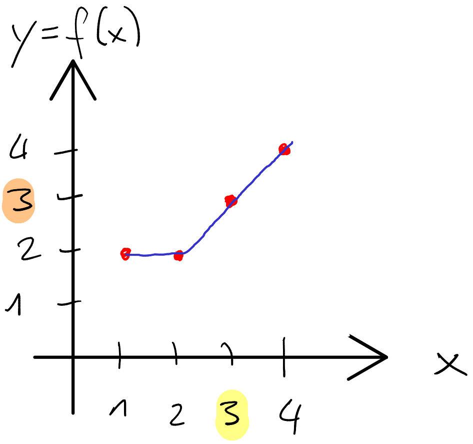
Einführung in die Differentialrechnung 23.5.2020

1.) Funktionen



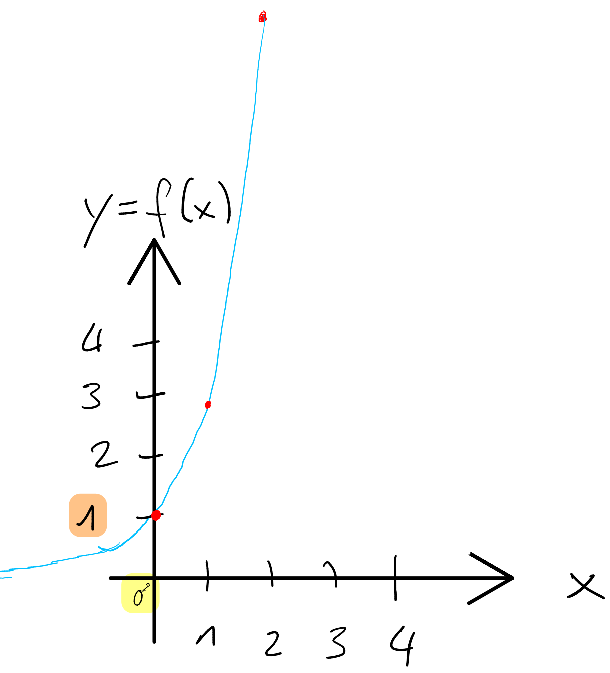
$$x \longrightarrow \boxed{\text{Funktion}} \longrightarrow f(x)$$

$x \xrightarrow{f}$	$f(x)=y$
1	2
2	2
3	3
4	4

$f(3) = 3$

$f(2) = 2$

1.a) Exponentialfunktionen und Logarithmusfunktion



$\exp(\cdot): \mathbb{R} \to \mathbb{R}^+, \quad x \mapsto e^x$

$$\boxed{e = 2{,}71828\ldots \qquad \text{(Eulersche Zahl)}}$$

$3^4 = \underbrace{3\cdot 3\cdot 3\cdot 3}_{4}$

$$\boxed{\begin{aligned} e^2 &= e\cdot e \\ e^0 &= 1 \end{aligned}}$$

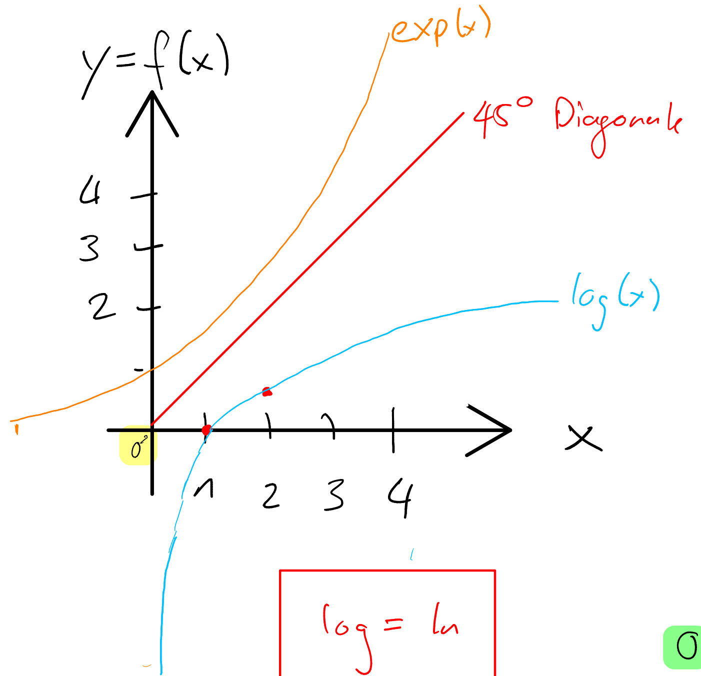
$7^0 = 1$

$e^0 = 1$

$f(0) = \exp(0) = e^0 = 1$

$f(1) = \ldots = e^1 = 2{,}718$

$f(2) = \ldots = e^2 = 7{,}39$



$$\boxed{\log = \ln}$$

$\ln(\cdot): \mathbb{R}^+ \to \mathbb{R}, \quad x \mapsto \ln(x)$

$\ln(3): \quad e^{\square} = 3$

$\log(7): \quad e^{\square} = 7$

$e^{\log(7)}: \quad e^{\log(7)} = 7$

$7\cdot \underbrace{\log(e)}_{1} = 7$

$\log(e^7) = 7$

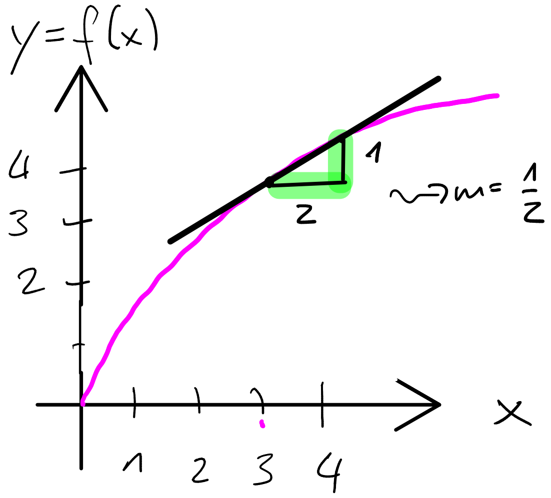
$0 = \log(1): \quad e^{\square} \overset{?}{=} 1, \quad \square = 0$

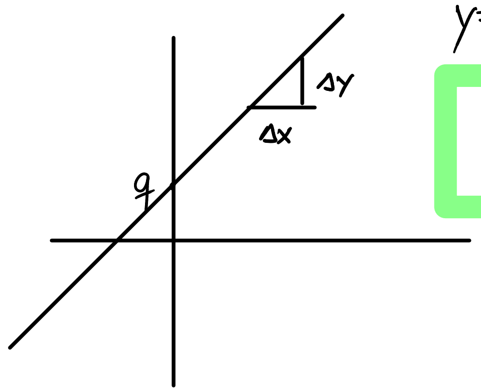
$0{,}693 = \log(2): \quad e^{\square} \overset{?}{=} 2$

$e^2 = 7{,}\ldots$

$e^1 = 2{,}7\ldots$

2.) Ableitung





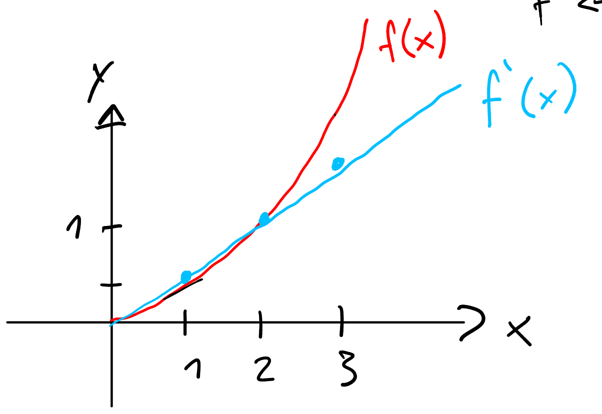
$y = mx + q$

$$\boxed{m = \frac{\Delta y}{\Delta x}}$$

$f'(3) = \frac{1}{2}$

f' ← Funktion

f ← Funktion



auswendig!

$$\boxed{\frac{d}{dx} e^x = e^x}$$

$$\boxed{\frac{d}{dx} \log(x) = \frac{1}{x}}$$

$$\frac{d}{dx} x^3 = 3x^2$$

$$\frac{d}{dx} 5x^{7} = 35x^6$$

$$\boxed{7^{-3} = \frac{1}{7^3}}$$

$$\frac{d}{dx}\frac{1}{x} = \frac{d}{dx} x^{-1} = -x^{-2} = -\frac{1}{x^2}$$

$$\frac{d}{dx}\frac{1}{x^2} = \frac{d}{dx} x^{-2} = -2 \cdot x^{-3} = -\frac{2}{x^3}$$

Ableitungsregeln

~> Polynom

~> Summen

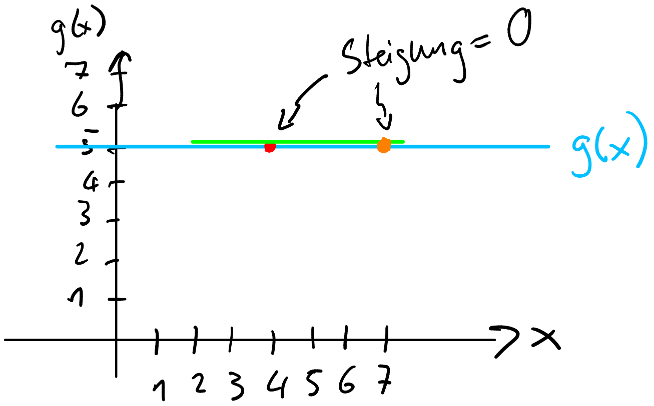
~> Produkt

~> Quotient

~> Kettenregel

~> spezielle Funktion (exp, log)

$g(x) = 5 \rightsquigarrow g'(x) = 0$



x	$g(x)$
1	5
2	5
3	5
4	5
5	5

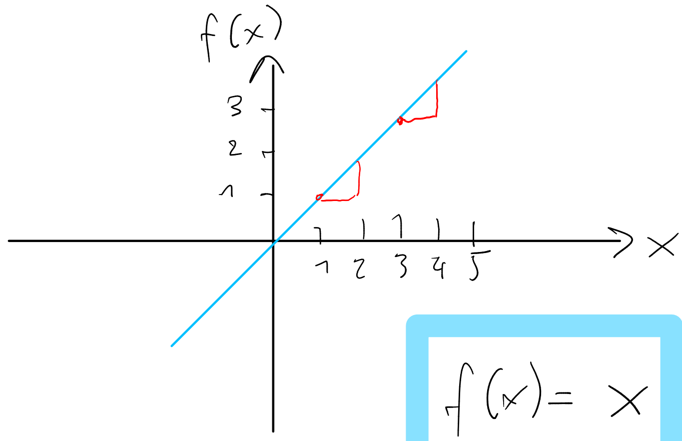
$p(x) = x^2 \rightsquigarrow p'(x) = \frac{dp}{dx} = 2x$

$f(t) = 3x^4 \rightsquigarrow f'(t) = \frac{df}{dt} = 0$

$f(t) = 5t^3 \rightsquigarrow f'(t) = \frac{df}{dt} = \underline{\underline{15t^2}}$

$f(t) = 3x^2 \rightsquigarrow f'(t) = 0$ $\qquad \frac{d}{dx} x^{\square} =$

$f(x) = 7x^3 + 27x^2 - x \Rightarrow f'(x) = 21x^2 + 54x -$



$$\boxed{f(x) = x}$$

x	$f(x)$
1	1
2	2
3	3
4	4
5	5

$$\Rightarrow \boxed{f'(x) = \frac{df}{dx} = 1}$$

$$f(x) = 3x - 7$$

Für lineare Funktionen $y = f(x) = mx + q$ gilt, dass die Ableitung $= f'(x) = m$

Ableitungsregeln

Summenregel

$$h(u) = 3u^4 - \frac{1}{2}u^2 + 3 \Rightarrow h'(u) = 12u^3 - u$$

$$f(x) = \sqrt[3]{x^2} + \frac{1}{x^2} - \frac{1}{\sqrt{x}}$$

$$\sqrt[3]{x^2} = x^{2/3} \overset{\frac{d}{dx}}{\rightsquigarrow} \frac{2}{3}x^{-1/3}$$

$$\frac{2}{3}\sqrt[3]{x^{-1}} = \frac{2}{3}\sqrt[3]{\frac{1}{x}}$$

$$\frac{1}{x^2} = x^{-2}$$

$$\overset{d/dx}{\rightsquigarrow} -2x^{-3} = -2 \cdot \frac{1}{x^3}$$

$$-\frac{1}{\sqrt{x}} = -\frac{1}{x^{1/2}} = -1 \cdot x^{-1/2} \overset{d/dx}{\Rightarrow} +\cancel{1}\cdot\left(+\frac{1}{2}\right) x^{-\frac{3}{2}} = \frac{1}{2}\frac{1}{x^{3/2}}$$

$$= \frac{1}{2}\frac{1}{\sqrt{x^3}}$$

$$f'(x) = \frac{2}{3}\sqrt[3]{\frac{1}{x}} - \frac{2}{x^3} + \frac{1}{2}\,\frac{1}{\sqrt{x^3}}$$

$$g(x) = 3\,\frac{1}{\sqrt{x^2}} - 7\,\frac{1}{\sqrt[3]{x^5}}$$

$$3\,\frac{1}{\sqrt[2]{x^2}} = \frac{3}{x^1} - 7\,\frac{1}{x^{5/3}}$$

$$= 3 \cdot x^{-1} - 7 \cdot x^{-5/3}$$

$$\overset{d/dx}{\rightsquigarrow} \quad = -3x^{-2} - 7 \cdot \left(-\frac{5}{3}\right) \cdot x^{-8/3}$$

$$= -3\,\frac{1}{x^2} + \frac{35}{3} \cdot \frac{1}{x^{8/3}} = -3\,\frac{1}{x^2} + \frac{35}{3}\,\frac{1}{\sqrt[3]{x^8}}$$

Kettenregel

$$f(x) = (3x^2 + 6x)^{79}$$

Quotientenregel

$$f(x) = \frac{x^3+1}{x-1}$$

Produktregel

$$f(x) = (3x+1)(x^2-1)$$

Ableiten von Polynomen

Ein Polynom ist definiert als eine Verkettung (plus oder minus) von Monomen der Form ax^b.

Beispiele für Monome: $3x$, x, $7x^2$, $13x^3$, $2x$

Beispiel für Polynom: $2x^2-7x+3$, $3x^3-11x^2+5x-13$, $2x^2-1$

Ableiten von Monom: $ax^b \xrightarrow{d/dx} abx^{b-1}$

Potenzgesetze Zusammenfassung für Ableiten

$$\boxed{\frac{1}{x^a} = x^{-a}} \qquad \boxed{\sqrt[a]{x^b} = x^{b/a}}$$

Beispiele:

$$\frac{1}{x^7} = x^{-7} \qquad \sqrt[3]{x^7} = x^{7/3}$$

$$\frac{1}{x^5} = x^{-5} \qquad \sqrt[4]{x^8} = x^{8/4} = x^2$$

$$\frac{1}{x} = x^{-1} \qquad \sqrt{x^3} = x^{3/2}$$

$$\frac{1}{x^{7/3}} = x^{-7/3} \qquad \sqrt{x^2} = x^{2/2} = x$$

$$\frac{1}{x^{25/5}} = x^{-5} \qquad \sqrt{x^{-3}} = x^{-3/2} = \frac{1}{x^{3/2}}$$

Beispiele Ableitungen mit Wurzel

$$f(x) = \sqrt[3]{x^2} + \frac{1}{x^2} - \frac{1}{\sqrt{x}}$$

$$f(x) = x^{2/3} + x^{-2} - \frac{1}{x^{1/2}}$$

$$f(x) = x^{2/3} + x^{-2} - x^{-1/2}$$

$$f'(x) = \frac{2}{3}x^{-1/3} - 2x^{-3} + \frac{1}{2}x^{-3/2}$$

$$f'(x) = \frac{2}{3}\frac{1}{x^{1/3}} - 2\frac{1}{x^{3}} + \frac{1}{2}\frac{1}{x^{3/2}}$$

$$f'(x) = \frac{2}{3}\frac{1}{\sqrt[3]{x}} - \frac{2}{x^{3}} + \frac{1}{2\sqrt{x^3}}$$

$$g(x) = 2x^2 - \frac{1}{\sqrt{x}} = 2x^2 - \frac{1}{x^{1/2}} = 2x^2 - x^{-1/2}$$

$$g'(x) = 4x + \frac{1}{2}x^{-3/2} = 4x + \frac{1}{2\,x^{3/2}} = \underline{\underline{4x \cdot \frac{1}{2\sqrt{x^3}}}}$$

$$h(x) = \frac{1}{3\sqrt{x^7}} - \frac{6}{5\sqrt{x}} = \frac{1}{3\,x^{7/2}} - \frac{6}{5\,x^{1/2}} = \frac{1}{3}x^{-7/2} - \frac{6}{5}x^{-1/2}$$

$$h'(x) = \frac{1}{3}\left(-\frac{7}{2}\right)x^{-9/2} - \frac{6}{5}\left(-\frac{1}{2}\right)x^{-3/2} = -\frac{7}{6}x^{-9/2} + \frac{6}{10}x^{-3/2}$$

$$= -\frac{7}{6\,x^{9/2}} + \frac{6}{10\,x^{3/2}} = \underline{\underline{\frac{-7}{6\sqrt{x^9}} + \frac{6}{10\sqrt{x^3}}}}$$

$$f(x) = \boxed{\frac{1}{\sqrt{x}}}\ \boxed{+\,3x^2} + \boxed{\frac{\sqrt{x}}{x}}$$

$$\frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} = x^{-1/2} \xmapsto{\frac{d}{dx}} -\frac{1}{2}x^{-3/2} = -\frac{1}{2x^{3/2}} = \boxed{-\frac{1}{2\sqrt{x^3}}}$$

$$3x^2 \xmapsto{d/dx} \boxed{6x}$$

$$\frac{\sqrt[2]{x}}{x} = \begin{cases} \dfrac{x^{1/2}}{x^1} = x^{1/2}\cdot x^{-1} = x^{-1/2} = \dfrac{1}{x^{1/2}} = \dfrac{1}{\sqrt{x}} \\[2ex] \dfrac{\sqrt{x}}{x} = \dfrac{\cancel{\sqrt{x}}}{\sqrt{x}\cdot\cancel{\sqrt{x}}} = \dfrac{1}{\sqrt{x}} \end{cases}$$

x^1

$$\frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} = x^{-1/2} \xmapsto{d/dx} -\frac{1}{2}x^{-3/2} = -\frac{1}{2\cdot x^{3/2}} = \boxed{-\frac{1}{2\sqrt{x^3}}}$$

$$f'(x) = -\frac{1}{2\sqrt{x^3}} + 6x - \frac{1}{2\sqrt{x^3}} = -\cancel{2}\cdot\frac{1}{\cancel{2}\sqrt{x^3}} + 6x = \underline{\underline{6x - \frac{1}{\sqrt{x^3}}}}$$

$$g(x) = \frac{\sqrt{x}}{\sqrt{x^3}} = \frac{x^{1/2}}{x^{3/2}} = x^{1/2}\,x^{-3/2} = x^{1/2-3/2} = x^{-2/2} = x^{-1}$$

$$x^{-1} \xmapsto{d/dx} \underline{\underline{-\frac{1}{x^2}}}$$

$$g'(x) = \underline{\underline{-\frac{1}{x^2}}}$$

Summenregel

$$\boxed{\begin{aligned} f(x) &= g(x) + h(x) \\ f'(x) &= g'(x) + h'(x) \end{aligned}}$$

Produktregel

$$\boxed{\begin{aligned} f(x) &= u(x) \cdot v(x) \\ f'(x) &= u'(x) \cdot v(x) + u(x) \cdot v'(x) \end{aligned}}$$

Rezept für Produktregel

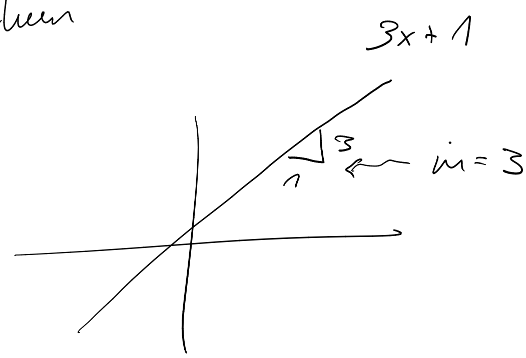
$$\boxed{f(x) = (3x+1)(7x^2+2)}$$

I: $u(x)$ und $v(x)$ identifizieren

$$\boxed{u(x) = 3x+1 \qquad v(x) = 7x^2+2}$$

II: $u'(x)$ und $v'(x)$ berechnen

$$\boxed{\begin{aligned} u'(x) &= 3 \\ v'(x) &= 14x \end{aligned}}$$

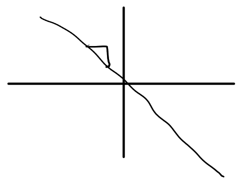


III. Produktformel einsetzen

$$f'(x) = u'(x)\, v(x) + u(x)\, v'(x)$$

$$\boxed{f'(x) = 3 \cdot (7x^2 + 2) + (3x+1)\, 14x}$$

$$f(x) = 27x\,(7x^2 - x)$$



$\rightsquigarrow$ I: $u(x) = 27x$ $\quad v(x) = 7x^2 - x$

$\rightsquigarrow$ II: $u'(x) = 27$ $\quad v'(x) = 14x - 1$

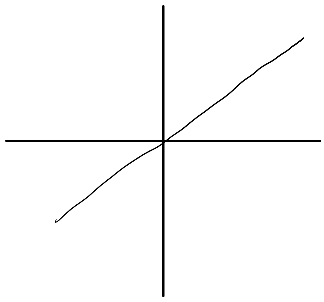
$\rightsquigarrow$ III: $27\,(7x^2 - x) + 27x\,(14x - 1)$

<u>Quotientenregel</u>

$$f(x) = \frac{u(x)}{v(x)}$$

$$f'(x) = \frac{u'(x)\, v(x) - u(x)\, v'(x)}{v(x)^2}$$

$$f(x) = \frac{3x+1}{x}$$



$\rightsquigarrow$ I: $u(x) = 3x+1$ $\qquad$ $v(x) = x$

$\rightsquigarrow$ II: $u'(x) = 3$ $\qquad$ $v'(x) = 1$

$\rightsquigarrow$ III: $$f'(x) = \frac{3x - (3x+1)\cdot 1}{x^2}$$

$$f'(x) = \frac{3x - (3x+1)}{x^2}$$

$$f'(x) = \frac{3x - 3x - 1}{x^2} = \underline{\underline{-\frac{1}{x^2}}}$$

$$f(x) = (7x+2)\,e^x$$

$\rightsquigarrow$ I: $u(x) = 7x+2$ $\qquad$ $v(x)\ e^x$

$\rightsquigarrow$ II: $u'(x) = 7$ $\qquad$ $v'(x) = e^x$

$\rightsquigarrow$ III: $$f'(x) = \frac{7(e^x) - (7x+2)\cdot e^x}{(e^x)^2}$$

$$f'(x) = \frac{7\cancel{e^x} - 7x\cdot \cancel{e^x} - 2\cancel{e^x}}{(e^x)^{\cancel{2}}} = \underline{\underline{\frac{7 - 7x - 2}{e^x}}}$$

Kettenregel

$$f(x) = u(v(x))$$

$$f'(x) = \underbrace{u'(v(x))}_{\text{äussere Ableitung}} \cdot \underbrace{v'(x)}_{\text{innere Ableitung}}$$

$$f(x) = (2x+3)^2$$

$\rightsquigarrow$ I: $u(x) = x^2$ $\quad v(x) = 2x+3$

$\rightsquigarrow$ II: $u'(x) = 2x$ $\quad v'(x) = 2$

$\rightsquigarrow$ III: $f'(x) = 2(2x+3) \cdot 2$

$$f(x) = (\underbrace{2x+3}_{})^2$$

$x^2 \rightsquigarrow 2x$

$$f'(x) = 2(2x+3) \cdot 2$$

$$f(x) = (5x+7)^3$$

$$f'(x) = 3(5x+7)^2 \cdot 5$$

$$f(x) = e^{2x}$$

$$f'(x) = e^{2x} \cdot 2$$

$$f(x) = e^{7x^2+3}$$

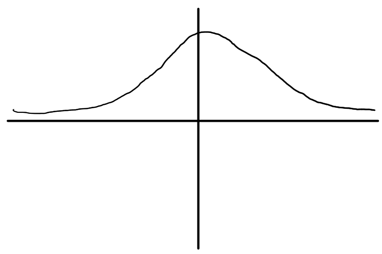
$$f'(x) = e^{7x^2+3} \cdot 14x$$

$$f(x) = e^{7x^2-x}$$

$$f'(x) = e^{7x^2-x} \cdot (14x-1)$$

<u>Mischaufgaben</u>

$$f(x) = e^{-\frac{x^2}{2}}$$



$$u(x) = e^{x} \qquad v(x) = -\frac{x^2}{2} = -\frac{1}{2} \cdot x^2$$

$$u'(x) = e^{x} \qquad v'(x) = -\frac{1}{\cancel{2}} \cdot \cancel{2}x = -x$$

$$f'(x) = e^{-\frac{1}{2}x^2} \cdot (-x) = -x e^{-\frac{1}{2}x^2}$$

$$u'(v(x))$$

$$f(x) = x \cdot e^{-x}$$

$$e^{x} \xrightarrow{d/dx} e^{x}$$

$$-x \xrightarrow{d/dx} -1$$

$\rightsquigarrow$ I: $u(x) = x$, $v(x) = e^{\boxed{-x}}$

$\rightsquigarrow$ II: $u'(x) = 1$, $v'(x) = e^{-x}(-1) = -e^{-x}$

$\rightsquigarrow$ III: $f'(x) = 1 \cdot e^{-x} + x e^{-x}(-1) = \underline{\underline{e^{-x} - x e^{-x}}} = e^{-x}(1-x)$

$$f(x) = \frac{x}{e^{x}}$$

$\rightsquigarrow$ I: $u(x) = x$, $v(x) = e^{x}$

$\rightsquigarrow$ II: $u'(x) = 1$, $v'(x) = e^{x}$

$\rightsquigarrow$ III: $f'(x) = \dfrac{1 \cdot \cancel{e^{x}} - x\cancel{e^{x}}}{(e^{x})^{\cancel{2}}} = \dfrac{1-x}{e^{x}} = \underline{\underline{e^{-x}(1-x)}}$

$$\frac{a}{e^{x}} = a \cdot \frac{1}{e^{x}} = a \cdot e^{-x}$$

$$\frac{a+b}{e^{x}} = (a+b) \cdot e^{-x}$$

$$f(x) = \frac{2x^2 - 1}{3x + 1}$$

$\rightsquigarrow$ I: $\quad u = 2x^2 - 1 \quad , \quad v = 3x + 1$

$\rightsquigarrow$ II: $\quad u' = 4x \quad , \quad v' = 3$

$\rightsquigarrow$ III: $\quad f'(x) = \dfrac{4x \cdot (3x+1) - (2x^2 - 1) \cdot 3}{(3x+1)^2}$

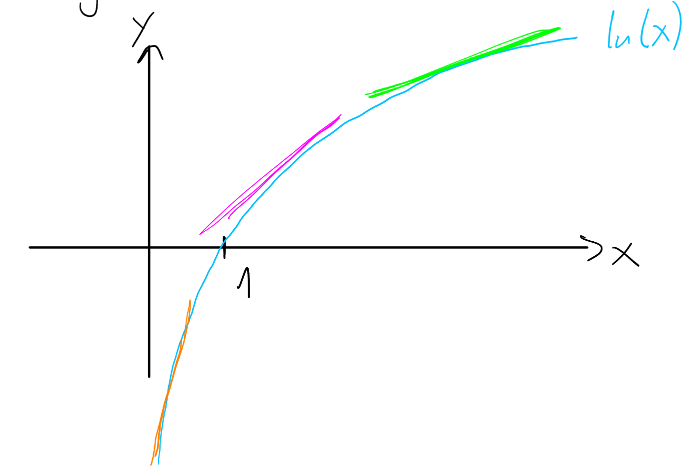
$$h(x) = \sqrt[3]{2x^2 - 1}$$

$\rightsquigarrow$ I: $\quad u(x) = \sqrt[3]{x} = x^{1/3} \quad , \quad v(x) = 2x^2 - 1$

$\rightsquigarrow$ II: $\quad u'(x) = \frac{1}{3} x^{-2/3} \quad , \quad v'(x) = 4x$

$\rightsquigarrow$ III: $\quad \underline{\underline{f'(x) = \frac{1}{3} (2x^2 - 1)^{-2/3} \cdot 4x}}$

Logarithmus

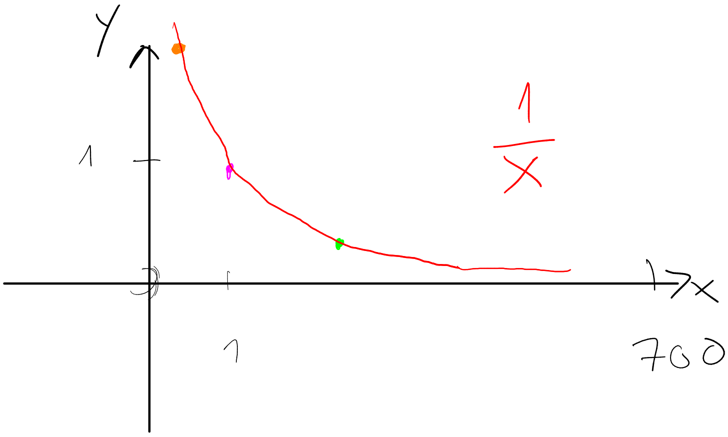


$\ln(7) \,\hat{=}\, \left(e^{\square} = 7\right)$

$\boxed{\ln(e) = 1}$

$e^{\overset{1}{\square}} = e$

$\boxed{a^0 = 1 \quad \text{für alle Zahlen}}$



$$\boxed{\frac{d}{dx} \ln(x) = \frac{1}{x}}$$

$$\boxed{\frac{d}{dx} \log_a x = \frac{1}{\ln(a)} \cdot \frac{1}{x}}$$

$$f(x) = \log_3(2\sqrt{x})$$

$\rightsquigarrow$ I: $\quad u(x) = \log_3(x) \quad , \quad v(x) = 2\sqrt[2]{x} = 2\cdot x^{1/2}$

$\rightsquigarrow$ II: $\quad u'(x) = \frac{1}{\ln(3)} \cdot \frac{1}{x} \quad , \quad v'(x) = x^{-1/2} = \frac{1}{\sqrt{x}}$

$\rightsquigarrow$ III: $\quad \frac{1}{\ln(3)} \cdot \frac{1}{2\sqrt{x}} \cdot \frac{1}{\sqrt{x}} = \frac{1}{2\cdot \ln(3)\, x}$

$$\boxed{f(x) = x^x} = e^{\ln(x^x)} = e^{x\ln(x)}$$

produktregel

Kettenregel $\rightsquigarrow$ I: $\quad u(x) = e^x \quad , \quad v(x) = x\cdot\ln(x)$

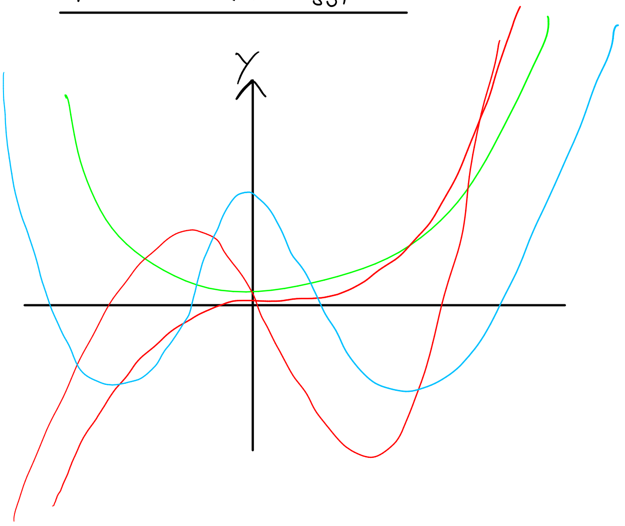
1. 2.

$\rightsquigarrow$ II: $\quad u'(x) = e^x \quad , \quad v'(x) = 1\cdot\ln(x) + \underbrace{x\cdot\frac{1}{x}}_{1} = \ln(x) + 1$

$\rightsquigarrow$ III: $\quad f'(x) = e^{x\ln(x)}(\ln(x)+1) = \underline{\underline{x^x(\ln(x)+1)}}$

Ist ihr bauch konvex, hatte sie sex, ist er konkav, war sie brav. + fraue sind obe.

Kurvendiskussion



x^3 x^4

$f(x) = 3x^2 - 5$

I: Monotonizität

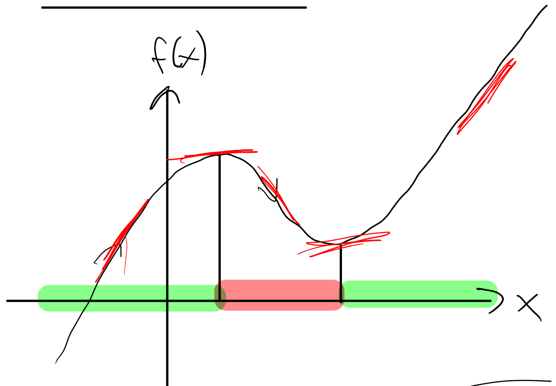
II: konvex / konkav

III: Extrema, lokal / global

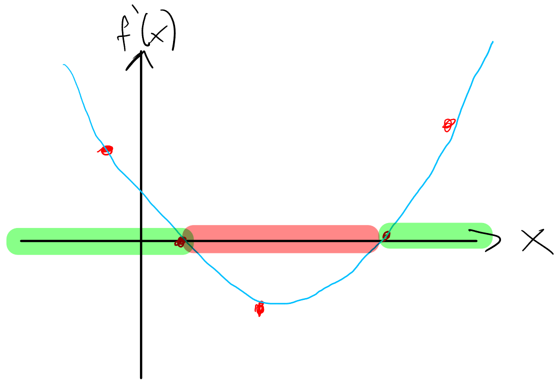
IV: Wendestell

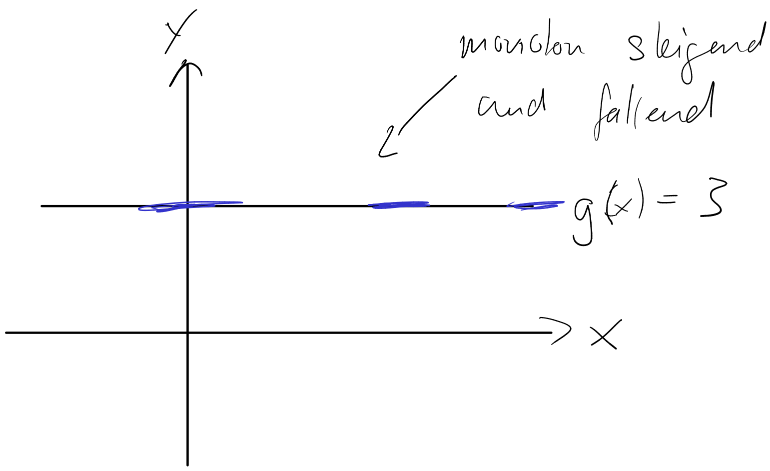
V: Graph zeichnen

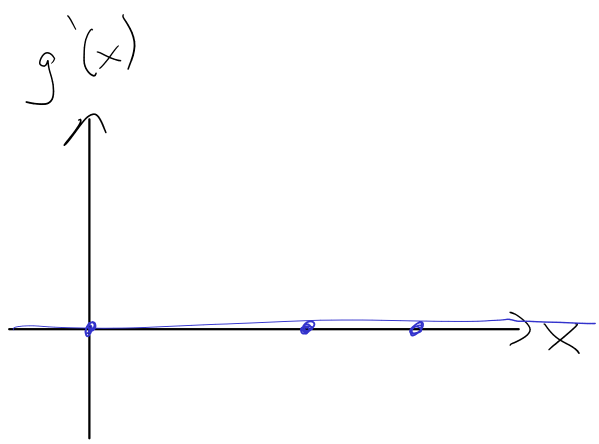
Monotonizität



steigend fallend steigend



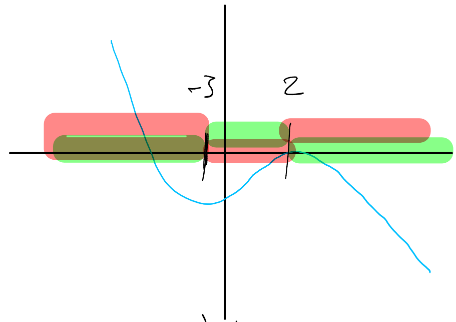




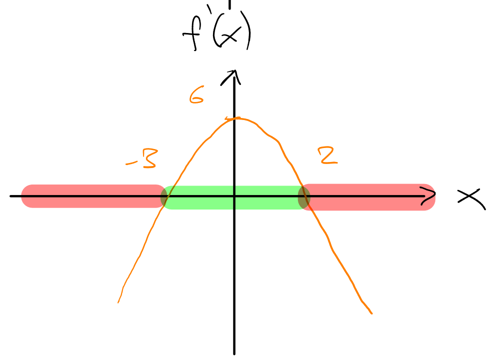
f monoton wachsend $\Leftrightarrow f' \geq 0$

f monoton fallend $\Leftrightarrow f' \leq 0$

$$f(x) = -\frac{1}{3}x^3 - \frac{1}{2}x^2 + 6x$$



$$f'(x) = -x^2 - x + 6$$



$$(x^2 + x - 6)$$

$$\frac{-1 \pm \sqrt{1+24}}{2}$$

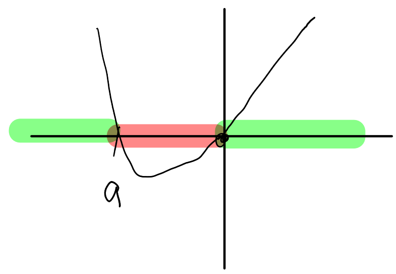
$$\frac{-1 \pm 5}{2} = \frac{4}{2} = \underline{\underline{2}}$$

$$\frac{-6}{2} = \underline{\underline{-3}}$$

$]-\infty; -3]$	monoton fallend
$[-3; 2]$	monoton steigend
$[2, \infty[$	monoton fallend

$$f(x) = 3x^3 + 5x^2 + 7$$

$$f'(x) = 9x^2 + 10x$$



Nullstellen: $\quad x(9x+10) \overset{!}{=} 0$

$\Rightarrow \quad]-\infty, a], \quad [a, 0], \quad [0, \infty[$

steigend, fallend, steigend

f konvex	**f konkav**
ist gleichbedeutend mit	ist gleichbedeutend mit
Steigung von f zunehmend	Steigung von f abnehmend
ist gleichbedeutend mit	ist gleichbedeutend mit
f' monoton steigend	f' monoton fallend
ist gleichbedeutend mit	ist gleichbedeutend mit
$\longrightarrow \quad f'' \geq 0$	$\longrightarrow \quad f'' \leq 0$

$$f(x) = -\frac{1}{3}x^3 - \frac{1}{2}x^2 + 6x$$

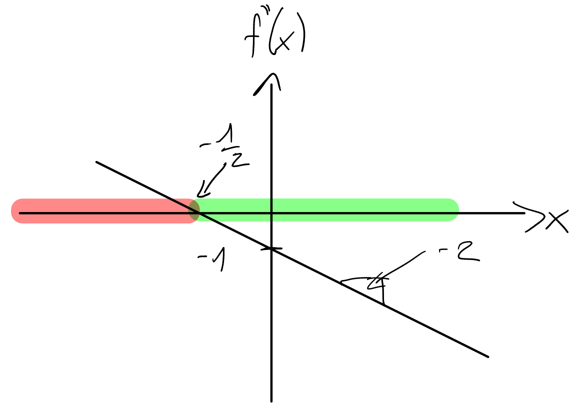
$$f'(x) = -x^2 - x + 6$$

$$f''(x) = -2x - 1$$

$$-2x - 1 \overset{!}{=} 0$$

$$\Rightarrow \quad -2x = 1$$

$$\Rightarrow \quad x = \frac{1}{-2}$$



konvex: $\left]-\infty, -\frac{1}{2}\right]$

konkav: $\left[-\frac{1}{2}, \infty\right[$

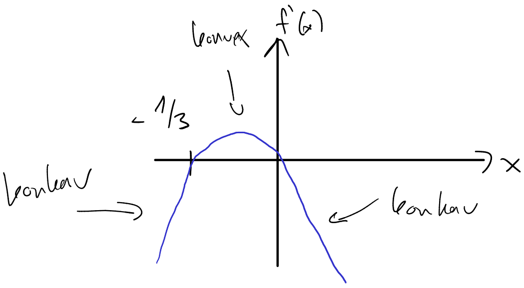
$$f(x) = -(x^3 + 2x^2 + 1) = -x^3 - 2x^2 - 1$$

R	F	
❑	☒	f ist auf ganz $\mathbb{R}$ monoton wachsend.
❑	☒	f hat die 3. Ableitung $f'''(x) = 6$. $\quad f'''(x) = -6$
❑	☒	f hat eine Wendestelle bei $x = 0$.
❑	☒	f hat eine Tangente mit Steigung -5 an der Stelle $x = -1$.

$\Rightarrow$ Betrachte 1ste Ableitung

$f'(x) = -3x^2 - 4x$

$\rightsquigarrow -\frac{1}{3}, \; 0$ (Nullstellen)

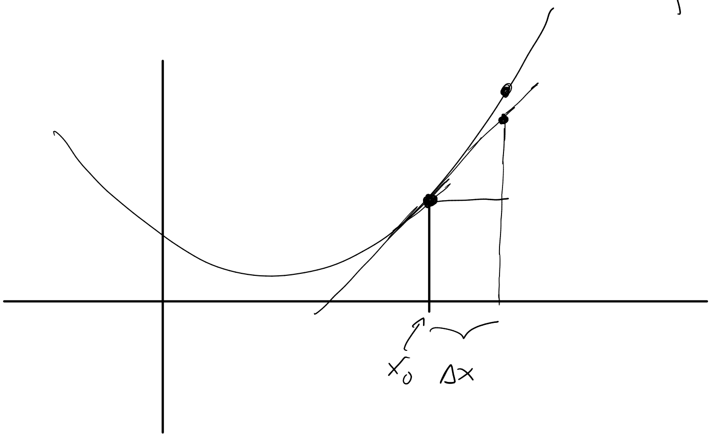


$f''(x) = -6x - 4$

$f'''(x) = -6$

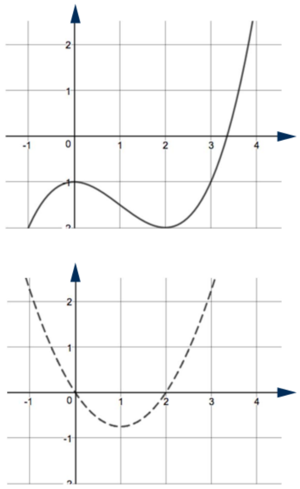
$f'(-1) = -6 + 4 = \underline{\underline{-2}}$

$f''(0) = -4 \neq 0 \implies$ keine Wendestelle



$f(x_0) + f'(x_0) \cdot \Delta x$

Integrale



$$f(x) = \frac{1}{4}x^3 - \frac{3}{4}x^2 - 1$$

Das Ableiten ist ein eindeutiger Prozess.

$$f'(x) = \frac{3}{4}x^2 - \frac{3}{2}x$$

Eine Konstante verschwindet beim Ableiten.

Eine Funktion F nennt man eine **Stammfunktion** der Funktion f, wenn ihre Ableitung F' mit der Funktion f übereinstimmt.

Es gilt also für alle x:

$$F'(x) = f(x)$$

$f(x)$	$F(x)$
0	c
1	$x + c$
x^r	$\frac{x^{r+1}}{r+1} + C$
$\frac{1}{x}$	$\ln(x) + C$
e^x	$e^x + c$

$c \in \mathbb{R}$

$$f(x) = x^3 \rightsquigarrow F(x) = \frac{x^{3+1}}{3+1} + c = \frac{1}{4}x^4 + c$$

$$g(x) = x^{-3} \rightsquigarrow F(x) = \frac{x^{-2}}{-2} + c$$

$$h(x) = x \cdot \sqrt{x} = x \cdot x^{1/2} = x^{3/2}$$

$$H(x) = \frac{x^{3/2+1}}{\frac{3}{2}+1} + c = \frac{x^{5/2}}{5/2} + c$$

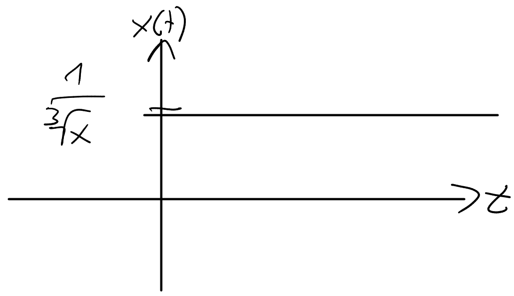
$$s(x) = \sqrt{\sqrt{x}} = \sqrt[2]{x^{1/2}} = x^{1/4}$$

$$S(x) = \frac{x^{1/4+1}}{1/4+1} + c = \frac{x^{5/4}}{5/4} + c$$

$$t(x) = \frac{1}{\sqrt[3]{x}} = \frac{1}{x^{1/3}} = x^{-1/3}$$

$$T(x) = \frac{x^{-1/3+1}}{-1/3+1} + c = \frac{x^{2/3}}{2/3} + c$$

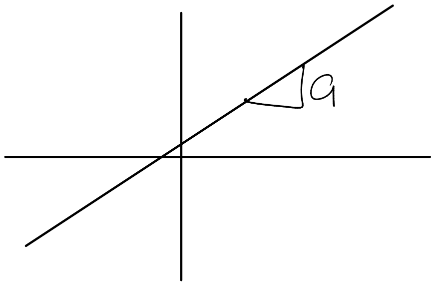
$$x(t) = \frac{1}{\sqrt[3]{x}}$$



$$X(t) = \frac{1}{\sqrt[3]{x}} t + C$$

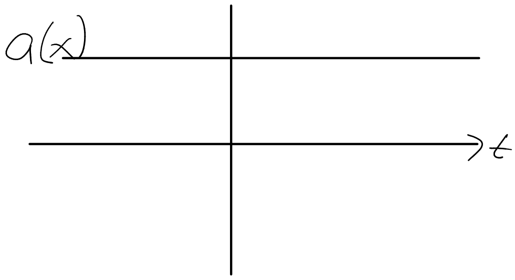
$f(x) = a \quad , \quad a \in \mathbb{R}$

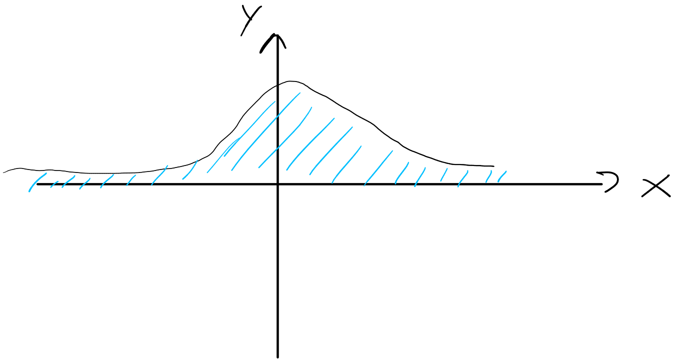
$F(x) = ax + c$

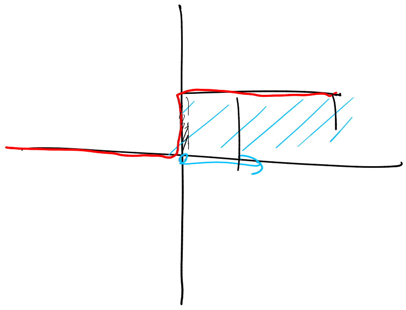


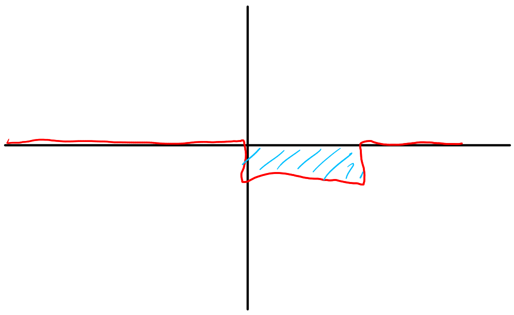
$f(x)$

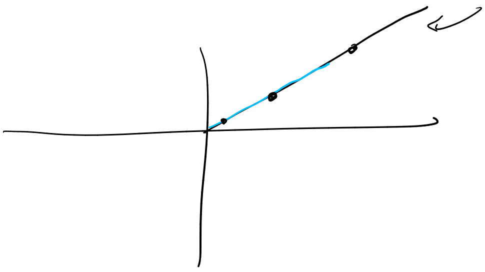
$f'(x)$





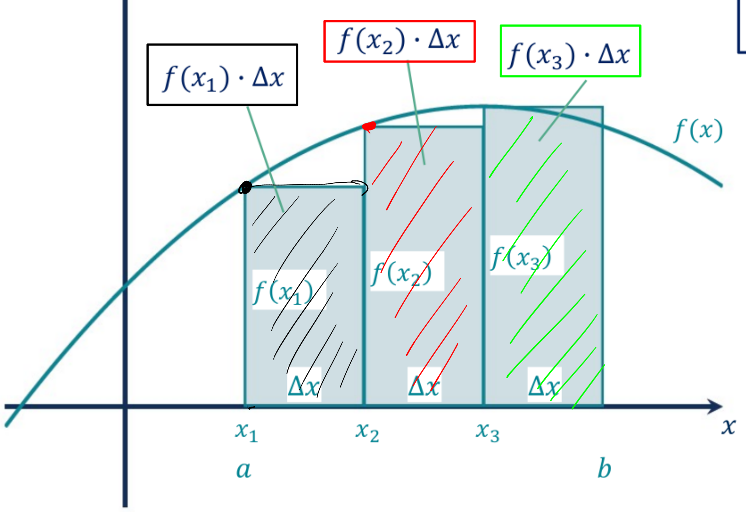






Fläche unter Kurve

Die Summe aller Rechtecksflächen ist eine Annäherung an die gesuchte Fläche A zwischen Kurve und x-Achse:

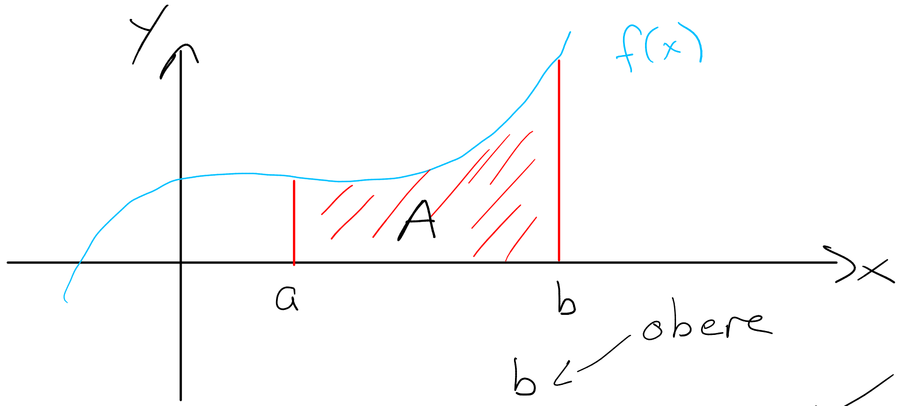


$$\boxed{A \approx \sum f(x_i) \cdot \Delta x}$$

$f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + f(x_3) \cdot \Delta x$

$A = \int f(x)\, dx$

Merke: Für sehr viele sehr schmale Rechtecke, d.h. für sehr kleines Δx, wird diese Näherung sehr genau.



$$A = \int_a^b f(x)\,dx$$

obere

Integrationsvariable

Integrand

untere Integrationsgrenze

$$\int_2^4 t^1 + 2\,dt = \frac{t^{1+1}}{1+1} + 2t \Big|_2^4 = \frac{1}{2}t^2 + 2t \Big)_2^4$$

$$= \frac{1}{2}(4)^2 + 2 \cdot 4 - \frac{1}{2}2^2 - 2 \cdot 2$$

$$= 8 + 8 - 2 - 4 = \underline{\underline{10}}$$

$$5x \rightsquigarrow 5 \rightsquigarrow 5x + C$$

$$7 \dashrightarrow \int 7 \cdot dt = 7t + C$$

$f(x) = 3x^2 + 7x \rightsquigarrow$ Finde Stammfunktion $F(x)$

$F(x) = \int f(x)\,dx$

F ____ $\uparrow \int dx$

$3x^2+7x$ ____ f

____ f' $\downarrow d/dx$

____ f''

$F(x) = \int 3x^2+7x\,dx$

$F(x) = \int 3x^2\,dx + \int 7x\,dx = 3\int x^2\,dx + 7\int x\,dx$

Summenregel

$= \cancel{3} \cdot \frac{x^3}{\cancel{3}} + C_1 + 7 \cdot \frac{x^2}{2} + C_2$

$= x^3 + C_1 + \frac{7}{2}x^2 + C_2 = x^3 + \frac{7}{2}x^2 + C$

$\int 25x^5 + 7\,dx = \int 25x^5\,dx + \int 7\,dx$

$= 25\underbrace{\int x^5\,dx} + 7\int 1\,dx$

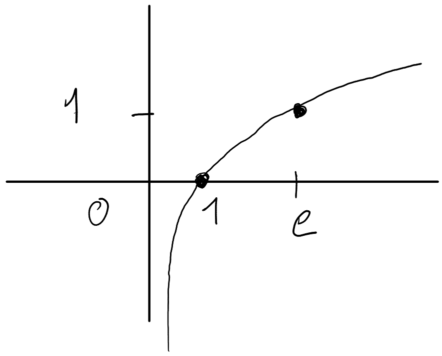
$= 25 \cdot \frac{x^6}{6} + 7 \cdot x + C$

$\int 8\,dx = 8\int 1\,dx$

$= 8x + C$

$$\int_1^e \frac{1}{x}\,dx = \ln(x)\Big|_1^e = \ln(e) - \ln(1) = 1 - 0 = \underline{\underline{1}}$$

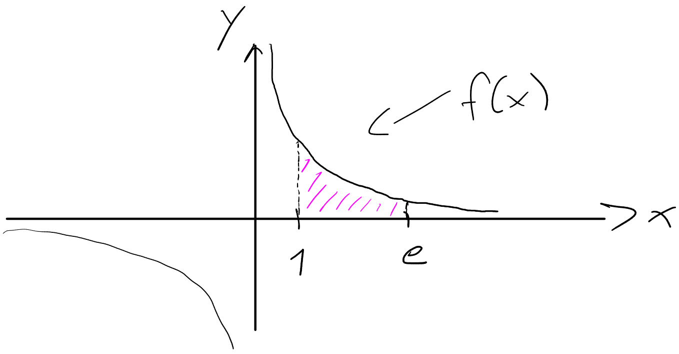
$F = \ln(x)$



$\rightsquigarrow 1$

$\uparrow \int dx$

$f(x) = \dfrac{1}{x}$



$f(x) = e^{3x}$

$$\int_0^{\ln(2)} e^{3x}\,dx = \frac{e^{3x}}{3}\Bigg|_0^{\ln(2)}$$

$\dfrac{e^{3x}}{3}$

e^{3x}

$e^{3x} \cdot 3$

$$= \frac{e^{3\ln(2)}}{3} - \frac{e^{0}}{3}$$

$$= \frac{8}{3} - \frac{1}{3} = \frac{8-1}{3} = \underline{\underline{\frac{7}{3}}}$$

$$\int 3x^2 + 5\,dx = \cancel{3}\left(\frac{x^3}{\cancel{3}}\right) + 5x + C = x^3 + 5x + C$$

$$\int 7x^2 + 7x\,dx = 7\left(\frac{x^3}{3}\right) + 7\frac{x^2}{2} + C$$

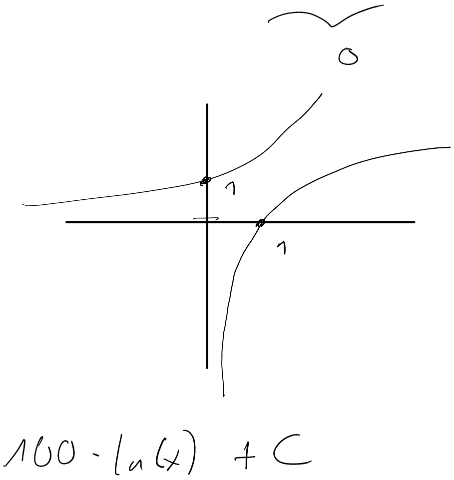
$$\int 3x^3 - 5\,dx = 3\left(\frac{x^4}{4}\right) - 5x + C$$

$$\int 1 \cdot x\,dx = 1\left(\frac{x^2}{2}\right) = \frac{1}{2}x^2 + C$$

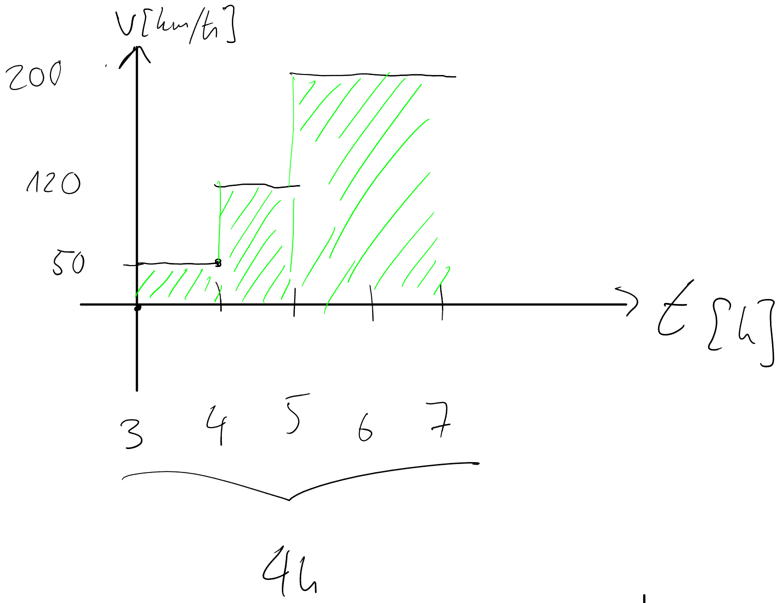
$$\int 1\,dx = x + C$$

$$\int_1^5 \frac{1}{x}\,dx = \ln(x)\Big|_1^5 = \ln(5) - \underbrace{\ln(1)}_{0}$$

mit Taschenrechner



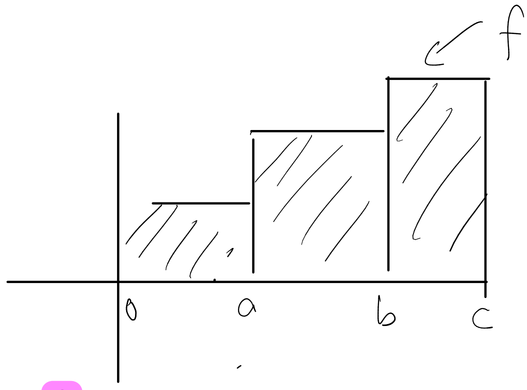
$$\int 3x^2 - 4x - \frac{100}{x}\,dx = 3\frac{x^3}{3} - 2x^2 - 100 \cdot \ln(x) + C$$



$$v = \frac{s}{t} \Rightarrow s = v \cdot t$$

fix 1h ⇝ 50km / 100km

fix 100k ⇝ 1h / 2h

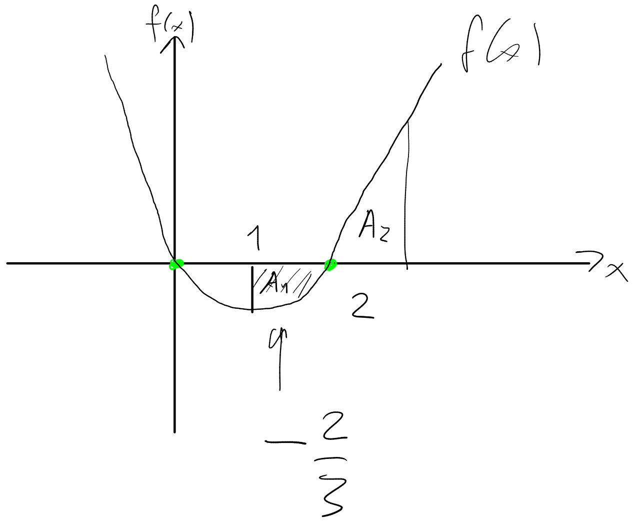


$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$F(a) = \int_a^a f(t)\,dt = -\int_a^a f(t)\,dt = 0$$

$$\int_1^4 3x^2\,dx = \frac{\cancel{3}x^3}{\cancel{3}}\Big|_1^4 = x^3\Big|_1^4 = 4^3 - 1^3 = 64-1=63$$

$$\int_2^4 (t+2)\,dt = \left[\frac{t^2}{2} + 2t\right]_2^4 = \frac{4^2}{2} + 2\cdot 4 - \frac{2^2}{2} - 2\cdot 2$$



$$f(x) = x^2 - 2x$$

I: Nullstelle:

$$f(x) = x^2 - 2x \overset{!}{=} 0$$

$$= x \cdot (x-2) \overset{!}{=} 0$$

$$\Rightarrow \quad 0, 2$$

$$A_1 = \int_1^2 f(x)\,dx = \int_1^2 x^2 - 2x\,dx = \left(\frac{x^3}{3} - x^2\right)\Big|_1^2$$

$$= \left(\frac{2^3}{3} - 2^2\right) - \left(\frac{1^3}{3} - 1^2\right) = \frac{8}{3} - 4 - \frac{1}{3} + 1$$

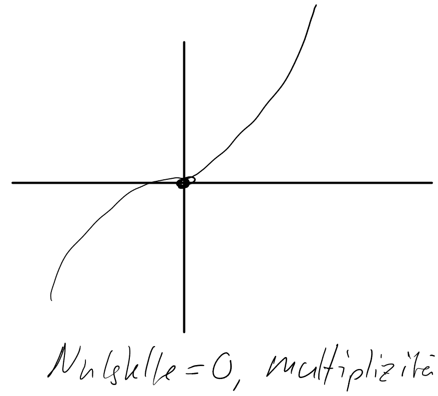
$$= \frac{7}{3} - 4 + 1 = \frac{7 - 12 + 3}{3} = -\frac{2}{3}$$

$$|A_1| = +\frac{2}{3}$$

$$A_2 = \int_2^3 x^2 - 2x\,dx = \frac{x^3}{3} - x^2 \Big|_2^3 = \frac{3^3}{3} - 3^2 - \frac{2^3}{3} + 2^2$$

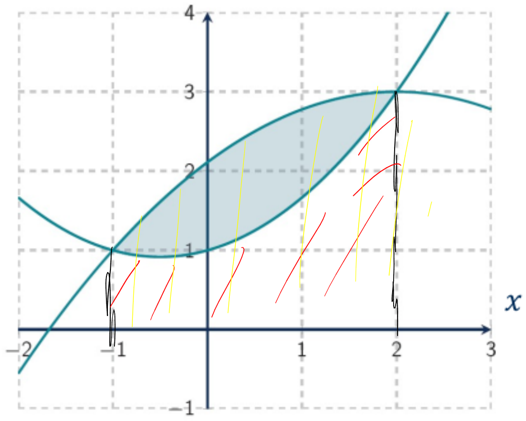
$$= \cancel{9} - \cancel{9} - \frac{8}{3} + 4 = \frac{-8 + 12}{3} = \frac{4}{3}$$

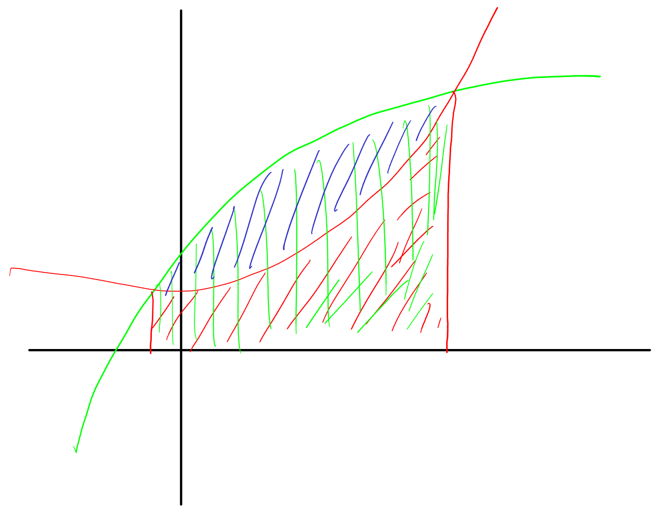
$$|A_1| + |A_2| = \frac{2}{3} + \frac{4}{3} = 2$$



$$f(x) = x^3 = \underset{\swarrow 0}{x} \cdot \underset{\searrow 0}{x} \cdot \underset{\searrow 0}{x} = 0$$

Nullstelle = 0, multiplizität = 3

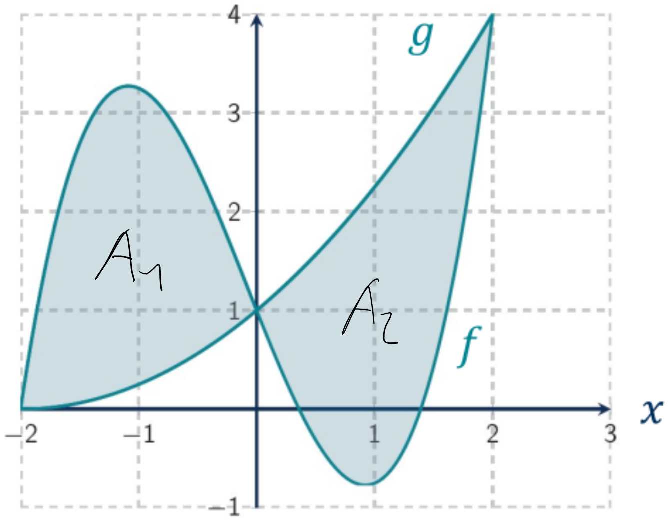




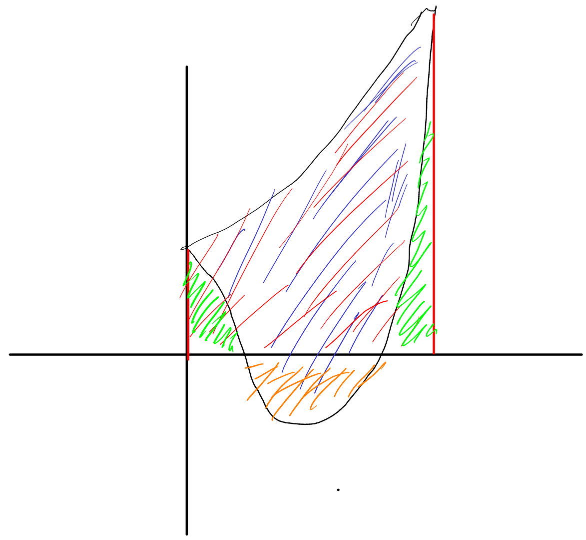
$$g(x) = \frac{1}{3}x^2 + \frac{1}{3}x + 1$$

$$f(x) = -\frac{2}{9}x^2 + \frac{8}{9}x + \frac{19}{9}$$

$$\int_{-1}^{2} f(x) - g(x)\,dx = \int_{-1}^{2} f(x)\,dx - \int_{-1}^{2} g(x)\,dx$$



$(-2, \ 0, \ 2)$

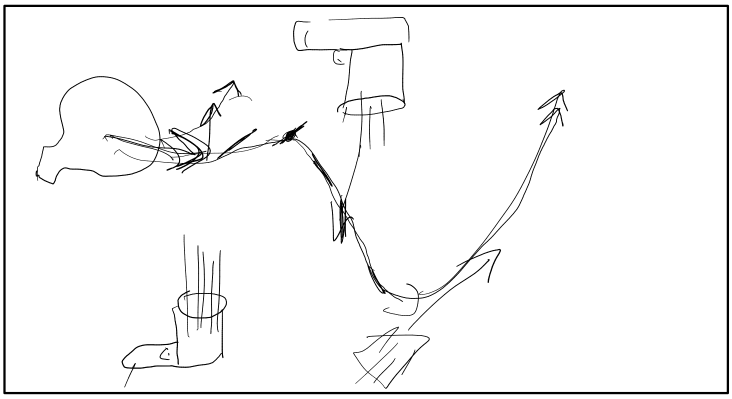


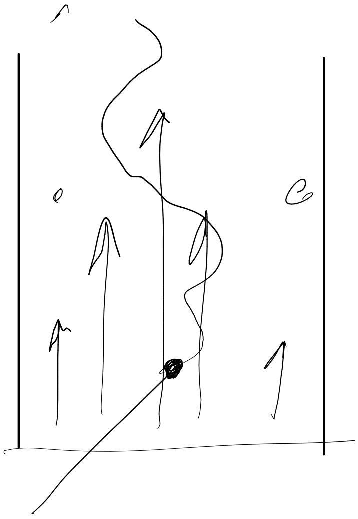
$$f(x) = x^3 + \frac{1}{4}x^2 - 3x + 1$$

$$g(x) = \frac{1}{4}x^2 + x + 1$$

$$A_1 = \int_{-2}^{0} f(x) - g(x)\,dx \quad , \quad A_2 = \int_{0}^{2} g(x) - f(x)\,dx = \ldots$$

Differentialgleichungen





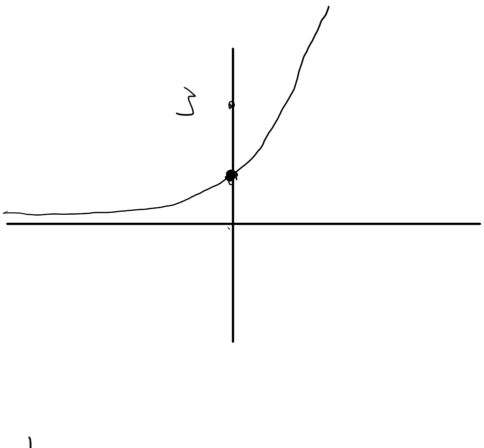
Bsp

$$f'(x) = f(x) \qquad f(x) = e^x$$

$$\frac{d}{dx} f(x) = f(x) \qquad f'(x) = e^x$$

$$\Rightarrow \quad e^x = \underbrace{f(x) = f'(x)} = e^x$$

$$\Rightarrow \quad f(x) = C \cdot e^x$$



$$f(x) \rightsquigarrow (0,1)$$

$$f(0) = C \cdot \underbrace{e^0}_{1} = C \overset{!}{=} 1$$

$$\Rightarrow \quad \underline{\underline{f(x) = e^x}}$$

Bsp. $y'(x) = 0{,}02 \cdot y(x)$

I: Schreibe $y' = \frac{dy}{dx}$

$$\frac{dy(x)}{dx} = 0{,}02 \cdot y(x)$$

II: Separation
y nach links
x nach recht

$$dy = 0{,}02\, y \cdot dx$$

$$\frac{1}{y}\, dy = 0{,}02 \cdot dx$$

III: Integration

$$\int \frac{1}{y}\, dy = \int 0{,}02\, dx$$

IV: Konstante

$$\Rightarrow \ln(y) = 0{,}02x + C$$

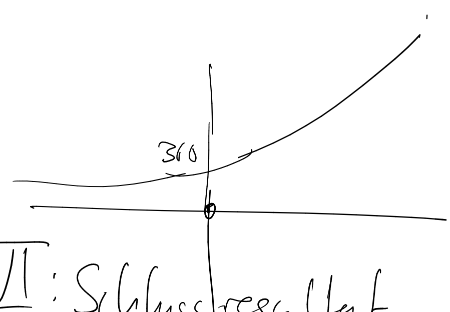
$e^{(\dots)}$

$$\Rightarrow y(x) = e^{0{,}02x + C} = e^{0{,}02x} \cdot \underbrace{e^{C_1}}_{C_2} = C e^{0{,}02x}$$

$$\Rightarrow y(x) = C e^{0{,}02x}$$

V: Anfangsbedingung mit Konstante
$y(0) = 300$

$$\Rightarrow y(0) = C \cdot \underbrace{e^{0{,}02 \cdot 0}}_{1} = 300$$



VI: Schlussresultat

$$\Rightarrow \underline{\underline{y(x) = 300 \cdot e^{0{,}02x}}}$$

$$y(0) = 300 \cdot \underbrace{e^{0,02 \cdot 0}}_{1} = 300$$ ✓ Anfangsbed?

$$y'(x) = 300 \cdot \boxed{e^{0,02x}} \cdot 0,02$$

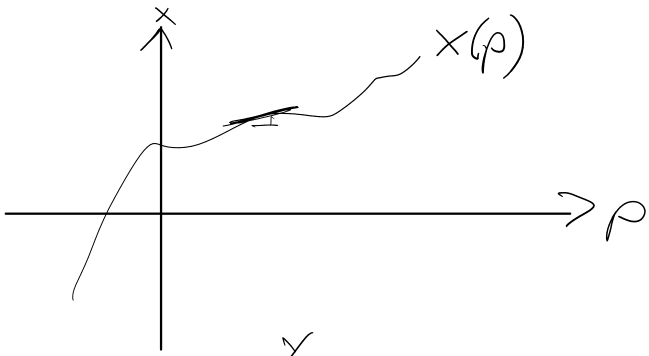
$$\overset{?}{=} 0,02 \cdot y(x)$$

DGL?

$$= 0,02 \cdot 300 \cdot \boxed{e^{0,02x}}$$ ✓

$$\boxed{\begin{aligned} y'(x) &= 307\, y \\ y(0) &= 2 \end{aligned}}$$

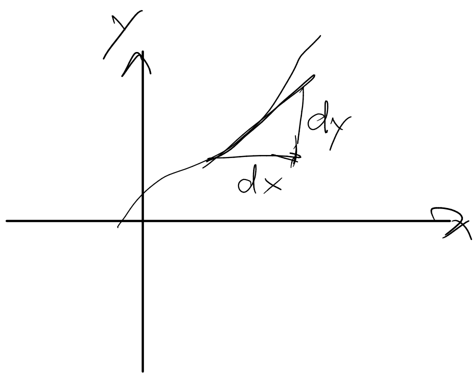
$$\boxed{\underbrace{x'(p)} \cdot \frac{p}{x(p)} = -1,1}$$



Elst

I: $\frac{dx}{dp} \cdot \frac{p}{x(p)} = -1,1$

Variable



$$\text{II:} \quad \overset{x}{} \frac{1}{x}\, dx = -1,1 \overset{p}{} \frac{1}{p}\, dp$$

III: $\int \frac{1}{x}\,dx = \int -1{,}1\ \frac{1}{p}\,dp$

$$\Rightarrow \ln(x) = -1{,}1\ \ln(p) + C$$

$$\Rightarrow x(p) = e^{-1{,}1 \cdot \ln(p) + C} = e^{-1{,}1 \ln(p)} \cdot \underbrace{e^{C}}_{C}$$

$$x(p) = \underbrace{e^{\ln(p)}}_{p}{}^{-1{,}1} \cdot C = \underline{\underline{C \cdot p^{-1{,}1}}}$$

d)	Betrachten Sie die folgende Differentialgleichung:	/3 P
	$$x'(p) \cdot \frac{p}{x(p)} = -1.1$$	
	Geben Sie für jede der folgenden Aussagen an, ob sie richtig oder falsch ist.	
	R F	
	☒ ☐ $x(p) = p^{-1.1}$ erfüllt die Differentialgleichung.	
	☒ ☐ $x(p) = 100p^{-1.1}$ erfüllt die Differentialgleichung.	
	☐ ☒ $x(p) = p^{-1.1} + 100$ erfüllt die Differentialgleichung.	
	☐ ☒ $x(p) = 100 - 1.1p$ erfüllt die Differentialgleichung.	

$$x' = -1{,}1\, p^{-2{,}1}$$

$$\stackrel{?}{=} \frac{x}{p}\,(-1{,}1)$$

$$= \frac{p^{-1{,}1} + 100}{p}\,(-1{,}1)$$

$$p^{-2{,}1} \sim p^{-2{,}1} + p^{-1}$$

$$x' = \frac{-1{,}1 \cdot x(p)}{p}$$

$x' - \frac{p}{x} = 1$	$x' \cdot \frac{p}{x} = 1$
$x' = 1 + \frac{p}{x}$	$x' = \frac{x}{p}$

$$x' \cdot \frac{p}{x} = 1 \qquad \Big|\Big| \cdot \frac{x}{p}$$

$$x' \cdot \frac{\cancel{p}}{\cancel{x}} \cdot \frac{\cancel{x}}{\cancel{p}} = 1 \cdot \frac{x}{p} \Rightarrow \underline{\underline{x' = \frac{x}{p}}}$$

$$x' = -1{,}1\,\frac{x}{p} \qquad x(p) = p^{-1{,}1}$$

$$x'(p) = -1{,}1\,p^{-2{,}1}$$

$$-1{,}1\,p^{-2{,}1} \overset{?}{=} -1{,}1 \cdot \boxed{\frac{p^{-1{,}1}}{p}} = -1{,}1 \cdot \underbrace{p^{-1{,}1} \cdot p^{-1}} = -1{,}1\,p^{-1{,}1+(-1)}$$

$$= -1{,}1\,p^{-2{,}1} \quad \checkmark$$

$\Rightarrow$ $x(p) = p^{-1{,}1}$ erfüllt DGL.

$$x' \cdot \frac{p}{x(p)} = -1{,}1 \quad \rightsquigarrow \quad x(p) = 100\,p^{-1{,}1}$$

$$x'(p) = -110\,p^{-2{,}1}$$

$$-110\,p^{-2{,}1} \cdot \frac{p}{100\,p^{-1{,}1}} \overset{?}{=} -1{,}1$$

$$\frac{p^{-2{,}1} \cdot p^{1}}{p^{-1{,}1}} \overset{?}{=} 1 \qquad \Bigg| \qquad \frac{p^{-2{,}1+1}}{p^{-1{,}1}} = p^{-2{,}1+1} \cdot p^{1{,}1} = p^{-2{,}1+1+1{,}1} = p^{0} = \underline{\underline{1}}$$

$$5x + 915 = -25$$

$$5x = -25 - 915$$

$$x = \frac{-25 - 915}{5}$$

$$5x \cdot 3y = 5$$

$x = \ldots$

$$5x = \frac{5}{3y}$$

$$x = \frac{\frac{\cancel{5}}{3y}}{\cancel{5}} = \underline{\underline{\frac{1}{3y}}}$$

$$7w \cdot z = 1$$

$w = \ldots$

$$w = \frac{1}{7z}$$

$$12 + 8yr = 25t - 7$$

$$(8r) \cdot y = 8yr = 25t - 19$$

$y = \ldots$

$$y = \frac{25t - 19}{8r}$$

$$y'(x) = x\,y(x)^2 + x \quad , \qquad \text{RW: } y(0) = 1$$

$$\rightsquigarrow \quad \frac{dy}{dx} = x\,y(x)^2 + x$$

$$\Rightarrow \quad dy = (x\,y(x)^2 + x) \cdot dx = x \cdot dx\,(y^2 + 1)$$

$$\Rightarrow \quad \frac{1}{y^2+1}\,dy = x \cdot dx$$

$$\Rightarrow \quad \int \frac{1}{y^2+1}\,dy = \int x\,dx = \frac{1}{2}x^2 + C$$

$$\arctan(y)$$

$$\boxed{y(0) \overset{!}{=} 1}$$

$$y = \underline{\underline{\tan\left(\frac{1}{2}x^2 + C\right)}}$$

$$\tan(0 + C) \overset{!}{=} 1 \Rightarrow \ldots$$

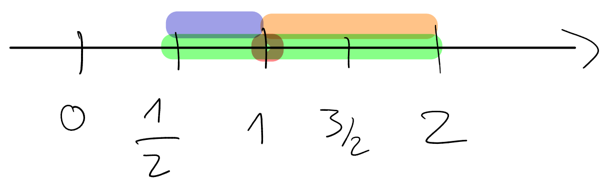
$$f(x) = -\frac{1}{2} + \frac{1}{2x^2} = -\frac{1}{2} + \frac{1}{2} x^{-2}$$

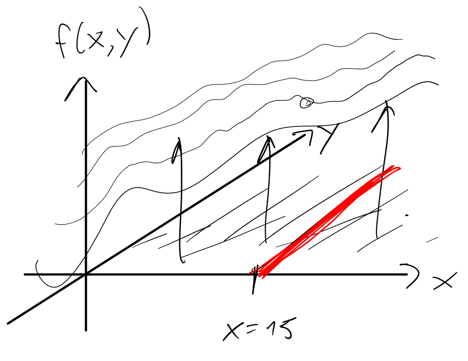
$$\Rightarrow \quad F(x) = -\frac{1}{2} x + \frac{1}{2} \frac{x^{-1}}{-1}$$

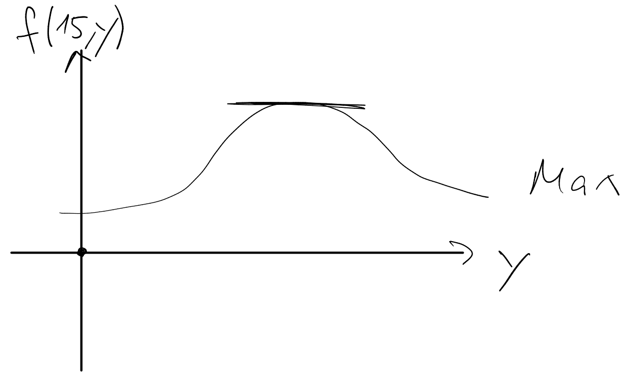
$$= \underline{\underline{-\frac{1}{2} x - \frac{1}{2} \frac{1}{x} + C}}$$

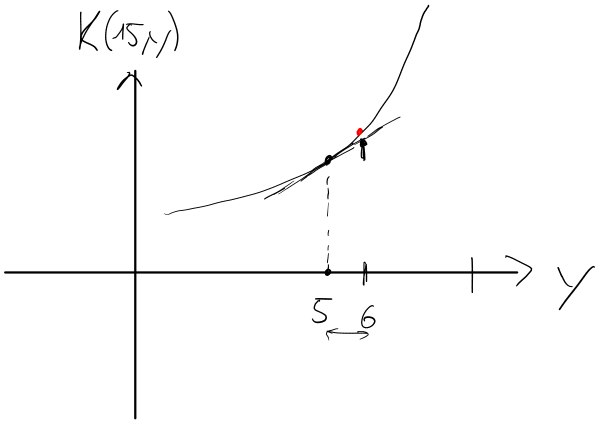
$$f(x) = -\frac{1}{2} + \frac{1}{2x^2} \stackrel{!}{=} 0 \qquad , -1, 1$$

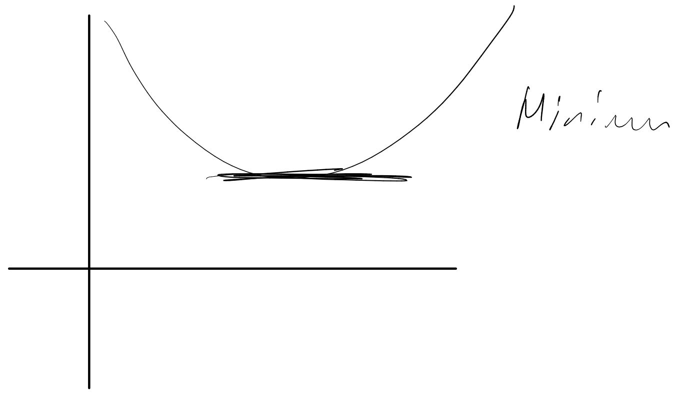
$$\frac{1}{2} \leq 2$$

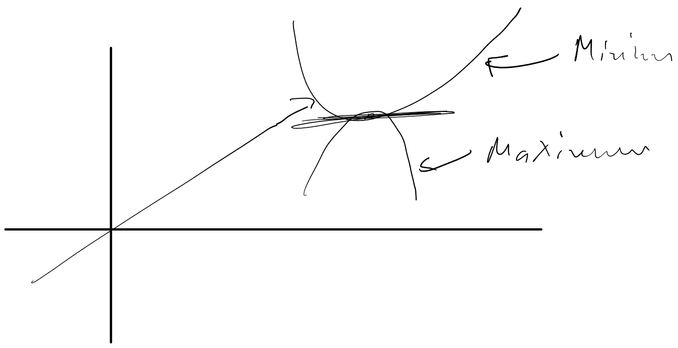












$$\frac{d}{dx}\,10x = 10 \cdot \underbrace{\frac{d}{dx}x}_{1} = 10$$

$$\frac{d}{dx}\,3xy^2 = 3y^2 \underbrace{\frac{d}{dx}x}_{1} = 3y^2$$

$K(x,y)$

$$\Rightarrow \left| \begin{matrix} K_x = 0 \\ K_y = 0 \end{matrix} \right|$$

2 unbekannte, 2 Gleichungen

$$K(x,y) = 20 + 10x + 0.5x^2 + 15y + 0.5y^2 - 0.3xy$$

$$E(x,y) = 20x + 30y$$

$$G(x,y) = E(x,y) - K(x,y)$$

$$G(x,y) = 10x + 15y - 0{,}5x^2 - 0{,}5y^2 + 0{,}3xy - 20$$

$$\left| \begin{array}{l} \frac{\partial G}{\partial x}(y) = 10 - x + 0{,}3y \overset{!}{=} 0 \\ \frac{\partial G}{\partial y}(x) = 15 - y + 0{,}3x \overset{!}{=} 0 \end{array} \right|$$

$$\Rightarrow \quad (x;y) = (15{,}9;\ 19{,}8) \rightsquigarrow \underline{\text{Extremalstelle}}$$

$$\boxed{\frac{\partial^2 G}{\partial x^2} = -1} \quad , \quad \frac{\partial^2 G}{\partial x \partial y} = 0{,}3 = \frac{\partial^2 G}{\partial y \partial x} = 0{,}3$$

$$\boxed{\frac{\partial^2 G}{\partial y^2} = -1}$$

Sattel
max
min

Lokale Extremwerte bei Funktionen mit zwei unabhängigen Variablen:	
Stationäre Stelle $P_0 = (x_0, y_0)$, falls:	$f_x(x_0,y_0) = 0 \wedge f_y(x_0,y_0) = 0$ (x_0, y_0)
P_0 ist ein Sattelpunkt, falls:	$f_{xx}(x_0,y_0) \cdot f_{yy}(x_0,y_0) < \left(f_{xy}(x_0,y_0)\right)^2$ ✗
P_0 ist ein Extremum, falls:	$f_{xx}(x_0,y_0) \cdot f_{yy}(x_0,y_0) > \left(f_{xy}(x_0,y_0)\right)^2$
– und zwar ein lokales Maximum, falls zudem	$f_{xx}(x_0,y_0) < 0$
– und zwar ein lokales Minimum, falls zudem	$f_{xx}(x_0,y_0) > 0$

$$\underbrace{G_{xx}(15,19)}_{-1} \cdot \underbrace{G_{yy}(15,19)}_{-1} \overset{?}{<} \left[G_{xy}(15,19)\right]^2$$

$$\underbrace{-1 \cdot -1}_{1} \overset{?}{<} 0{,}09$$

$$1 \overset{?}{<} 0{,}09$$

↯ ⇝ kein Sattelpunkt

$G_{xx}(15, 19) = -1 \overset{?}{<} 0$ ✓ $\rightsquigarrow$ Maximum

Bsp.

$$\boxed{f(x,y) = x^3 - 3x + 3xy^2}$$

Stationäre Stelle

I: Funktion f nach ableiten, und f nach y ableiten

II: $f_x \overset{!}{=} 0 \quad \wedge \quad f_y \overset{!}{=} 0$

III: Gleichungssystem lösen

IV: Lösungen des Gleichungssystems sind stationäre Stellen

Bsp

I: $f_x(y) = 3x^2 - 3 + 3y^2$

$f_y(x) = 3x \cdot 2y = 6xy$

II: $\left| \begin{array}{l} 3x^2 - 3 + 3y^2 = 0 \\ 6xy \overset{!}{=} 0 \end{array} \right|$

$(0,1)$

$x = 0$ $\qquad$ $y = 0$

$3y^2 = 3$ $\qquad$ $3x^2 = 3$

$y^2 = 1$ $\qquad$ $x^2 = 1$

$y = \pm 1$ $\qquad$ $x = \pm 1$

~~$(0,0)$~~

$\left.\begin{array}{l} (0,1) \\ (0,-1) \\ (1,0) \\ (-1,0) \end{array}\right\}$ 4 Stat. Stelle

Max, Min, Sattelpunkt

I: f_{xx}, f_{yy}, $f_{xy} = f_{yx}$

II: Stationäre Pkt. einsetzen

III: Siehe Schema

Bsp. $f_{yy} = 6x$, $f_{xy} = 6y$, $f_{xx} = 6x$

$(1,0) \rightsquigarrow \cancel{0 \cdot 0} < 0$

3.) Integrale