{\vec{S}}_{\vec{r}+\vec{R}} - \sum_{\vec{r}} g\mu_B H\, \hat{S}^z_{\vec{r}} \quad , \quad J>0 \qquad \text{A.F.}$$

$$\left.\begin{array}{l} \hat{S}^z_{\vec{r}} = \hat{a}^+_{\vec{r}}\hat{a}_{\vec{r}} - \frac{1}{2} \quad , \quad \hat{S}^+_{\vec{r}} = \hat{a}^+_{\vec{r}} \quad , \quad \hat{S}^-_{\vec{r}} = a_{\vec{r}} \end{array}\right\} \text{Matsubara-Masuda transformation}$$



• $|\uparrow\rangle$ occupied

• $|\downarrow\rangle$ empty

$$|\uparrow\uparrow\rangle \xrightarrow{\hat{a}} |\uparrow\rangle \xrightarrow{\hat{a}} |0\rangle$$
$$|0\rangle \xrightarrow{a^+} |\uparrow\rangle \xrightarrow{a^+} |\uparrow\uparrow\rangle$$
$$|\uparrow\rangle \xrightarrow{\hat{a}} |0\rangle \xrightarrow{\hat{a}} 0 \in \mathbb{C}$$

nicht im Hilbertraum

hard-core constraint : $\hat{\mathcal{H}}_{HC} = \sum_{\vec{r}} u\, \hat{a}^+_{\vec{r}}\hat{a}^+_{\vec{r}}\hat{a}_{\vec{r}}\hat{a}_{\vec{r}}$

$u \to \infty$

forbids on site magnetization larger than $|\uparrow\rangle$

all states can be empty...

→ ascrible the a-particles a bosonic statistics. But no assumption of Ground State so far.

$$\hat{\mathcal{H}} = \frac{J}{2}\sum_{\vec{r},\vec{R}} [\hat{a}^+_{\vec{r}}\hat{a}^+_{\vec{r}+\vec{R}} + \hat{a}^+_{\vec{r}+\vec{R}}\hat{a}_{\vec{r}}] + J\sum_{\vec{r},\vec{R}} [\hat{a}^+_{\vec{r}}\hat{a}_{\vec{r}} - \tfrac{1}{2}][\hat{a}^+_{\vec{r}+\vec{R}} a_{\vec{r}+\vec{R}} - \tfrac{1}{2}] - \underbrace{g\mu_B H \sum_{\vec{r}} [\hat{a}^+_{\vec{r}} a_{\vec{r}} - \tfrac{1}{2}]}_{\text{Chemical potential}} + \hat{H}_{HC}$$

→ particle-hole symmetry

→ $(a^+ \leftrightarrow a)$ and $(H \leftrightarrow -H)$ invariant (comes from $(H \to -H)$ and $(\vec{S} \to -\vec{S})$ invariance)

F.T.

$$\Rightarrow \hat{\mathcal{H}} = \sum_{\vec{q}} [\hbar\omega(\vec{q}) - \mu]\, \hat{a}^+_{\vec{q}} a_{\vec{q}} + \frac{1}{2N}\sum_{\vec{q},\vec{q}',\vec{k}} V_{\vec{k}}\, \hat{a}^+_{\vec{q}+\vec{k}}\, \hat{a}^+_{\vec{q}'-\vec{k}}\, \hat{a}_{\vec{q}}\, \hat{a}_{\vec{q}'}$$

??

$$\Rightarrow \quad \hbar\omega(\vec{q}) = J \sum_{\vec{R}} (1 + \cos\vec{q}\cdot\vec{R})$$

Should be concave???

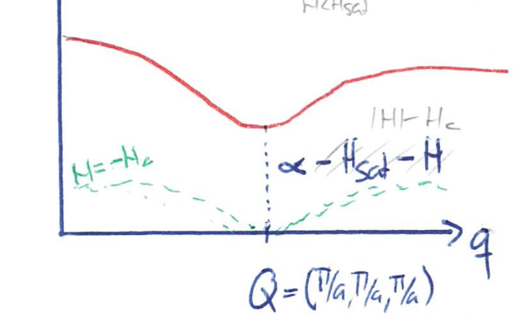
$\rightsquigarrow$ Hamiltonian of interacting Bose gas

Bose gas analogy

Assume $H \simeq -H_c$ (just a definition), $T=0$, for $H \geq -H_c \Rightarrow$ particles have to appear

$$\xrightarrow[h>0]{h\le -H_c} \quad E(\vec{q}) = \underbrace{J \sum_{\vec{R}} (1 + \cos\vec{q}\cdot\vec{R})}_{\hbar\omega(\vec{q})} + \underbrace{g\mu_B(H+H_c)}_{\mu}$$

Here $\vec{Q} = (\pi/a, \pi/a, \pi/a)$ minimizes $E(Q)$ which becomes zero at $H = -H_c$



small parameter

$\mu = -g\mu_B \cdot h$

$$E(q) = \frac{\hbar^2 q^2}{2m}, \quad \frac{m}{\hbar^2} \propto g\mu_B H_c$$

applicable in dilute limit

$$\overset{\text{dilute}}{1 \gg} \rho = \frac{1}{N} \langle\langle \hat{a}^+_{\vec{Q}} \hat{a}_{\vec{Q}} \rangle\rangle \neq 0 \quad (h<0) \quad \Rightarrow \text{interacting gas with hardcore bosons}$$

result of BEC

$$\Rightarrow \quad T_{BEC} \propto \frac{\hbar^2}{m} \cdot \rho^{2/3}, \quad \langle\langle \hat{a}^+_{\vec{Q}} \rangle\rangle = \sqrt{N\rho}\, e^{i\varphi} \quad \text{condensate wavefunction}$$

Energy (rotational invariant)

$$\langle\langle \hat{a}^+_{\vec{r}} \rangle\rangle = \langle\langle \hat{S}^+_{\vec{r}} \rangle\rangle = \langle\langle \hat{S}^x_{\vec{r}} + i\hat{S}^y_{\vec{r}} \rangle\rangle = \sqrt{\rho_0}\, e^{i\vec{Q}\cdot\vec{r} + i\varphi}$$

$\times$ magnetization, $\times$ direction

(transverse a.f. order parameter

The appearance of Bose-Einstein condensate of a-particles simply corresponds to the familiar antiferromagnetic order that sets in perpendicular to the field ~~below H_c~~. / above $-H_c$

Nonlinear sigma model

LSWT has assumption about ground state with long-range order.

Assume short-range ordering with $\xi \gg a$ (but not $\to\infty$) $\Rightarrow$ avoid dealing with lattice by neglecting details of short wavelength behaviour. Focus on long wavelength properties (hydrodynamic approach), insensitive to microscopic details. Instead, continuous field approximates the orientations of the actual spins on the lattice. $\rightsquigarrow$ Non-linear sigma model (low energy behaviour without the assumption of long range order).

Mapping to a rotor

$$\hat{\mathcal{H}} = J(\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2) \qquad \text{(A.F.)}$$

$$\xRightarrow[\text{classical}]{\text{Semi-}} \quad \vec{S}_{1,2} = \langle\langle \hat{\vec{S}}_{1,2} \rangle\rangle \quad \xRightarrow[\text{field}]{\text{effective}} \quad g\mu_B \vec{H}_1^{\text{eff}} = -J\vec{S}_2 \quad (1 \leftrightarrow 2)$$

$$\Rightarrow \quad \dot{\vec{S}}_1 = \frac{J}{\hbar}[\vec{S}_2 \times \vec{S}_1], \qquad \dot{\vec{S}}_2 = \frac{J}{\hbar}[\vec{S}_1 \times \vec{S}_2] \qquad \text{(Euler's equation)}$$

$\vec{M} = \vec{S}_1 + \vec{S}_2$ (couples to $\vec{H}^{ext}$)

$\vec{N} = \vec{S}_1 - \vec{S}_2$ (description of A.F.)

$$\Rightarrow \begin{cases} \dot{\vec{M}} = 0 \\ \dot{\vec{N}} = J/\hbar\,[\vec{M} \times \vec{N}] \end{cases}$$

Rotation of staggered magnetization $\vec{N}$ around $\vec{M}$

$\Rightarrow \ddot{\vec{N}} = 0 \Rightarrow N = |N| = \text{const.}$

$\Rightarrow$ rotation of fixed-length vector $\vec{N} = N\vec{n}$ (rigid rotor)

$\Rightarrow$ quantum-mechanical rotor $\hat{\vec{L}} \leftrightarrow \hat{\vec{M}}$.

Many body version

AF $\rightarrow$ two sublattices A & B

$$\langle\langle \hat{\vec{S}}^A_{\vec{r}} \rangle\rangle = \vec{S}^A_{\vec{r}} = \vec{M}(\vec{r}) + \vec{N}(\vec{r}), \quad \langle\langle \hat{\vec{S}}^B_{\vec{r}} \rangle\rangle = \vec{S}^B_{\vec{r}} = \vec{M}(\vec{r}) - \vec{N}(\vec{r})$$

If groundstate $\simeq$ AF

$\Rightarrow$ $\vec{M}(\vec{r})$ small, $|\vec{M}(\vec{r})| \ll |\vec{N}(\vec{r})|$

$\Rightarrow$ $\vec{N}(\vec{r})$ varies slowly in space, $|\vec{N}(\vec{r}) - \vec{N}(\vec{r}+d\vec{r})| \ll |\vec{N}(\vec{r})|$

shift by one lattice constant a

$\Rightarrow$ $\vec{M}(\vec{r})$ varies slowly in space, $|\vec{M}(\vec{r}) - \vec{M}(\vec{r}+d\vec{r})| \ll |\vec{M}(\vec{r})|$

Beware: No assumption that $\vec{M}, \vec{N}$ are uniform or periodic.

$$\dot{\vec{S}}^A_{\vec{r}} = \frac{J}{\hbar}\left(\sum_{d\vec{r}} \vec{S}^B_{\vec{r}+d\vec{r}}\right) \times \vec{S}^A_{\vec{r}}$$

$$\dot{\vec{S}}^B_{\vec{r}} = \frac{J}{\hbar}\left(\sum_{d\vec{r}} S^A_{\vec{r}+d\vec{r}}\right) \times \vec{S}^B_{\vec{r}}$$

$\vec{M}(\vec{r}+d\vec{r}) + \vec{M}(\vec{r}-d\vec{r}) \simeq 2\vec{M}(\vec{r})$

$\vec{N}(\vec{r}+d\vec{r}) + \vec{N}(\vec{r}-d\vec{r}) \simeq 2\vec{N}(\vec{r}) + a^2\nabla^2\vec{N}(\vec{r})$

$$\Rightarrow \quad \dot{\vec{N}}(\vec{r}) = 4J\hbar^{-1}d\,[\vec{M}(\vec{r}) \times \vec{N}(\vec{r})]$$
$$\dot{\vec{M}}(\vec{r}) = -\frac{J\hbar^{-1}a^2}{2}[\nabla^2\vec{N}(\vec{r}) \times \vec{N}(\vec{r})]$$

As all original spins had same length S, $\Rightarrow$ $N = 2S$ everywhere, $\vec{N}(\vec{r}) \Rightarrow 2S\vec{n}(\vec{r})$

$\vec{n}(\vec{r})^2 = 1$

$$\frac{\hbar^2}{2}\frac{\partial^2\vec{n}(\vec{r},t)}{\partial t^2} = 4(SJda)^2\nabla^2\vec{n}(\vec{r},t)$$

(non-linear sigma model)

$$\mathcal{S} = \frac{\hbar S}{2}\int dt \int d\vec{r}\left\{\frac{1}{JS^2}\left(\frac{\partial\vec{n}}{\partial t}\right)^2 - 8J(da)^2(\nabla\vec{n})^2\right\}$$

dynamics reduced to minimization of classical action corresponding to a fixed length vector field.

Relevance to electron spin resonance

$T=0,\ M \ll N$:

$$\mathcal{L} = \frac{\chi_\perp}{2\gamma^2}\left(\dot{\vec{n}} + \gamma[\vec{H}\times\vec{n}]\right)^2 - U_{anisotropy}(\vec{n}) - U_{inhomogeneity}(\nabla\vec{n})$$

useful for ESR

$\chi_\perp = \frac{(g\mu_B)^2}{J}$ $\gamma = g\mu_B/\hbar$

$\sim (\nabla n)^2$ generalization of potential energy

$$U_{anisotropy}(\vec{n}) = \frac{1}{2}\sum_{\alpha,\beta} A_{\alpha\beta}n^\alpha n^\beta, \quad \underset{\text{tetragonal}}{A = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}, \quad \underset{\text{orthorombic}}{A = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

ESR measures the uniform dissipative susceptibility $\chi''_{\alpha\beta}(\vec{q}=0,\omega)$, which has peaks at certain frequencies $\omega_0(\vec{H})$.

For $\vec{q}=0 \Rightarrow \cancel{U_{inhomogenity}} \to 0$, $\vec{n} \rightsquigarrow \vec{n}(\theta,\varphi) \Rightarrow \mathcal{L} = \mathcal{K}(\dot{\varphi},\dot{\theta},\varphi,\theta) - \mathcal{U}(\varphi,\theta)$

$\overset{\text{Euler-Lagrange}}{\underset{\text{isotropic}}{\Longrightarrow}}\ \mathcal{L} = \frac{\chi_\perp}{2\gamma^2}\left(\dot{\vec{n}} + \gamma[\vec{H}\times\vec{n}]\right)^2 \ \overset{\text{is minimized by}}{\Longrightarrow}\ \dot{\vec{n}} = -\gamma[\vec{H}\times\vec{n}]$ (larmore theorem)

$\Rightarrow \omega_0(\vec{H}) = \gamma H$

short range $\Rightarrow$ gap ??

p.496 Chaikin

2D XY model

XY ferromagnet on square lattice, $\hat{\vec{S}}_{\vec{r}} = S\begin{pmatrix} \cos\varphi_{\vec{r}} \\ \sin\varphi_{\vec{r}} \end{pmatrix}$

$$\hat{\mathcal{H}} = -|J| \sum_{\vec{r}, d\vec{r}} (\vec{S}_{\vec{r}} \cdot \vec{S}_{\vec{r}+d\vec{r}}) = -|J| S^2 \sum_{\vec{r}, d\vec{r}} \cos(\varphi_{\vec{r}} - \varphi_{\vec{r}+d\vec{r}})$$

- cancels φ_r

φ doesnt change much $\Rightarrow$ $\varphi_{\vec{r}+d\vec{r}} \overset{\text{Taylor}}{\simeq} \varphi_{\vec{r}} + (\nabla\varphi_{\vec{r}} \cdot d\vec{r})$, $\cos(\nabla\varphi_{\vec{r}} \cdot d\vec{r}) = 1 - \frac{1}{2}(\nabla\varphi_{\vec{r}} \cdot d$

from $-\frac{1}{2}$ ...

continuum limit $\Rightarrow$ $$\hat{\mathcal{H}} = \underbrace{E_{GS}}_{-|J| S^2 \sum_{\vec{r}, d\vec{r}} 1} + \frac{|J| S^2}{2} \int d\vec{r}\, (\nabla\varphi_{\vec{r}})^2$$ has $U(1)$ symmetry

Hamiltonian describes deviations from the ferromagnetic state of almost parallel spin

Assume decomposition in plane waves: $\varphi_{\vec{r}} = \varphi_0 + \sum_{\vec{k}} \delta\varphi e^{i\vec{k}\cdot\vec{r}} + \delta\varphi^* e^{-i\vec{k}\cdot\vec{r}}$ and recall that $|\delta\varphi|^2 \propto L^{-2}$

in Ferromagnet: $\omega_{\vec{k}} \propto k^2$

$$\Rightarrow \quad \hat{\mathcal{H}} - E_{GS} \propto \sum_k k^2$$

spin waves, low energy

$$N = \int_0^T \nu(E)\, n(E)\, dE \xrightarrow[E \ll T]{} N \propto \int_0^T \frac{T}{E}\, dE$$ diverges at lower limit

$\Rightarrow$ HMW holds, assumption of plane waves wrong, no LRO

Vortices and antivortices

characterized by winding number m, $\oint \vec{\nabla}\varphi_{\vec{r}}\, d\vec{r} = 2\pi m \begin{cases} >0 & \text{Vortex} \\ <0 & \text{antivortex} \\ =0 & \text{core \& contour} \end{cases}$

ensures that initial position is reached

Vortex and antivortex are the robust topological objects: they can not be turned into one another by a uniform rotation of the spins. (topological charge)
Creation of vortex-antivortex pair would respect the topological charge conservation
As the energy of a vortex-antivortex lowers the energy the interaction between them is attractive. A strongly bound pair closely resembles the uniform ferromagnetic state. At high temperatures the pairs would be broken apart by the fluctuations. ⇒ essence of the transition

On a closed contour of radius R around the core, the spins rotate by $\pm 2\pi$.

$$\Rightarrow \quad \nabla\varphi_{\vec{r}} = \frac{d\varphi}{dr} = \frac{\pm 2\pi}{2\pi R} \quad \Rightarrow \quad E_{\text{vortex}} = \frac{|J|S^2}{2}\int_a^L \frac{1}{R^2}\, 2\pi R\, dR = \pi |J| S^2 \log\left(\frac{L}{a}\right)$$

charge

length of charge

polar coordinate

$(\nabla\varphi)^2$

as the structure does not change, the derivative can be taken as the total charge divided by total length

Vortices are clearly energetically expensive objects. However, entropy

$$S_{\text{vortex}} \sim 2\log\left(\frac{L}{a}\right), \quad W \sim \left(\frac{L}{a}\right)^2$$

$$\Rightarrow \quad F_{\text{vortex}} = \left(\pi |J| S^2 - 2T\right)\log\left(\frac{L}{a}\right) \quad \Rightarrow \quad T_{BKT} = \frac{\pi |J| S^2}{2}$$

(Beresinskii-Kosterlitz-Thouless transition)

It corresponds to a spontaneous dissociation of vortex-antivortex pairs as $T > T_{BKT}$. The resulting state is still lacking LRO as $\langle\langle \vec{S}_{\vec{r}} \cdot \vec{S}_{\vec{r}'} \rangle\rangle \xrightarrow[|\vec{r}-\vec{r}'|\to\infty]{} 0$, but

below T_{BKT} $\longrightarrow$ $\langle\langle \vec{S}_{\vec{r}} \cdot \vec{S}_{\vec{r}'} \rangle\rangle \propto |\vec{r}-\vec{r}'|^{-T/2\pi\Gamma}$ ← helicity modulus

the order is the pairing and hiding, "ordered by non existing"
is a form of hidden order as vortex-antivortex pair almost indistinguishable.
No symmetry is broken at T_{BKT}. → topological phase transition

Vortices and antivortices part 2

Remember: no LRO, (HMW)

has its own universality class?

$$\xi \propto e^{\frac{A}{\sqrt{T-T_{BKT}}}} \quad \text{diverges at } T_{BKT}$$

$$\chi \propto e^{\frac{B}{\sqrt{T-T_{BKT}}}} \quad \text{diverges at } T_{BKT}$$

grow faster than any power law

⇝ makes numerical and experimental investigation more challenging

Below T_{BKT} both correlation length ξ and uniform magnetic susceptibility χ remain infinite. This is another highly unusual feature of the BKT transition. Infinitely small applied field would turn the system into a field-induced ferromagnet at $T < T_{BKT}$.

XY-Antiferromagnets

Similar to the ferromagnet. Can be mapped to the ferromagnet problem by applying same trick as in LSWT: rotate the spin operators at certain positions and afterwards manipulate with ferromagnetically aligned pseudospin objects.

Easy-plane Hamiltonian:

$$\eta = \begin{cases} <0 & \text{Ising} \\ 0 & \text{Heisenberg} \\ 1 & \text{XY} \end{cases}$$

$$\hat{\mathcal{H}} = J \sum_{\vec{r}, d\vec{r}} \left[(\vec{S}_{\vec{r}} \cdot \vec{S}_{\vec{r}+d\vec{r}}) - \eta S^z_{\vec{r}} S^z_{\vec{r}+d\vec{r}} \right]$$

$\eta = 1 \Rightarrow \mathcal{H} = J \sum_{\vec{r}, d\vec{r}} S^x_{\vec{r}} S^x_{\vec{r}+d\vec{r}} + S^y_{\vec{r}} S^y_{\vec{r}+d\vec{r}}$

As $|\vec{S}| =$ const, energetically favourable to be in x-y plane as no lowering in energy when z-component $\neq 0$

almost Heisenberg

For $\eta \ll 1$ (analytically): $T_{BKT} = \dfrac{4\pi J S^2}{\log \frac{\pi^2}{\eta}} \overset{\text{Heisen-berg}}{=} 0$

Beware A.F. in magnetic field

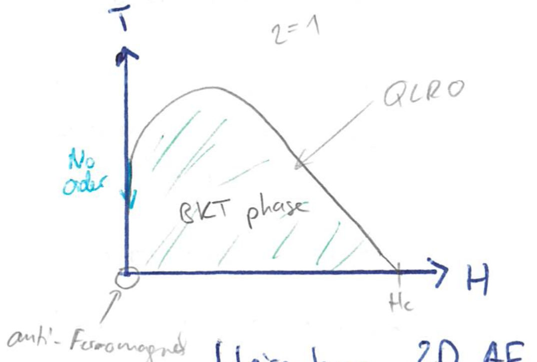
but

$$\left.\begin{aligned} \xi &\propto e^{A'/T} \\ \chi &\propto e^{B'/T} \end{aligned}\right\} \text{still fast diverging}$$

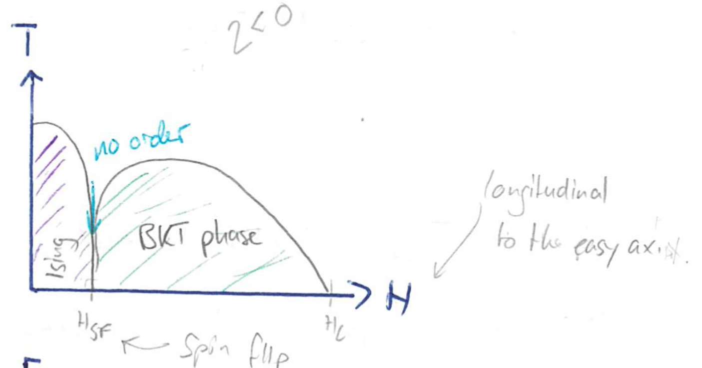
/ slightly modified

$\frac{1}{\sqrt{T}} \Leftrightarrow \frac{1}{T}$

$S=\frac{1}{2}$ square lattice AF 2D Heisenberg has long-range Néel order at $T=0$ with $\langle \hat{S}^z \rangle \simeq 0{,}65$. LSWT applicable to some extent. One needs to take into account the reduced magnetic moment in a sublattice, causing the renormalization of the dispersion.



Heisenberg 2D A.F.



Easy axis A.F.
weakly Ising

Antiferromagnets in a field

The Heisenberg system may display a BKT transition in the presence of magnetic field. The reason is the staggered magnetization preferring being perpendicular to the infinitesimally weak external field $\rightsquigarrow$ BKT behaviour.

p.225 Landau QM (Kramers)

Nonmagnetic magnets

Do uncompensated ionic spins following the external magnetic field make a material magnetic? This however does not guarantee, that the microscopic moments react to the field, especially at very low temperatures it may happen that the ground state of a particular magnetic ion with even number of electrons is a singlet.

for odd no formation of singlet possible

Example: (Ni^{2+} in easy-plane environment)

- $S=1$, singlet ground state $|S^z=0\rangle$ and doublet excited states $|S^z_{=\pm 1}\rangle$ seperated by energy gap.
- $\langle\langle \hat{\vec{S}} \rangle\rangle = \langle 0|\hat{\vec{S}}|0\rangle = 0 \Rightarrow$ magnetic moment quenched at low temperature and the external magnetic field has nothing to couple to.
- $\Delta \ll k_B T$?? at high temperature all states are present with equal probability and the material would behave like a normal $S=1$ paramagnet

Example: ($S=\frac{1}{2}$ A.F. dimer)

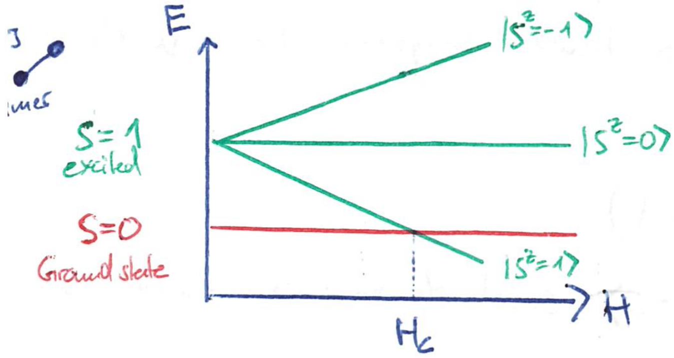
- for A.F. interaction $\rightsquigarrow$ singlet $|0,0\rangle$ and triplet $|0,(\pm 1,0)\rangle$ as ground state of dimer not degenerate $\rightsquigarrow$ spins decouple from external field
  i.e. only in singlet
- For $T >$ gap (singlet/triplet) coupling to $\vec{H}^{ext}$ restored
- For $T \gg$ gap, $\rightsquigarrow$ trivial $S=1/2$ paramagnet

An even number of electrons in the "magnetic molecule" bears risk of ending up in a non-degenerate non-magnetic ground state. For odd number of electron when Kramers theorem (p.225 Landau QM) would guarantee that every level is at least doubly degenerate. is spin different? Yes...

complex conjugate spinor, different state with the same energy

The mechanism of this effect is: the states that can couple to the magnetic field are at the same time energetically expensive and can not be reached at low enough temperatures.

$\Rightarrow$ magnetization remains zero in the ground state

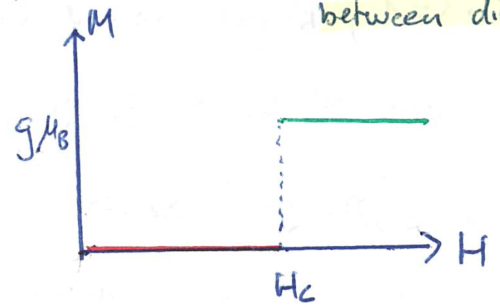


The energy of the excited states are field dependent

$|S^z=1\rangle$ is lowered by Zeeman $\Rightarrow$ ground state

$\Rightarrow$ At H_c, magnetization is restored abruptly

More complex magnetic molecules $\Rightarrow$ several steps until full saturation. Assumes no interaction J between dimer/m.molecules.



Discreteness of the magnetization curve stems from Quantum mechanics.

100% quantum effect $\Rightarrow$ quantum paramagnets

Collective behaviour

Magnetic Bose-Einstein condensation

For weak interaction between magnetic molecules (weaker than singlet gap) no order occurs (not even mean field). As long as singlet picture valid

Assuming interaction $J'_{\vec{R}}$ (between m. molecules, can be weak) and a magnetic field. The excited states are no longer localized and are able to hop from site to site. High-energy states start to behave like quasiparticles and form a band. Dispersion law

$$\hbar\omega = \sqrt{\Delta^2 + 2\Delta J'(\vec{q})} \rightarrow \begin{cases} \sim q \\ \sim q^2 \end{cases} \quad \Delta \gg q$$

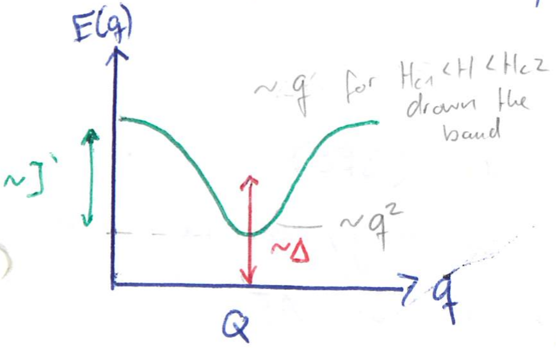
Δ: ground state energy gap of free m. molecule; $J'(\vec{q})$: F.T.

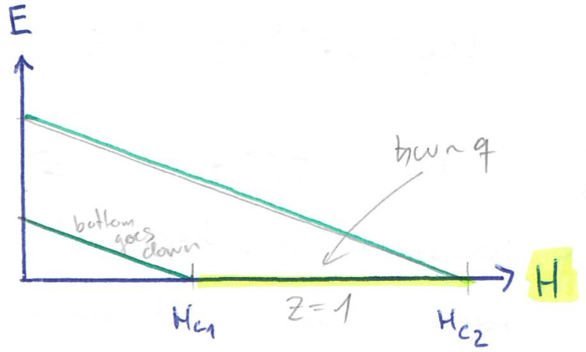
⟹ "level-crossing" event in magnetic field becomes extended to a finite field interval $H_{c1} - H_{c2}$, no jump-like manner anymore as the interaction would resist to it. (as above)

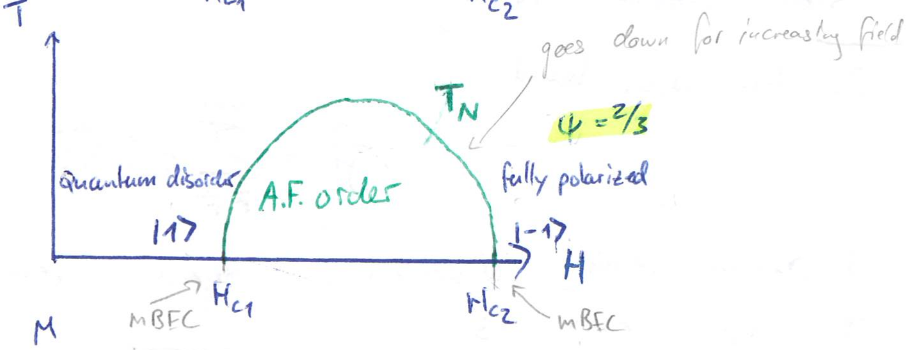
⟹ $\langle\hat{S}\rangle \neq 0$ (magnetic molecule can couple) magnetic field makes parallel, interaction prefers antiparallel

⟹ uniform along the field but antiparallel transverse to it

⇝ field induced quantum phase transition



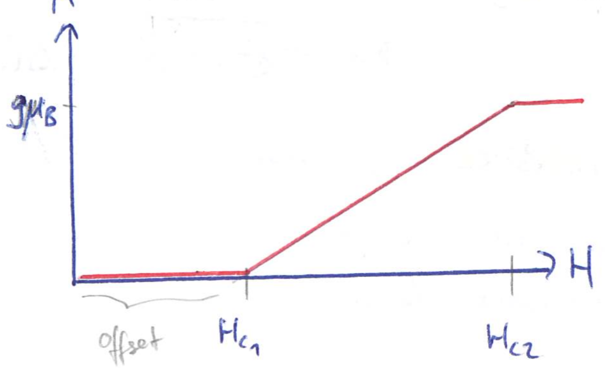




⟹ magnetic Bose Einstein universality class

z = 2

z = 1 after BEC (in A.F. phase)



$z=1$ criticality

lecture 6
36 min

singlet & triplet gap

For large interactions the intrinsic energy scale does not matter. Such systems should be an ordinary magnet. On the other hand, weak interactions there is quantum disorder. Therefore $\xrightarrow[\text{Q.D.} \quad | \quad \text{Magnet}]{} J$, and can be driven by applying pressure and hence altering superexchange geometries. Not a magnetic BEC transition due to different symmetry.

no coupling to field

$$\hat{\mathcal{H}}_0 = \sum_{\vec{r},\vec{r}'}^{\text{dimers}} J \hat{\vec{S}}_{\vec{r}} \hat{\vec{S}}_{\vec{r}'} + \sum_{\vec{r},\vec{r}'}^{\text{inter}} J' \hat{\vec{S}}_{\vec{r}} \hat{\vec{S}}_{\vec{r}'} \quad , \quad \hat{\mathcal{H}}' = -g\mu_B \sum_{\vec{r}} \hat{S}^z_{\vec{r}}$$

$[\hat{\mathcal{H}}_0, \hat{\mathcal{H}}'] = 0 \Rightarrow$ triplet not distorted but degeneracy is removed:

$$\hbar\omega = \sqrt{\Delta^2 + (cq)^2} \pm g\mu_B H$$

field compensating the gap

At $H_c = \dfrac{\Delta}{g\mu_B}$: $\quad \hbar\omega = \dfrac{c^2}{2\Delta} q^2 \quad \Rightarrow z = 2$ (BEC)

Only after the transition has happened the low-energy spectrum would start gradually turning linear, as it should be for an A.F. $\hbar\omega \sim q$

F.M. $\rightsquigarrow \hbar\omega \sim q^2$

On the other hand:

$$\hat{\mathcal{H}}' = \sum_{\vec{r},\vec{r}'}^{\text{inter}} \delta J' \hat{\vec{S}}_{\vec{r}} \hat{\vec{S}}_{\vec{r}'} \quad \text{does not commute with } \hat{\mathcal{H}}_0$$

Dispersion relation distorted but degeneracy remains. $\mathcal{H}_{tot}$ remains Heisenberg and symmetric.
The triplet softens as

$$\hbar\omega = \sqrt{\Delta^2 + (cq)^2}; \quad \Delta(p) \longrightarrow 0 \quad \Rightarrow \hbar\omega = cq$$

pressure

$\Rightarrow z = 1$ not the BEC for pressure induced transition

Problem: ordered components are vanishingly small $\quad \langle S^z \rangle = 0$

$\rightsquigarrow$ Generalized spin wave theory

Generalized spin wave theory

Ansatz
$|\psi\rangle = \prod_{\vec r} |\psi\rangle_{\vec r}, \quad |\psi\rangle_{\vec r} = \sum_{\lambda} m_\lambda |\lambda\rangle_{\vec r}$

Holstein-Primakoff and Matsubara-Masuda map spins to bosons. It is possible to construct spin operators using a few bosonic particles of different "flavors". (Schwinger bosons)

Assume $S=1/2$ with states $|\uparrow\rangle$ and $|\downarrow\rangle$, vacuum $|0\rangle$

$$|\uparrow\rangle = a_\uparrow^\dagger |0\rangle, \quad a_\uparrow|\uparrow\rangle = |0\rangle, \quad a_\uparrow|\downarrow\rangle = 0$$
$$|\downarrow\rangle = a_\downarrow^\dagger |0\rangle, \quad a_\downarrow|\downarrow\rangle = |0\rangle, \quad a_\downarrow|\uparrow\rangle = 0$$

Operators: $A_{\alpha\beta} = \langle\alpha|\hat A|\beta\rangle \Rightarrow \hat A = \sum_{\alpha,\beta=\uparrow,\downarrow} a_\alpha^\dagger A_{\alpha\beta} a_\beta$

Here: $\hat S^\gamma = \sum_{\alpha,\beta=\uparrow,\downarrow} a_\alpha^\dagger \sigma^\gamma_{\alpha\beta} a_\beta$, where σ pauli matrices

$$|\psi\rangle = \lambda|\uparrow\rangle + \mu|\downarrow\rangle \Rightarrow \underbrace{|\lambda|^2 + |\mu|^2 = 1}_{\text{normalized}} \Rightarrow a_\uparrow^\dagger a_\uparrow + a_\downarrow^\dagger a_\downarrow = 1 \;\; (=2S)$$

is a constraint of possible number of Schwinger bosons of different flavours. In case of higher S, the constraint has to be increased as $2S$.

$$\Rightarrow \hat S^+ = a_\uparrow^\dagger a_\downarrow, \quad S^- = a_\uparrow a_\downarrow^\dagger$$

Thus, replacing "up" and "down" particle switch between the states of Zeeman basis $|S,m\rangle$. The complete Hilbert state of an arbitrary magnetic ion is accessible.

S=1 model ground state (Following the lecture)

$|0\rangle, |{-1}\rangle, |1\rangle$ — 3 states

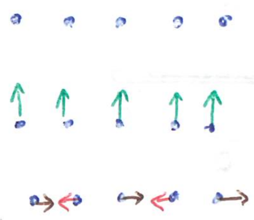
$a_{0,r} \quad a_{1,r} \quad a_{-1,r}$

$$|\psi_{\vec r}\rangle = \cos\theta\,|0\rangle_{\vec r} + \sin\theta\, e^{i\vec Q\cdot\vec r}\left(\cos\varphi\,|1\rangle_{\vec r} + \sin\varphi\,|{-1}\rangle_{\vec r}\right)$$

$|QD\rangle$: $\theta=0$, $\varphi = ?$ (arbitrary), $|\psi\rangle = \prod_{\vec r} |0\rangle_{\vec r}$

fully polarized $|FP\rangle$: $\theta = \pi/2$, $\varphi=0$, $|\psi\rangle = \prod_{\vec r} |1\rangle_{\vec r}$

$|AF\rangle$: $\theta=\pi/2$, $\varphi = \pi/4$, $|\psi\rangle_{\vec r} = \pm(|1\rangle + |{-1}\rangle)$



$$\Rightarrow |\psi\rangle_{\vec r} = (\cos\theta\, \hat a^\dagger_{\vec r,0} + \dots)\underbrace{|0\rangle}_{\text{vacuum}} \xleftrightarrow[\text{convenient}]{\text{more}} |\psi\rangle_{\vec r} = \hat b^\dagger_{\vec r,\psi}|0\rangle$$

single boson creating the whole single ground state

$$\begin{pmatrix} b^+_{\vec r,\lambda_1} \\ \vdots \\ b^+_{\vec r,\lambda_N} \end{pmatrix} = \hat{U} \begin{pmatrix} a^+_{\vec r,\lambda_1} \\ \vdots \\ a^+_{\vec r,\lambda_N} \end{pmatrix} \Rightarrow \hat{A}_{\vec r} = \sum_{\alpha\beta} (\hat{U}\hat{A}\hat{U}^+)_{\alpha\beta} \, \hat{b}^+_{\vec r \alpha} \hat{b}_{\vec r \beta}$$

$\hat{U}(\varphi,\theta)$

unitary

MF approximation

$$\langle\langle \hat{b}_{\psi,\vec r} \rangle\rangle = \langle\langle \hat{b}^+_{\psi,\vec r} \rangle\rangle = \sqrt{1 - \underbrace{\sum_{\lambda\neq\psi} b^+_{\lambda,\vec r} b_{\lambda,\vec r}}_{\text{not in condensate}}}$$

normalization condition

other states

groundstate

total # bosons/site $\stackrel{!}{=} 1$

/ most of them in condensate

$\Rightarrow$ little residue

Taylor

$$= 1 - \frac{1}{2} \sum_{\lambda\neq\psi} b^+_{\lambda,\vec r} b_{\lambda,\vec r} + \dots$$

reduced number of bosons by 1

$\hat{\mathcal{H}}_{spin} \rightarrow \mathcal{H}_{N-1}$ $\hat{b}$ flavours, dilute $\Rightarrow$ no interaction between b

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \cancel{\hat{\mathcal{H}}_1}^{\,0} + \hat{\mathcal{H}}_2$$

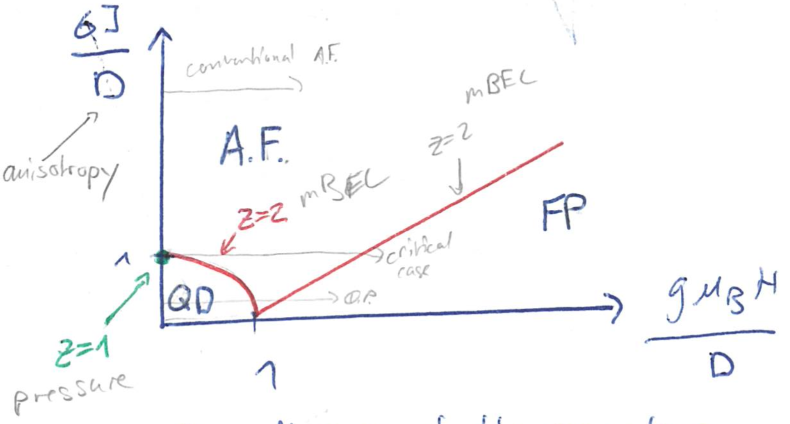
$E_{MF}(\varphi,\theta)$

minimize $\Rightarrow \varphi_0, \theta_0$

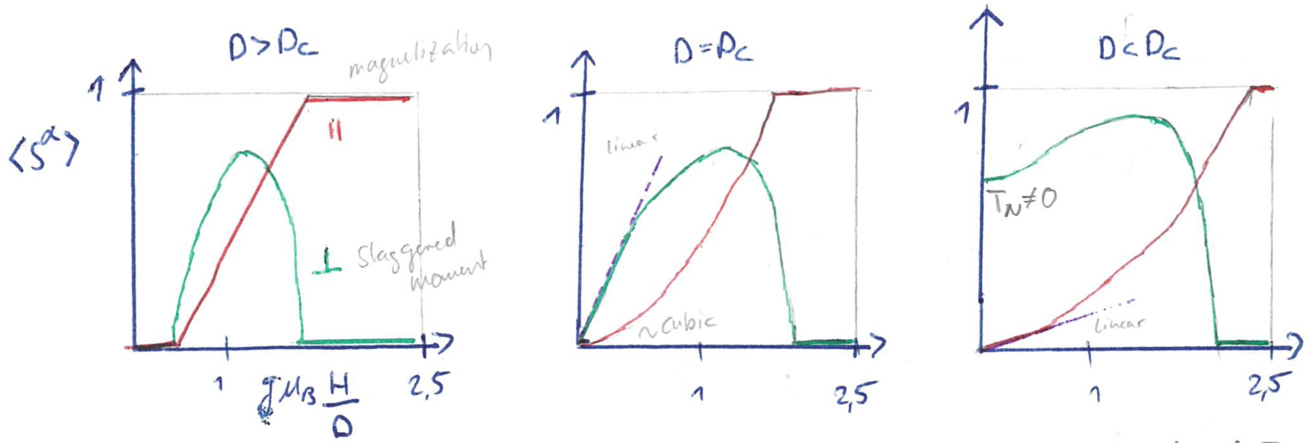
$\Rightarrow$ we get $|GS\rangle$

biquadratic

anisotropy



Phase diagram of the easy plane S=1 A.F.



MF approximation Part II

$$\hat{\mathcal{H}}_2 = \sum_{\vec{q}} \sum_{\alpha,\beta \neq 4} E_{\alpha\beta}(\vec{q})\, \hat{b}^{\dagger}_{\alpha q}\, \hat{b}_{\beta \vec{q}} + c.c.$$ (3.56 script)

$$+ \Gamma_{\alpha\beta}(\vec{q})\, \hat{b}^{\dagger}_{\alpha \vec{q}}\, \hat{b}_{\beta q} + c.c.$$

Not yet diagonalized

Bogolyubov transformation

We have to diagonalize the Hamiltonian with bosons of a few different flavours.

- make new particles from the superpositions of full and empty states of b-bosons.

$$M = \begin{pmatrix} E & \Gamma \\ -\Gamma & -E \end{pmatrix}$$ $2(N-1) \times 2(N-1)$ matrix

↳ eigenvalues → $\hbar\omega(q)$ $N-1$

↳ eigenvectors $X = \begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix}$

complex conjuge, No transposition

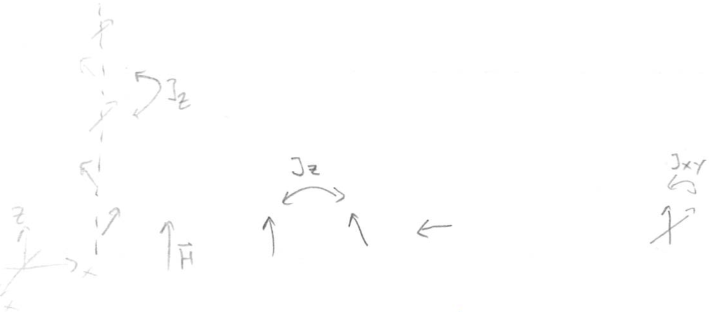
$$\hat{b}_{\vec{q}\alpha} = \sum_{\beta} u^{\alpha\beta}_{\vec{q}}\, \hat{c}_{\vec{q},\beta} + v^{*\alpha\beta}_{-\vec{q}}\, \hat{c}^{\dagger}_{-\vec{q},\beta}$$ (multimode Bogoliubov transfor)

zero point fluctuations

$$\Rightarrow \hat{\mathcal{H}}_2 = \sum_{\lambda} \left(\hbar\omega^{\lambda}(\vec{q}) + \frac{1}{2}\right) \hat{c}^{\dagger}_{\lambda q}\, \hat{c}_{\lambda \vec{q}}$$

See animation in slide

$S = 1/2$ spin chains



Jordan-Wigner transformation

XXZ spin chain, magnetic field applied along the special direction z:

chain direction ??

$$\hat{\mathcal{H}} = \sum_n^{L=\infty} J_{xy}\left(\hat{S}_n^x \hat{S}_{n+1}^x + \hat{S}_n^y \hat{S}_{n+1}^y\right) + J_z \hat{S}_n^z \hat{S}_{n+1}^z - g\mu_B H \hat{S}_n^z$$

chain site

not general

Naive approach and its failure

$$\hat{S}_n^+ = \hat{c}_n^+ \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \hat{S}_n^- = \hat{c}_n \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \hat{S}_n^z = \hat{c}_n^+ \hat{c}_n - \frac{1}{2}$$

(Matsubara-Masuda)

Remember: hard-core repulsion $\Rightarrow$ quasiparticles bosonic

all states can be empty

Without this constraint:

p. 222 Landau QM

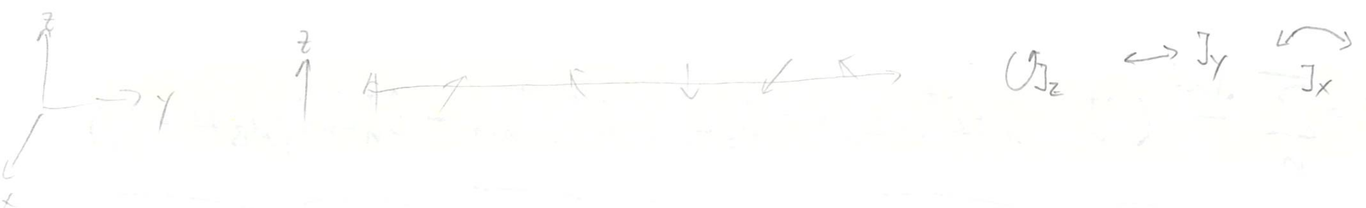
$$\{\hat{c}_n, \hat{c}_n^+\} = 1, \quad \{\hat{c}_n, \hat{c}_n\} = \{\hat{c}_n^+, \hat{c}_n^+\} = 0 \qquad \text{(fermionic)}$$

However, $[\hat{c}_n, c_{n'}^+] \overset{n \neq n'}{=} 0$ is boson-like but $\{\hat{c}_n, \hat{c}_{n'}^+\} = 0$ is required!

fermionic

interchange problematic if fermions

$\Rightarrow$ introduced mapping not enough. The modification of our transformation has to be non-local.



Introducing non locality

p.9 Schollwöck

The appropriate non-local transformation was suggested by Jordan and Wigner.
$\Rightarrow$ attaching an infinite string of $\hat{S}^z$ operators to every c-particle.

$$\hat{c}_n = \hat{S}_n^- \prod_{m<n} (-2\hat{S}_m^z)$$

$$\hat{c}_n^+ = \hat{S}_n^+ \prod_{m<n} (-2\hat{S}_m^z)$$

Particles and holes possess a modified phase that depends on the local magnetization in the infinite half-chain to the left.

exercise
$\Rightarrow$ c-particles possess the right fermionic statistics.

Due to this non-locality it is difficult to translate back into spin

Jordan-Wigner particles are topological as they "know" the state of infinite number of other sites. The reverse transformation

$$\hat{S}_n^+ = \hat{c}_n^+ e^{i\pi \sum_{m=-\infty}^{n-1} c_m^+ c_m}$$

$$\hat{S}_n^- = \hat{c}_n e^{-i\pi \sum_{m=-\infty}^{n-1} c_m^+ c_m}$$

$$\hat{S}_n^z = \hat{c}_n^+ \hat{c}_n - \frac{1}{2}$$

simply the particle density

$\vec{H}$ along z-direction

$\sim S^z$ How much

potential

$$\Rightarrow \hat{\mathcal{H}} = \sum_n \underbrace{\frac{J_{xy}}{2}(\hat{c}_n^+ \hat{c}_{n+1} + \hat{c}_{n+1}^+ \hat{c}_n)}_{\text{hopping term}} + \underbrace{J_z \hat{c}_n^+ \hat{c}_n \hat{c}_{n+1}^+ \hat{c}_{n+1}}_{\text{interaction}} - \underbrace{\hat{c}_n^+ \hat{c}_n (g\mu_B H + J_z)}_{\text{Chemical potential}}$$

$\Rightarrow$ Solving the problem of interacting fermions in one dimension would automatically solve the problem of XXZ spin chain.

XY chain

Free fermions

there is no interaction term
it is not generally "free"

$J_z = 0 \rightsquigarrow$ Hamiltonian of free fermions:

along chain

$$\hat{\mathcal{H}} = \sum_n \underbrace{\frac{J_{xy}}{2}\left(\hat{c}_n^{+}\hat{c}_{n+1} + \hat{c}_{n+1}^{+}\hat{c}_n\right)}_{\text{hopping term}} + \hat{c}_n^{+}\hat{c}_n\, g\mu_B H$$

1) FT
2.) Harmonic oscillato
3) find dispersio

$$\hat{c}_n^{+} = \frac{1}{\sqrt{L}}\sum_q e^{iqna}\hat{c}_q^{+} \overset{\text{F.T.}}{\Longrightarrow} \hat{\mathcal{H}} = \sum_q \left(\epsilon_q + \tfrac{1}{2}\right)\hat{c}_q^{+}\hat{c}_q$$

dispersion: $\epsilon_q = J_{xy}\cos qa - g\mu_B H \Rightarrow g\mu_B H_{sat} = J_{xy}$

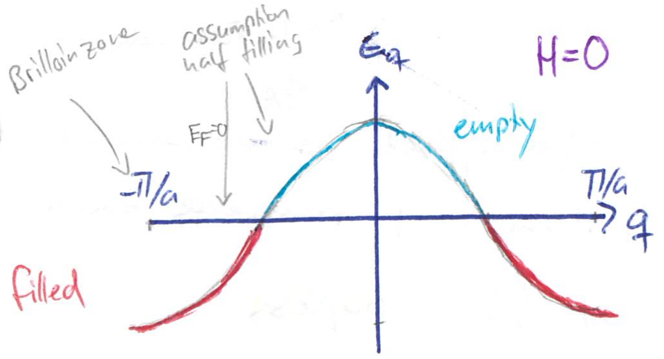
HMW

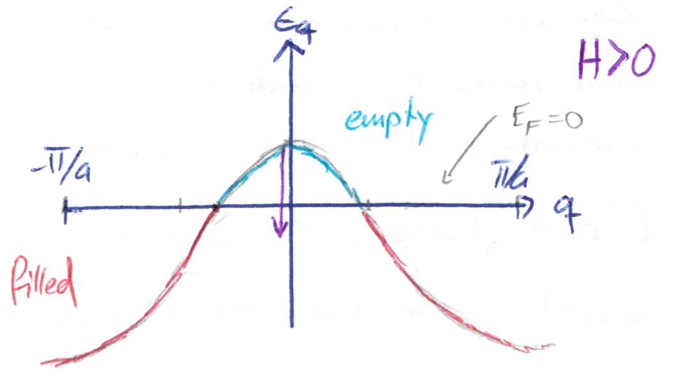
$\Rightarrow$ no magnetization in zero magnetic field $\Rightarrow$ # particles = # holes

$\pi/2$

at half filling, $\langle\langle \hat{c}_i^{+}\hat{c}_i\rangle\rangle = 1/2 \Rightarrow E_F = 0$, $f(\epsilon_q) = \dfrac{1}{1+e^{\epsilon_q/T}}$, $v_F = \left.\dfrac{\partial \epsilon}{\partial q}\right|_{\epsilon=0} = J_{xy}$

A magnetic field would push the band further below $E_F = 0$ thus increasing the number of present particles (that is magnetization).





At the critical field H_{sat}, the full band is below $E_F \Rightarrow$ all the states are occupied $\Rightarrow$ fully polarized

$\rightsquigarrow$ similar to 2D/3D Fermi gas

XY model response functions

p. 11 Schollwöck

Absence of interactions, ideal electron gas model results can be borrowed. Susceptibility per site

Lindhard

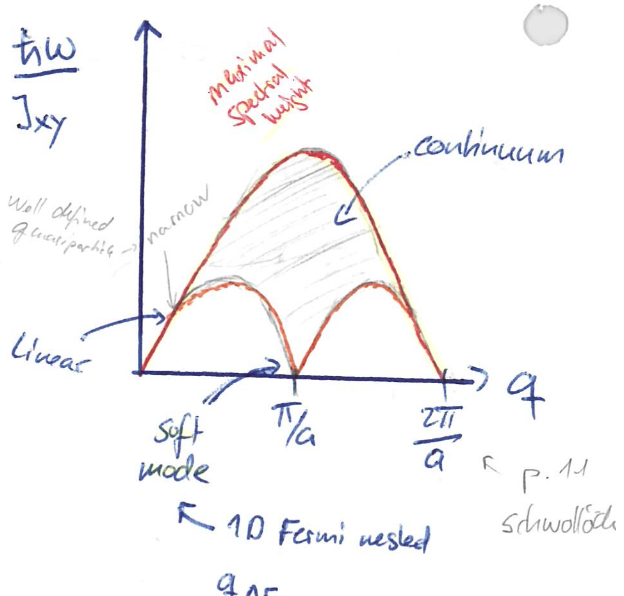
$$\chi(q,\omega) = -\mu_B^2 \sum_k \frac{f(\epsilon_k) - f(\epsilon_{k+q})}{\epsilon_k - \epsilon_{k+q} - \hbar(\omega + i\varepsilon)}, \quad \varepsilon \to 0$$

$$\Rightarrow \quad S(q,\omega) = -L\mu_B^2 \sum_k [f(\epsilon_k) - f(\epsilon_{k+q})]\, \delta(\epsilon_k - \epsilon_{k+q} - \hbar\omega)$$

Describes picking a particle from the Fermi sea and promoting it to a non-occupied state above E_F.

Describes $\Delta S = 0$ processes.

For $\Delta S = \pm 1$ this happens at Fermi level $\Rightarrow$ total momentum change by π/a.



One Fermion added or removed the excited higher energy state can be constructed by creating a particle-hole excitation. The only difference would be the offset in momentum (from change in fermion number). Continuum of transverse excitations would look like above but shifted by π/a. Gapless

$U = J_{xy}$

Nature of the ground state

For $T=0$, the correlation functions for the transverse and longitudinal spin components is (final result quoted):

$$\langle GS| \hat{S}^z_n \hat{S}^z_{n+L} |GS\rangle \propto \begin{cases} 1/L^2, & L \text{ odd} \\ 0, & L \text{ even} \end{cases}$$

← algebraic decay

$$\langle GS| \hat{S}^x_n \hat{S}^x_{n+L} |GS\rangle \propto \frac{1}{\sqrt{L}}$$

← very slow

Slow decay is a fingerprint of a critical state, and $T=0 \Rightarrow$ quantum critical state

finite interchain coupling $\Rightarrow$ semiclassical Néel state

extra anisotropy $\Rightarrow$ makes it gapped, pushes away from order

Ising Chain

T=0 order and domain walls

is chain still along z?? and if so, ⇒ ⇐ → ← instead of ↑↓↑↓

pure Ising ⟺ only J_z term left, ground state: Néel antiferromagnetic

$|\uparrow\downarrow\uparrow\downarrow\ldots\rangle$ minimizes $\hat{H} = J_z \sum_n \hat{S}^z_n \hat{S}^z_{n+1}$ ⇒ Néel state is robust

The domain wall (phase slip) is the lowest-energy excitation. In the pure Ising case with $J_{xy}=0$ the domain walls have no mean of propagating along the chain, are localized and the momentum-idependant energy is $\epsilon_q = \frac{J_z}{2}$. The entropy is $\sim \log L$. For any $T \neq 0$ the probability to have one is $\rho_{DW} \propto e^{-\frac{J_z}{2T}}$

↳ no term to flip the spins

$$\Rightarrow \quad \xi(T) \propto \frac{1}{\rho_{DW}} \propto e^{\frac{J_z}{2T}} \quad \text{(length of domain)}$$

Thus the transition to the ordered state at $T=0$ is reminiscent of the BKT universality class. Gapped

erinnernd

Non-ideal Ising model: mobile domains

S^+, S^- in the term

If a small J_{xy} exchange component is added the domain wall becomes mobile. For $J_{xy} \neq 0$ the staggered magnetization is reduced and some moving domain walls are inevitably present even at $T=0$. As long as $J_{xy} \ll J_z$ the deviation from the Néel state is small and we can treat them as weakly-interacting particles

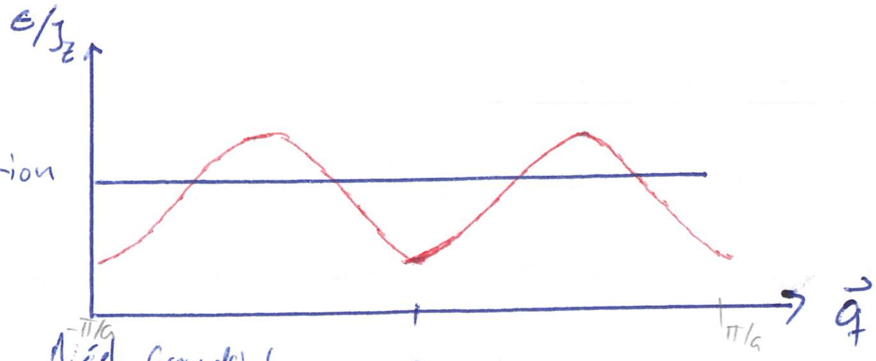
no derivation

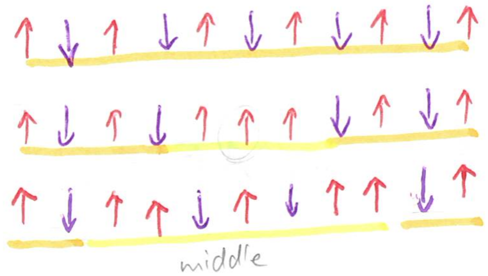
$$\Rightarrow \quad \epsilon_q = \sqrt{\left(\frac{J_z}{2}\right)^2 - J_z J_{xy} \cos(2qa)}$$

Non-ideal Ising model: mobile domains part II

$J_{xy} = 0{,}45\, J_z$

in Random Phase Approximation





Néel Groundstate

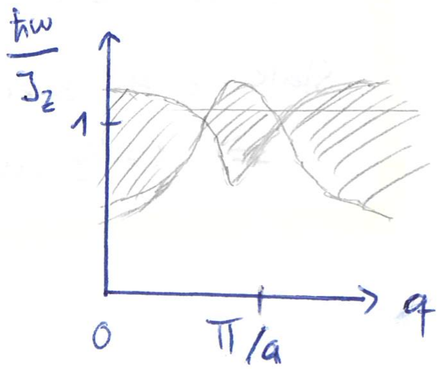
$\Delta S = \pm 1$ (spin flip) — $\pm 1/2$ for flip and $\pm 1/2$ for initial state

Flipping pairs comes at no energy cost

- Energy does not depend on length L of the middle domain
- There will be two domain walls, each carrying $S = 1/2$
- Excitations are fermions, we can only excite them in pairs (cannot change spin state by half-integer)
- if L even $\Rightarrow$ $\Delta S = 0$, L odd $\Rightarrow$ $\Delta S = \pm 1$
- continuum in dynamics response of the system by

$$\hbar\omega(q) = \epsilon_k + \epsilon_{k+q} \quad , \quad k \text{ open parameter}$$

Beware: approximation is under the assumption that $J_{xy} < J_z$



butterfly continuum

Quasi particles cannot avoid each other

Fermi Gas description not appropriate

Non ideal $\Rightarrow$ levels in the continuum p.57 Auerbach

Heisenberg case

No analytical approach

Lieb-Schultz-Mattis theorem

For $S=1/2$ dimer the "semiclassical" degenerate states $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ are not the true eigenstates of the problem. Instead, it is

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

separated from the excited states by a gap Δ. For ring A.F., # spins = even = N

$$\hat{\mathcal{H}} = J\sum_{n=1}^{N} \hat{\vec{S}}_n \hat{\vec{S}}_{n+1} + J\hat{\vec{S}}_1 \hat{\vec{S}}_N$$

Marshall's theorem $\Rightarrow \sum_{n=1}^{N} \hat{S}^z_n |\Psi_0\rangle = 0$ singlet and unique ground state

$\Rightarrow \Delta \neq 0$ for any finite ring, but what happens at $N \to \infty$ (therm. limit)

For $S=1/2$, Δ will vanish in the thermodynamic limit.

Heisenberg A.F.:
For half-odd integer spin chains there exists an excited state with energy that vanishes as $N\to\infty$.
$\Rightarrow$ $S=1/2$ chain would have a gapless spin.

proof in script

Some Heisenberg chain properties

Heisenberg chain gapless ⇒ quantum critical and

$$\langle \Psi_0 | \hat{S}^x_n \hat{S}^x_{n+L} | \Psi_0 \rangle \overset{\text{isotropic}}{=} \langle \Psi_0 | \hat{S}^z_n \hat{S}^z_{n+L} | \Psi_0 \rangle \propto \frac{(-1)^L}{L} \qquad \text{QLRO}$$

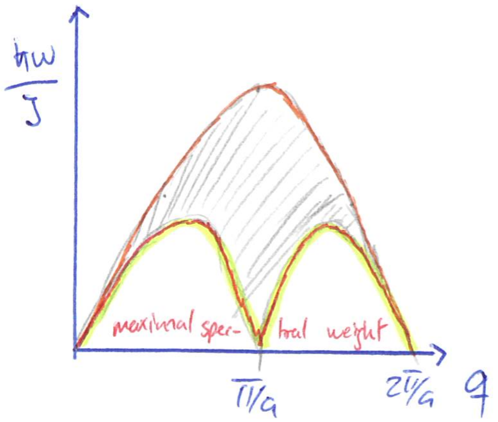
There are $S=1/2$ spinons similar to DW or particle-hole excitations in XY chain

We are only able to excite pairs of spinons,

Dispersion is LSWT solution renormalized by $\frac{\pi}{2}$

The dynamic structure factor $S^{\alpha\alpha}(q,\omega)$ is divergent along $\hbar\omega_L(q)$, just as for normal spin wave.

Bottom of continuum is descendant of the classical spectrum without power-law tails.



Ansatz for two-spinon picture

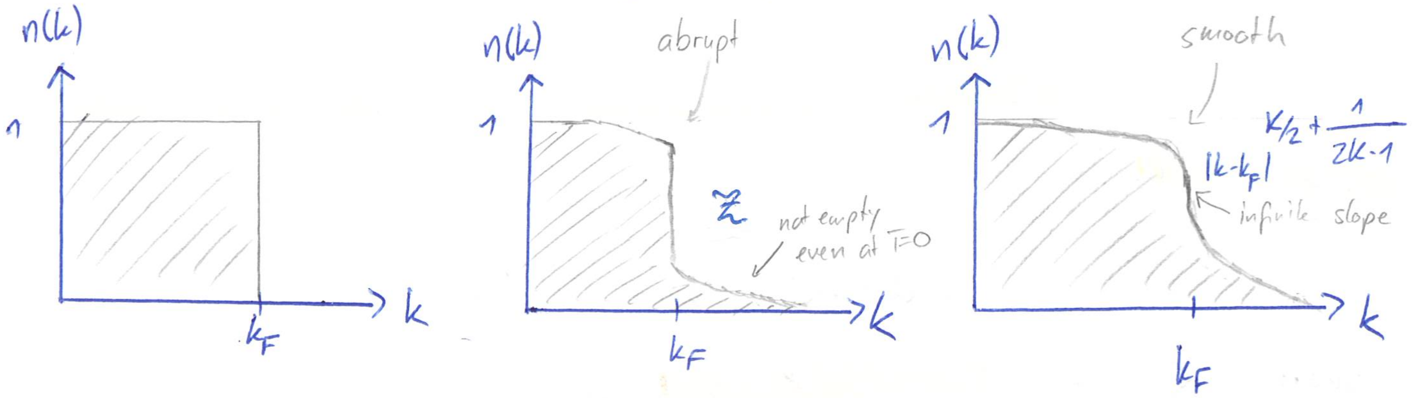
$$S(q,\omega) \propto \frac{1}{\sqrt{\omega^2 - \omega_L(q)^2}} \Theta(\omega - \omega_L(q)) \Theta(\omega_u(q) - \omega) \qquad \text{(Müller ansatz)}$$

- intensity diverges as $\frac{1}{\sqrt{\cdot}}$ ⇒ $S(\pi,\omega) \propto \frac{1}{\omega}$
- not exact, upper boundary should be more washed as higher order excitations are neglected
- covers about 98% of the total spectrum, sufficient for practical purposes
- The spin of a spin wave in the Heisenberg A.F. chain of spins 1/2 is equal to $\frac{1}{2}$ rather than 1.

Tomonaga - Luttinger spin liquid

Fermi liquid versus Tomonaga - Luttinger liquid

$T=0$



Fermi gas

Fermi Liquid
2D/3D

Tomonaga - Luttinger Liquid
1-D

In the ideal gas: $n(k) = \frac{1}{1+e^{\frac{\epsilon_k - E_F}{T}}}$, abrupt cutoff at $T=0$ at $\epsilon_{k_F} = E_F$

Close to the Fermi surface: $\epsilon_k \simeq E_F + \hbar v_F (k-k_F)$ dispersion, not $n(k)$!

Due to absence of interactions such excitations are absolutely well-defined and have infinite life time.

Fermi-Liquid:

abrupt cutoff at E_F, but even at $T=0$ the states above k_F not empty.

Reduction of 1 to $Z<1$ (the stronger the interactions, the smaller Z)
The beauty is that it can handle very strong interactions (e.g. Coulomb) but $Z \sim 1$ leaving gaseous picture for the excitations. The landau quasiparticles are characterized by a simple linearized dispersion relation, but the life time τ of the quasiparticles are now finite. However, the damping decreases close to E_F

$$\frac{1}{\tau} \sim (k-k_F)^2$$

$\Rightarrow$ particle well defined at Fermi surface. Quasiparticles present also for $T=0$. Excited electrons are dressed by weak density fluctuations.

Interacting fermions in 1D

Due to the space restrictions the "dressing" density fluctuations become infinitely large, leading to a collapse of the Fermi surface jump in $n(k)$, which is now smooth. Singular behaviour in $n(k)$ is in the slope at k_F which becomes infinite. The Tomonaga-Luttinger liquid is the analogue of the Fermi liquid in 1-D. The singular behaviour can be approximated by

$$n(k) \propto |k-k_F|^{K/2 + \frac{1}{2K-1}}$$

where K is the Luttinger exponent and positive.

$$K = \begin{cases} >1 & \text{attractive interactions} \\ =1 & \text{non interacting} \\ <1 & \text{repulsive interaction} \end{cases}$$

The low-energy excitations at $k \simeq k_F$ are not quasiparticles with well-defined ϵ_k dispersion but rather a continuum of available states for every k within the cone:

$$E_F + \underbrace{u}_{\simeq v_F}|k-k_F|$$

u is the analogue of the Fermi velocity. and together with K fully characterize the low-energy physics of the Tomonaga-Luttinger liquid. The observables are proportional to $\left(\frac{T}{u}\right)^{\phi(K)}$. For spin systems described by TLL, $\frac{\omega}{T}$ kind of scaling will be found that was proposed for the QCP. It is clearly non MF-like as $d_{eff} = 1+1 = 2$ as far as possible from the mean field threshold $d_{eff} = 4$.

Bosonisation

From lattice fermions to fields

The goal is to represent the interacting fermionic particle operators in terms of some continuous quantum bosonic fields. Jordan-Wigner representation of XXZ:

$\vec{H}=0$

$$\hat{\mathcal{H}} = \sum_i \frac{J_{xy}}{2}\left(\hat{c}_i^{+}\hat{c}_{i+1} + \hat{c}_{i+1}^{+}\hat{c}_i\right) + J_z \hat{c}_i^{+}\hat{c}_i \hat{c}_{i+1}^{+}\hat{c}_{i+1} - \hat{c}_i^{+}\hat{c}_i J_z$$

see chapter introducing non locality

The operator $\hat{S}_i^z = \hat{c}_i^{+} c_i - \frac{1}{2}$ is simply the particle density $\hat{\rho}_i - \frac{1}{2}$, which can be written as a continuous function

$$\hat{S}^z(x) = \hat{\rho}(x) - \frac{1}{2} = \sum_n \delta(x - x_n) - \frac{1}{2}$$

in 1D there is a unique correspondance → particles and not lattice sites

$$= \sum_n |\nabla f(x)|\, \delta(f(x) - 2\pi n) \quad , \quad f(x_n) = 2\pi n$$

makes labelling field $f(x)$ well defined

particles

$$\Rightarrow \quad \hat{\rho}(x) = \frac{\nabla f(x)}{\pi} \sum_{p=-\infty}^{\infty} e^{ipf(x)}$$

Introducing a new field $\varphi(x)$ which describes the deviations of the filling

$$f(x) = \pi x - 2\varphi(x)$$

$$\Rightarrow \hat{S}^z(x) = \left[\frac{1}{2} - \frac{1}{\pi}\nabla\varphi(x)\right] \sum_{p=-\infty}^{\infty} e^{ip(\pi x - \varphi(x))} - \frac{1}{2}$$

long-wavelength limit (large x): $\hat{S}^z(x) = -\frac{1}{\pi}\nabla\varphi(x)$, $\rho(x) \simeq \frac{1}{2} - \frac{1}{\pi}\nabla\varphi(x)$

As the creation and annihilation operators can always be written in terms of a phase and amplitude

$$\hat{c}_n^+ = \sqrt{\rho_n}\, e^{-i\theta} \quad , \quad \hat{c}_n = \sqrt{\rho_n}\, e^{i\theta}$$

$\Rightarrow$ $\hat{c}^+(x) = \sqrt{\hat{\rho}(x)}\, e^{-i\theta(x)}$, where the phase field $\theta(x)$ is introduced

Meaning of φ and θ

In order to satisfy the fermionic nature of $\hat{c}$-operators the fields φ and θ must obey

$$[\varphi(r), \tfrac{1}{\pi}\nabla\theta(r')] = i\delta(r-r')$$

coordinate/position   canonical momentum

Oscillator analogue:

$\Rightarrow$ System of identical quantum oscillators that exist everywhere

$\Rightarrow$ ensemble of bosons (Groundstate: no bosons, each state adds a boson)
In the bosonic basis the $\hat{H}$ can be easily diagonalized.

In one dimension: We can switch between the bosonic and fermionic descriptions at our convenience, but the interactions between the quasiparticles would be affected. In a sense we can trade particle statistics for interactions.

$\varphi \hat{\simeq}$ density of fermions $\hat{\equiv}$ polar canting angle of the spin in respect to z axis

$\theta \hat{=}$ phase $\hat{=}$ orientation of spin in xy plane

$\Rightarrow$ it is clear why they are not commuting as for spin rotations do not commute either.

The final form

more profound derivation $\Rightarrow$

$$\hat{S}^z(x) = -\frac{1}{\pi}\nabla\varphi(x) + \sqrt{2A_z}\,(-1)^x \cos(2\varphi(x))$$

string in $\hat{c}^+$ $\Rightarrow$

$$\hat{S}^\pm(x) = e^{\mp i\theta(x)}\left[\sqrt{2A_x}\,(-1)^x + 2\sqrt{B_x}\cos(2\varphi(x))\right]$$

A_x, B_x, A_z are non-universal and depend on the details of the Hamiltonian such as J_z/J_{xy} ratio or magnetic field. Further, a magnetic field rescales $\varphi(x)$ by

$$\varphi(x) \to \varphi(x) - \pi m^z x$$

The TLL Hamiltonian

The most primitive Tomonaga-Luttinger Hamiltonian is given by

overall energy scale

LRO in spin

$$\hat{H} = \frac{u}{2\pi}\int \left[K(\nabla\theta(x))^2 + \frac{1}{K}(\nabla\varphi(x))^2\right] dx$$

relative weight of φ, θ reflects the interaction

constant fermion density, important for repulsive interaction

In the majority case this is the only part of the Hamiltonian resposible for the low-energy behaviour.

Correlations and response

$\sim$ magnetic field

incommensurate

longitudinal: $$\langle \hat{S}^z_n \hat{S}^z_{n+L} \rangle = (m^z)^2 + A_z \cos(\overbrace{\pi[1+2m^z]L}^{\text{incommensurate}}) \left(\frac{1}{L}\right)^{2K} - \frac{K}{2\pi^2 L^2}$$

see 2.)

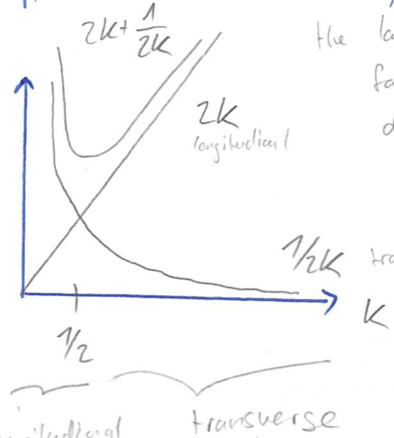
always fastest

transverse: $$\langle \hat{S}^+_n \hat{S}^-_{n+L} \rangle = \underbrace{A_x \cos(\pi L) \left(\frac{1}{L}\right)^{1/2K}}_{\text{commensurate Neel transverse}} - \underbrace{B_x \cos(2\pi m^z L)}_{\text{incommensurate}} \left(\frac{1}{L}\right)^{2K + \frac{1}{2K}}$$

$\Rightarrow$ groundstate of TLL model is critical (power law correlations)

$\Rightarrow$ A_x, B_x, A_z are amplitude prefactors for different correlation types

$$\Rightarrow K \begin{cases} > 1/2 , & \text{transverse dominant} \\ = 1/2 , & \text{correlations isotropic} \\ \text{else} , & \text{longitudinal dominant} \end{cases}$$



the larger, the faster the decay

$\Rightarrow$ not only spatial but also temporal correlations, as the spectrum is relativistic (linear) $\Rightarrow r = \sqrt{x^2 + (iut)^2}$

Therefore, we can get exact $\chi(\vec{q}, \omega, T)$

FT of correlation in time and space

1.) $q \to 0$, $i\langle \hat{S}^z \rangle^2 \delta(q)\delta(\omega)$ describes the "FM" Bragg peak due to the presence of the static magnetization in the system

2.) $\underset{\text{F.T.}}{\overset{\text{sokhotskii}}{\rightsquigarrow}}$ $$\mathcal{P} \frac{\pi k u q^2}{(uq)^2 - (\hbar\omega)^2} + i\pi^2 k u q^2 \delta((uq)^2 - (\hbar\omega))$$

$q = 0, w = 0 \Rightarrow$

$\chi^{zz}(0,0) = \frac{\pi k}{u}$

$\hbar\omega = uq$

„sound-wave"

3.) $$q \to \frac{\pi \pm 2\pi \langle S^z \rangle}{q_0}$$

$$\chi^{zz}(q, \omega, T) \propto \frac{1}{u} \left(\frac{T}{u}\right)^{2K-2} F_{2K}\left(\frac{u(q-q_0)}{T}, \frac{\omega}{T}\right)$$

$q = q_0$: $\chi \propto T^{\gamma} \mathcal{F}\left(\frac{\omega}{T}\right)$ (Quantum critical dynamics)

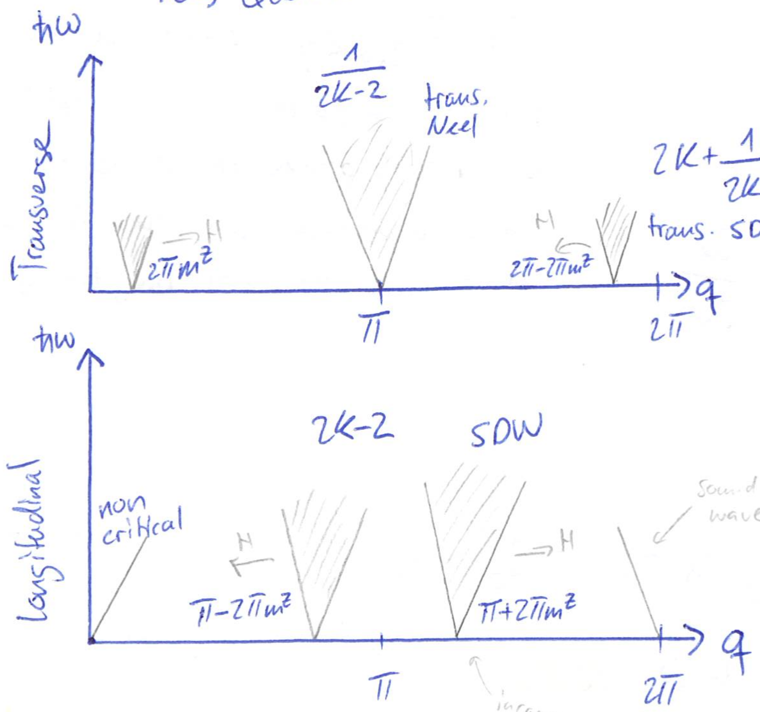
4.) Transverse:

$\chi^{xx}(q,\omega,T)$: $q \to \pi$

$$\chi^{xx}(q,\omega,T) \propto \frac{1}{u}\left(\frac{T}{u}\right)^{\frac{1}{2K}-2} F_{\frac{1}{2K}}\left(\frac{u(q-q_0)}{T}, \frac{\omega}{T}\right)$$

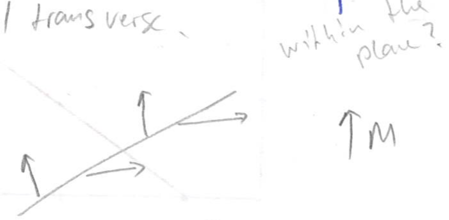
5.) $q \to 2\pi\langle\hat{\vec{S}}^z\rangle$, and $2\pi - 2\pi\langle\hat{\vec{S}}^z\rangle$

$\rightsquigarrow$ Quantum critical too



Line slope is $\hbar\omega = uq$

Neel transverse.



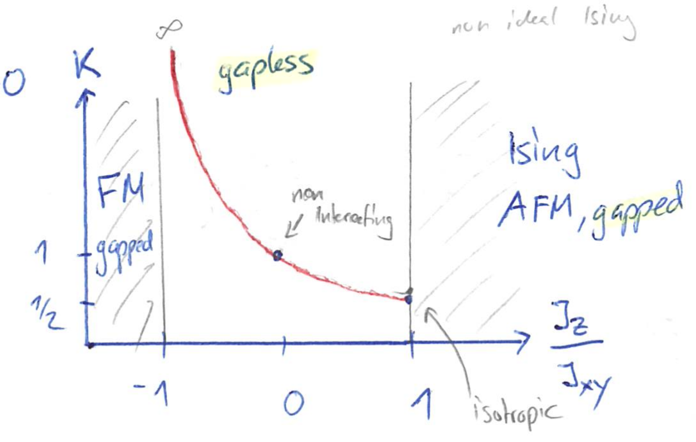
$K=\frac{1}{2}$ isotropic

Beware: Semiclassically length of spin constant but in magnetized 1st? chain, this can happen

Gapless

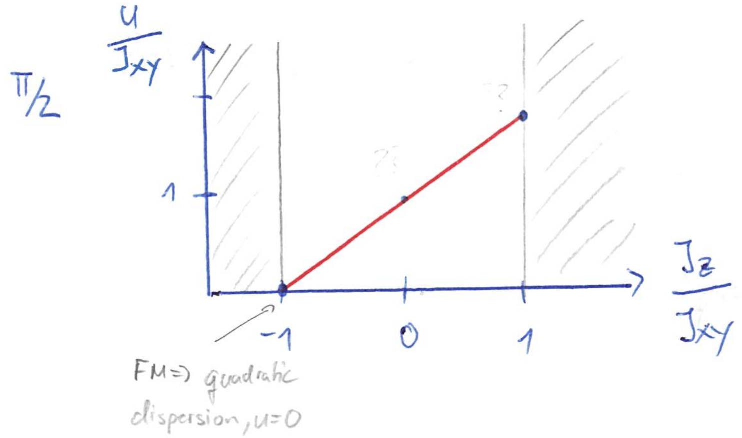
XXZ chain revisited

$H = 0$



Once J_z exceeds J_{xy}, the Tomonaga-Luttinger liquid description breaks down and the Jordan-Wigner c-fermi end up forming a crystal, that corresponds to the Ising magnetic order.

For $J_z = -J_{xy}$, $K \to \infty$, $u = 0$
spectrum stops being linear, and TLL description breaks down as the system is a ferromagnet and the dispersion relation is not linear but, but quadratic.



$\vec{H} \parallel \hat{z}$

1.) Saturation field: $g \mu_B H_{sat} = J_z + J_{xy}$

$u = 0$, $K = 1$ — non interacting as field induced ferromagnet fermionic band depleted → interaction negligible

linear → quadratic

2.) Ising destabilized: $\Delta = g \mu_B H$ — „spin flop"

Ising gap

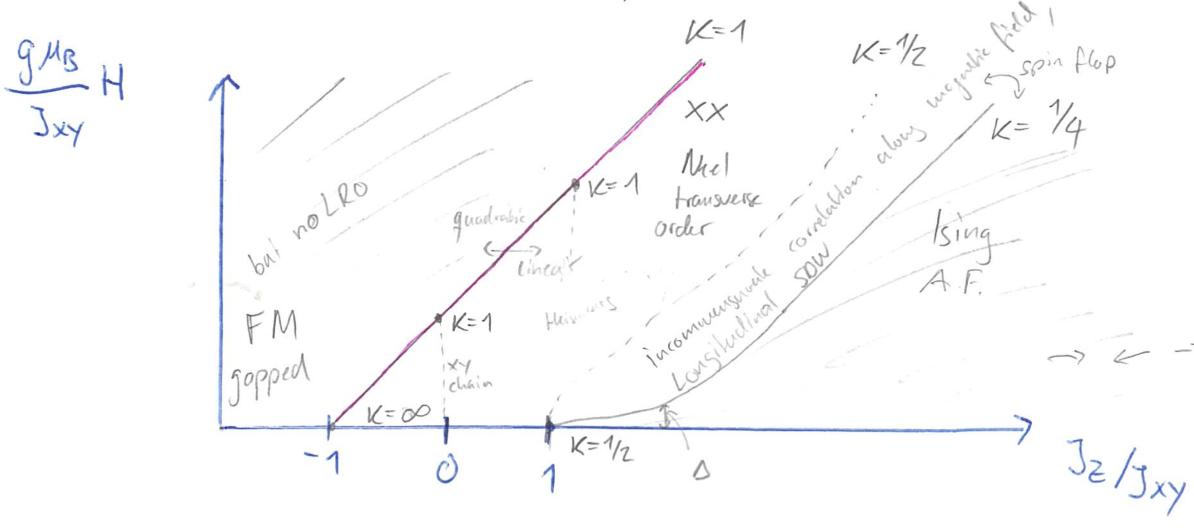
SF → TLL

opens in the spectrum

$\Delta = J_z - J_{xy}$ for $J_z \gg J_{xy}$

$$\Delta \propto e^{-\pi \sqrt{\frac{J_z}{J_z - J_{xy}}}} \quad J_z \to J_{xy}$$

Remember! insulating magnet! and not a metal



Zero field

In zero magnetic field there exists an analytical solution relating the Tomonaga-Luttinger spin liquid parameters K and u to the exchange constants of the Hamiltonian:

$$K = \frac{\pi}{2\arccos\left(\frac{-J_z}{J_{xy}}\right)} \qquad \arccos(0) = \frac{\pi}{2}$$

$$u = \frac{1}{1-\frac{1}{2}K} \sin\left(\pi\left[1-\frac{1}{2}K\right]\right)\frac{J_{xy}}{2}$$

??

$\Rightarrow J_z = 0 \Rightarrow K = 1 \quad , \quad u = J_{xy}$ in agreement with free fermion model

renormalization with respect to $\hbar\omega(q) = J\sum_{\vec{R}}(1+\cos\vec{q}\cdot\vec{R})$
Batyev-Braginskii approach

$\Rightarrow J_z = J_{xy} \Rightarrow K = \frac{1}{2} \quad , \quad u = \frac{\pi}{2}J$, "spin wave" velocity renormalized

in agreement with previous discussion
correlations and response chapter

Non-zero field

A uniform magnetic field would not couple to the magnetic order, therefore not changing the nature of the TLL state but modifying the parameter K and u instead. In strong magnetic field the TLL ends as $g\mu_B H = J_{xy} + J_z$. At H_{sat} the fermionic band becomes nearly full and thus the interaction just vanish (no states to scatter to). $\Rightarrow K = 1$

?? $u=0$, quadratic not linear

Exception: $J_z = -J_{xy}$ where K is non-analytical $\Rightarrow K = \infty$

Coupled chains

The idealized situation of perfetly 1-dimensional chain is not realistic as interchain coupling J' is always present. Therefore, in realistic systems there will be order at $T=0$. The TLL description provides a precise answer to this question.

Considering the predictions of Mean field approach and the susceptibilities, at least one of the susceptibilities will diverge at low temperatures. It follows that χ will diverge not at $T=0$ but earlier thanks to the non-zero coupling betweeen the chains.

$$\chi^{MF}(T) = \frac{\chi^{(0)}}{1 - J'\chi^{(0)}}$$

$$T_N \propto (J')^{\lambda} \quad , \quad \lambda = \frac{2K}{4K-1} = \begin{cases} 1 & \text{Heisenberg} \\ 2/3 & \text{XY model} \end{cases}$$

Not ladders! $\rightsquigarrow$ 1D

Chains and Ladders

Spin ladder basics

The one dimensional $S=1/2$ chain will develop long range order at very low temperatures in the presence of a small interchain coupling.

On the other hand for strong coupling $\rightsquigarrow$ AF. It is impossible to identify a chain direction and the ordering temperature $\sim J$

For # chains $\xrightarrow{\text{coupled}} \infty$, semiclassical story at low enough temperatures.

What happens for a few coupled chains?

$\longrightarrow$ depends on number of chains, even/odd. ← single

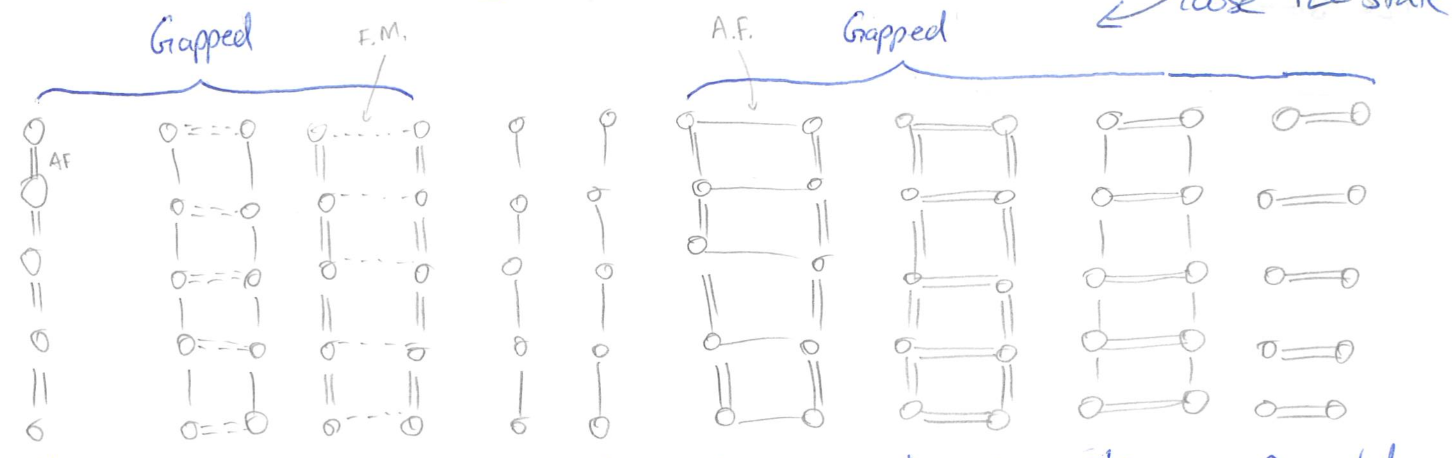
The extreme even case is the spin ladder.

$$\hat{\mathcal{H}} = \sum_n \underbrace{J_{\parallel}}_{\text{leg}} \left(\hat{\vec{S}}_{n,1} \underbrace{\hat{\vec{S}}_{n+1,1} + \hat{\vec{S}}_{n,2}}_{\text{legs}} \hat{\vec{S}}_{n+1,2} \right) + \underbrace{J_{\perp}}_{\text{rung}} \hat{\vec{S}}_{n,1} \hat{\vec{S}}_{n,2}$$

Special cases: $J_{\parallel} = 0 \Rightarrow$ non interacting dimers

$J_{\perp} = 0 \Rightarrow$ uncoupled chains

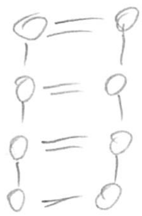
$J_{\perp} \to -\infty \Rightarrow$ Haldane chain



Haldane chain | Coupled FM dimers | FM rung Ladder | Decoupled chains | Strong leg case | isotropic ladder | strong rung ladder | Decoupled dimers

The TLL state can be restored by the magnetic field. However, the properties of such field-induced TLL may not be anywhere near the original Heisenberg chain.

Strong rung case

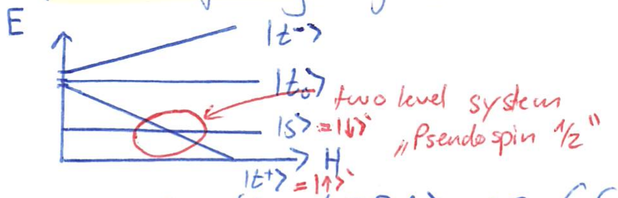


$J_\perp \gg J_\parallel$

dimerized systems $\Rightarrow$

$$\begin{cases} |s\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \\ |t_+\rangle = |\uparrow\uparrow\rangle \\ |t_0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |t_-\rangle = |\downarrow\downarrow\rangle \end{cases}$$

While the magnetic field is small, the groundstate remains the direct product of rung singlets.



can be calculated

The dispersion by means of Random phase approximation (RPA) or GSWT (Remember they are the same in the gapped regime)

$\Rightarrow$ triplet of excitations, split by magnetic field

$|t_+\rangle$ going to be lowest in energy. As $H \to H_{c1}$, the gap closes and the QPT happens. In 1D, no long range order due to long-wavelength fluctuations.

ladder to chain $\Rightarrow$ TLL state

XXZ chain mapping strong rung

In the applied magnetic field the only low-energy states are the singlet and the lowest triplet. We can take only two lowest energy states $|s\rangle$ and $|t^+\rangle$ and pretend that they correspond

$|s\rangle = |\tilde{\uparrow}\rangle,\quad |t_+\rangle = |\tilde{\downarrow}\rangle$

to some effective spin-1/2 chain system in a fictious magnetic field.

$\Rightarrow$ each rung of the ladder corresponds to a single pseudospin object.

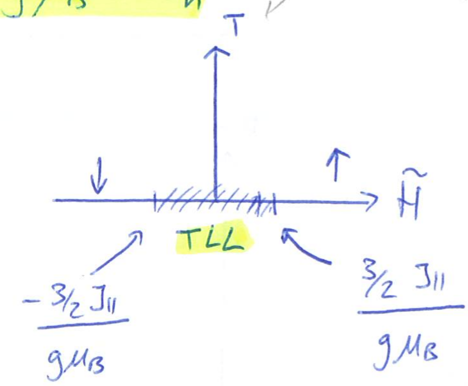
ladder to chain mapping

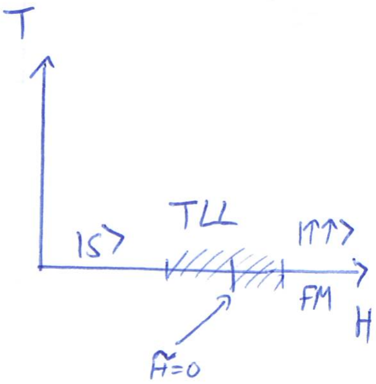
$$\boxed{\hat{S}^+_{n,1} = -\frac{1}{\sqrt{2}}\hat{\tilde{S}}^+_n, \quad \hat{S}^+_{n,2} = \frac{1}{\sqrt{2}}\hat{\tilde{S}}^+_n, \quad \hat{S}^z_{n,2} = \frac{1}{4}\left(1 + 2\hat{\tilde{S}}^z_n\right)}$$

$$[\hat{\tilde{S}}^\alpha, \hat{\tilde{S}}^\beta] = i\varepsilon_{\alpha\beta\gamma}\hat{\tilde{S}}^\gamma \qquad \tilde{H} = H - \frac{J_\perp + \frac{1}{2}J_\parallel}{g\mu_B}$$

$$\hat{\mathcal{H}}_{XXZ} = \sum_n J_\parallel\left(\hat{\tilde{S}}_n\hat{\tilde{S}}_{n+1} - \frac{1}{2}\tilde{S}^z_n\tilde{S}^z_{n+1}\right) - g\mu_B\tilde{H}\hat{\tilde{S}}^z_n$$

which corresponds to a $J_z/J_{xy} = \frac{1}{2}$ chain.



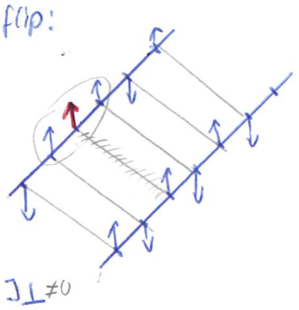


maybe symmetric $\Rightarrow$ permutation leads to bound state same

and antisymmetric $\Rightarrow$ bound state switches side

Weak $J_\perp$ rung

spin flip:



extra cost $\sim J_\perp$

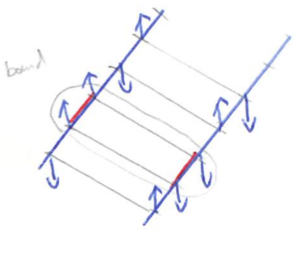
Domain: $\sim J_\perp \cdot L$

$\Rightarrow$ spin flip remains

$J_\perp$ is the "glue"

$\Rightarrow$ antisymmetric triplet, $S=1$

$q_\perp = \pi$



$q_\perp = 0$

singlet, $S=0$

$\Rightarrow$ symmetric

single DW not possible

$\Rightarrow$ somewhere far away there are 2 more spinons

$\Rightarrow$ 2 spin flips on each leg

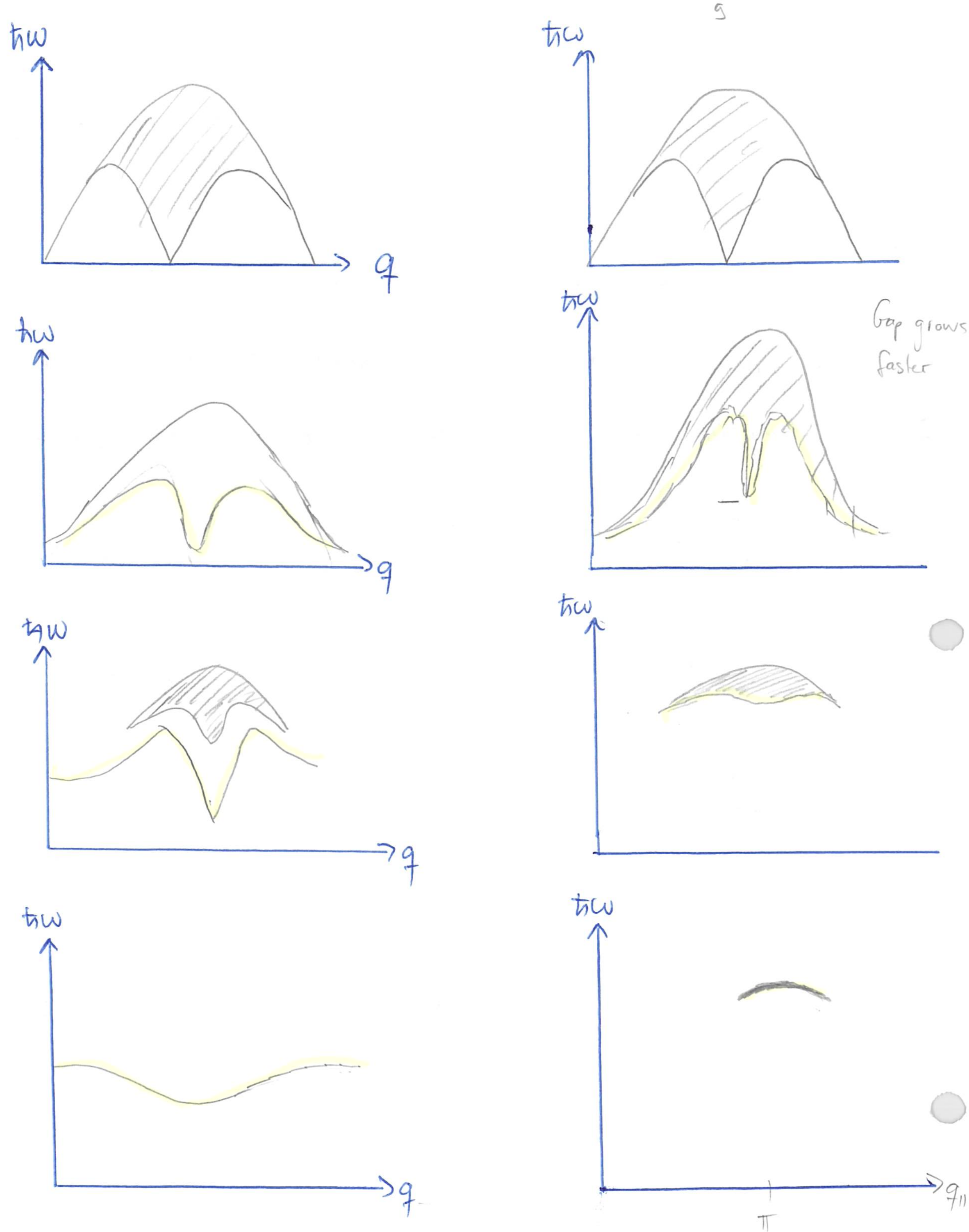
$\rightsquigarrow$ 2 ... oscillations

$J_\perp = 0$
decoupled

gap opening due
to $J_\perp > 0$

continua weaker

$J_\perp \gg J_\parallel$



Antisymmetric

$q_\perp = \pi$

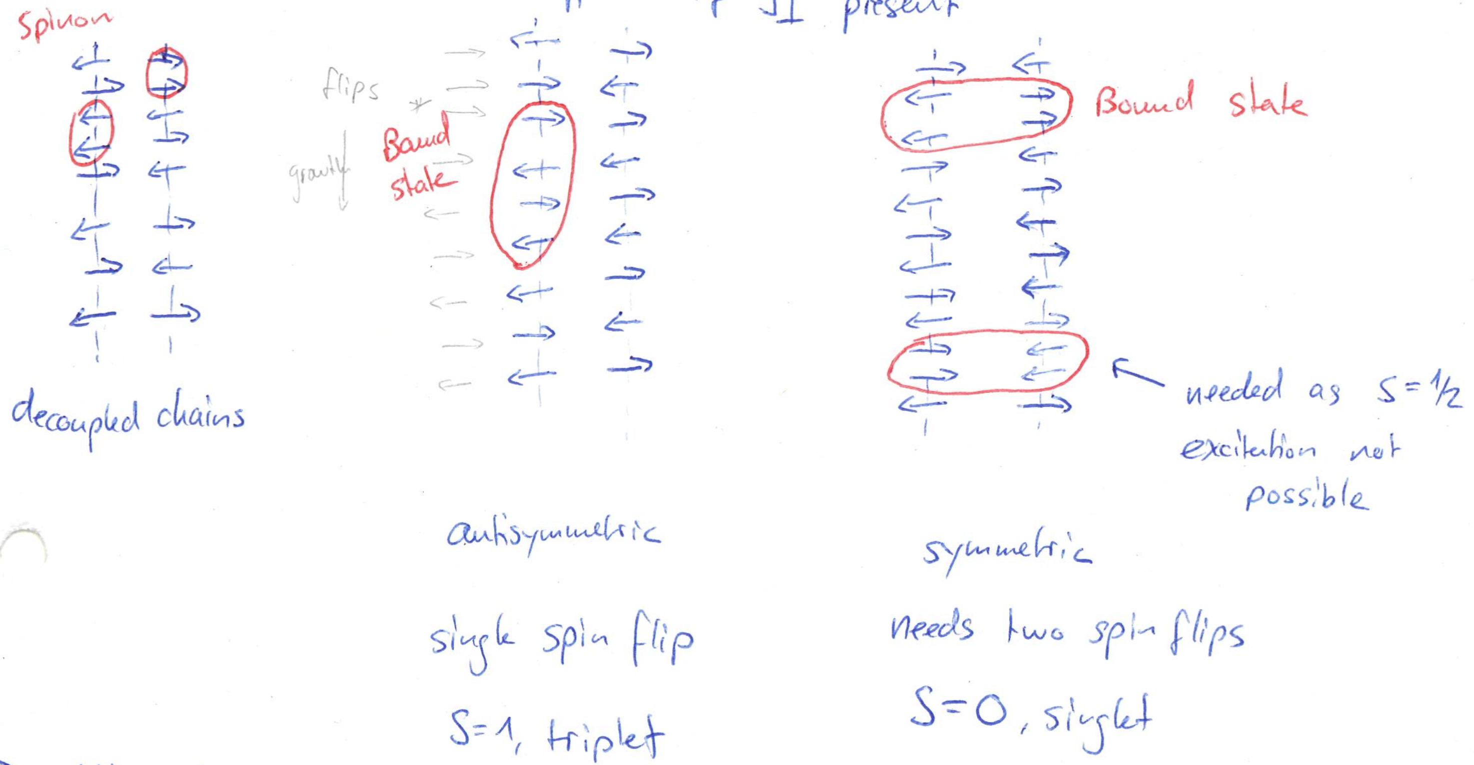
Symmetric

$q_\perp = 0$

Sharp modes

Weak rung coupling

For single chain, a spinon (domain wall of wrong orientation) can move freely as the size does not matter. different if $J_\perp$ present



$\Rightarrow$ additional quantum number: Symmetric/antisymmetric

Haldane chains

⇝ SRO ?

Haldane showed that the $S=1$ chain features only short-range correlations and is gapped. It was found that

$$\xi \propto e^{\pi S} a \quad , \qquad a \text{ the chain period}$$

$\Rightarrow S \to \infty$, $\xi \to \infty \Rightarrow$ difference between integer and half integer vanishes. Lieb Schult Mattis

For $S=1$: $\xi \simeq 6a$ (short ranged)

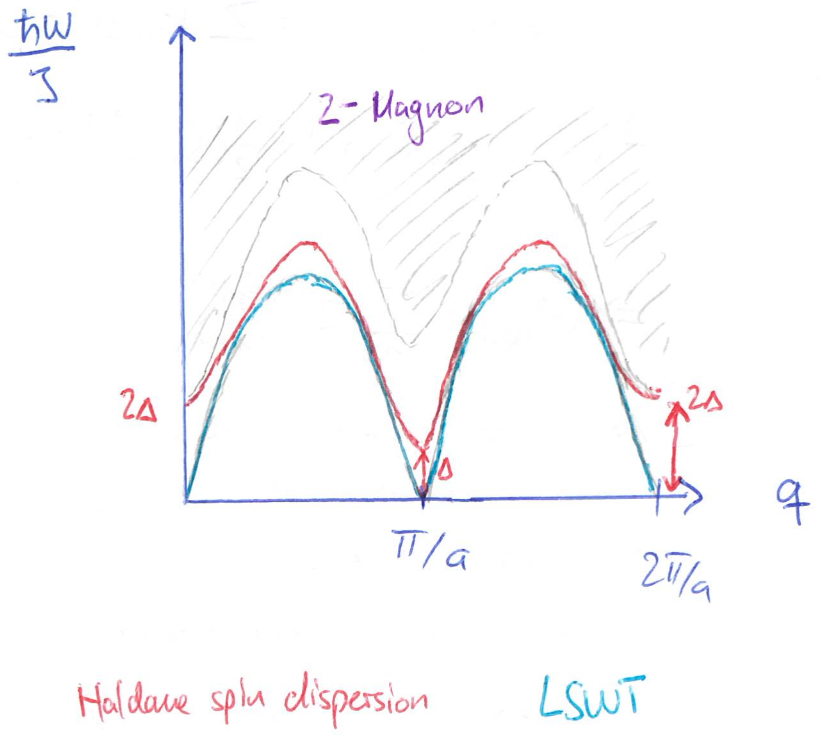
Only for integer spins because for half odd integer spins → Lieb-Schultz-Mattis theorem. →gapless From NLσM, a short range order implies a gap in the excitation spectrum, which can be estimated by the NLσM

NLσM ⇒ first formula

$$\Delta = 2Ja\,\xi^{-1}$$

The spectrum of excitations features a proper magnon-like $S=1$ quasiparticle (unlike the $S=1/2$ chain). where $S=\pm1,0$ As usual near the $q=\pi/a$ wavevector

$$\hbar\omega(\vec{q}) = \sqrt{\Delta^2 + (cq)^2} \quad , \qquad c = 2Ja \quad ,$$

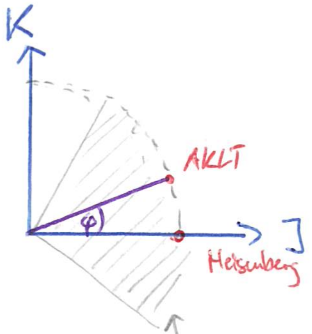


AKLT model

$$\hat{\mathcal{H}} = \sum_n J(\hat{\vec{S}}_n \cdot \hat{\vec{S}}_{n+1}) + \underbrace{K(\hat{\vec{S}}_n \cdot \hat{\vec{S}}_{n+1})^2}_{\text{biquadratic term}} \quad , \quad S=1$$

AKLT: $K = \frac{1}{3}J$

↳ exact solution



$$\tan\varphi = \frac{K}{J}$$

same thermodynamic ground state, gapped "phase"

Idea:

For $K = \frac{1}{3}J$, the Hamiltonian becomes equivalent to an operator that "projects" many body spin states to a special subset with the total spin of any pairs of neighbourgs necessarily being 2. Then a in which no pair of neighbours has a total spin 2 would automatically become an eigenstate of zero energy. As the Hamiltonian is equivalent to a projector, it has no negative eigenvalues and hence such a state would necessarily be the ground state.

Projection operators:

$\hat{\vec{S}}_1, \hat{\vec{S}}_2$ spins-1 , σ - total spin $= 0, 1, 2$

$$|\Psi\rangle = \sum_{\sigma=0,1,2} \sum_{m=-\sigma}^{\sigma} a_{m\sigma} |m, \sigma\rangle$$

$$\hat{P}^{(2)}_{1,2} |\Psi\rangle = \sum_{m=-2}^{2} a_{m2} |m, 2\rangle$$

erased $|0\rangle, |1\rangle$ state

projects on the restricted part of the Hilbert space

$$\langle\Psi| \hat{P}^{(2)}_{1,2} |\Psi\rangle = \sum |a_{m2}|^2 \geq 0$$

$$\Rightarrow \hat{P}^{(2)}_{1,2} = \frac{1}{6}(\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2)^2 + \frac{1}{2}(\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2) + \frac{1}{3}$$

→ irrelevant

↳ looks like AKLT

$$\Rightarrow \hat{\mathcal{H}}_{AKLT} = \sum_n 2J \cdot \hat{P}^{(2)}_{n,n+1}$$

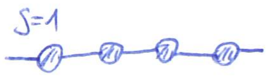
Assume state where neighbouring spins are not completely parallel, $\hat{P}^{(2)}_{1}$ will wipe out such a state, eigenvalue zero.

$\xrightarrow{\text{non negativity}}$ yields the groundstate

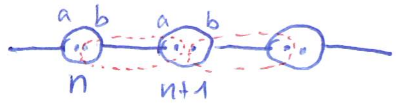
$$\hat{\mathcal{H}}_{AKLT} |\Psi_0\rangle = 0 \Rightarrow |\Psi_0\rangle \text{ - ground state}$$

AKLT : Building the ground state

Singlet on one site not possible as S=1



$\Rightarrow$



S=1 ion physically consists of 2 spin 1/2 electrons

S=1 is formed by ferromagnetic coupling of rung exchange

dimers

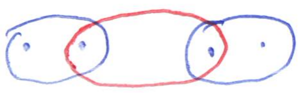
$$|S(n,n+1)\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle^a_{n+1}|\downarrow\rangle^b_n - |\downarrow\rangle^a_{n+1}|\uparrow\rangle^b_n\right)$$

$$|0\rangle_n = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle^a_n|\downarrow\rangle^b_n + |\downarrow\rangle^a_n|\uparrow\rangle^b_n\right)$$

$$|+1\rangle_n = |\uparrow\rangle^a_n|\uparrow\rangle^b_n$$

$$|-1\rangle_n = |\downarrow\rangle^a_n|\downarrow\rangle^b_n$$

$$|\tilde{\Psi}_0\rangle = \prod_n |S(n,n+1)\rangle$$ the desired property



locked in singlet state

initially: $4 \times S = \frac{1}{2}$

now: $2 \times S = \frac{1}{2}$ $\Rightarrow$ total spin $\sigma = 0, 1$

locking in singlet

ends can be alligned or not

Valence bond solid state:

By forming dimerized pairs the translational invariance is broken, without actually breaking the translational invariance. Periodicity remains the same, but dimerization

$$|\tilde{\Psi}_0\rangle \Rightarrow |\Psi_0\rangle, \quad g_n = \frac{1}{\sqrt{3}}\begin{pmatrix} -|0\rangle_n & -\sqrt{2}|-1\rangle_n \\ \sqrt{2}|+1\rangle_n & |0\rangle_n \end{pmatrix}$$

matrix wave function associated with each site

a product of two matrices will only contain $\sigma = 0, 1$ states

$$\Rightarrow |\Psi_0\rangle = \mathrm{tr}\left(\prod_n g_n\right)$$ is the ground state

degenerate, gapped and features a form of topological order.

4, spins at ends

Singlet $\leftrightarrow$ triplet

Excitations in AKLT

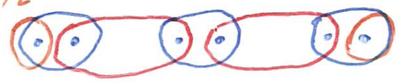
$$|s(n,n+1)\rangle \Rightarrow |t(n,n+1)\rangle$$

singlet → triplet

costs energy $\Rightarrow$ gapped

$\Rightarrow \xi_{AKLT} \sim a$ very SRO

1.) $s=1/2$ $s=1/2$



"orphane", spin 1/2 dof due to cut of chain

How will the two orphane spins pair together? $\rightsquigarrow$ 4 possibilities

but close in energy cause spins very far away

$\Rightarrow$ 4x degeneracy

2.) Hidden order: (String order)

$\rightarrow$ Néel state is allowed $|\ldots\uparrow\downarrow\uparrow\ldots\rangle$

$\rightarrow$ with zeros are possible $|\ldots\uparrow 0\downarrow\ldots\rangle$ and $|\ldots\downarrow 0\uparrow\ldots\rangle$

$\rightarrow$ with multiple zeros are possible $|\ldots\uparrow 0 0\downarrow\ldots\rangle$

$\rightarrow$ changing hidden alternation $|\ldots\uparrow\downarrow\downarrow\uparrow\downarrow\ldots\rangle$ not allowed

$\rightarrow$ change including zeros $|\ldots\uparrow\downarrow 0\downarrow\uparrow\ldots\rangle$ not allowed

$\Rightarrow$ looks like Néel order diluted with random zeros

$$\langle \hat{S}^z_n \hat{S}^z_{n+L}\rangle \propto e^{-\frac{La}{a}} \rightarrow 0 \quad L\to\infty \quad \text{due to the random zeros}$$

$$O^{zz}_{String} \overset{\lim_{L\to\infty}}{=} \langle \hat{S}^z_n \, e^{i\pi\sum_{m=n+1}^{n+L-1}\hat{S}^z_m}\, \hat{S}^z_{n+L}\rangle \neq 0$$

doesn't correspond to something easy measurable, doesn't break symmetry as in Landau

$|\pm\rangle = -1$

$|0\rangle = 1$

$\Big\}$ ignores zeros in between and sees hidden Néel order

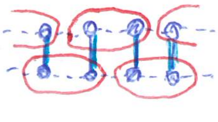
$\Rightarrow$ topological order

$$O^{zz}_{AKLT} = \frac{4}{9} > 0{,}37 \approx O^{zz}_{Haldane}$$

Ladders and hidden order

AKLT

FM: $J_\perp < 0$



cut

$S = 1/2$

Singlet

One singlet bond is cut

odd number

"odd" topological order

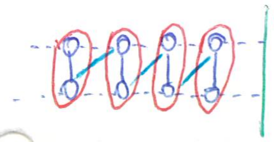
cross one singlet

$\sigma = 1$

singlet along ladder

$$O^{zz}_{odd} = \lim_{L\to\infty} \langle \hat{\sigma}^z_n \exp\left(-i\pi \sum_{n+1}^{n+L-1}\right) \hat{\sigma}^z_{n+L} \rangle \neq 0 \quad \text{for} \quad J_\perp < 0$$

AF: $J_\perp > 0$



cut

RS-like (Rung singlet like)

Singlet across ladder

No broken singlet, "Even" type
zero

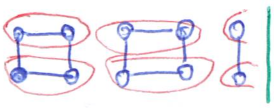
Heisenberg case

$|s\rangle$ $\sigma = 1$

$$O^{zz}_{even} = \lim_{L\to\infty} \langle \hat{\sigma}^z_n \exp\left(-i\pi \sum_{n+1}^{n+L-1} \hat{\sigma}^z_m\right) \hat{\sigma}^z_{n+L} \rangle \neq 0 \quad \text{for} \quad J_\perp > 0$$

Mutually exclusive (see slide)

Case no hidden order



2x singlets broken   assume no overlap for singlet legs

No string order, but still gapped (no translational invariance)

modulated ladder

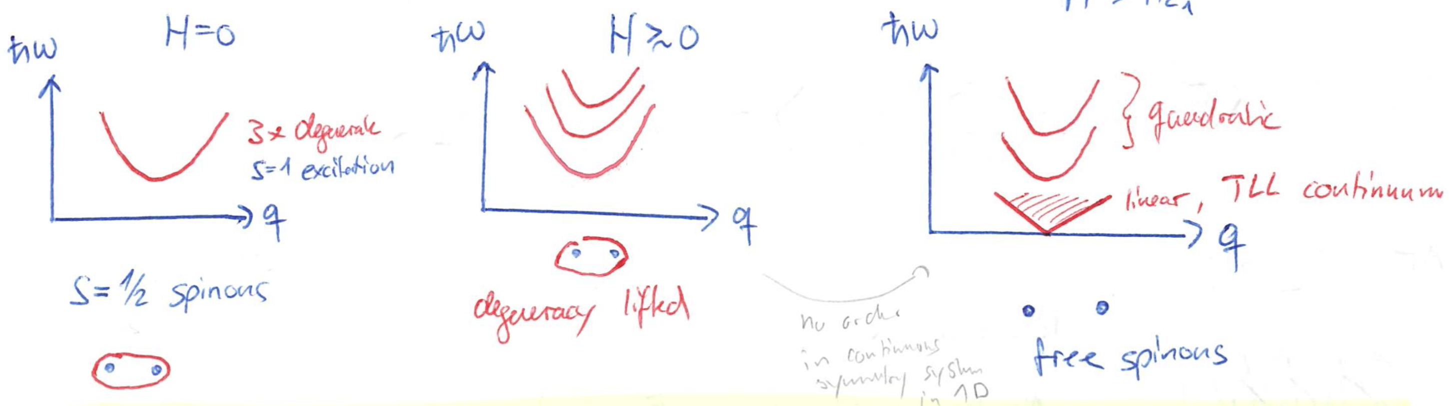
Ladders in a field

Some ideas already mentioned:

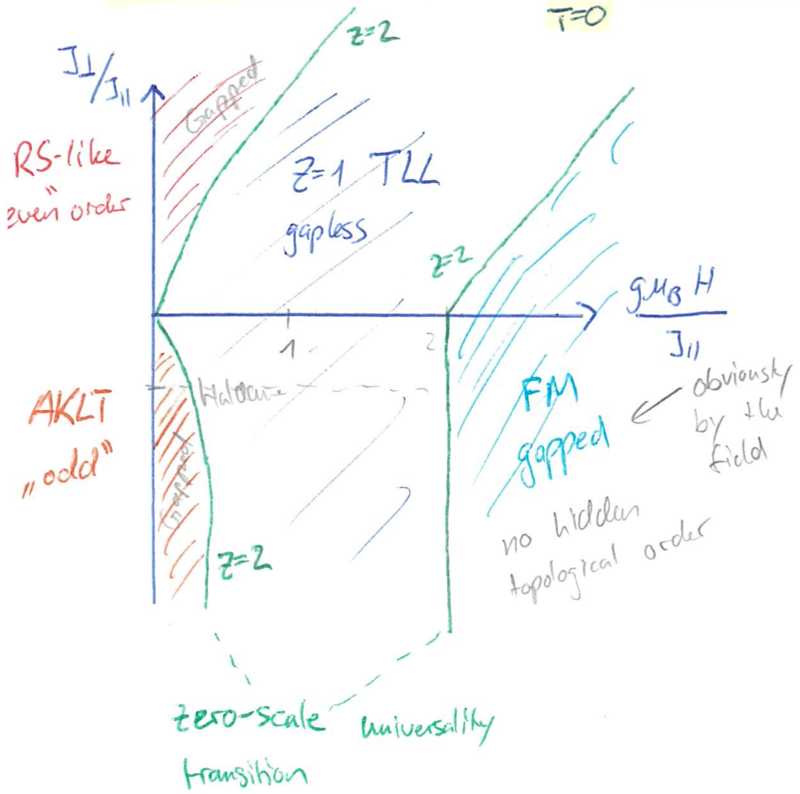
- $\vec{H}$ large, gap closed
- TLSL (replaces AF in 1-D) no LRO in 1D!!!

Heisenberg XY ⟶ HMW

Ising ⟶ T=0



Haldane ⟹ All the gapless phases of 1-dimensional spin systems as long as they have linearly dispersion excitation at low energies, all of them are going to be TLL with no exception.



same as $J_\perp < 0$ only "helps" stabilizing

$$g\mu_B H_{c2} = \begin{cases} 2J_\parallel & \text{for } J_\perp < 0 \\ 2J_\parallel + J_\perp & \text{for } J_\perp > 0 \end{cases}$$

← need to overcome both

$$\Delta = \begin{cases} \sim J_\perp, & J_\perp \ll J_\parallel, \text{ AF} \\ \sim J_\perp - J_\parallel, & J_\perp \gg J_\parallel, \text{ AF} \end{cases}$$

weakly coupled dimers approach } energy cost of $|s\rangle \to |t\rangle$ minus actual bandwith of excitation

$$\Delta = \begin{cases} \sim |J_\perp|, & |J_\perp| \ll J_\parallel \\ 0{,}4 J_\parallel, & J_\perp \to -\infty \end{cases}$$

→ slides

$z=2$ QCP in 1D

Remember BEC cannot exist in 1D as it would imply a breaking of a continuous symmetry

HMW, or by $\rho \to \infty$ in lower dim. dos

see internet for sources

$\Rightarrow$ TLL

but approximation of gapped parabolic dispersion still holds

μ - chemical potential

Gapped side: $\hbar\omega(\vec{q}) = \dfrac{(\hbar q)^2}{2m} + g\mu_B(H_{c_1} - H)$

see previous figure

gapped

effective mass only system related parameter

interacting bosons by hard-core repulsion

picture $\rightsquigarrow$ cannot excite two triplet on the same dimer

1 quasiparticle per site

miracle

$\Rightarrow$ Hard-core bosons $\hat{=}$ free fermions in 1-D

$$f(q) = \frac{1}{1 + e^{\left(\frac{(\hbar q)^2}{2m} + \mu\right)/T}} \qquad k_B = 1$$

$\mu \sim H$

(Fermi-Dirac).

$$f(q) = \tilde{f}\left(\frac{q^2}{Tm}, \frac{H-H_c}{T}\right), \quad E = \frac{1}{2\pi}\int_{-\infty}^{\infty} dq\, \hbar\omega(q)\, f(q)$$

$q^2 \sim T$

$q^3 \sim T^{3/2}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dq \left(\frac{(\hbar q)^2}{2m} + g\mu_B(H - H_c)\right)\tilde{f}\left(\frac{q^2}{Tm}, \frac{H-H_c}{T}\right)$$

$$\propto T^{3/2}\, \text{const.}\left(\frac{H-H_c}{T}\right) \Rightarrow \varphi = \nu z = 1$$

$z=2$ parabolic $\Rightarrow \nu = \frac{1}{2}$

Magnetization:

$$M = \frac{1}{2\pi}\int_{-\infty}^{\infty} dq\, \tilde{f}\left(\frac{q^2}{2mT}, \frac{H-H_c}{T}\right) \overset{??}{\propto} \sqrt{T}\cdot \text{const}\left(\frac{H-H_c}{T}\right)$$

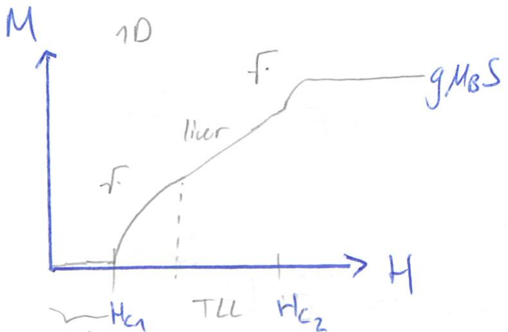
, $T=0 \Rightarrow M \propto \sqrt{H_c - H}$

for higher dimension: $M \propto H - H_c$

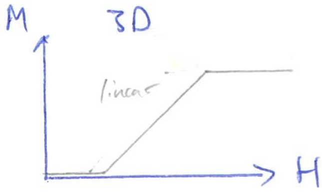
no tunable parameters

fully characterized by energy scale

Zero-scale universality (not TLL as 2 parameters)



gapped phase of spin ladder



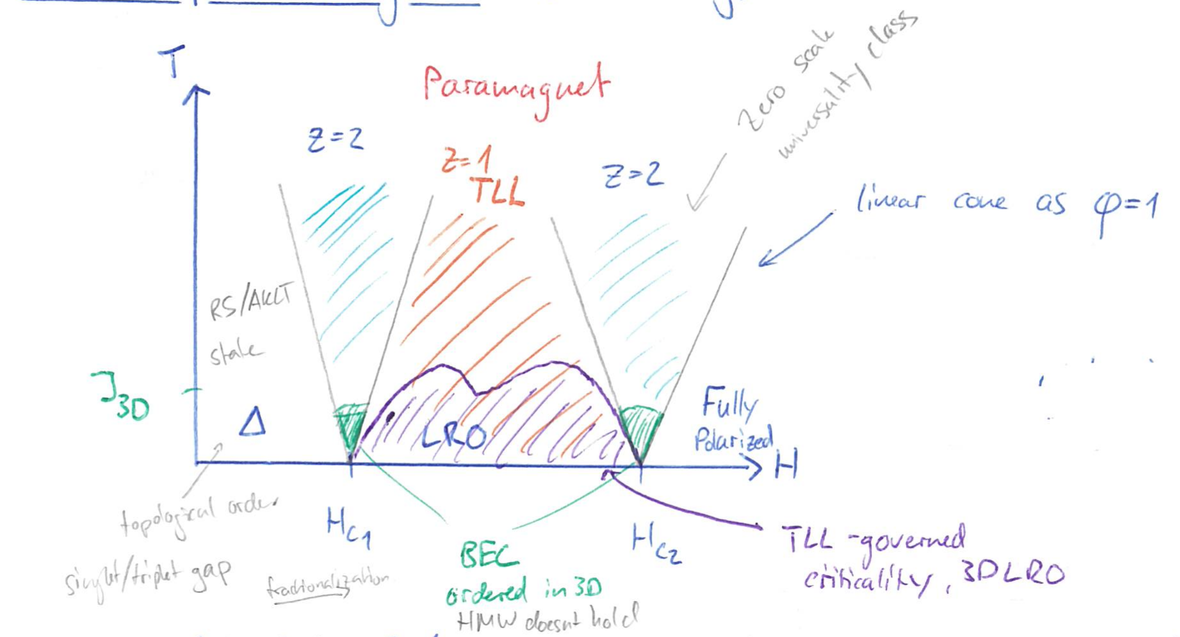
$$S^{\pm}(q,\omega) = \mathcal{A}\left(\frac{q}{\sqrt{mT}}, \frac{\omega}{T}, \frac{H-H_c}{T}\right)$$

$\mathcal{A}$: not analytically known

universal function without dependance on other parameters

Haldane chain, spin ladder, $S=1/2$ at H_{sat} $\Rightarrow$ at critical point (separating gapped and gapless state) universal behaviour parametrised by the curvature of the parabola

<u>Universal phase diagram</u> (Heisenberg)



For 1-D chain $S=1/2$, the phase diagram starts from TLL in $\vec{H}=0$.

Case: 3-D interactions present:

1.) Gapped states persist, $J_{3D} \ll \Delta$

2.) $z=2$ QCP $\Rightarrow$ 3D BEC for $T \lesssim J_{3D}$, $T_N \propto (H-H_c)^{2/3}$

3.) TLL state $\Rightarrow$ TLL critical correlations, often transverse

$$\overset{MF}{\chi}(q,T) = \frac{\chi^0(q,T)}{1 + J_{3D}(q)\,\chi^0(q,T)} \quad \text{- MF susceptibility}$$

Can turn ∞ for $T \neq 0$

$q=\pi$ for transverse

$$\chi^{\pm}_{\pi}(T) \propto \frac{A_x}{u} \cdot \left(\frac{T}{u}\right)^{\frac{1}{2k}-2}$$

MF criterion of order: $J_{3D} \cdot \chi^{\pm 0}(T_N) = 1$

$\Rightarrow \chi \to \infty \Leftrightarrow$ phase transition

$$T_N \propto u \left(\frac{A_x J_{3D}}{u}\right)^{\frac{2k}{4k-1}}$$

Dirty Quantum Magnets

Harris' criterion

x - defect concentration $\sim 10^{-2} - 10^{-3}$ typically

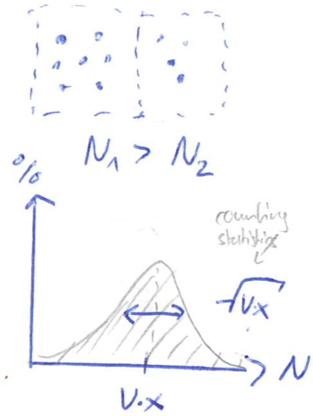
1.) Rare region effects (strong local modifications)

2.) Disorder is irrelevant (averaged out)

$$T_c(x) \overset{\text{Taylor}}{\approx} T_c(0) + x\left(\frac{\partial T}{dx}\right) \quad \text{(temperature behaviour)}$$

$$T_{c_V} \approx T_c(x) \pm \frac{\overbrace{\sqrt{Vx}}^{\text{uncertainty}}}{\underbrace{V}_{\text{Volume}}}\left(\frac{\partial T}{\partial x}\right) \longrightarrow \delta T_V \propto \sqrt{\frac{x}{V}}$$

$\sim \delta x_{loc}$



$\xi \propto |T - T_c(x)|^{-\nu}$ ← correlation length as in ideal system

$V \propto \xi^d \propto |T - T_c(x)|^{-\nu d}$

$$\frac{\delta T_V}{|T - T_c(x)|} \begin{cases} \to 0 & \text{for } T \to T_c(x) \Rightarrow \text{disorder irrelevant} \\ \to \text{const.}/\infty & \Rightarrow \text{disorder is relevant} \end{cases}$$

$= |T - T_c(x)|^{\frac{\nu d}{2} - 1}$

$$\boxed{\begin{array}{ll} \text{for } \nu d > 2 & \text{disorder irrelevant} \\ \text{for } \nu d < 2 & \text{disorder relevant} \end{array}}$$

→ $\nu(x)$ has to be renormalized so that $\nu(x) d > 2$

→ no transition, crossover

→ new phase, rare region effect, disorder acts as a ~~ph~~ driver for the phase transition

Harris' criterion $\nu d > 2$; no dependence on x quantitatively
(otherwise $T_c(x)$ becomes washed out)

• clean hyperscaling $2 - \alpha = \nu d \Rightarrow \alpha < 0$ • applicable to QPT, $d \neq d_q$

Rare regions: depleted magnets

Example: Cu^{2+} $(S=1/2)$ $\longleftrightarrow$ Zn^{2+} $(S=0)$ (depletion)

$Cu_{1-x}Zn_x$ system, $x \simeq 1$, as in magnetic nearest neighbourgh the interactions are nearest neighbourgh $\Rightarrow$ no interactions left

$\Rightarrow$ No order, diluted paramagnet

$x \simeq 0 \Rightarrow$ AF order

As there is order, there must be a critical concentration x_c

$\Rightarrow$ percalation theory

$L \to \infty$ cluster of Cu^{2+}, $x_c = 0{,}4$ is needed for 2D square lattice

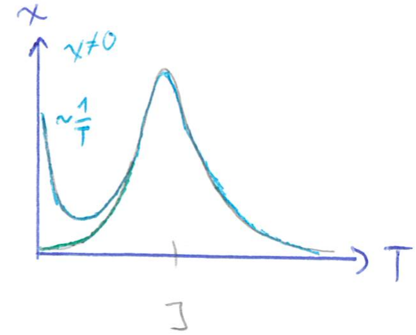
$\xi \propto |x - x_c|^{-\nu}$, $\chi_{\pi,\pi} \propto |x - x_c|^{-\gamma}$ $\Rightarrow$ second order phase transition

square lattice

Creating the order by impurities

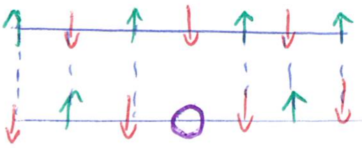
$\equiv |t\rangle$

$-|s\rangle$ ← non magnetic, if one spin is removed we are going to be left with a free $S=1/2$.



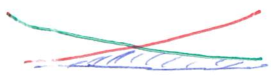
finite ξ in the system

$$\langle S_r^z \rangle \propto (-1)^{\frac{r}{a}} e^{-\frac{r}{\xi}}$$



spin island (correlated part close to impurity)

$$\sum_{\vec{r}} S^z_{\vec{r}} = \frac{1}{2}$$ smeared out over the island



Overlap ~ spin amplitude

co-aligned ⇝ odd number of spins in between ⇒ FM

for even N steps ⇒ A.F.

1.) sign-alternating interaction between islands, depends on # steps

2.) exp. decay $\propto e^{-r_{12}/\xi}$ the exchange and visible in the profile

3.) typical amplitude for the exchange between the islands is $\sim J'$

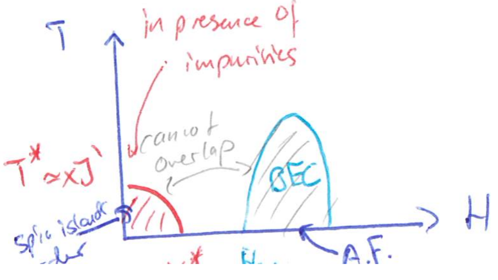
→ if close, then two free spins interacting with J'

$$J(\vec{r}_{ij}) = (-1)^{\frac{|\vec{r}_i - \vec{r}_j|}{a}} e^{-\frac{|\vec{r}_1 - \vec{r}_2|}{\xi}}$$

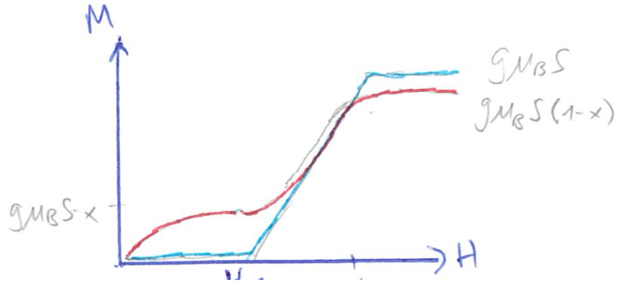
total spin of the islands centred at $\vec{r}_j$

$$\hat{\mathcal{H}}_{islands} = \sum_{i,j} J(\vec{r}_{ij}) \vec{\mathcal{S}}_i \vec{\mathcal{S}}_j$$

$\langle J_{ij} \rangle \sim J' \cdot x$ leads to order at $T^* \simeq \langle J \rangle$ (creates magnetic order)



$$H^* \simeq \frac{\langle J \rangle}{g \mu_B}$$



Still BEC washed out. Bose glass phase

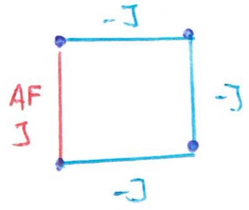
Magnetic Frustration

If possible to split into two identical sublattices $\Longleftrightarrow$ bipartite

For AF $\Rightarrow$ collinear state with lowest possible energy (Marshall's theorem)

Not possible for non-bipartite lattice, does not support collinear states.

But beware: even in bipartite frustration can occur $\longrightarrow$ villain lattice



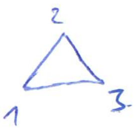
villain lattice

Toulouse criterion: $\prod\limits_{\text{contour}} \text{sign}(-J) = -1 \Rightarrow$ frustrated

Degeneracy in triangular lattice

$$\hat{\mathcal{H}} = J \sum_{\langle i,j \rangle} \hat{\vec{S}}_i \cdot \hat{\vec{S}}_j - \sum_i g\mu_B \vec{H} \cdot \hat{\vec{S}}_i$$

(Heisenberg A.F on Δ-lattice)



$$\hat{\vec{S}}_\Delta = \hat{\vec{S}}_1 + \hat{\vec{S}}_2 + \hat{\vec{S}}_3$$

$$(\hat{\vec{S}}_\Delta)^2 = 3S(S+1) + 2(\vec{S}_1 \cdot \vec{S}_2) + 2(\vec{S}_2 \cdot \vec{S}_3) + 2(\vec{S}_1 \cdot \vec{S}_3)$$

$$\hat{\mathcal{H}} = \sum_\Delta \frac{J}{4} (\hat{\vec{S}}_\Delta)^2 - \frac{g\mu_B}{6} \vec{H} \cdot \vec{S}_\Delta$$

- factorized
- Spins on Δ not independent

factor of 2 since each bond is shared by 2 Δ.

every site shared by 6 Δ

$\frac{\partial}{\partial S_\Delta} = 0$

$$\boxed{\vec{S}_\Delta = \frac{g\mu_B \vec{H}}{3J}}$$

local constraint for Ground state

$\Rightarrow$ As there are many ways to arrive at this result it includes massive degeneracy. $\leftarrow$ Special! doesn't occur due to symmetry in Hamiltonian, but from geometric frustration.

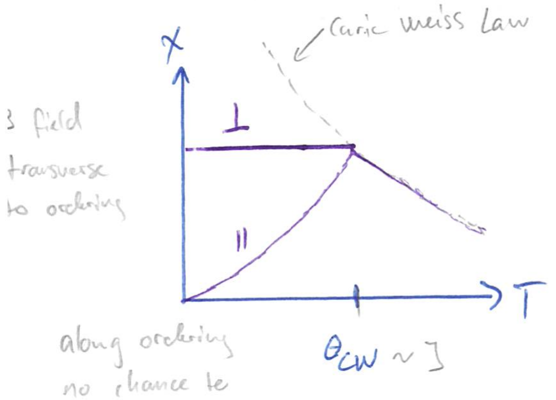
$\Rightarrow$ accidental degeneracy

$\hat{\mathcal{H}}$ symmetry: four continuous variables, $\theta_1, \theta_2, \dots$

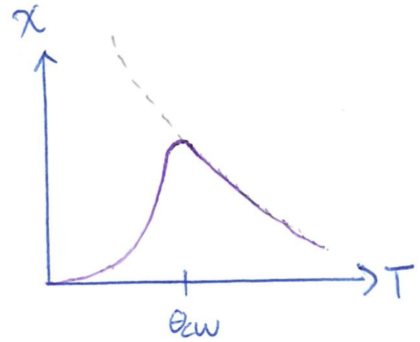
accidental : θ_n for every few triangles

$\# \theta_n \propto N_{spins}$ macroscopic number

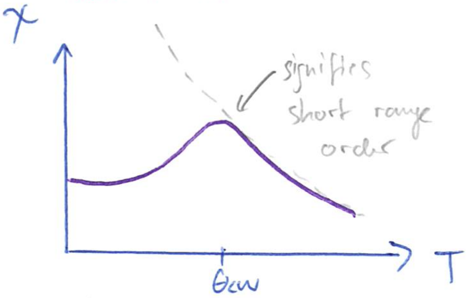
Problem with the order formation. visible in for e.g. χ_m



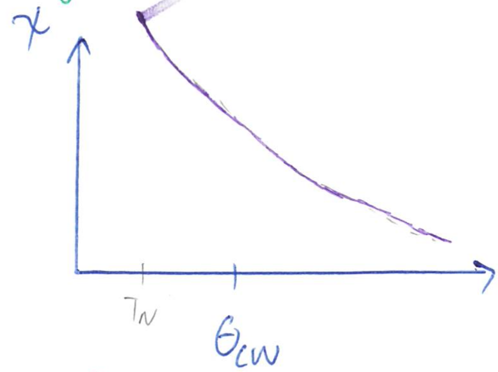
normal AF



gapped AF

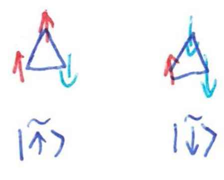


Low-D AF



frustrated system

fails to order, has SR interaction



Ramirez Criterion

Beware: not suitable for lower D problem, as in those ordering is anyway suppressed

$$f = \frac{\theta_{CW}}{T_N}$$ large in frustrated system

$$\cong \frac{J}{T_N}, \quad \frac{\text{energy scale of interaction}}{\text{ordering temperature}}$$

Order from disorder

Previously only the internal energy was considered for finding the groundstate. Here:

$$F = \underbrace{E - TS}_{\text{max } S \text{ for given groundstate}} \quad \text{is minimized}$$

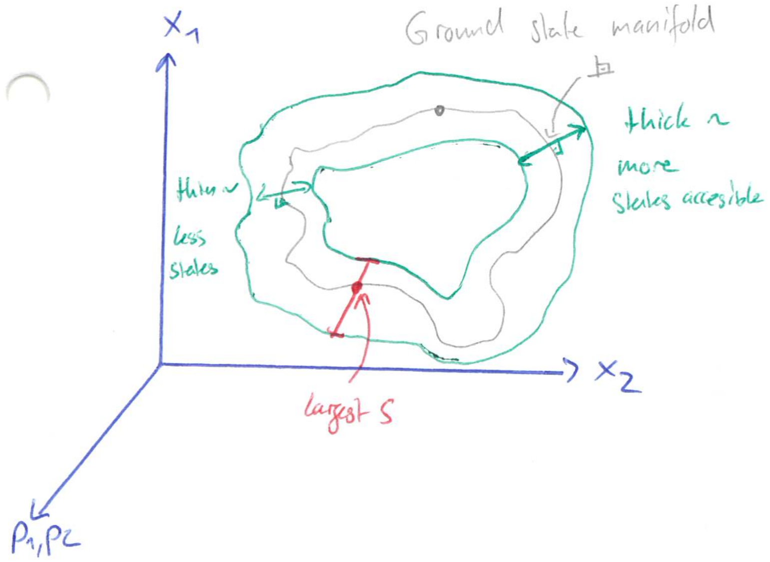
Example: classical oscillator

$$\mathcal{H}(x,p) = \frac{p^2}{2m} + \frac{kx^2}{2}$$

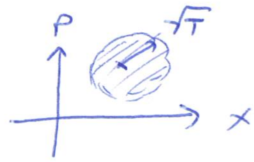
for: $T=0$: GS $\Rightarrow$ $p = x = 0$

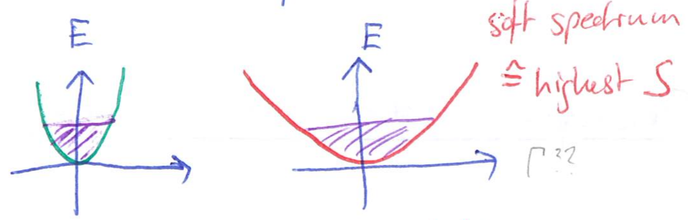
$$\left\langle \frac{p^2}{2m} \right\rangle = \left\langle \frac{kx^2}{2} \right\rangle = T \qquad \text{(equipartition theorem)}$$

$$\Rightarrow \sqrt{\langle p^2 \rangle}, \sqrt{\langle x^2 \rangle} \propto \sqrt{T}$$



$\Rightarrow$





will be selected by order from disorder mechanism

Classical case

single continuous parameter θ defines ground state manifold $|\lambda(\theta)\rangle$

$$\mathcal{H} = E_{GS}^{\lambda} + \sum_{\underset{\text{excitation mode}}{m}} \sum_{\underset{\text{momentum}}{\vec{q}}} \hbar\omega_{\vec{q}}^{m}(\theta)\, \overbrace{n_{\vec{q}}^{m}(\theta)}^{\text{population number}}$$

$$Z = \sum_{\text{conf.}} e^{-\mathcal{H}/T} = \ldots\ldots = e^{-\frac{E_0}{T}} \prod_{m\vec{q}} \frac{T}{\hbar\omega_{\lambda}^{m}(\vec{q})}$$

$$F = -T \log Z = E_0 - T \sum_{m} \sum_{\vec{q}} \left[\log T - \log(\hbar\omega_{\lambda}^{m}(\vec{q})) \right] = E_0 - T \underbrace{\sum \log T}_{\text{small}} + \underbrace{T \sum_{mq} \log \hbar\omega_{\lambda}^{m}(q)}_{\text{has to be minimal}}$$

Quantum case:

important

$T \to 0$ $\to 0$

$$\hat{\mathcal{H}} = E_0 + \sum_{m,\vec{q}} \hbar \omega_\lambda^m(\vec{q}) \left[\cancel{\hat{n}_\lambda^m(\vec{q})} + \frac{1}{2} \right]$$

$$\Rightarrow E_{tot} = E_0 + \underbrace{\frac{1}{2} \sum_{m,q} \hbar \omega_\lambda^m(\vec{q})}_{\substack{\text{minimize} \\ \text{zero point} \\ \text{fluctuations}}}$$

$\Rightarrow$ the state with the "softest" spectrum minimizing $\sum_m \sum_{\vec{q}} \hbar\omega_{\vec{q}}^m$ will be realized.

Quantum fluctuations can drive the order from disorder mechanism.

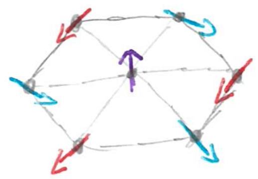
However,

	Classical	Quantum
	$\sum \log \omega$	$\sum \omega$

Triangular lattice in magnetic field

$$\vec{S}_\Delta = \frac{g\mu_B}{3J} \vec{H} \qquad \text{local constraint}$$

For $\vec{H}=0$: 120° structure, planar

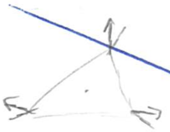


#dof : • 2 for closing the plane

• 1 angle for initial spin $\hat{=}$ phase

• 1 discrete symmetry : chirality

For $\vec{H} \neq 0$,



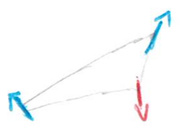
Umbrella structure, losing the first two dof as fixed by the magnetic field

chirality and phase remains

$U(1) \times \mathbb{Z}_2$

wrong, does not include the degeneracy

Triangular lattice in a magnetic field



Y state · coplanar

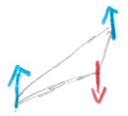
$U(1) \times \mathbb{Z}_3$

phase ↖ which one points down

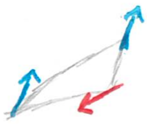
example of order from disorder mechanism

↳ UUD state · collinear ?? ↑↑↓

has continuous symmetry, soft spectrum

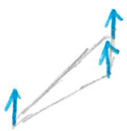


$\mathbb{Z}_3$, $M_{uud} = \frac{1}{3} M_{sat}$ ← $-\frac{1}{3}$ for not ↑ and $-\frac{1}{3}$ for ↓, stable in a range of $\vec{H}$



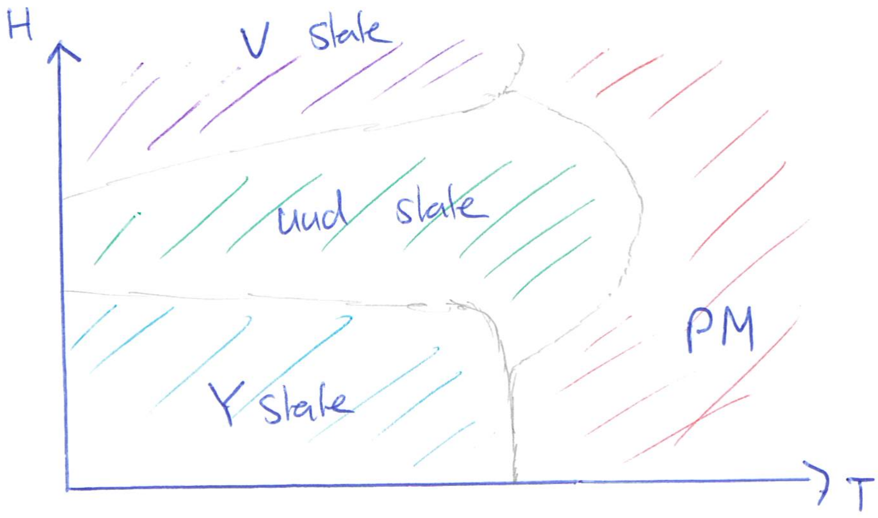
V state · coplanar ↖↗↗

$U(1) \times \mathbb{Z}_3$



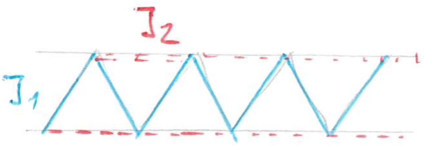
Saturated

Softening the structure by creating artificial symmetries ⟹ some modes are silenced. e.g, planar configuration, some modes associated with out of plane motion may be softened, or collinear (minimal amount of symmetries to be broken).



Non-classical ground states

Frustrated spin chains

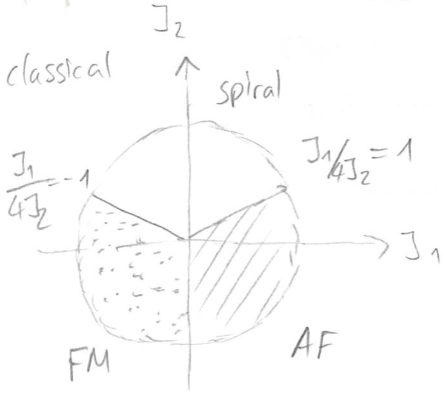


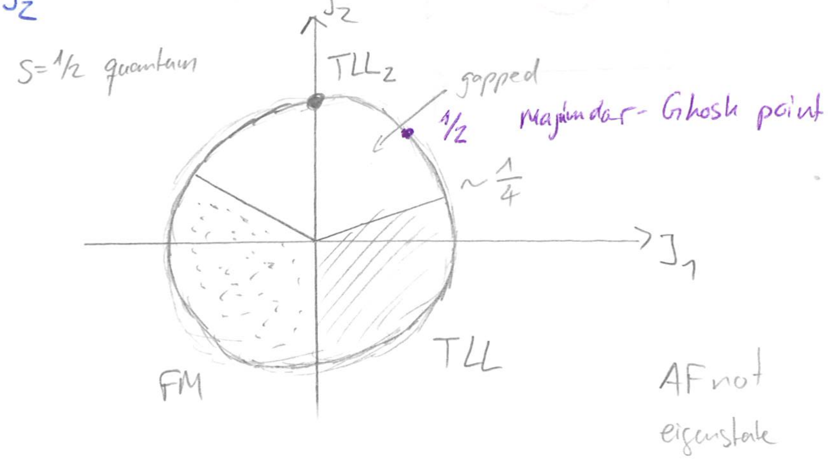
$$\mathcal{H} = \sum J_1 (\hat{\vec{S}}_n \hat{\vec{S}}_{n+1}) + J_2 (\hat{\vec{S}}_n \hat{\vec{S}}_{n+2})$$

$\Rightarrow$ spiral with angle $\varphi = Q \cdot a$

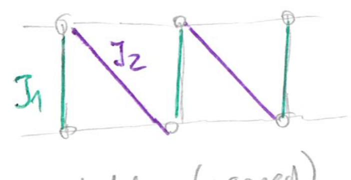
$$\Rightarrow \quad E(Q) = J_1 S^2 \cos(Qa) + J_2 S^2 \cos(2Qa)$$

$$\overset{\text{minimize}}{\Longrightarrow} \quad \cos Qa = -\frac{J_1}{4J_2} \in \mathbb{R} \quad \Rightarrow \text{if } |\text{rhs}| > 1 \Rightarrow \cos Qa = \pm 1$$





Spiral not possible because they spontaneous break a number of continuous symmetry. Quantum fluctuation dont allow that.

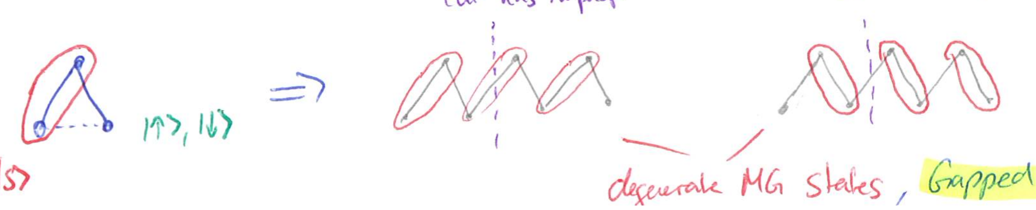


spin ladder (gapped)

$$\hat{\mathcal{H}}_{MG} = J \sum_n 2 (\hat{\vec{S}}_n \hat{\vec{S}}_{n+1}) + (\hat{\vec{S}}_n \hat{\vec{S}}_{n+2})$$

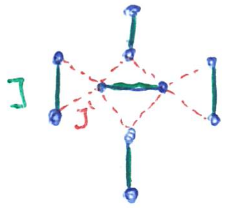
J = 1

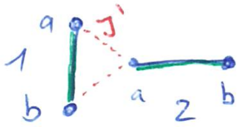
$$\Rightarrow \hat{\mathcal{H}}_{MG} = \sum_n \frac{J}{2} (\hat{\vec{S}}_{\Delta_n})^2 \quad , \quad \hat{\vec{S}}_{\Delta n} = \hat{\vec{S}}_n + \hat{\vec{S}}_{n+1} + \hat{\vec{S}}_{n+2} : \frac{1}{2} \text{ or } \frac{3}{2}$$



degenerate MG states, Gapped as locked in singlet

Shastry - Sutherland model



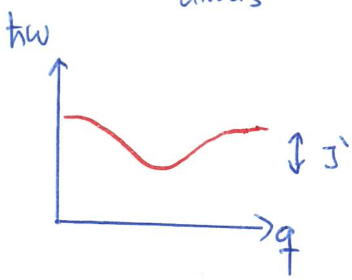


$$|s\rangle_1 = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_{a_1}|\downarrow\rangle_{b_1} - |\downarrow\rangle_{a_1}|\uparrow\rangle_{b_1}\right)$$

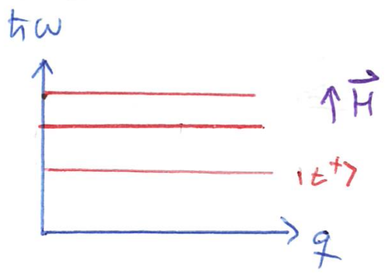
$$\hat{\mathcal{H}}` = J` \cdot \left(\hat{\vec{S}}_{1a} \cdot \hat{\vec{S}}_{2a}\right) + J`\left(\hat{\vec{S}}_{1b} \cdot \hat{\vec{S}}_{2a}\right)$$

$$\langle s_1 | \hat{\mathcal{H}}` | s_1 \rangle = 0$$

$$|GS\rangle = \prod_{\text{dimers}} |s_i\rangle$$ – good GS even for $J` \sim J$



„normal" dimerized AF



Massive degeneracy

Quadrupolar order

How to characterize the spin if not by $\langle S_{\vec{r}} \rangle$? ← generally $\neq 0$ for $\langle\langle S_{\vec{r}}^\alpha S_{\vec{r}}^\alpha \rangle\rangle \neq 0$

Spin coherent state: $|S(\varphi,\theta)\rangle$ ← wave function corresponding to a state pointing into a particular direction

$$\langle S(\varphi,\theta)| \hat{\vec{S}} |S(\varphi,\theta)\rangle = S \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix}$$

rotate from fully along z

$$|S(\varphi,\theta)\rangle = e^{i\hat{S}_z\varphi} e^{i\hat{S}_x\theta} |S,S\rangle$$

If $\langle S_{\vec{r}} \rangle \neq 0 \Rightarrow$ local magnetic field, detectable with μSR, NMR, ...
referred as dipolar. They break time reversal symmetry and rotational symmetry of the spin space.

However, frustration may provoke spontaneous breaking of spin rotational symmetry without formation of dipolar magnetic moments. Invisible to conventional magnetism probes even though magnetic.

The quadrupolar components of the magnetic moments go ordered which is a time reversal-invariant rank 2 tensor. $\Rightarrow$ tensorial order parameter

Spin rotational symmetry broken $\Rightarrow$ outcomes for measuring spin along different directions would not be the same.

Analogy to $S=1$ paramagnet with easy plane anisotropy: $\hat{z}$

Ground state is $|S_z\rangle = 0$ singlet and at D (energy) $|\pm 1\rangle$ with $S^z = \pm 1$

Singlet $\rightsquigarrow \langle \vec{S}_{\vec{r}} \rangle = 0$, However some matrix elements not identical

Qudrupolar components

$$\hat{Q}^{\alpha\beta}_{\vec r} = \underbrace{\hat S^{\alpha}_{r} \hat S^{\beta}_{r} + \hat S^{\beta}_{r} \hat S^{\alpha}_{r}}_{\text{symmetric}} \underbrace{- \frac{2}{3} S(S+1)\delta_{\alpha\beta}}_{\text{makes it traceless}}$$

, symmetric and time reversal

⇒ 5 dof

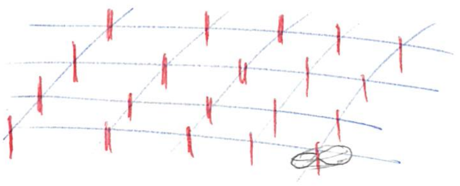
⇒ Quadrupolar "order" is not a conventional order as no spontaneous symmetry breaking ⇝ toroid with vanishingly small hole and no z-component

<u>Bilinear - biquadratic model</u>

Heisenberg + bilinear on square lattice: $\vec{\mathcal{H}} = \sum_{\vec r, \vec R} J(\hat S_r \cdot \hat S_{r+R}) + K(\vec S_{\vec r} \cdot \vec S_{\vec r + R})^2$

full rotational symmetry and time reversal symmetry present.

$J = \frac{k}{2}$ ⇝ Quadrupolar order

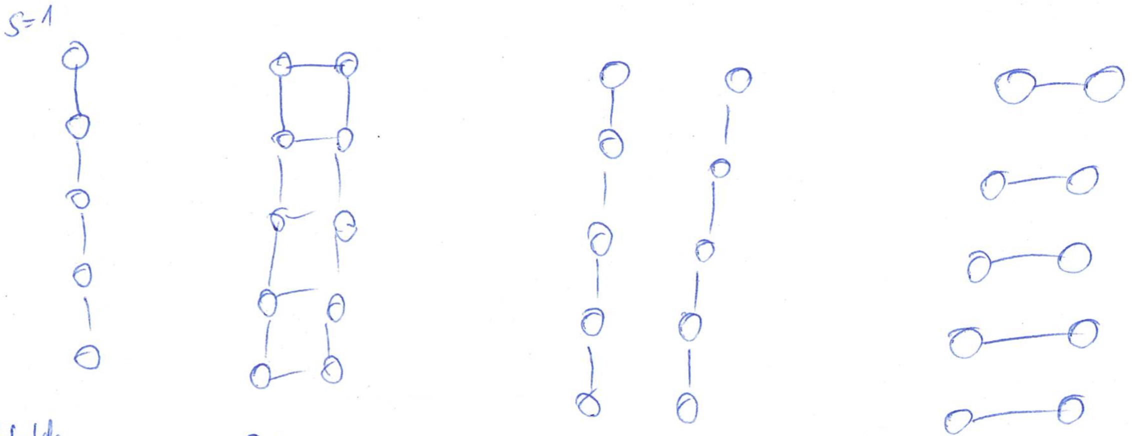


director, ferroquadrupolar order

spontaneously emerging quadrupolar moments directors are coaligned but the direction of directors is selected spontaneously. Still invariant w.r.t. time reversal

What are spin ladders, how are they different from spin chains and what are the important limits of the model?

19



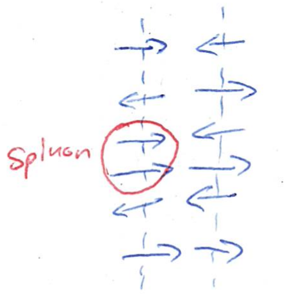
Haldane FM rung decoupled dimerized

decoupled $\rightsquigarrow$ no LRO $\begin{cases} \text{HMW} \\ \text{TLL soft} \\ \text{Ising domain walls} \end{cases}$ $\Rightarrow$ gapless

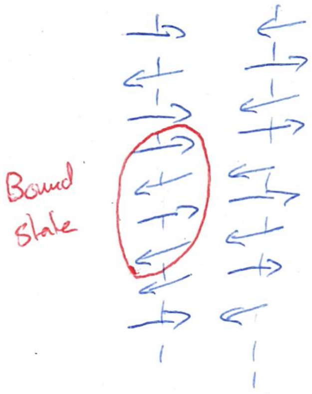
$J_\perp \neq 0 \rightsquigarrow$ LRO

How can we understand the presence of the gap for weak rung coupling?
How are the excitations in the ladders related to the excitations in chains?
What is the additional quantum number in the ladder case? 20

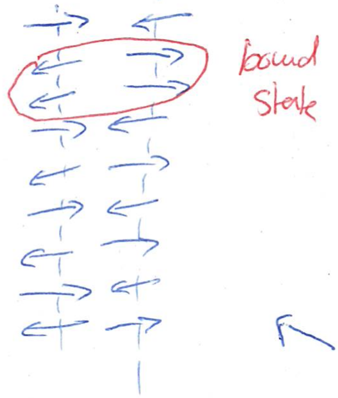
The presence of interchain coupling leads to LRO making the excitations gapped.



decoupled chains



antisymmetric
single spin flip
$S=1$, triplet



symmetric
needs two spin flips
$S=0$, singlet

farther bound state far away
$S=1/2$ not possible

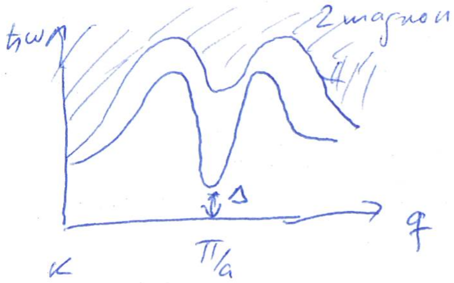
$\longrightarrow$ additional quantum number: symmetric / antisymmetric

What is the AKLT model and how can it help us to understand the nature of the Haldane state?

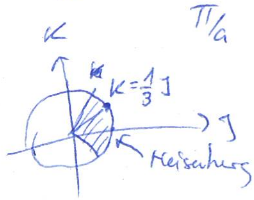
21

$J_\perp = -\infty$, $J_\parallel$ a.F.

Haldane ⇝ $S=1$, short ranged and gapped. Lieb Schultz Mattis not applicable, $\Delta \sim e^{-\pi S}$



$$\hat{\mathcal{H}} = \sum_n J(\vec{S}_n \cdot \vec{S}_{n+1}) + K(\vec{S}_n \cdot \vec{S}_{n+1})^2 \quad , \quad S=1$$



$K = \frac{1}{3}J$ ⇝ exact solution

Heisenberg same thermodynamic ground state, but has some excitations present, but captures the essential physics of the $S=1$ Haldane chain.

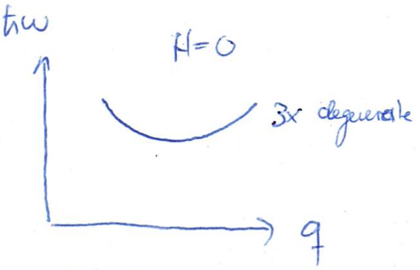
1.) Gapped, singlet / triplet

2.) Degeneracy due to spins at end

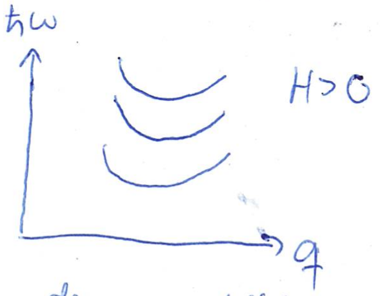
3.) String order, zero diluted Néel state, but reduced

What is the nature of the field-induced transition between gapped and gapless phase in 1D? Sketch a generic quasi 1-D phase diagram.

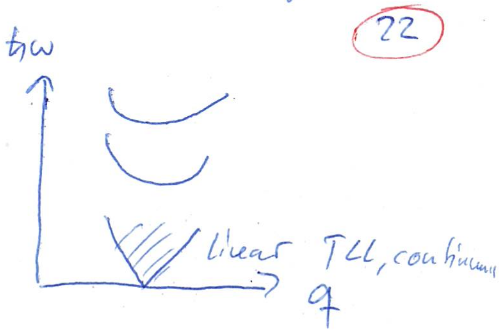
22



S = 1/2 spinons



degeneracy lifted



free spinons