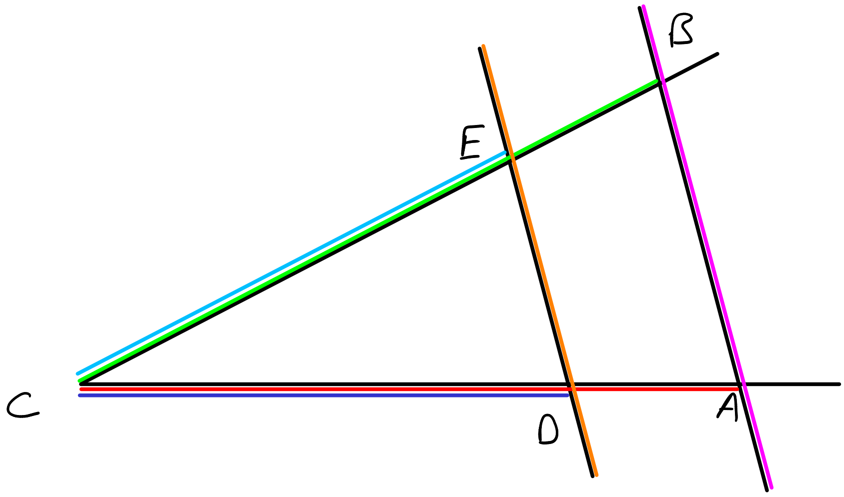
Proportionality Theorems



$\overline{ED} \parallel \overline{AB} \Rightarrow \dfrac{\overline{CA}}{\overline{CD}} = \dfrac{\overline{CB}}{\overline{CE}}$	First Proportionality Theorem
$\dfrac{\overline{CA}}{\overline{CD}} = \dfrac{\overline{CB}}{\overline{CE}} \Rightarrow \overline{ED} \parallel \overline{AB}$	Inverse of First Proportionality Theorem
$\overline{ED} \parallel \overline{AB} \Rightarrow \dfrac{\overline{ED}}{\overline{AB}} = \dfrac{\overline{CE}}{\overline{CB}}$	Second Proportionality Theorem

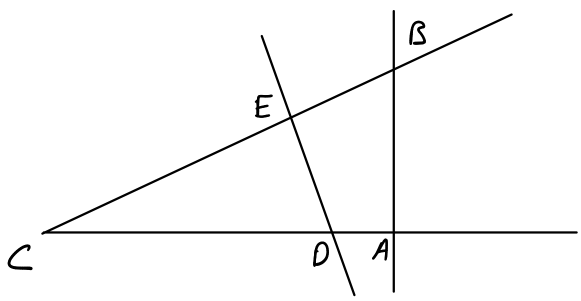
Proof by Contradiction

Our goal is to prove $A \Rightarrow B$. In this case we will do this via a proof by contradiction, where we assume A to be true and B to be false $A \wedge \neg B$ We then try to derive a contradiction.

Let us prove the inverse of the First Proportionality Theorem by contradiction. It is important to note that we have already shown the First Proportionality Theorem and will make us of it in this proof. Assume that

$$\boxed{\frac{\overline{CA}}{\overline{CD}} = \frac{\overline{CB}}{\overline{CE}}} \text{ and } \boxed{\overline{ED} \nparallel \overline{AB}}.$$

In words we assume that the equality of ratios are true while the lines are not parallel to each other. The Sketch below describes our initial situation.



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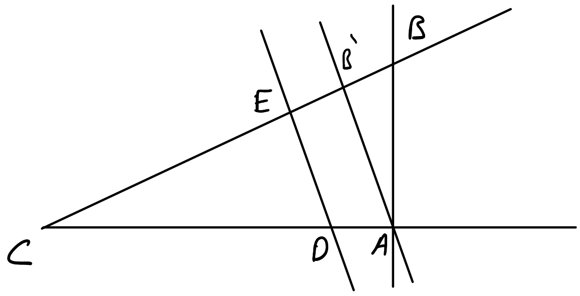
We now try to derive a contradiction by translating DE through A and getting the intersection point B' as shown in the sketch below.



We have simply constructed a new line which is parallel. We can now use the First Proportionality Theorem as we have assumed it to be true at the beginning of this proof. It says that if the lines are parallel, we get the relation

$$\frac{\overline{CA}}{\overline{CD}} = \frac{\overline{CB'}}{\overline{CE}}. \qquad (1)$$

Note that we did NOT use the inverse of the First Proportionality Theroem in this step. We have constructed a parallel line and used the First Proportionality Theorem which states that for parallel lines we can use the equality of the ratios. In this proof however, we want to show that if the equality of ratios are given, then they are parallel, which is exactly the inverse statement.

Don't forget the assumption we made at the beginning. We said that

$$\frac{\overline{CA}}{\overline{CD}} = \frac{\overline{CB}}{\overline{CE}} \qquad (2)$$

is true. We can now cast Equation 1 into Equation 2 resulting in

$$\frac{\overline{CB'}}{\overline{CE}} = \frac{\overline{CB}}{\overline{CE}}$$

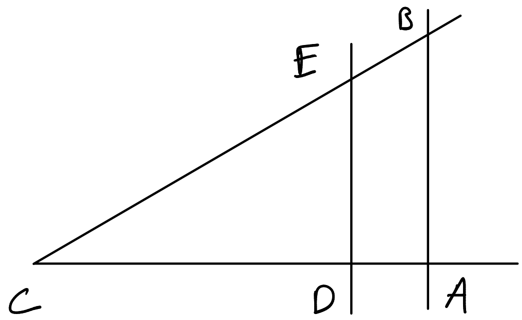
and hence $\overline{CB'} = \overline{CB}$. We know that B and B' are on the same ray and therefore they must be identical. But if they were identical, then the lines were already parallel in the first place which contradicts our assumption, that the lines are not parallel. This is a contradiction and therefore proving the inverse of the First Proportionality Theorem.

$\square$

We now show a counterexample that the inverse of the Second Proportionality Theorem is not true. First, let us remind ourselves what the inverse would be:

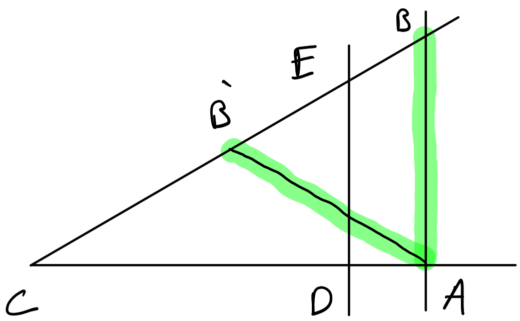
$$\boxed{\frac{\overline{ED}}{\overline{AB}} = \frac{\overline{CE}}{\overline{CB}}} \Longrightarrow \boxed{\overline{ED} \parallel \overline{AB}}.$$

Let us sketch a situation where the equality of the ratios holds. For this we simply us the Second Proportionality Theorem to constuct one. We remind ourselves that the assumption for the Second Proportionality Theorem was, that the lines are parallel to each other.



Just to be clear: $\boxed{\dfrac{\overline{ED}}{\overline{AB}} = \dfrac{\overline{CE}}{\overline{CB}}}$ is already fulfilled because the lines are parallel!

Let's translate Point B on the ray where CE while keeping the same distance $\overline{AB}$. This does certainly not violate the equality of ratios.



This is not possible in general, however we are trying to construct a counterexample which proves that the inverse of the Second Proportionality Theorem is not always true. Looking at the sketch above we see that the green lines are not parallel. We have therefore found a counterexample.

What have we actually done?
Imagine your friend telling you that all birds can fly. You can then disprove his statement by mentioning that the emperor penguin is not capable of flying. You therefore falsified the statement that all birds can fly. You however did not say anything about the other birds which of course can fly. In the counterexample above we have done something similar. We showed that the inverse of the Second Proportionality Theorem in general is not true. We however did not make any statement whether there might be a special case where this.

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