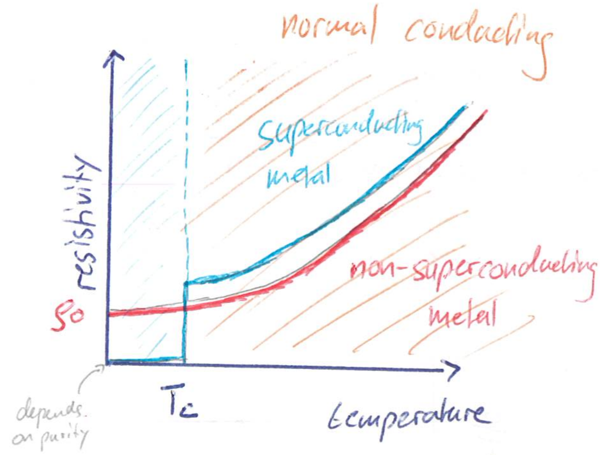
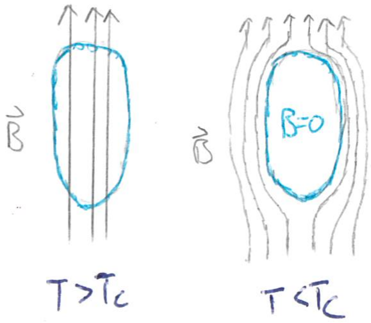
Superconductivity

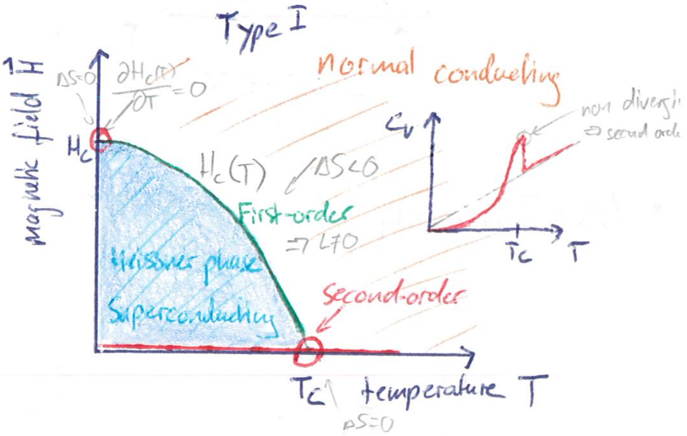
The two basic phenomena:

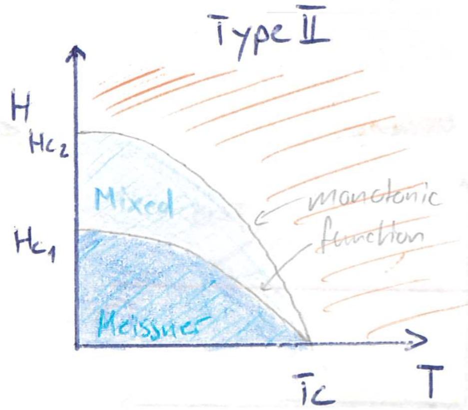
- Electrical resistivity vanishes
- Perfect diamagnetism
  (Meissner-Ochsenfeld effect)

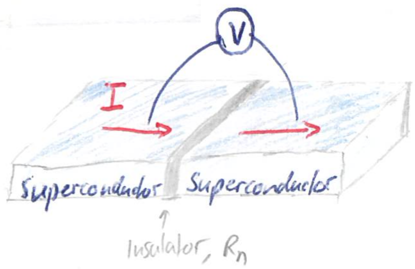




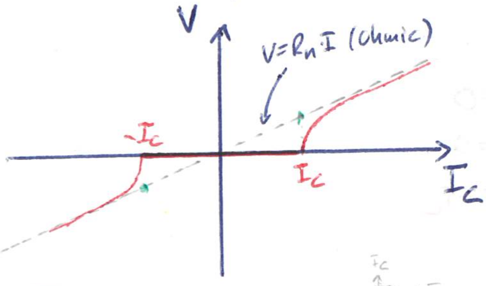
Irrespective whether ZFC or FC.





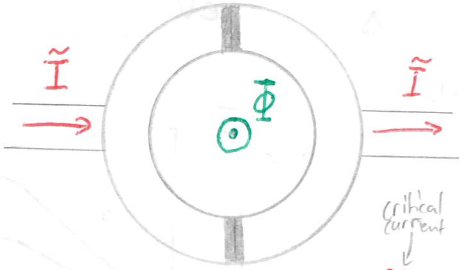


Josephson junction



$$I_c R_n \sim \frac{k_B T_c}{e} \begin{cases} 7\left(1 - \frac{T}{T_c}\right) & T \lesssim T_c \\ 3 & T \to 0 \end{cases}$$

Apply finite $|V| < V \Rightarrow$ ac current with $\omega = \frac{2e}{\hbar} V$
(Universal ac-Josephson effect)

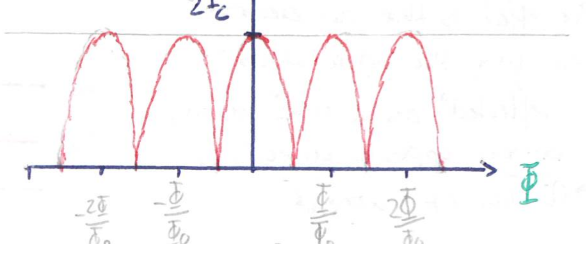


SQUID

$\Phi_0 = \frac{hc}{2e}$ Cooper pair $2e^-$

critical current

$$\tilde{I}_c(\Phi) = 2 I_c \left|\cos\left(\frac{\pi \Phi}{\Phi_0}\right)\right|$$

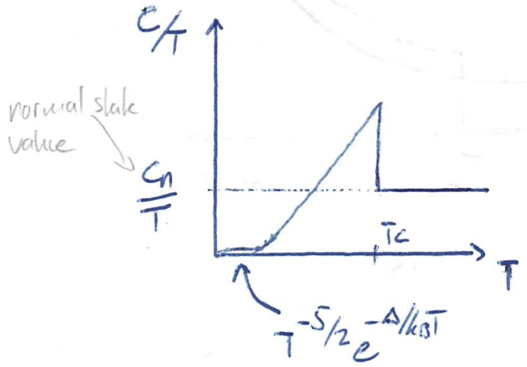


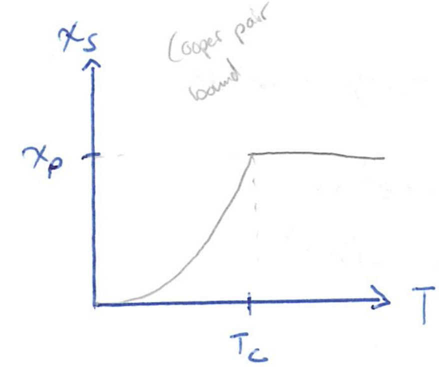
• Flux quantization: The magnetic flux which can thread the loop is quantized. $\sim \Phi_0$

$\Rightarrow$ current around the loop is quantized and cannot decay continuously.

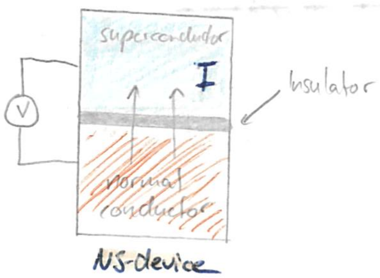
• Electronic energy spectrum has a gap: There are no excitations at very low energies.

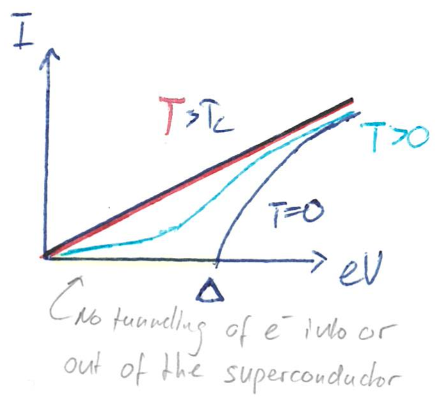
e^--contribution $\sim e^{-\Delta/k_BT}$

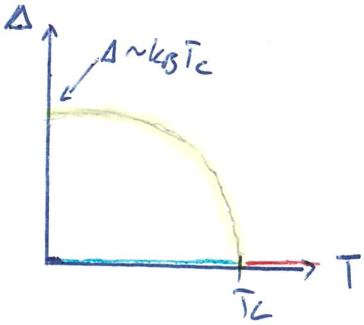


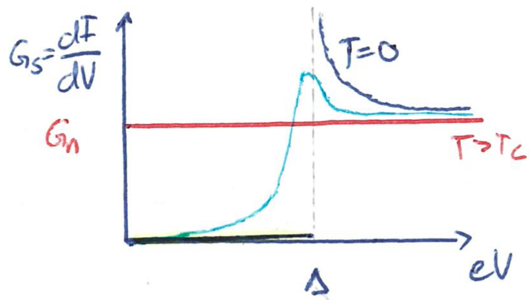


• NS-Tunneling:

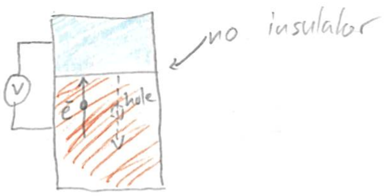


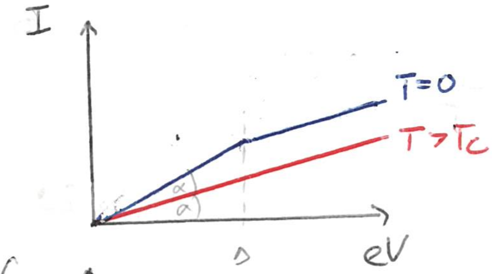


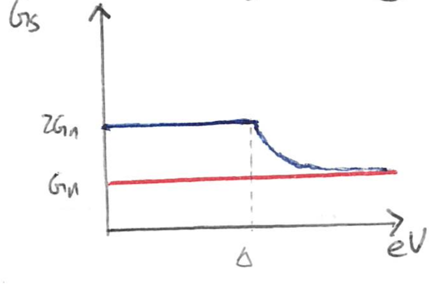




• Andreev reflection:







The effect is that an electron impinging the superconductor is "reflected" as a "hole" moving in exactly opposite direction.

$\rightarrow$ Doubling of current

Thermodynamic properties

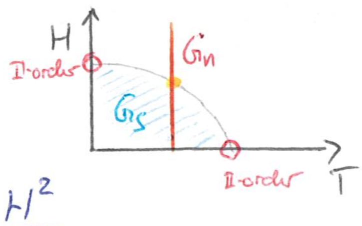
→ p.159 Lifshitz
p.349 Abrikosov

magnetic induction, applied field

$$dG = -S\,dT - \frac{1}{4\pi} B\,dH \qquad \text{Gauss units}$$

Assume $B=\mu H$, $\mu \approx 1$ (normal metal state) $\Rightarrow$ $B(H) = \begin{cases} H & \text{normal conducting} \\ 0 & \text{superconducting} \end{cases}$ (Type I)

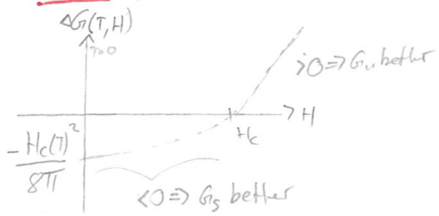
For T fix: $G(T,H) - G(T,0) \overset{T=\text{fix}}{=} -\frac{1}{4\pi}\int_0^H B(H')\,dH'$



$\vec{B}=0$: $G_S(T,H) - G_S(T,0) = 0$

$-\frac{1}{4\pi}\int_0^H \mu H' dH'$: $G_n(T,H) - G_n(T,0) = -\frac{H^2}{8\pi}$

Superconducting phase transition $\overset{\text{def}}{\Longleftrightarrow}$ $G_S(T,H_c) = G_n(T,H_c)$



$\Rightarrow \Delta G(T,H) = G_S(T,H) - G_n(T,H) = \frac{1}{8\pi}\left[H^2 - H_c(T)^2\right]$ — condensation energy

For $H=0 \Rightarrow \Delta G(T,0) = -\frac{H_c(T)^2}{8\pi} \overset{!}{\underset{T=T_c}{=}} 0 \Rightarrow H_c(T_c) = 0$

$S = \left(-\frac{\partial G}{\partial T}\right)_H \Rightarrow \Delta S(T,H) = \underbrace{S_S(T,H)}_{-\frac{\partial}{\partial T}\frac{1}{8\pi}H^2 = 0} - S_n(T,H) = \frac{1}{4\pi} H_c(T) \frac{dH_c(T)}{dT} \xrightarrow[T\to 0]{\text{III. Law}} 0$

(∂_T linear; G_S = const; $\neq 0$, $C = T\frac{\partial G}{\partial T}$; $\frac{dH_c}{dT} \to 0$ and hence)

$\frac{C}{T} = -\frac{\partial^2 G}{\partial T^2}\Big|_H \Rightarrow \Delta C(T,H) = \underbrace{C_S(T,H)}_{=0} - C_n(T,H) = \frac{T}{4\pi}\left[\left(\frac{dH_c}{dT}\right)^2 + H_c \frac{d^2 H_c}{dT^2}\right]$

(∂_T^2 linear; finite at T_c; would be 0 for I order)

use ΔS for distinguishing

phase transition $= \begin{cases} \text{I order}; & T = 0, T_c \text{ as } \Delta S = 0 \text{ and } \Delta C > 0 \\ \text{II order}: & \text{elsewhere as } \Delta S < 0 \Rightarrow L = T_c \Delta S \end{cases}$

T=0 case is second order by III law thermodynamics, no Cv consideration

II-order: e^- contribution $\Rightarrow C_n(T,0) = \gamma T$, where $\gamma = \pi^2 \nu(\varepsilon_F) k_B^2 / 3$ (sommerfeld coefficient)

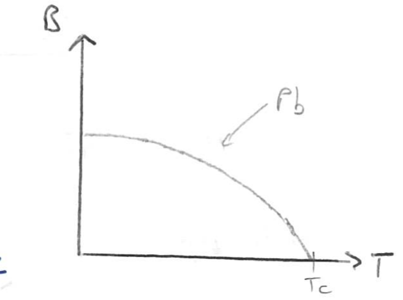
$$\Rightarrow \Lambda \sim \frac{\Delta C}{C_n}\Big|_{T=T_c} = \frac{1}{4\pi\gamma}\Big(\underbrace{\frac{dH_c}{dT}}_{\sim -2H_c(0)/T_c}\Big)^2_{T=T_c} > 0 \quad {=1{,}43 \atop \sim 1} \Rightarrow H_c(0)^2 \sim \pi\gamma T_c^2$$

can be "seen" geometrically

Specific heat:

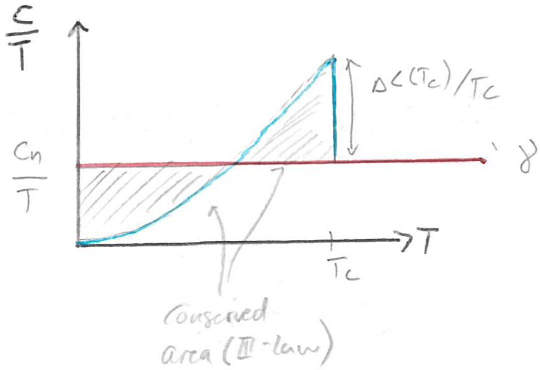
Assume $H_c(T) = H_c(0)\left[1-\left(\frac{T}{T_c}\right)^2\right]$

$$\Rightarrow C_s(T,0) = \gamma T - \frac{H_c(0)^2 T}{2\pi T_c^2}\left[1-3\left(\frac{T}{T_c}\right)^2\right], \quad 0 \le T < T_c$$



Cancel e^- contribution T-linear Term $\Rightarrow$

$$C_s(T,0) = 3\gamma T_c\left(\frac{T}{T_c}\right)^3$$



Meissner-Ochsenfeld effect - London theory

p. 329 Abrikosov
p. 4 Tinkham

After London:

I: $n_0 = n_n + n_s$ (two fluid model)

II: n_s ideal incompressible fluid, $\vec{v}(\vec{r},t)$, $\vec{\nabla}\cdot\vec{v} = 0$

$\vec{j}(\vec{r},t) = n_s e \vec{v}(\vec{r},t)$ (dissipation free supercurrent)

Drude maxwell $\rightarrow$ III: $\vec{w}(\vec{r},t) = \underbrace{\text{rot}\,\vec{v}(\vec{r},t)}_{\text{"existing" vortices}} + \underbrace{\frac{e}{m_e c}\vec{B}(\vec{r},t)}_{\text{"driver of" vortices}} = 0$ for $\vec{r} \in$ supercondu

Screening:

$\vec{w}=0 \Rightarrow$

$$\underbrace{\text{rot}\,\vec{j} = -\frac{n_s e^2}{mc}\vec{B}}_{\text{london equation}} \qquad (\text{II}, \text{III})$$

london penetration depth

$\omega_p = \frac{4\pi n_0 e^2}{m}$

Maxwell $\Rightarrow$ $$\nabla^2\vec{B} = \lambda^{-2}\vec{B}, \quad \lambda^{-2} = \frac{4\pi n_s e^2}{mc^2} = \frac{\omega_p^2}{c^2}\frac{n_s}{n_0}$$

I No assumption about T!
$\Rightarrow$ Valid for all $T<T_c$

Casimir Gorter

$$n_s(T) = n_0\left[1 - \left(\frac{T}{T_c}\right)^4\right]$$

$\Rightarrow$ $T=0$, $\lambda = \delta = \frac{c}{\omega_p}$ (Debye length)
$T=T_c$, $\lambda = \infty$

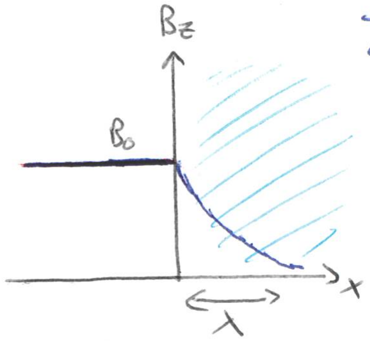
Lec. 18 solid state theory

Application: superconducting half space

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ H_0 \end{pmatrix} \Rightarrow \frac{d^2 B_z}{dx^2} - \theta(x)\lambda^2 B_z = 0$$

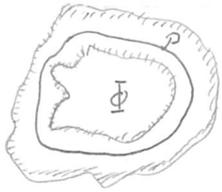
$$\Rightarrow B_z(x) = \begin{cases} B_0 = H_0 & x<0 \\ B_0 e^{-x/\lambda} & x \geq 0 \end{cases}$$ and screened by

maxwell $\rightsquigarrow$ $$j_y(x) = \begin{cases} 0 & x<0 \\ \frac{cB_0}{4\pi\lambda}e^{-x/\lambda} & x\geq 0 \end{cases}$$, would emerge in a perfect conductor



Persistent current:

P path is deep ($\vec{j}=0$) and F is area encircled



$$\frac{mc}{e}\int_F d\vec{S}\cdot\vec{w} \overset{\text{Stokes, def } w}{=} \oint_P d\vec{s}\cdot\Big[\vec{A} + \underbrace{\cancel{\tfrac{mc}{e}\vec{v}}}_{0,\ j=0}\Big] = \Phi$$

$$\overset{\vec{p}=m\vec{v}+\frac{e}{c}\vec{A}}{=} \frac{c}{e}\oint_P d\vec{s}\cdot\vec{p} \overset{\text{Bohr Sommerfeld}}{=} \frac{c}{e}h\cdot n, \quad n=0,\pm 1,\pm 2,\ldots$$

Aharanov-Bohm quantization

$e \Longleftrightarrow 2e$

Ginzburg-Landau theory

p. 382 Abrikosov

p. 178 Lifshitz

p. 104 Tinkham

OP has to vanish above T_c and grow continuously below. Further, spontaneous symmetry breaking is required.

Ginzburg, Landau $\Longrightarrow$ Ψ as OP with $|\Psi(\vec{r},T)|^2 = \tilde{\gamma}\, n_s(\vec{r},T)$

describes a charged superfluid and transforms as

(3.177) Blatter QM

$$\hat{\Phi}_\chi \Psi(\vec{r},T) = \Psi(\vec{r},T)\, e^{i\chi(\vec{r})}, \qquad \hat{K}\Psi(\vec{r},T) = \Psi(\vec{r},T)^*$$

$$\hat{\Phi}_\chi \vec{A}(\vec{r}) \overset{\text{"def."}}{=} \vec{A}(\vec{r}) + \frac{\hbar c}{e^*}\vec{\nabla}\chi(\vec{r}), \qquad \hat{K}\vec{A}(\vec{r}) = -\vec{A}(\vec{r})$$

p. 124 Blatter QM

U(1)-operation — Time reversal

18.34 QM Blatter

$$F[\Psi,\vec{A};T] = F_N(T) + \int d\vec{r}\left[a|\Psi(\vec{r})|^2 + \frac{b}{2}|\Psi(\vec{r})|^4 + \frac{1}{2m^*}|\vec{\Pi}\Psi(\vec{r})|^2 + \frac{\vec{B}(\vec{r})^2}{8\pi}\right]$$

free energy of normal conductor

$\frac{p^2}{2m}$

covariant Gradient, maintains Gauge invariance

The free energy functional is a scalar under all valid symmetry operations. $T\sim T_c$ $\Rightarrow$ OP small

Variation $\Psi \Longrightarrow$ $$\frac{1}{2m^*}\vec{\Pi}^2\Psi + a\Psi + b|\Psi|^2\Psi = 0,$$

$$\vec{\Pi} = \frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}(\vec{r})$$

use functional derivative $\vec{A} \Longrightarrow$ $$\frac{e^*}{2m^* c}\left[\Psi^*\vec{\Pi}\Psi + \Psi(\vec{\Pi}\Psi)^*\right] - \frac{1}{4\pi}\operatorname{rot}\vec{B} = 0$$

Ginzburg-Landau equations (17.6) Abrikosov

p. 181 Lifshitz

BC: $\vec{n}\cdot(\vec{\Pi}\Psi(\vec{r}))\big|_{\text{boundary}} = 0 \quad \Rightarrow \quad \vec{n}\cdot\vec{j} = 0$

$$\vec{n}\times[\vec{B}(\vec{r}) - \vec{H}(\vec{r})]\big|_{\text{boundary}} = 0$$

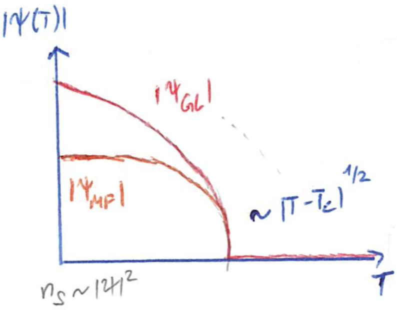
Uniform superconducting phase

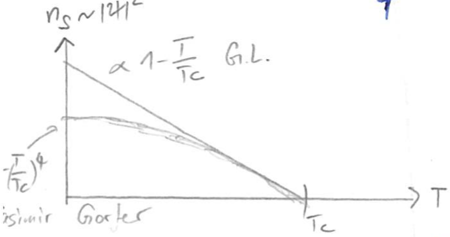
$\vec{H}=0,\ \vec{\Pi}\psi=0 \Rightarrow a(T)\psi + b|\psi|^2\psi = 0$

$G\vec{A}=0$. uniform $\Rightarrow \nabla\psi=0$ already $\partial_\phi=0$

$= a'\left(\frac{T}{T_c}-1\right)$

$$\Rightarrow |\psi(\vec{r},T)|^2 = \begin{cases} 0 & , T>T_c \\ \frac{a'(T_c-T)}{bT_c} & , T\leq T_c \end{cases}$$





Only fixes absolute value and not the phase $\psi=|\psi|e^{i\phi}$, $\phi\in[0,2\pi)$. In the equilibrium state the superconductor spontaneously chooses a phase constant everywhere. Spontaneous $U(1)$ Gauge symmetry violation.

$\Rightarrow |\psi(T)| \propto \left(1-\frac{T}{T_c}\right)^{1/2}$, $\beta = 1/2$, $|\psi|^2 \propto 1-\frac{T}{T_c}$, $n_s(T) \overset{\text{casimir gorter}}{\propto} 1-\left(\frac{T}{T_c}\right)^4 \xrightarrow{T\to T_c} \overset{\text{correct}}{\propto}\left(1-\frac{T}{T_c}\right)$ 4

see script

Close to T_c, $f(T,H=0) = f_n(T,0) - \frac{a^2}{2b} \underset{(2.7)}{=} f_n(T,0) - \frac{H_c(T)^2}{8\pi}$

use $|\psi|^2 = -\frac{a}{b}$ in $F[\psi]$ ← condensation energy

$H_c(T) = \sqrt{\frac{4\pi}{b}}\,a$

$H_c \sim \pm a$

$\Rightarrow H_c(T) \propto 1-\frac{T}{T_c}$ ← good behaviour as anticipated in $H_c(T) \propto 1-\left(\frac{T}{T_c}\right)^2 \xrightarrow{T\to T_c} 1-\frac{T}{T_c}$

$S(T) = -\frac{\partial f}{\partial T} = S_n(T) + \frac{aa'}{bT_c} \overset{\text{above}}{=} S_n(T) + \frac{a'^2}{bT_c}\left(\frac{T}{T_c}-1\right)$ is continuous but

$$\frac{C}{T} = \frac{\partial S}{\partial T} = \begin{cases} \gamma & T>T_c \\ \gamma + \frac{a'^2}{bT_c^2} & T\leq T_c \end{cases}$$

metallic contribution, low temperature

$\Rightarrow \frac{\Delta C}{C_n} = \frac{a'^2}{b\gamma T_c^2} = 1{,}43$ (2.13) Sigrist (2.59) Tinkham

London equation and flux quantization $\qquad \nabla\psi = 0$

$\text{rot}\,\vec{B} = \frac{4\pi}{c}\vec{j}$

Using the Maxwell equation in the Ginzburg Landau equation

solve for rot B, do multiplication

factor of 2 somewhere from cooper pair??

assume $\psi = \text{const}$

covariant gradient

p. 181 Lifshitz

p 438 Blatter QM

$$\vec{j}(\vec{r}) = \frac{e^*\hbar}{2m^* i}\left\{\psi^*\vec{\nabla}\psi - \cancel{\psi\vec{\nabla}\psi^*}\right\} - \frac{e^{*2}}{m^* c}|\psi|^2\vec{A}$$

c because $\frac{4\pi}{c}$

rot

$$\Rightarrow \quad \nabla^2\vec{B} = \underbrace{\frac{4\pi e^{*2}}{m^* c^2}|\psi|^2}_{\lambda^{-2}}\vec{B} \quad \overset{\text{will follow}}{\Rightarrow} \quad \lambda^{-2} = \frac{4\pi e^2}{mc^2} n_s$$

$e^{*2} = 4e^2$

$\gamma_s = \frac{1}{2}$

Flux quantization $\quad$ p 401 Abrikosov

Assume $\quad \psi(\vec{r}) = |\psi| e^{i\phi(\vec{r})}$, thick cylinder $\gg \lambda$

$$\Rightarrow \quad \vec{j}(\vec{r}) = \frac{e^*\hbar}{m^*}|\psi|^2\vec{\nabla}\phi(\vec{r}) - \frac{e^{*2}}{m^* c}|\psi|^2\vec{A}(\vec{r}) = \frac{e^*\hbar}{m^*}|\psi|^2\left\{\vec{\nabla}\phi - \frac{e^*}{\hbar c}\vec{A}\right\}$$

integrate over cylinder

$$\Rightarrow \quad \oint_P d\vec{s}\cdot\vec{A} + \frac{m^* c}{e^{*2}}\cancel{\oint_P d\vec{s}\cdot\frac{\vec{j}}{|\psi|^2}} = \frac{\hbar c}{e^*}\underbrace{\oint_P d\vec{s}\cdot\nabla\phi}_{2\pi n}, \quad n = 0, \pm 1, \ldots$$

$j = 0$, $\to 0$

as ψ must be single valued, "periodic boundary"

$$\Rightarrow \quad \Phi = \Phi_0 \cdot n, \quad \Phi_0 = \frac{hc}{2e} \quad \Rightarrow \text{ energy barrier, persistent supercurrent}$$

p. 403 Abrikosov

Little-Parks effect

Cylinder with wall $\ll \lambda$ $\Rightarrow$ flux not quantized, $H = \frac{\Phi}{\pi R^2}$ applied

$$|\vec{j}| = \frac{e\hbar}{m}|\psi|^2\frac{1}{2\pi R}\oint_P d\vec{s}\cdot\left[\vec{\nabla}\phi - \frac{2e}{\hbar c}\vec{A}\right] = \frac{e\hbar}{mR}|\psi|^2\left(n - \frac{\Phi}{\Phi_0}\right)$$

Radius cylinder

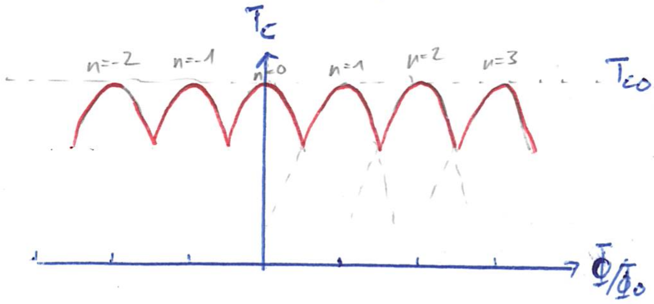
$$\frac{1}{2m^*}|\vec{\pi}\psi|^2 \xrightarrow[\text{average}]{\text{spatial}} \frac{\hbar^2|\psi|^2}{4mR^2}\left(n - \frac{\phi}{\phi_0}\right)^2$$

$$\Rightarrow \quad \bar{f} = \underbrace{a|\psi|^2 + \frac{b}{2}|\psi|^4 + \frac{\hbar^2|\psi|^2}{4mR^2}\left(n - \frac{\Phi}{\Phi_0}\right)^2}_{} + \frac{H^2}{8\pi}$$

unrenormalized

$$a + \frac{\hbar^2}{4mR^2}\left(n - \frac{\Phi}{\Phi_0}\right) \overset{!}{=} 0 \quad \Rightarrow \quad T_c = T_{c_0}\left[1 - \frac{\hbar^2}{4mR^2 a'}\left(n - \frac{\Phi}{\Phi_0}\right)^2\right]$$

The critical temperature is reduced by the magnetic flux through the cylinder in an oscillatory way.



follow the highest possible T_c by choosing the appropriate winding number n.

Spatial dependent order parameter - coherence length   p.390 Abrikosou
p.184 Lifschitz

$\vec{A}=0$

$$\psi(x=0)=0 \quad , \quad \psi(x\to\infty)=\psi_0=\sqrt{\frac{-a}{b}}$$

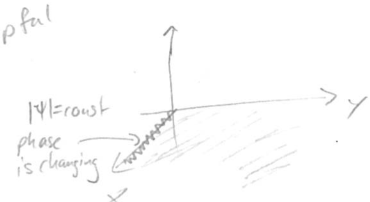
$$\psi(\vec{r})=\psi_0 f(x) \xRightarrow{G.L.E.} \frac{\hbar^2}{4ma} f''(x)-f(x)+f(x)^3=0$$

$$\Rightarrow f(x)=\tanh\left(\frac{x}{\sqrt{2}\xi}\right) \quad , \quad \xi(T)^2=\frac{\xi_0^2 T_c}{T_c-T} \Rightarrow \xi(T)\propto |T_c-T|^{-1/2} \; , \; \nu=1/2$$

$$\xi^2=-\frac{\hbar^2}{4ma}$$

Critical current in a thin film   p.173 Lifshitz helpful

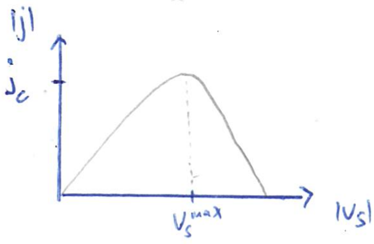
$$\text{Thin film}, \; \psi(\vec{r})=|\psi|e^{iqx} \Rightarrow \nabla\phi=(q,0,0)$$



$\frac{\Pi}{\hbar}$ "nev"

$$\vec{j}=\frac{e\hbar}{m}|\psi|^2\left(\nabla\phi-\frac{2e}{\hbar c}\vec{A}\right)=2e|\psi|^2\vec{v}_s$$

2, kürzen sich



$$\Rightarrow f=a|\psi|^2+\frac{b}{2}|\psi|^4+|\psi|^2 m\vec{v}_s^{\,2}$$

$$\xi^2=-\frac{\hbar^2}{4ma}$$

$\partial_\psi f=0$ $\Rightarrow$ solve for $|\psi|^2$

$$|\psi|^2=|\psi_0|^2\left[1-\left(\frac{2m\xi\vec{v}_s}{\hbar}\right)^2\right]$$

$$\Rightarrow \vec{j}=2e|\psi_0|^2\left[1-\left(\frac{2m\xi\vec{v}_s}{\hbar}\right)^2\right]\vec{v}_s$$

$$\Rightarrow v_s^{max}=\frac{1}{\sqrt{3}}\frac{\hbar}{2m\xi} \quad , \quad \phi'(x)=q_c=\frac{1}{2\sqrt{3}\xi} \Rightarrow q_c\xi\sim 1 \quad \text{critical current}$$

Londons critical current tied to condensation energy, reached if superfluid kinetic energy density coincides with condensation energy.

$$\Rightarrow j_c^{London}=\frac{\hbar e}{\sqrt{2}m\xi}n_s=\left(\frac{3}{2}\right)^{3/2} j_c$$

p. 364 Abrikosov
→ p. 190 Lifshitz

Two types of superconductors – mixed phase

Role of length scales in a superconductor

Ψ small $\Rightarrow$ $$\frac{1}{4m}\underbrace{\left(\frac{\hbar}{i}\vec{\nabla}-\frac{2e}{c}\vec{A}\right)^2}_{\pi^2}\Psi(\vec{r})+a\,\Psi(\vec{r})=0$$ (Ginzburg-Landau) Linear in Ψ

S.E
$a \leftrightarrow -E$
$e \leftrightarrow 2e$
$m \leftrightarrow 2m$

p. 458 Landau QM

$\vec{A}=(0,Hx,0)$
$\Rightarrow B=(0,0,H)$

$$\Longrightarrow\quad E_{n,k_z}=\frac{\hbar^2k_z^2}{4m}+\frac{\hbar e H}{mc}\left(\frac{1}{2}+n\right)=-a=-a'\left(\frac{T}{T_c}-1\right)$$

The highest critical temperature is given by E_{00}: $\dfrac{\hbar e H}{2mc}=a'\left(1-\dfrac{T}{T_c}\right)$

Lifshitz p. 151 $$\Longrightarrow\quad H_{c2}(T)=\frac{2mca'}{e\hbar}\left(1-\frac{T}{T_c}\right)=\frac{\Phi_0}{2\pi\xi^2}$$

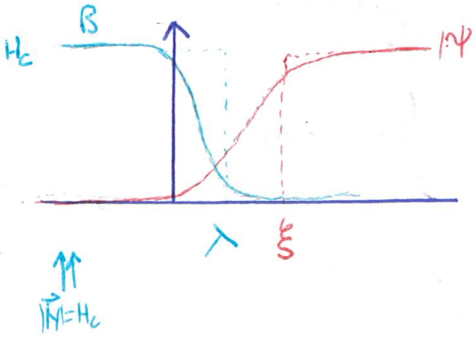
Compare to $$H_c(T)=\frac{\Phi_0}{2\sqrt{2}\pi\lambda\xi}$$

$$H_{c2}(T)=\kappa\sqrt{2}\,H_c(T)$$

where $\kappa=\dfrac{\lambda}{\xi}$ $\Rightarrow$ $\begin{cases}\text{type I}, & H_c>H_{c2} \quad \kappa<\frac{1}{\sqrt{2}}, \text{ first order}\\ \text{type II}, & H_c<H_{c2} \quad \kappa>\frac{1}{\sqrt{2}}, \text{ second order}\end{cases}$

Domain wall energy

p. 185 Lifshitz   p. 364 Abrikosov



approx.
$\Rightarrow$ $|\Psi|=\begin{cases}0 & x\le\xi\\ |\Psi_0| & \xi<x\end{cases}$, $|B|=\begin{cases}H_c & x\le\lambda\\ 0 & \lambda<x\end{cases}$

$$G_{DW}=G-G_0\approx(\xi-\lambda)\frac{H_c^2}{8\pi}=\begin{cases}>0, & \xi>\lambda \quad \text{type I}\\ <0, & \xi<\lambda \quad \text{type II}\end{cases}$$

Superconducting part of Gibbs in uniform

$\Rightarrow$ exact: $\xi=\sqrt{2}\lambda$, the ratio of length scales distinguishes the type

$\Rightarrow$ type I: avoid domain walls, type II: energetically beneficial

but both cases: loss of condensation energy.

Intermediate state for a type I superconductor

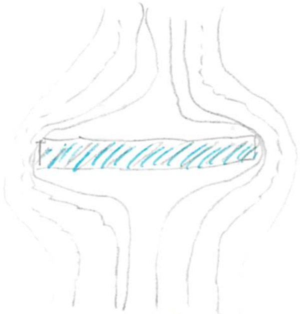
Superconducting sphere $\Rightarrow \vec{H}(\vec{r}) = \vec{H}_0 + \frac{H_0 R^3}{2} \vec{\nabla} \frac{\cos\theta}{r^2}$

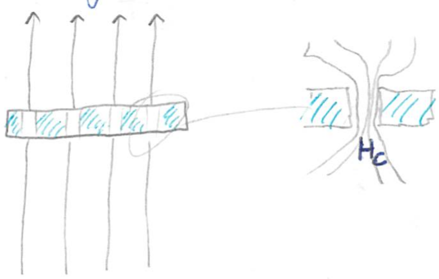
tangential field: $H_\theta(\vec{r})\big|_{r=R} = \frac{3}{2} H_0 \sin\theta$

$\Rightarrow H_0 = \frac{2}{3} H_c$ at the equator $\Rightarrow$ superconductivity disappears there

at the expense of creating domain walls

$\rightsquigarrow$ demagnetization effect





field lines are compressed to generate H_c

No wandering to the center, abrupt change

Meissner state

intermediate state

Mixed state for type II superconductor

p. 195 Lifshitz

p. 420 Abrikosov...

Here $G_{\Omega N} < 0$, $H_{c2} > H_c$, OP grows continuously, magnetic field enter the s.c. as vortices in the mixed state. Find a nice argument why vortices

Vortex as a flux line

$\Psi(\vec{r}) = |\Psi_0| g_n(r) e^{in\theta}$ cylindrical coordinate, n phase winding number

$$\overset{\text{G.L.}}{\Longrightarrow} \quad g_n(r) = \begin{cases} \propto r^n & r \ll \xi \quad , \quad A(r) \approx \frac{\pi}{\Phi_0} H(0)\, r \\ \propto 1 - e^{-x\sqrt{2}/\xi} & r \gg \xi \quad , \quad A(r) \approx \frac{n}{r} + \delta A(r) \end{cases}$$

$\hookrightarrow$ leads to exponential suppression

$$\Phi = \oint_{r \gg \xi} \vec{A} \cdot d\vec{s} = \frac{\Phi_0}{2\pi} \oint r \frac{n}{r} d\phi = n \Phi_0 \quad , \quad n = \pm 1 \text{ energetically favourable}$$

p. 409 Abrikosov

def

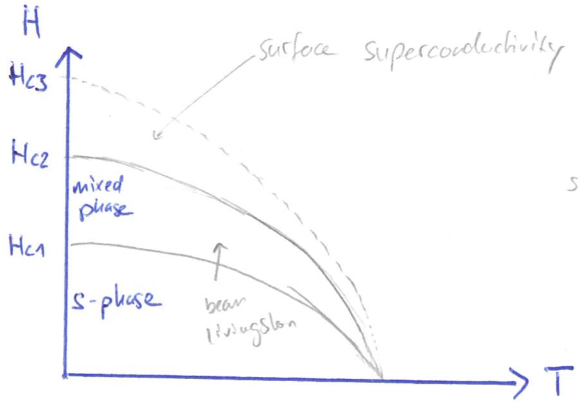
$\overset{\text{approx}}{\Longrightarrow} \quad g_1(r) = \tanh\left(\frac{r}{\xi}\right)$

supercurrent

$$\Rightarrow \quad j_\theta(r) = \frac{e\hbar}{m} |\Psi_0|^2 \tanh\left(\frac{r}{\xi}\right)\left(\frac{1}{r} - \frac{2\pi}{\Phi_0} A_\theta(r)\right) \overset{\text{maxwell}}{\Longrightarrow} \text{fancy London equation}$$

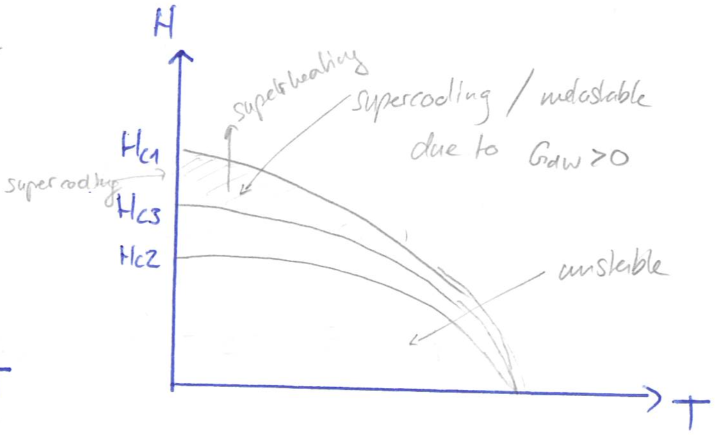
Nucleation*

p. 128 Tinkham
p. 190 Lifshitz



no superheating/supercooling since $\alpha < 0$ but bean livingston

type II

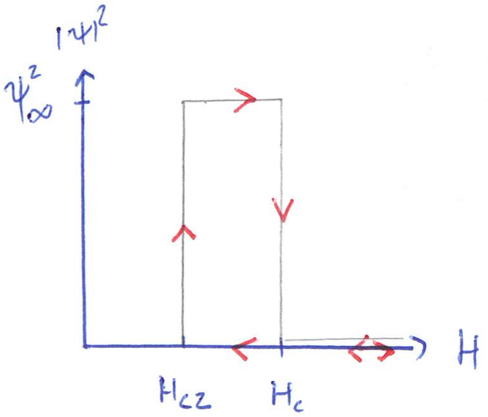


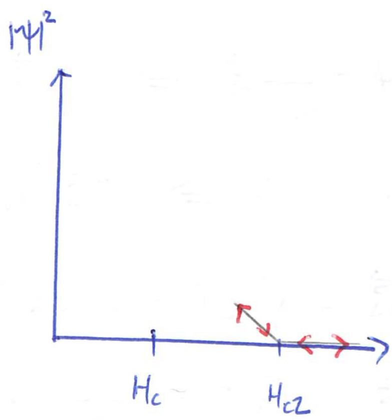
type I

H_{c2} is the highest field at which superconductivity can nucleate in the interior of a large sample in decreasing field.

The sign of the surface tension: $\alpha = \begin{cases} >0 & \kappa < \frac{1}{\sqrt{2}} \quad \text{type I} \\ <0 & \kappa > \frac{1}{\sqrt{2}} \quad \text{type II} \end{cases}$

$H_{c3} = \frac{1}{0{,}59} H_{c2} > H_{c2}$





type I

type II

second-order

Supercool, at H_{c2} nucleation occurs
discontinuous and irreversible jump
hysteretic

reversible

The structure of the mixed state*

p 193 Lifshitz

$H \gtrsim H_{c1}$. To attain the maximum thermodynamic favourability the nuclei of the normal phase within the superconducting phase, they must have (with negative surface tension) the largest possible surface. The structure expected is therefore one in which the n-phase nuclei are filaments parallel to the field.

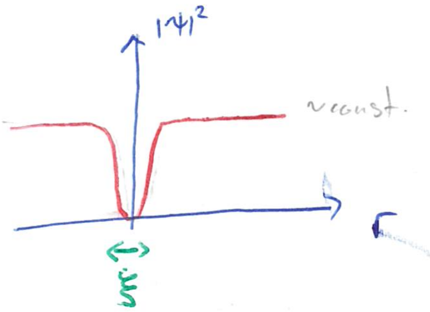
Free energy of filament

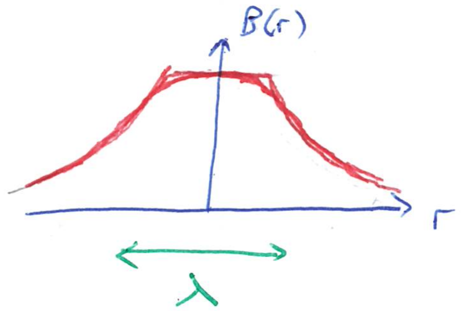
$$\tilde{F} = \hat{F}_s + L\varepsilon - \int \underbrace{\vec{H}}_{\text{const}} \cdot \underbrace{\vec{B}\, \frac{dV}{4\pi}}_{\int B\,dV = L\Phi_0}$$

$\vec{B} \neq 0$ close to vortex

$$= \tilde{F}_s + \underbrace{L\varepsilon - \frac{L\Phi_0 H}{4\pi}}_{=0} \quad \Rightarrow \quad H_{c1} = \frac{4\pi\varepsilon}{\Phi_0}$$

Assume $\kappa \gg 1 \Leftrightarrow \lambda \gg \xi$. The length ξ determines the order of magnitude of the radius of the "core" of the filament in which $|\psi|^2$ varies from 0.





The whole of the magnetic flux passes through the region outside the core, where $|\psi|^2 = \text{const}$. $\Rightarrow$ use London equation (London applicable) — assumed $\nabla\psi = 0$ in GL.

$$\vec{j} = \frac{e\hbar}{2m} n_s \left(\vec{\nabla}\phi - \frac{2e}{\hbar c}\vec{A}\right) \Rightarrow \quad \vec{A} + \lambda^2 \operatorname{rot}\vec{B} = \frac{\Phi_0}{2\pi}\vec{\nabla}\Phi$$

$\oint \nabla\phi = 2\pi$

Integrate along contour C $\overset{\text{Stokes}}{\Rightarrow}$ $\int \vec{B}\cdot d\vec{o} + \lambda^2 \oint \operatorname{rot}\vec{B}\cdot dl = \Phi_0$

my contour

$$\Rightarrow \quad \vec{B} - \lambda^2 \Delta \vec{B} = \phi_0\, \delta(r)$$

distances $\sim \xi$ are regarded as zero

For $\lambda \gg r \gg \xi$: $\quad B(r) = \dfrac{\phi_0}{2\pi\lambda^2} \log \dfrac{\lambda}{r}$

For $r \gg \xi,\ r \gtrsim \lambda$ $\quad B(r) = \dfrac{\phi_0}{2\pi\lambda^2} K_0\left(\dfrac{r}{\lambda}\right)$, $\quad B(r) \sim e^{-r/\lambda} \quad r \to \infty$

$$F_v = \frac{1}{8\pi}\left(\int \vec{B}^2 + \lambda^2 (\operatorname{rot}\vec{B})^2\, dV\right) \Rightarrow \quad \varepsilon = \left(\frac{\Phi_0}{4\pi\lambda}\right)^2 \log\left(\frac{\lambda}{\xi}\right)$$

$\Rightarrow$ $n = \pm 1$ thermodynamically favourable $\Rightarrow$ "repulsion"

London-Limit

B p.196 Lifshitz

Assume $\lambda \gg \xi$ $\Rightarrow$ $\xi \sim \text{const}$ (London) $\Rightarrow$ $\vec{B} - \lambda^2 \vec{\nabla}^2 \vec{B} = \hat{e}_z \Phi_0 \delta^{(2)}(r) = e_z \Phi_0 \frac{\delta(r)}{2\pi r}$

$\Rightarrow$ London applicable ← from section above $\Leftrightarrow \xi \sim 0$

macDonalds function

$$\Rightarrow \quad B_z(r) = \frac{\Phi_0}{2\pi\lambda^2} K_0\left(\frac{r}{\lambda}\right) = \frac{\Phi_0}{2\pi\lambda^2} \begin{cases} \sqrt{\frac{\pi\lambda}{2r}}\, e^{-r/\lambda} & r \gg \lambda \\ \log\left(\frac{\lambda}{r}\right) & r \ll \lambda, \quad B_z(0) \sim \frac{\Phi_0}{2\pi\lambda^2}\log\kappa \end{cases}$$

due to δ

use cutoff

$H = \mu B = B$??

$$\approx \frac{1}{8\pi}\int d^2r \left[B^2 + \lambda^2(\nabla\times B)^2\right] = \frac{1}{8\pi}\int d^2r\, \vec{B}\cdot\left[\vec{B} - \lambda^2\vec{\nabla}^2\vec{B}\right]$$

$$F_v = \int d\vec{r}\left[\frac{m}{4e^2}|\psi|^{-2}|\vec{j}|^2 + \frac{\vec{B}^2}{8\pi}\right] = \underbrace{\left(\frac{\Phi_0}{4\pi\lambda}\right)^2}_{\sim B^2??}\log\kappa$$

(line energy of single vortex) ignoring core energy

$$F_{core} \approx \left(\frac{\Phi_0}{4\pi\lambda}\right)^2 \frac{c}{2}$$ small

legendre?

B?

$$G_v(T,H) = F_v(T,H) - \int d^2r\, \frac{\vec{H}\cdot\vec{B}}{4\pi} = F_v(T,H) - \frac{\Phi_0}{4\pi} H_z \stackrel{!}{=} 0$$

p.198 Lifshitz

similar to $B_z(0)$ but factor $\frac{1}{2}$ due to $\frac{1}{4\pi}$ / $\frac{1}{8\pi}$

$$\Rightarrow \quad H_{c1} = \frac{\Phi_0}{4\pi\lambda^2}\log\kappa \quad \stackrel{T\to T_c}{\propto} |T - T_c| \qquad < H_c \stackrel{(4.4)}{=} \frac{\Phi_0}{2\sqrt{2}\,\pi\lambda\xi}$$

$\frac{\log\kappa}{\kappa} \to 0$

smallest external field compensating the line energy

condensation beneficial, vortex too expensive ??

The occurence of vortex filaments becomes thermodynamically favourable at H_{c1}. p. 194 Lifshitz

assumption that

$$H_{c1} = H_c \underbrace{\frac{\log\kappa}{\kappa\sqrt{2}}}_{\text{small}} \quad \left.\begin{matrix} T \to T_c \\ \kappa \gg 1 \end{matrix}\right| \text{ London limit}$$

Mixed phase

For $H < H_{c1}$ a flux line is too expensive $\Rightarrow$ remains in Meissner phase

$H_{c2} > H > H_c$ vortices enter from the boundary (mixed phase)

$H > H_{c2}$ cores overlap, sc. suppressed, $r \approx \xi\sqrt{2}$ p.407 Abrikosov

Vortex interaction and vortex lattice

p. 423 Abrikosov

Assume general winding number in London limit

$$\Rightarrow \vec{B} - \lambda^2 \vec{\nabla}^2 \vec{B} = \hat{e}_z \Phi_0 n \, \delta^{(2)}(r) \qquad \text{(linear)}$$

see above

p. 409 Abrikosov

$$\Rightarrow F_{v,n} \approx \left(\frac{\Phi_0 n}{4\pi\lambda}\right)^2 \log\kappa > n \cdot F_v \approx n \left(\frac{\Phi_0}{4\pi\lambda}\right)^2 \log\kappa$$

Flux lines with higher winding number favour to split up into smaller ones.

Understood as a repulsive force.

Assume solution to be $B_z(\vec{r}) = \frac{\Phi_0}{2\pi\lambda^2} \sum_{i=1}^{N} K_0\left(\frac{|\vec{r} - \vec{r}_i|}{\lambda}\right)$

superposition

$$\Rightarrow \quad F_{v,N} \overset{(4.29)}{=} \frac{1}{8\pi} \int d^2r \, \vec{B}(\vec{r}) \cdot [\vec{B} - \lambda^2 \vec{\nabla}^2 \vec{B}]$$

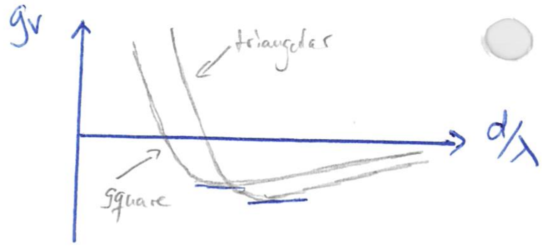
small $r \to$ large contribution

$$F_{v,N} = \underbrace{N\left(\frac{\Phi_0}{4\pi\lambda}\right)^2 \log\kappa}_{\text{line energy } N \text{ flux lines}} + \underbrace{\left(\frac{\Phi_0}{4\pi\lambda}\right)^2 \sum_{i \neq j} K_0\left(\frac{|\vec{r}_i - \vec{r}_j|}{\lambda}\right)}_{\text{repulsive interaction}}$$

$\Rightarrow$ in a homogeneous we get a vortex lattice. square or triangular

$$\underset{\text{enthalpy density}}{g_v}(H, T, \overset{\text{distance}}{d}, \underset{\text{density}}{\nu}) = \nu F_v(B,T) - \nu \frac{\Phi_0 H}{4\pi}$$

$$= \nu \frac{\Phi_0 H}{4\pi} (H_{c_1} - H) + \nu z \left(\frac{\Phi_0}{4\pi\lambda}\right)^2 K_0(d/\lambda)$$

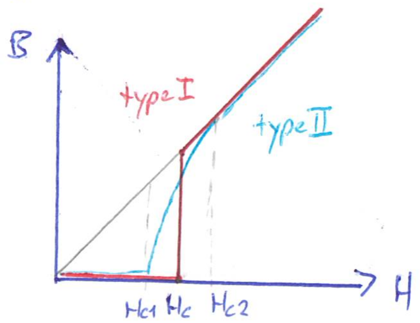


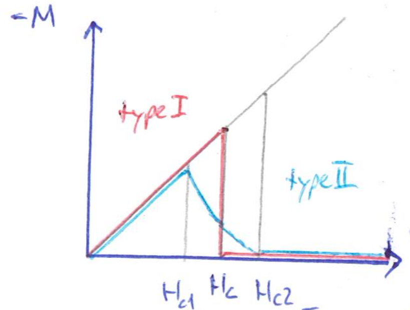
Restriction to nearest neighbour is sensible for $d \gg \lambda$, K_0 falls of exponentially

$\Rightarrow$ most often triangular lattice

Magnetization of the type II superconductor

→ p. 439 Abrikosov
p. 157 Tinkham
beware p. 190 Lifshitz
superheating / supercooling effect





← resistivity measurement
not clear
↳ measure H_{c1} via diamagnetic property

Bean-Livingston barrier:

Vortex has to enter from surface at special places, therefore H_{c1} not clearly visible as above. Further, no supercurrent normal to the sample surface: ← GL boundary condition
Vortex and its mirror vortex attract each other ⇝ penetration more difficult. The magnetic field is given by

along z

$$B_z(\vec{r}) = \frac{\Phi_0}{2\pi\lambda^2} K_0(|\vec{r}-\vec{r}_0|/\lambda) - \frac{\Phi}{2\pi\lambda^2} K_0(|\vec{r}-\vec{r}_0'|/\lambda) + He^{-x/\lambda}$$

decays when going into x

F_v calculation (4.29)

$$\Rightarrow \quad V(d,H) = G_v = \frac{\Phi_0}{4\pi}(H_{c1}-H) - \left(\frac{\Phi_0}{4\pi\lambda}\right)^2 K_0\left(\frac{2d}{\lambda}\right) + \frac{\Phi_0 H}{4\pi} e^{-d/\lambda} \quad (\text{for } d \gg \xi)$$

(4.31) G_v calculation

We may consider $V(d,H)$ as a potential for the vortex.

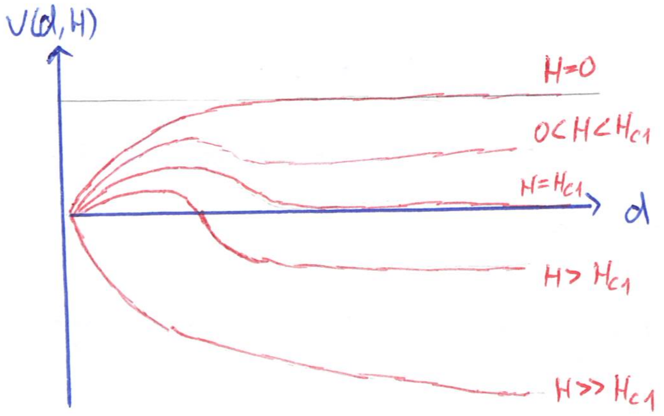


Fig 120 Abrikosov

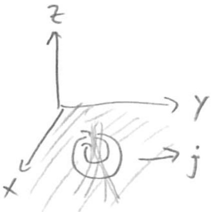
For $H \sim H_{c1}$ the Bean-Livingston barrier appears.
⇝ hysteretic behaviour

Flux line motion and Bean model

p 157 Tinkham

Vortices are subject to the Lorentz force, flux flow with dissipative behaviour.

Flux flow resistance



$$\vec{f} = \frac{1}{c}\vec{j} \times \vec{z}\,\Phi_0$$, if vortex not pinned $\Rightarrow$ moves $\perp j$, but in superconducting condensate

$\Rightarrow$ viscous drag, friction force $\eta \vec{v}$

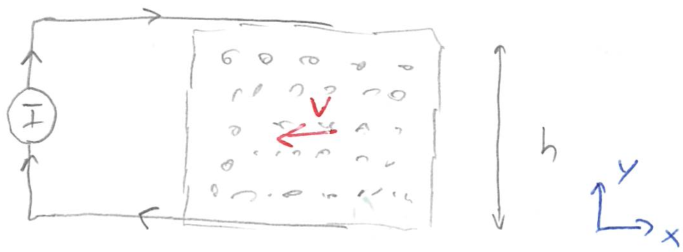
$$\Rightarrow \quad \frac{j\Phi_0}{c} = \eta v$$ dissipation of current.

change in flux

density / single Area

$$\hookrightarrow \Delta\Phi = n\Phi_0 v h \,\Delta t$$

$$\Rightarrow \quad Eh = \frac{n\Phi_0^2 h}{\eta c^2} j$$



$E = \rho_{ff}\, j$

$$\Rightarrow \quad \rho_{ff} = \frac{n\Phi_0^2}{\eta c^2} \quad , \quad \overset{B = n\Phi_0}{\Rightarrow} \quad \rho_{ff} = \rho_n \frac{B}{B_{c2}} \quad \text{(empirical)}$$

normal metal

ratio of system being threaded by normal core region

If flux lines are free $\Rightarrow$ no supercurrent in mixed phase

In reality: pinning centers (not every flux line needs to be pinned $\rightsquigarrow$ lattice)

Flux line depinning determines the critical supercurrent.

As long as they are pinned, no flux flow and hence no resistivity.

Idea: if free then all vortices move into the circuit and increasing the flux in the circuit, by Lenz rule a electromotive force acts against it.

BCS Theory – Microscopic Understanding

p 333 Abrikosov
→ p.17 Tinkham

Cooper instability

Consider free e^-

$$\mathcal{H}_e = \frac{1}{2m}\int d^3r \left(\frac{\hbar\vec{\nabla}}{i}\hat{\psi}_s(\vec{r})\right)^\dagger \left(\frac{\hbar\vec{\nabla}}{i}\hat{\psi}(\vec{r})\right) = \sum_{\vec{k},s} \overbrace{\epsilon_{\vec{k}}}^{\frac{\hbar^2\vec{k}^2}{2m}} c^\dagger_{\vec{k},s} c_{\vec{k},s}$$

$\Rightarrow$ plane waves

Normal metal: $n_{\vec{k},s} = \dfrac{1}{e^{\beta(\epsilon_{\vec{k}}-\mu)}+1}$, $\epsilon_F = \dfrac{\hbar^2 k_F^2}{2m}$, $k_F = (3\pi^2 n)^{1/3}$, $k_B T_F = \epsilon_F$

Ground state: $|\psi_0\rangle = \prod_{\vec{k}}^{|\vec{k}|\le k_F} \prod_{s=\uparrow,\downarrow} c^\dagger_{\vec{k},s}|0\rangle$

Two attractive electrons

p 338 Abrikosov

in the ground state the momentum of the pair has to vanish.

$$\left[-\frac{\hbar^2}{2m}\left\{\vec{\nabla}_1^2+\vec{\nabla}_2^2\right\} + V(\vec{r}_1-\vec{r}_2) - E\right]\psi(\vec{r}_1,\vec{r}_2;s_1,s_2) = 0$$

$V(\vec{r}_1-\vec{r}_2) = g\,\delta(\vec{r}_1-\vec{r}_2)$ contact interaction

$\vec{k}$-space

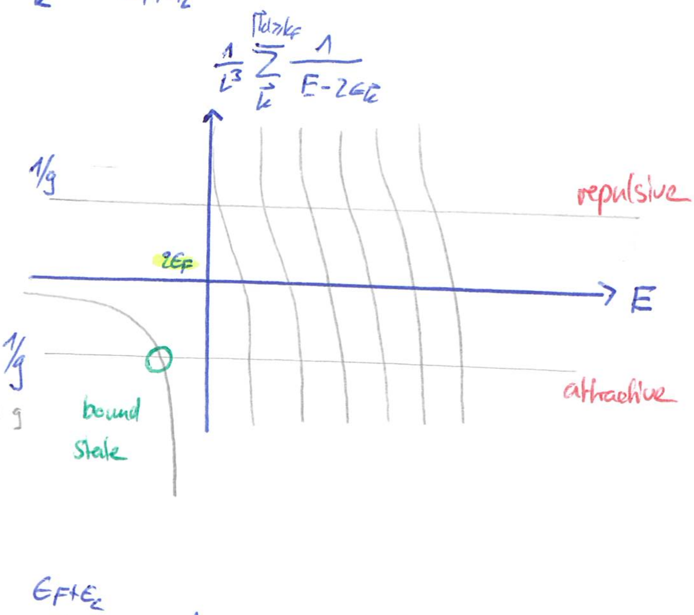
$$\Rightarrow \quad \psi(\vec{r}_1,\vec{r}_2,s_1,s_2) = \sum_{\vec{k}}{}^{'} e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)} \underbrace{w_{\vec{k}}}_{\text{orbital}} \underbrace{\chi_{s_1,s_2}}_{\text{spin}}$$

$|k|>k_F$

Separated spin part

$$\Rightarrow \quad (2\epsilon_{\vec{k}} - E)w_{\vec{k}} + \sum_{\vec{k}'}{}^{'} V_{\vec{k}\vec{k}'} w_{\vec{k}'} = 0 \qquad V_{\vec{k}\vec{k}'} = \frac{g}{L^3}$$

$\Rightarrow \quad w_k = \dfrac{\sum'..}{E-2\epsilon_k} = \dfrac{\Gamma}{E-2\epsilon_k}$



$$\Rightarrow \quad \frac{1}{g} = \frac{1}{L^3}\sum_{\vec{k}}^{|\vec{k}|>k_F} \frac{1}{E-2\epsilon_{\vec{k}}} = \int_{\epsilon>\epsilon_F} d\epsilon\, \frac{N(\epsilon)}{E-2\epsilon}$$

negative g

because $\epsilon_F \gg \epsilon_c$? and $N(\epsilon)$ varies upon ϵ_F?

Assume g small, $N(\epsilon) \approx N(\epsilon_F)$

cutoff for convergence

$$\Rightarrow \quad \frac{1}{g} \approx N(\epsilon_F)\int_{\epsilon_F}^{\epsilon_F+\epsilon_c} \frac{d\epsilon}{E-2\epsilon} = N(\epsilon_F)\int_{\epsilon_F}^{\epsilon_F+\epsilon_c}\frac{d\epsilon}{\Delta+2\epsilon_F-2\epsilon} \approx \frac{N(\epsilon_F)}{2}\log\frac{\Delta}{-2\epsilon_c}$$

energy of the two e^-

unperturbated plane wave p.18 Tinkham

$$\Rightarrow \quad \Delta = -2\epsilon_c e^{-2/|g| N(\epsilon_F)}$$ valid for $|g| N(\epsilon_F) \ll 1$

$\Rightarrow \Delta \ll \epsilon_c$

For this state: $E = 2\epsilon_F + \Delta < 2\epsilon_F$

$$\Psi(\vec{r}_1, \vec{r}_2; s_1, s_2) = -\underbrace{\Psi(\vec{r} = \vec{r}_1 - \vec{r}_2)}_{\substack{\text{even parity} \\ L=0}} \underbrace{\chi_{s_1 s_2}}_{}$$ antisymmetric

$$\Rightarrow |S=0, S_z=0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

follows from $\Psi(\vec{r})$ wavefunction consideration

Instability

Cooper pair with energy $< 2\epsilon_F$, happens for any coupling constant g

Pairing states with higher angular momentum L behave as $\Psi(r) \propto r^L$ such that two e^- avoid each other.

Coherent BCS state

p23 Tinkham

could be filled fermi sea of other e^-

Cooper pair state: $$|\Psi_1\rangle = \sum_{\vec{k}} w_{\vec{k}} \hat{c}^{+}_{\vec{k}\uparrow} \hat{c}^{+}_{-\vec{k}\downarrow} |vac\rangle$$

no e^-

$\forall k$

BCS $\Rightarrow$ $$|\Psi_{BCS}\rangle = \prod_{\vec{k}} \{ u_{\vec{k}} + v_{\vec{k}} \hat{c}^{+}_{\vec{k}\uparrow} \hat{c}^{+}_{-\vec{k}\downarrow} \} |0\rangle, \quad |u_{\vec{k}}|^2 + |v_{\vec{k}}|^2 = 1$$

Example: Fermi sea:
$$\underbrace{u_{\vec{k}} = \begin{cases} 0, & |\vec{k}| < k_F \\ 1, & |\vec{k}| > k_F \end{cases}}_{|u_k|^2 \hat{=} \text{prob. not occupied}} \leftarrow \neq 0 \text{ for normalization} \qquad \underbrace{v_{\vec{k}} = \begin{cases} 1, & |\vec{k}| \le k_F \\ 0, & |\vec{k}| > k_F \end{cases}}_{|v_k|^2 \hat{=} \text{prob. occupied}}, \quad u_{\vec{k}} v_{\vec{k}} = 0$$

Fermi sea: $$|\phi_0\rangle = \prod_{k \le k_F, \sigma} c^{+}_{\vec{k}\sigma} |0\rangle = \prod_{k \le k_F} c^{+}_{k\uparrow} c^{+}_{k\downarrow} |0\rangle = \prod_{k \le k_F} \{ 0 + 1 \cdot c^{+}_{k\uparrow} c^{+}_{-k\downarrow} \} |0\rangle$$

Two attractive electrons*

Ground state is filled Fermi sea and add two electrons whose energies are above ε_F and have opposite momentum.

$$\left[-\frac{\hbar^2}{2m}\left(\vec{\nabla}_1^2+\vec{\nabla}_2^2\right)+V(\vec{r}_1-\vec{r}_2)-E\right]\Psi(\vec{r}_1,\vec{r}_2;s_1,s_2)=0$$

Assume $V(\vec{r}_1-\vec{r}_2)=g\,\delta(\vec{r}_1-\vec{r}_2)$. In k-space

$$\Psi(\vec{r}_1,\vec{r}_2;s_1,s_2)=\sum_{\vec{k}}{}' e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}\,w_{\vec{k}}\,\chi_{s_1,s_2}$$

Recast into Hamiltonian

$$\Rightarrow\quad (2\varepsilon_{\vec{k}}-E)w_k+\sum_{\vec{k}'}{}' V_{\vec{k},\vec{k}'}w_{\vec{k}'}=0$$

dirty derivation

divide by w_k

$V=\frac{g}{L^3}$

$$\Rightarrow\quad w_{\vec{k}}=\frac{\sum_{\vec{k}'}' V_{\vec{k},\vec{k}'}w_{\vec{k}'}}{E-2\varepsilon_{\vec{k}}}=:\frac{\Gamma}{E-2\varepsilon_{\vec{k}}}$$

$$\text{with}\quad \Gamma=\sum_{\vec{k}'}{}' V_{\vec{k},\vec{k}'}w_{\vec{k}'}=\underbrace{\frac{g}{L^3}}_{V}\sum_{k}^{k>k_F}\frac{\Gamma}{E-2\varepsilon_k}$$

Dividing by Γ yields

$\int dk=\int d\varepsilon\, N(\varepsilon)$

$$\frac{1}{g}=\frac{1}{L^3}\sum_{k}^{k>k_F}\frac{1}{E-2\varepsilon_k}=\int_{\varepsilon>\varepsilon_F}d\varepsilon\,\frac{N(\varepsilon)}{E-2\varepsilon}$$

$$\approx N(\varepsilon_F)\int_{\varepsilon_F}^{\varepsilon_F+\varepsilon_c}\frac{d\varepsilon}{E-2\varepsilon}=:N(\varepsilon_F)\int_{\varepsilon_F}^{\varepsilon_F+\varepsilon_c}\frac{d\varepsilon}{\Delta+2\varepsilon_F-2\varepsilon}\approx\frac{N(\varepsilon_F)}{2}\log\frac{\Delta}{-2\varepsilon_c}$$

only for $g<0$

$$\Rightarrow\quad \Delta=-2\varepsilon_c\,e^{-2/|g|N(\varepsilon_F)}$$

$\Rightarrow$ added particles form a bound state of lower energy.

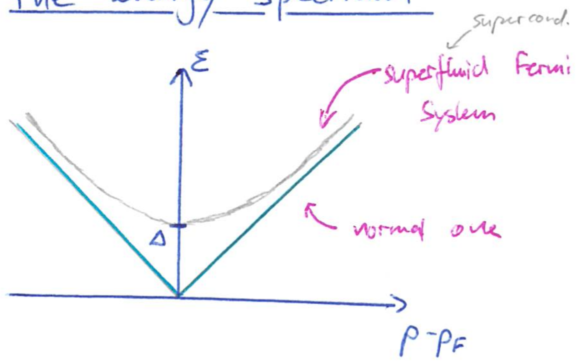
Energy scale of binding energy *

p. 337 Abrikosov

The critical temperature is a measure of the binding energy ~ K, which is 10^4 times lower than kinetic energy of electrons $\sim \varepsilon_F$. A bound state can not be produced if the interaction is not strong enough. Exceptions to this are the purely one- ~~one~~ two dimensional case.
Pairs are not formed by free electrons but by quasiparticles of the Fermi liquid, where the Fermi sphere is occupied. This actually leads to a replacement of the three-dimensional problem by a two dimensional one. In a two dimensional model any attraction will be sufficient for the particles to be bound.

The energy spectrum *

→ p. 157 Lifshitz



$$E_k = \begin{cases} \sqrt{\xi_k^2 + |\Delta|^2} & |\xi_k| \leq \varepsilon_c \\ |\xi_k| & |\xi_k| > \varepsilon_c \end{cases}$$

The excited states are seperated from the ground state by an energy gap. Δ decreases and becomes zero at T_c where the system passes from superfluid to normal state.
The quantity 2Δ may be regarded as the binding energy of the Cooper pair which would have to be expanded in order to break it up.

p.25 Tinkham

Variational minimization

$$\hat{\mathcal{H}}_{pair} = \frac{1}{2}\sum_{s,s'}\int d\vec{r}^3 d\vec{r}'^3 \; \hat{\psi}_s(\vec{r})^+ \hat{\psi}_{s'}(\vec{r}')^+ V(\vec{r}-\vec{r}')\, \hat{\psi}_{s'}(\vec{r}')\, \hat{\psi}_s(\vec{r})$$

$V(\vec{r}-\vec{r}') = g\,\delta^3(\vec{r}-\vec{r}')$

negative

$$= \frac{g}{2L^3}\sum_{s,s'}\sum_{\vec{k},\vec{k}',\vec{q}} \hat{c}^+_{\vec{k}+\vec{q},s}\, \hat{c}^+_{\vec{k}'-\vec{q},s'}\, \hat{c}_{\vec{k}',s'}\, \hat{c}_{\vec{k},s}$$

additional kinetic energy of pair

$$\hat{\mathcal{H}}_{kin} = \sum_{\vec{k},s}(\epsilon_{\vec{k}}-\mu)\,\hat{c}^+_{\vec{k}s} c_{\vec{k}s} = \sum_{\vec{k},s}\xi_{\vec{k}}\,\hat{c}^+_{\vec{k},s} c_{\vec{k},s} \quad , \quad \xi_{\vec{k}} = \epsilon_{\vec{k}} - \mu$$

k full, k' empty $\rightsquigarrow$ $v_k^* u_{k'}^*$? $u_k v_k^* u_{k'}^* v_{k'}$

k empty, k' full $\rightsquigarrow$ $u_k v_{k'}$

$\langle \hat{n}_k \rangle = 2|v_{\vec{k}}|^2$

$$\langle \Psi_{BCS}|\hat{\mathcal{H}}_{pair}|\Psi_{BCS}\rangle = \frac{g}{L^3}\sum_{\vec{k},\vec{k}'} u_{\vec{k}} v^*_{\vec{k}} u^*_{\vec{k}'} v_{\vec{k}'} \quad , \quad \langle \Psi_{BCS}|\mathcal{H}_{kin}|\Psi_{BCS}\rangle = 2\sum_{\vec{k}}\xi_{\vec{k}}|v_{\vec{k}}|^2$$

maybe find geometric interpretation

$E_{BCS} = E_{kin} + E_{pair}$

From $|u_{\vec{k}}|^2 + |v_{\vec{k}}|^2 = 1 \Rightarrow u_{\vec{k}} = \sin\Theta_{\vec{k}}$, $v_{\vec{k}} = \cos\Theta_{\vec{k}}$, $\tan(2\Theta_k) = -\frac{\Delta}{\xi_k}$

$\partial_{\Theta_k} = 0$ assume $\Delta_k = \Delta$ — definition p.26 Tinkham — definition

$$\Longrightarrow \quad \Delta = -\frac{g}{2L^3}\sum_{\vec{k}}\sin(2\Theta_{\vec{k}}) = -\frac{g}{2L^3}\sum_{\vec{k}}\frac{\Delta}{E_k} \quad , \quad E_{\vec{k}}^2 := \xi_{\vec{k}}^2 + \Delta^2$$

self consistency equation

$\Delta = 0$ solution $\hat{=}$ Slater, Fermi sea $T=0$

Solved by
$$1 = -\frac{g}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{\xi_{\vec{k}}^2+\Delta^2}} \approx \frac{N(\varepsilon_F)g}{2}\int_{-\varepsilon_c}^{\varepsilon_c}\frac{d\xi}{\sqrt{\xi^2+\Delta^2}}$$

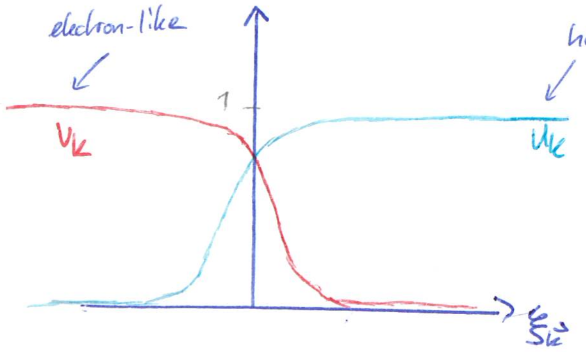
$$\Longrightarrow \quad \Delta = \frac{\varepsilon_c}{\sinh\left(\frac{1}{gN(\varepsilon_F)}\right)} \approx 2\varepsilon_c e^{-1/|g|N(\varepsilon_F)}$$

assuming $|g|N(\varepsilon_F) \ll 1$

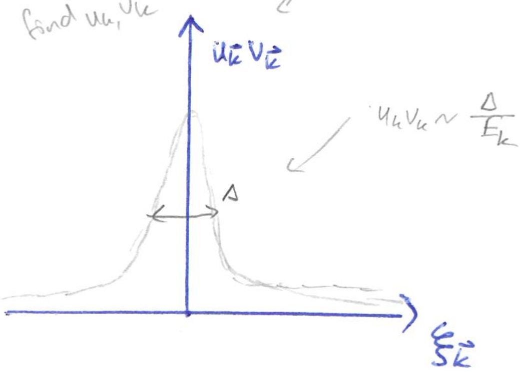
$\Longleftrightarrow \Delta \ll \varepsilon_c$

use this to find u_k, v_k

$T=0$ p.29 Tinkham



would be 0 in Fermi sea



$\langle \Psi_{BCS}|n_{\vec{k}}|\Psi_{BCS}\rangle = 2|v_{\vec{k}}|^2$ no longer a step function

measure of overlap of the 2 functions

$$u_k^2 = \frac{1}{2}\left(1+\frac{\xi}{\sqrt{\xi^2+\Delta^2}}\right)$$

Idea: destroying cooper pair costs Δ in energy? How does this relate to coherent states in QM, $a|\phi\rangle = a_k|\phi\rangle$

$$\langle \psi_{BCS} | \hat{c}_{-\vec{k}\downarrow} \hat{c}_{\vec{k}\uparrow} | \psi_{BCS} \rangle = u_{\vec{k}} v_{\vec{k}} = \frac{\Delta}{2E_{\vec{k}}} \neq 0$$

This expectation value is "off-diagonal" as it does not conserve the particle number. But

$$\langle \psi_{BCS} | \hat{c}_{\vec{k},s} | \psi_{BCS} \rangle = 0$$

$\Rightarrow$ not possible to remove / add a single particle.

Ground state energy

p. 29 Tinkham

normal

$$\frac{E_{BCS} - E_n}{L^3} = \frac{2}{L^3} \sum_{\vec{k}}^{|\vec{k}| \geq k_F} \left\{ \xi_{\vec{k}} - \frac{\xi_{\vec{k}}^2}{E_{\vec{k}}} \right\} + \frac{\Delta^2}{g} \approx -\frac{1}{2} N(\varepsilon_F) \Delta^2 = -\frac{H_c(0)^2}{8\pi} < 0$$

density of pairs near ε_F is $\sim N(\varepsilon_F)\Delta$ whereby each pair contributes $-\Delta$.

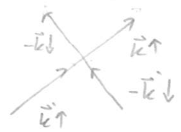
Decoupling of many-body Hamiltonian

p. 337 Abrikosov (concept)

p. 157 Lifshitz

reduced Hamiltonian

$$\mathcal{H}_{BCS} = \sum_{\vec k,s} \xi_{\vec k}\, \hat c^{\dagger}_{\vec k,s}\hat c_{\vec k,s} + \sum_{\vec k,\vec k'} V_{\vec k,\vec k'}\, \hat c^{\dagger}_{\vec k\uparrow}\hat c^{\dagger}_{-\vec k\downarrow}\hat c_{-\vec k'\downarrow}\hat c_{\vec k'\uparrow}$$



- only includes scattering among pairs of electrons carrying zero total momentum.

and $$V_{\vec k,\vec k'} = \begin{cases} \underbrace{g}_{<0}/L^3 & |\xi_k|, |\xi_{k'}| \le \underbrace{\epsilon_c}_{\text{cutoff}} = \hbar\omega_D = k_B\Theta_D \quad \text{(see later)} \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \Delta_k = \begin{cases} \Delta & |\xi_k| \le \epsilon_c \\ 0 & |\xi_k| > \epsilon_c \end{cases}$$

Decoupling scheme

Amplitude: create pair $\langle \hat c^{\dagger}_{k\uparrow}\hat c^{\dagger}_{-k\downarrow}\rangle$; anihilate pair $\langle \hat c_{-k'\downarrow}\hat c_{k'\uparrow}\rangle$

$$\Rightarrow \mathcal{H}'_{BCS} = \sum_{\vec k,s}\xi_{\vec k}\,\hat c^{\dagger}_{\vec k,s}\hat c_{\vec k,s} + \sum_{\vec k,\vec k'} V_{\vec k,\vec k'}\left\{ b^*_{\vec k}\,\hat c_{-\vec k',\downarrow}\hat c_{\vec k'\uparrow} + b_{\vec k'}\,\hat c^{\dagger}_{\vec k\uparrow}\hat c^{\dagger}_{-\vec k\downarrow} - b^*_{\vec k} b_{\vec k'}\right\}$$

Define: $\Delta_{\vec k} = -\sum'_{\vec k'} V_{\vec k\vec k'}\, \underbrace{b_{\vec k'}}_{\text{mean field}}$ and $\Delta^*_{\vec k'} = -\sum'_{\vec k} V_{\vec k,\vec k'}\, b^*_{\vec k}$

lead to gap

$$\boxed{\Rightarrow \mathcal{H}'_{BCS} = \sum_{\vec k}\left(\hat c^{\dagger}_{\vec k\uparrow}, \hat c_{-\vec k\downarrow}\right)\begin{pmatrix} \xi_k & -\Delta_{\vec k} \\ -\Delta^*_{\vec k} & -\xi_k \end{pmatrix}\begin{pmatrix} \hat c_{\vec k\uparrow} \\ \hat c^{\dagger}_{-\vec k\downarrow}\end{pmatrix} + K_0}$$

Hybidization of electron and hole like Fermions around ϵ_F

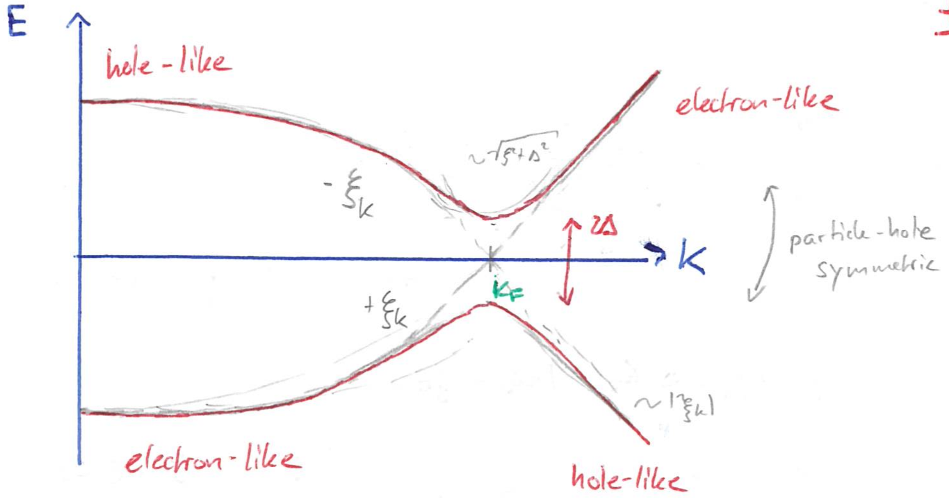
The electron-hole space $(\hat c^{\dagger}_{k\uparrow}, \hat c_{-\vec k\downarrow})$ is called Nambu space.

Bogolyubov transformation $\Rightarrow$

$$\mathcal{H} = \sum_{\vec k} E_{\vec k}\left\{\hat\gamma^{\dagger}_{\vec k1}\hat\gamma_{\vec k1} + \hat\gamma^{\dagger}_{-\vec k2}\hat\gamma_{-\vec k2}\right\} + \tilde K_0$$, $\hat\gamma$ fermionic

The quasi-particle excitations can be simply described as Fermions created by the $\gamma^{\dagger}$ which are in one-to-one correspondance with $\hat c^{\dagger}_k$ of the normal metal. p. 41 Tinkham

p. 42 Tinkham? see extra sheet above



$\pm E_{\vec{k}}$ energy of Bogolyubov quasiparticles $\hat{\gamma}^{(+)}_{k\alpha}$

K_0 corresponds to filling of lower band

$$E_{\vec{k}} = \begin{cases} \sqrt{\xi_{\vec{k}}^2 + |\Delta|^2} & |\xi_k| \leq \epsilon_c \\ |\xi_k| & |\xi_k| > \epsilon_c \end{cases}$$

Thermodynamic equilibrium

$$Z = e^{-\beta \tilde{K}_0} \prod_{\vec{k}} \left\{1 + e^{-\beta E_{\vec{k}}}\right\}^2 \Rightarrow \Omega = \tilde{K}_0 - 2k_BT \sum_k \log\left\{1 + e^{-\beta E_k}\right\}$$

$$\Rightarrow 0 \overset{!}{=} \frac{\partial \Omega}{\partial \Delta^*} = -\frac{\Delta}{g} L^3 - \sum_k{}' \frac{\Delta}{2E_{\vec{k}}} \tanh\left(\frac{E_{\vec{k}}}{2k_BT}\right)$$

$\frac{\partial \Omega}{\partial \Delta}$ self consistently

$$\Rightarrow \quad \Delta = -\frac{g}{L^3} \sum_{\vec{k}}{}' \frac{\Delta}{2E_{\vec{k}}} \tanh\left(\frac{E_{\vec{k}}}{2k_BT}\right)$$

$|\xi_k| < \epsilon_c$

Gap equation

Temperature dependence of Δ

p. 345 Abrikosov

p. 34 Tinkham

$T = T_c$:
$\Delta \to 0$

Start from the gap equation: $\Delta = -\frac{g}{L^3} \sum_{\vec{k}}' \frac{\Delta}{2E_k} \tanh\left(\frac{E_{\vec{k}}}{2k_BT}\right)$

$$\Rightarrow \quad 1 = -gN(\varepsilon_F) \log\left(\frac{1{,}13\,\varepsilon_c}{k_BT_c}\right)$$

$$\Rightarrow \quad k_BT_c = 1{,}13\,\varepsilon_c\, e^{-1/|g|N(\varepsilon_F)}$$

for any finite g

$T \to 0$: $\tanh\left(\frac{x}{T}\right) \xrightarrow{T \to 0} \mathrm{sgn}(x)$

$$\Delta(0) = 2\varepsilon_c\, e^{-1/|g|N(\varepsilon_F)} = \underbrace{1{,}76}_{2/1{,}13}\, k_BT_c$$

$T \to T_c$:

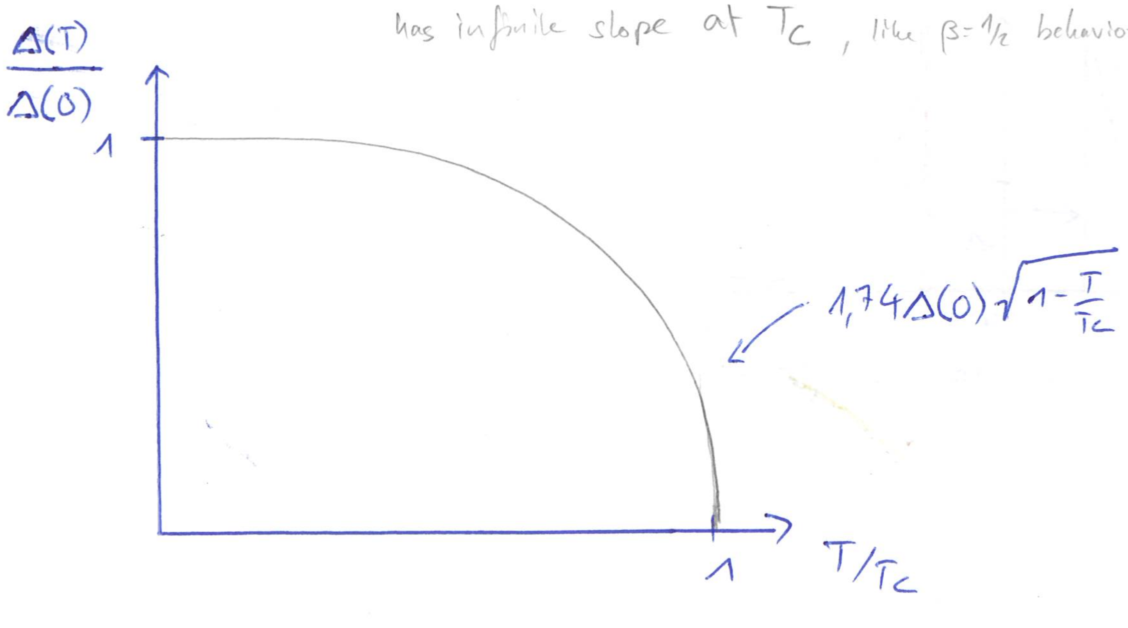
expand self consistency equation in small Δ

mean field dependence

$$\Delta(T) = 1{,}74\,\Delta(0)\left(1 - \frac{T}{T_c}\right)^{1/2} \quad , \quad \beta = 1/2$$

approximation: $\Delta(T) \approx \Delta(0) \tanh\left(1{,}74\sqrt{(T_c - T)/T}\right)$

has infinite slope at T_c, like $\beta = 1/2$ behaviour



<u>Ground state properties</u> p.41 Tinkham

Quasi-particle density of states: The spectrum of quasiparticles opens a gap at the Fermi energy, $E_{\vec{k}} = \pm\sqrt{\xi_{\vec{k}}^2 + \Delta^2}$

$\frac{1}{L^3}\sum_k \delta(E - E_{\vec{k}})$

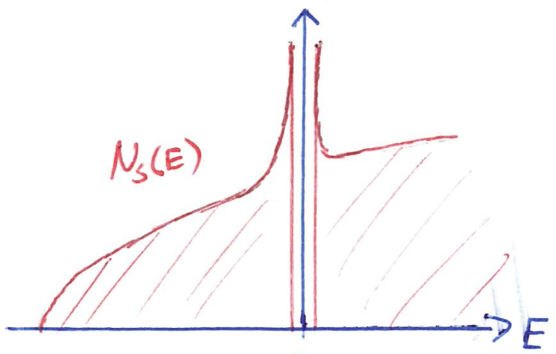
$$N_s = \begin{cases} N(\varepsilon_F + \sqrt{E^2-\Delta^2}) \dfrac{|E|}{\sqrt{E^2-\Delta^2}}, & E > +\Delta \\ 0 & E \leq \Delta \\ N(\varepsilon_F - \sqrt{E^2-\Delta^2}) \dfrac{|E|}{\sqrt{E^2-\Delta^2}}, & E < -\Delta \end{cases}$$

as $\Delta \ll \varepsilon_F$

$N(\xi + \varepsilon_F) \approx N(\varepsilon_F)$

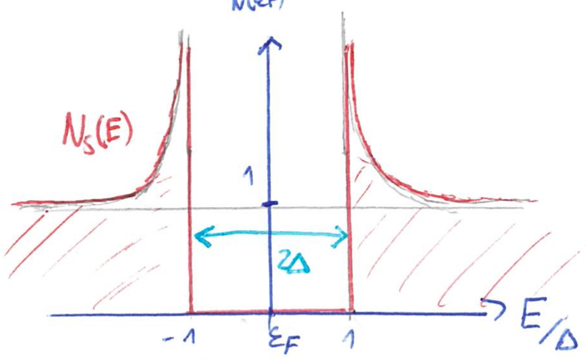
$\Rightarrow$

$$N_s(E) \approx \begin{cases} N(\varepsilon_F) \dfrac{|E|}{\sqrt{E^2-\Delta^2}}, & |E| > \Delta \\ 0 & E \leq \Delta \end{cases}$$

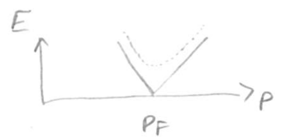


forbidden region

No states are lost, the states removed within the gap are pushed outside.



energy close to ε_F $(E=0)$



diverges at p_F

$$N(E) = \frac{2}{(2\pi\hbar)^3} \int_{\xi = E} \frac{dS}{|v_F|}$$

Gap for $T \ll T_c$: p. 35 Tinkham

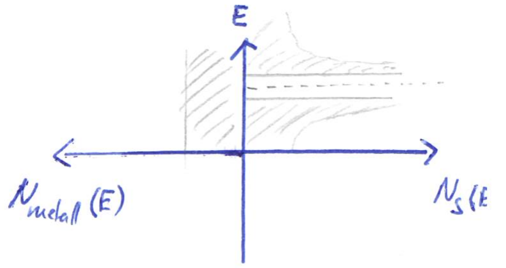
$$\tanh\left(\frac{E_{\vec{k}}}{2k_BT}\right) = 1 - 2n_F(E_{\vec{k}}) = 1 - \frac{2}{e^{\beta\varepsilon}+1}$$

$$\Rightarrow \quad \Delta(T) \approx \Delta(0)\left(1 - \frac{2e^{-\Delta(0)/k_BT}}{|g|\,N(\varepsilon_F)}\right)$$

Low temperature behaviour is governed by thermal activation over the excitation gap $\Delta(0)$. For low temperatures ($T \lesssim 0{,}2T_c$) the gap $\Delta(T)$ is good approximation for $\Delta(0)$.

Condensation energy:

Quasiparticles removed from $\varepsilon_F \Rightarrow$ all states occupied in the ground state are moved towards lower energy for $-\epsilon_c < \xi < 0$. This energy gain is called condensation energy.



$$E_{cond} = |E_S - E_{metall}| = \underbrace{\frac{1}{2}}_{\text{Cooper pair}} \underbrace{N(\varepsilon_F)}_{\frac{3}{\pi^2}\gamma \text{ Greshkenbein}} \Delta(0)^2 = 0{,}46\,\gamma T_c^2 \quad \left(= \frac{H_c^2(0)}{8\pi}\right)$$

$\frac{1}{2} \Delta\, N_S = \#\text{states}$

$\Rightarrow \quad \Delta(0) \propto k_BT \Rightarrow \epsilon_c$ drops out

$\Rightarrow$ weak coupling theory

BCS Theory - Pairing Interaction and critical temperature

p. 334 Abrikosov

Lecture 14. SST Geshkenbein

p. 19 Tinkham

p. 177 Lifshitz (short)

Electronic renormalization of the Coulomb interaction

Lindhard function

$$\varepsilon(\vec{q},\omega) = 1 - \frac{4\pi e^2}{q^2}\chi_0(\vec{q},\omega)$$

C.15 Geshkenbein solid state theory

Assuming $V_{ext}(\vec{r},t) = V_0(\vec{q},\omega)e^{i\vec{q}\cdot\vec{r}-i\omega t}$ with monochromatic modulation

applied field

$$\Rightarrow V_{ren}(\vec{q},\omega) = \frac{V_0(\vec{q},\omega)}{\varepsilon(\vec{q},\omega)} \quad , \quad V_0(\vec{q},0) = \frac{4\pi e^2}{q^2} \quad \text{(Coulomb)}$$

resulting potential

- Static Thomas-Fermi screening:

$$\varepsilon(\vec{q},0) = 1 + \frac{k_{TF}^2}{q^2} \quad , \quad k_{TF}^2 = \frac{6\pi e^2 n}{\varepsilon_F}$$

$$\Rightarrow V_{ren}(\vec{q},0) = \frac{4\pi e^2}{q^2 + k_{TF}^2} \Rightarrow V_{ren}(r) = \frac{e^2}{r}e^{-r/l_{TF}}$$

$\Rightarrow$ short ranged, effectively a contact interaction.

- Plasma excitation:

$$\omega \to \infty, \quad \varepsilon(0,\omega) = 1 - \frac{\omega_p^2}{\omega^2} \quad , \quad \omega_p^2 = \frac{4\pi e^2 n}{m}$$

The electronic plasma frequency is too high for the electrons to screen the Coulomb potential.

Ionic renormalization of the Coulomb interaction

$T_c, H_c \sim M^{-1/2} \Rightarrow$ phonons important

Ions, $m \ll M$

$$\Omega_p^2 = \frac{m}{M}\omega_p^2 \Rightarrow e^- \text{ react instantaneously to ionic motion}$$

static, k Thomas Fermi, $\omega = c k$, $c_s^2 = \frac{\Omega^2}{k_{TF}^2} = \frac{m}{6M}v_F^2 \ll v_F^2$

normal modes of ions: $\omega_q^2 \approx \dfrac{\Omega_p^2}{\varepsilon(\vec{q},0)} \xrightarrow{q \ll k_{TF}} \dfrac{\Omega_p^2 q^2}{k_{TF}^2}$, $c_s^2 \ll v_F^2$

energy scale of phonons: $\hbar\omega_D \approx \sqrt{\frac{m}{M}}\,\varepsilon_F \ll \varepsilon_F \Rightarrow \Theta_D \ll T_F$

k_{Fermi}

$$\omega_{q_D} = c_s q_D \sim c_s k_F$$

Electronic and ionic polarization:

Define the bare electronic dielectric function ε^{el}, V^{el}
and the ionic dielectric function ε^{ion}, V^{ion}

induced by displacements

bare

ion polarization

$$\varepsilon^{el} V_{ren} = V_0 + V_{ion} \qquad V_{ren} = \frac{V_0 + V_{ion}}{\varepsilon^{el}}$$

$$\varepsilon^{ion} V_{ren} = V_0 + V_{el}$$

$$\left.\right\} \quad V_{ren} = V_0 + V_{ion} + V_{el}$$

$$\Rightarrow \quad \varepsilon = \varepsilon^{el} + \varepsilon^{ion} - 1$$

renormalized potential

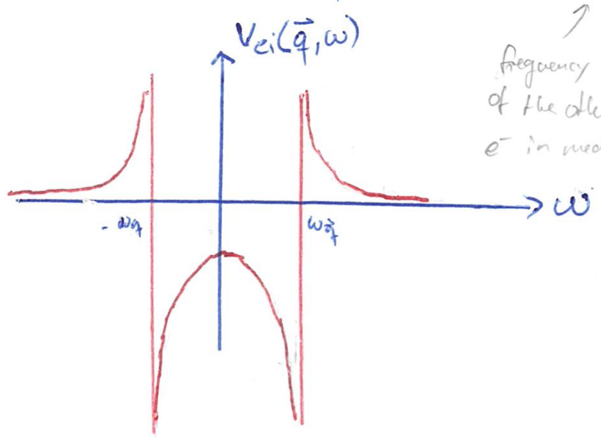
repulsive $e^- - e^-$ short ranged

V_{ei} dynamical interaction through ion polarization

$$\Rightarrow V_{ren}(\vec{q},\omega) = \frac{4\pi e^2}{q^2 + k_{TF}^2}\left\{1 + \frac{\omega_{\vec{q}}^2}{\omega^2 - \omega_{\vec{q}}^2}\right\} = V_{ee} + V_{ei} \qquad \text{(2-10) Tinkham}$$

frequency of the other e^- in media

phonon frequency



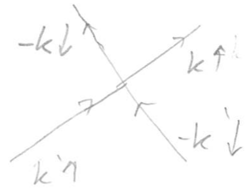
Pairing interaction

$V_{ei}(\vec{q},\omega)$ attractive within $-\omega_{\vec{q}} \leq \omega \leq \omega_{\vec{q}}$

$\omega \rightarrow$ energy transfer $\qquad \xi_{\vec{k}'} - \xi_{\vec{k}} = \hbar\omega$

$\vec{q} \rightarrow$ momentum transfer , $\vec{k}' - \vec{k} = q$

restrict to cooper channel: $|\vec{k}\uparrow; -\vec{k}\downarrow\rangle \longrightarrow |\vec{k}'\uparrow; -\vec{k}'\downarrow\rangle$



$$\mathcal{H}_{pair} = \frac{1}{2} \sum_{\vec{k},\vec{k}'} V_{\vec{k}\vec{k}'} \hat{c}^+_{\vec{k}\uparrow} \hat{c}^+_{-\vec{k}\downarrow} \hat{c}_{-\vec{k}'\downarrow} \hat{c}_{\vec{k}'\uparrow}$$

same but $V = -\frac{g}{L^3}$

$$\Rightarrow \quad \Delta_{\vec{k}} = -\sum_{\vec{k}'} V_{\vec{k},\vec{k}'} \langle \hat{c}_{-\vec{k}'\downarrow} \hat{c}_{\vec{k}'\uparrow} \rangle = -\sum_{\vec{k}'} V_{\vec{k},\vec{k}'} \frac{\Delta_{\vec{k}'}}{2E_{\vec{k}'}} \tanh\left(\frac{E_{\vec{k}'}}{2k_BT}\right)$$

$$V_{\vec{k},\vec{k}'} = V(\xi_k, \xi_{k'}) = \begin{cases} g/L^3, & |\xi_k|, |\xi_{k'}| \leq \hbar\omega_D \\ 0, & \text{otherwise} \end{cases}$$

largest possible frequency for $\omega_{\vec{q}}$

$$\Delta_{\vec{k}} = \Delta(\xi_{\vec{k}}) = \begin{cases} \Delta, & |\xi_k| \leq \hbar\omega_D \\ 0, & \text{otherwise} \end{cases}$$

natural cut off: $\epsilon_C = \epsilon_D$ $\sim 10^2$-$10^3 K$

$\epsilon_F \sim 10^4 K$

Coupling constant for e^--phonon

$$\Rightarrow \quad k_BT_C = 1{,}13\, \epsilon_D e^{-1/\lambda} = 1{,}13\, k_B \Theta_D e^{-1/\lambda} \quad , \quad \lambda = |g|\, N(\epsilon_F)$$

$g \approx -\frac{\epsilon_F}{6n} \Rightarrow \lambda \approx \frac{1}{4}$, $\lambda \sim 1$, but often $\lambda \ll 1$

Isotope effect:

$$\omega_D \propto \frac{1}{\sqrt{M}} \Rightarrow T_c M^\alpha = \text{const} \quad \text{with} \quad \alpha = 1/2$$

BCS Theory - Thermodynamics

p. 348 Abrikosov
$\rightarrow$ p. 35 Tinkham

Specific heat

for microscopic derivation of G.L.-functional

$\Omega = \tilde{K}_0 - 2k_B T \sum_k \log(1+e^{-\beta E_k})$, $E_k(\Delta)$

Ω as particle number not constant (5.74)

$$S_S(T) = -\frac{d\Omega(T,\Delta(T))}{dT} = -\underbrace{\frac{\partial\Omega}{\partial\Delta}}_{\text{minimized by definition}=0}\frac{\partial\Delta}{\partial T} - \frac{\partial\Omega(T,\Delta(T))}{\partial T}$$

(5.75)

$$\Rightarrow S_S(T) = -2k_B \sum_{\vec{k}} \left[f_{\vec{k}} \log f_{\vec{k}} + (1-f_{\vec{k}}) \log(1-f_{\vec{k}}) \right]$$

Quasiparticles — Quasi-holes

$$C_S(T) = \frac{T}{L^3}\frac{dS(T,\Delta(T))}{dT} = -\frac{2}{TL^3}\sum \frac{\partial f_k}{\partial E_k}\left(E_k^2 - \frac{T}{2}\frac{d\Delta(T)^2}{dT}\right)$$

redistribution of quasi-particles among energy states as T changes $\longrightarrow C_n(T)$

effect of temperature-dependent gap in changing the energy levels $\longrightarrow \Delta(T)$

Compare to $C_n(T) = \frac{2\pi^2}{3}N(\varepsilon_F)k_B^2 T = \gamma T$

For $T \ll T_c$: exponentially small, $\Delta \gg k_B T$

for $T \to T_{c-}$

$$\Rightarrow \Delta C = C_S(T_{c-}) - C_n(T_c) = \boxed{\frac{8\pi^2}{7\zeta(3)}N(\varepsilon_F)k_B^2 T_c} = \frac{a'^2}{bT_c^2}$$

$$\Rightarrow \frac{\Delta C}{C_n(T_c)} = 1{,}43 \text{ is universal}$$ $\Rightarrow$ second order transition

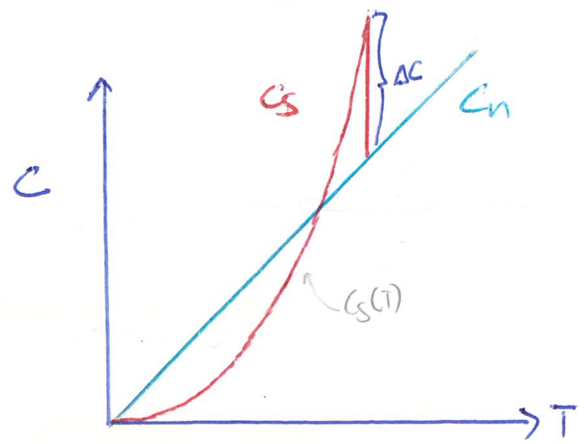


Fig 10.4 Abrikosov p351

Spin susceptibility

for microscopic derivation of G.L. functional

p. 62 Tinkham

Zeeman term to $\mathcal{H}_{BCS}$: $\mathcal{H}_Z = -\mu_B H_z \sum_{\vec{k}} \left\{ \hat{c}^{\dagger}_{\vec{k}\uparrow} \hat{c}_{\vec{k}\uparrow} - \hat{c}^{\dagger}_{-\vec{k}\downarrow} \hat{c}_{-\vec{k}\downarrow} \right\}$

$$\Rightarrow \quad \hat{H}_{\vec{k}} = \begin{pmatrix} \xi_{\vec{k}} - \mu_B H_z & -\Delta_{\vec{k}} \\ -\Delta^*_{\vec{k}} & -\xi_{\vec{k}} - \mu_B H_z \end{pmatrix}$$

Bogolyubov

split

$$\Rightarrow \quad \mathcal{H} = \sum_{\vec{k}} \left\{ E_{\vec{k}+} \hat{\gamma}^{\dagger}_{\vec{k}1} \hat{\gamma}_{\vec{k}1} + E_{\vec{k}-} \hat{\gamma}^{\dagger}_{-\vec{k}2} \hat{\gamma}_{-\vec{k}2} \right\} + \tilde{K}_0$$

$$E_{\vec{k}\pm} = \sqrt{\xi^2_{\vec{k}} + \Delta^2} \mp \mu_B H_z$$

H_z small

$$\Rightarrow \quad M_z \overset{\text{"def"}}{=} \mu_B \sum_{\vec{k}} \left\{ \langle \hat{c}^{\dagger}_{\vec{k}\uparrow} \hat{c}_{\vec{k}\uparrow} \rangle - \langle \hat{c}^{\dagger}_{-\vec{k}\downarrow} \hat{c}_{-\vec{k}\downarrow} \rangle \right\} \approx -2\mu_B^2 H_z \sum_{\vec{k}} \frac{\partial n_F(E_{\vec{k}})}{\partial E_{\vec{k}}}$$

Pauli

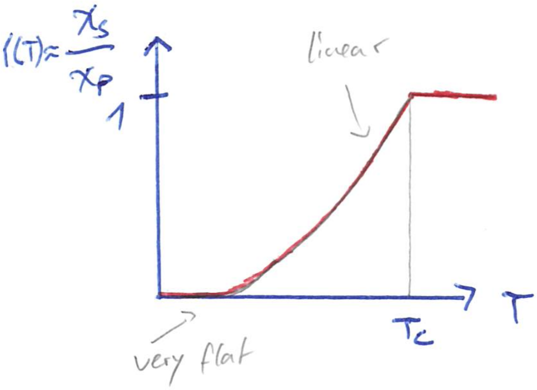
$$\chi = \frac{M}{H} \quad \Rightarrow \quad \chi_S(T) \approx \chi_P \, Y(T) \quad , \quad \chi_P = 2\mu_B^2 N(\varepsilon_F)$$

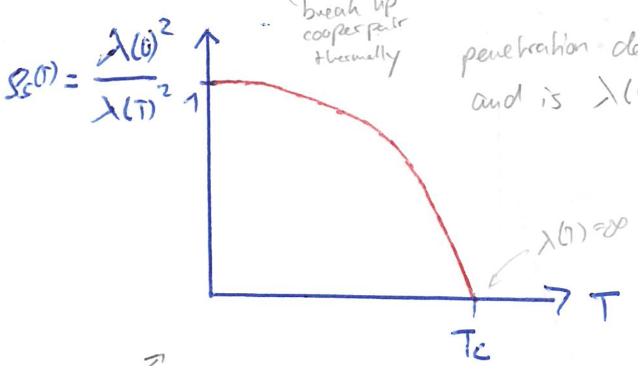
Yosida function, corresponds to the fraction of normal e^-

For $T \to T_{c-}$: $Y(T) = 1 - 2\left(1 - \frac{T}{T_c}\right)$ linear $\qquad Y(T) = \frac{n_n(T)}{n_0}$

For $T \to 0$: $Y(T) \approx \sqrt{2\pi\beta\Delta(0)}\, e^{-\beta\Delta(0)}$ exp $\qquad \Leftrightarrow \; n_S(T) = n_0(1 - Y(T))$

break up cooper pair thermally





penetration depth goes to ∞ for $T = T_c$ and is $\lambda(0)$ for $T = 0$

reflection

$T \lesssim T_c$

$$n_S(T) = 2n_0\left(1 - \frac{T}{T_c}\right)$$

$$\Rightarrow \quad |\psi|^2 = -\frac{a}{b} = \frac{1}{2} n_S = n_0 \frac{T_c - T}{T_c}$$

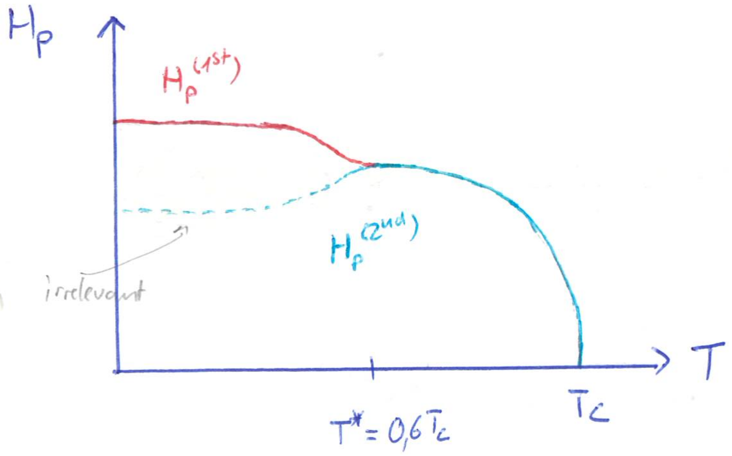
Paramagnetic limiting: p. 408 Abrikosov

Pauli - Clogston - Chandrasekar: break up of spin-singlet configuration due to strong magnetic field

$$\mu_B H_p^{(2nd)} \overset{\text{def.}}{=} \frac{\Delta(0)}{2} \quad , \quad G_s(H,T) - G_n(H,T)\Big|_{T=0} = -\frac{1}{2} N(\varepsilon_F)\Delta(0)^2 + \frac{1}{2}\chi_p H^2$$

$$= -\frac{1}{2} N(\varepsilon_F)\Delta(0)^2 + \mu_B^2 N(\varepsilon_F) H^2 \overset{!}{=} 0$$

$$\Rightarrow \mu_B H_p^{(1st)} = \frac{\Delta(0)}{\sqrt{2}} > \mu_B H_p^{(2nd)}$$



$$\frac{H_p(0)}{T_c} \approx 2\,\frac{T}{K}$$

Beware: only considers coupling to external field via e^--spin, while e^--charge through vector potential is neglected. Ignores orbital depairing. ← H_{c2}

Close to T_c: $H_{c2}(T) \propto |T_c - T|$, $H_p(T) \propto |T_c - T|^{1/2}$

WHH $\Rightarrow$ $\alpha = 4{,}34\,\dfrac{T_c}{T_F}$ $\begin{cases} \ll 1 & \Rightarrow \text{paramagnetic irrelevant} \\ \sim \underbrace{4{,}5}_{1??} & \Rightarrow \text{paramagnetic limiting } (CeCoIn_5) \end{cases}$

Bianchi: heavy electron mass suppresses orbital supercurrents

⤳ shows transition from first to second

Meissner screening and London penetration depth

Assume vector potential $\vec{A}$ $\Rightarrow$ $\mathcal{H}_{kin} = \frac{1}{2m} \sum_s \int d\vec{r} \{\vec{D}^* \hat{\psi}_s(\vec{r})^+\}\{\vec{D}\psi_s(\vec{r})\}$

$\vec{D} = \frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}$

$\frac{\partial \mathcal{H}}{\partial A}$
$$\Rightarrow \hat{\vec{j}} = \underbrace{\frac{\hbar e}{2mi} \sum_s [\hat{\psi}_s(\vec{r})^+ \{\vec{\nabla}\hat{\psi}_s(\vec{r})\} - \{\vec{\nabla}\psi_s(\vec{r})^+\}\psi_s(\vec{r})]}_{\hat{\vec{j}}_{para}} - \underbrace{\frac{e^2}{mc}\hat{\rho}(\vec{r})\vec{A}(\vec{r})}_{\vec{j}_{dia}}$$

19.20) Blatter QM

$n_0 = n_n + n_s$

$$\Rightarrow \langle \hat{j}_{para} \rangle = \frac{n_0 e^2}{mc} Y(T) \vec{A}_0 \quad , \quad \langle \vec{j} \rangle = \frac{n_0 e^2}{mc}[\underset{para}{Y(T)} \underset{dia}{-1}]\vec{A}_0 = -\frac{n_s(T)e^2}{mc}\vec{A}_0$$

$\mathrm{rot}\, B = \frac{4\pi}{c} j$

$$\Rightarrow \lambda(T)^{-2} = \frac{4\pi e^2}{mc^2} n_s(T) = \frac{4\pi e^2}{mc^2} n_0 \{1 - Y(T)\}$$

Paramagnetic current is gradually suppressed with lowering temperature since only normal electrons can contribute. For $T=0$, only diamagnetic contribution survives. For $T>T_c$, j_{para} cancels j_{dia} leading to the ohm's law. p.500 Blatter QM

a term in

Ginzburg-Landau functional

p.382 Abrikosov
p.179 Lifshitz footnote

$$F[\psi, \vec{A}; T] = F_n(T) + \int d^3r \left[a|\psi(\vec{r})|^2 + \frac{b}{2}|\psi(\vec{r})|^4 + \frac{1}{4m}|\vec{\pi}\psi(\vec{r})|^2 + \frac{\vec{B}(\vec{r})^2}{8\pi} \right]$$

$$\Rightarrow \Delta C = \frac{a'^2}{bT_c} = \frac{4\pi^2}{7\zeta(3)} N(\varepsilon_F) k_B^2 T_c$$

$n_s(T)$ at T_c susceptibility

$$\Rightarrow |\psi(T)|^2 = n_0 \frac{T_c - T}{T_c} = \frac{a'}{b}\frac{(T_c - T)}{T_c}$$

$|\psi(T)|^2 = \hat{\rho} n_s = \hat{\rho} n_0 (1 - Y(T)) \overset{T \approx T_c}{=} \hat{\rho}\, 2 n_0 \frac{T_c - T}{T_c}$, $\hat{\rho} = \frac{1}{2}$

$$\Rightarrow a' = \frac{4\pi^2}{7\zeta(3)} \frac{N(\varepsilon_F)}{n_0}(k_B T_c)^2 , \quad b = \frac{a'}{n_0}$$

$$\xi_0^2 = \frac{\hbar^2}{4ma'}$$

$$\Rightarrow F[\psi, \vec{A}; T] = F_n(T) + \int d\vec{r} \left[f_0 \left\{ \underbrace{\left(\frac{T}{T_c} - 1\right)}_{\tau} |\psi|^2 + \frac{1}{2n_0}|\psi|^4 + \xi_0^2 \left| \left(\frac{1}{i}\vec{\nabla} - \frac{2\pi}{\Phi_0}\vec{A}\right)\psi(\vec{r})\right|^2 \right\} + \frac{\vec{B}(\vec{r})^2}{8\pi} \right]$$

a' ; $\frac{b}{2} = \frac{1}{2n_0}, b = \frac{a'}{n_0}$

$$a' = f_0 = 1{,}51 \frac{N(\varepsilon_F)\Delta(0)^2}{n_0} \quad , \quad F - F_n = -1{,}51 \frac{N(\varepsilon_F)\Delta(0)^2}{2}\left(1 - \frac{T}{T_c}\right)^2$$

Ginzburg criterion and critical fluctuation region

Aluminum obeyes behaviour expected from our mean-field treatment (BCS and Ginzburg Landau).

Ginzburg criterion $\Rightarrow$ $$\frac{\Delta T}{T_c} = \frac{|T-T_c|}{T_c} \sim (k_F \xi(0))^{2d/d-4} \sim \left(\frac{T_F}{T_c}\right)^{2d/d-4}$$

For $d=3$: $$\frac{\Delta T}{T_c} \sim \left(\frac{T_F}{T_c}\right)^{-6} \ll 1$$

Al: $\frac{\Delta T}{T_c} \sim 10^{-31}$, normally $\sim 10^{-18} - 10^{-12}$, validity due to $k_F \xi(0) \gg 1$

mean spacing between electrons much smaller than the extension of the cooper pairs $\xi(0)$. $\Rightarrow$ Cooper pair overlaps with extremely many other cooper pairs and is in a strong average field.

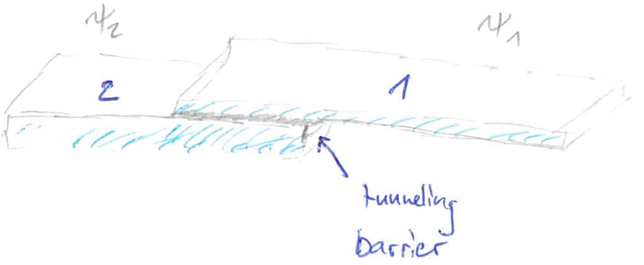
Beware, $k_F \xi(0) \sim 10$ for some superconductors $\Rightarrow$ critical region becomes visible. Cuprate high temperature superconductors among them.

Josephson effect

p. 192 Tinkham
p. 204 Lifshitz
p. 541 Abrikosov

DC and AC Josephson effect

$\Psi = \sqrt{N_{s,1}}\, e^{i\phi_1}$, $N_s = \frac{n_s}{2}$ density of Cooper pairs, satisfies $i\hbar \frac{\partial \Psi}{\partial t} = E\Psi$



$$\Rightarrow i\hbar \frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \mathcal{H}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}E_1 & K\\ K & E_2\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

K coupling constant

if we apply a voltage $U \Rightarrow$ Cooper pair tunneling gains/looses $2eU = E_1 - E_2$

$\varphi = \phi_1 - \phi_2$

DC-Josephson effect: Notice that $\frac{\partial N_{s1}}{\partial t} = -\frac{\partial N_{s2}}{\partial t}$

$$\Rightarrow I = 2e\frac{\partial N_{s1}}{\partial t} = \frac{4eK}{\hbar}\sqrt{N_{s1}N_{s2}}\,\sin\varphi = I_c \sin\varphi$$

Josephson critical current

if identical superconductor: $I_c = \frac{2eK}{\hbar} n_s$ phenomenological

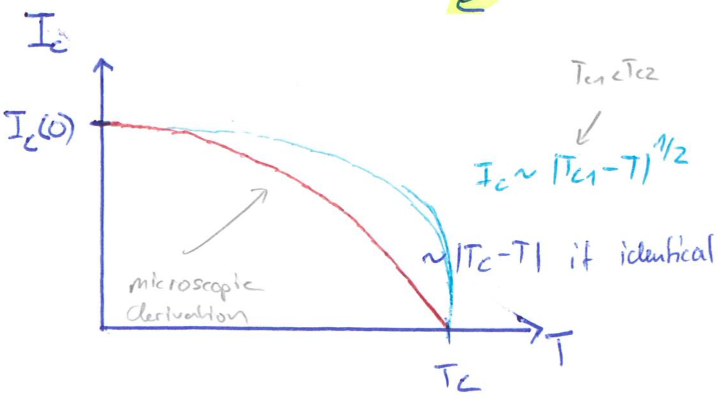
microscopic derivation $\Rightarrow$ $I_c(T) = \frac{\pi}{2eR_n}\Delta(T)\tanh\left(\frac{\Delta(T)}{2k_BT}\right)$ not derived

$T \to 0$, $\tanh \to 1$, $n_0 \to n_s$ $\Rightarrow K = \frac{\hbar \pi \Delta(0)}{4e^2 R_n n_0}$

R_n: resistance of the junction when in normal state

$T=0$: $I_c = \frac{\pi\Delta}{2eR}$

$\Rightarrow I_c R_n|_{T=0} \simeq 2{,}76\,\frac{k_B T_c}{e}$ independent of details of junction



AC Josephson effect: $\frac{\partial\varphi}{\partial t} = \frac{2e}{\hbar}V = \frac{E_1 - E_2}{\hbar}$ p.557 Abrikosov

Gauge invariance $\Rightarrow$

$\Rightarrow$ $I(t) = I_c \sin\left(\frac{2e}{\hbar}Vt\right)$ high oscillation frequency $\sim 10^{14}$ Hz

$\Rightarrow$ Josephson contact under a voltage generates electromagnetic radiation

$\nu = \frac{1}{2\pi}\frac{\partial\varphi}{\partial t}$, if $j > j_c \Rightarrow$ voltage

Junction energy

$E = \int P\,dt = \int U \cdot I\,dt$

See above

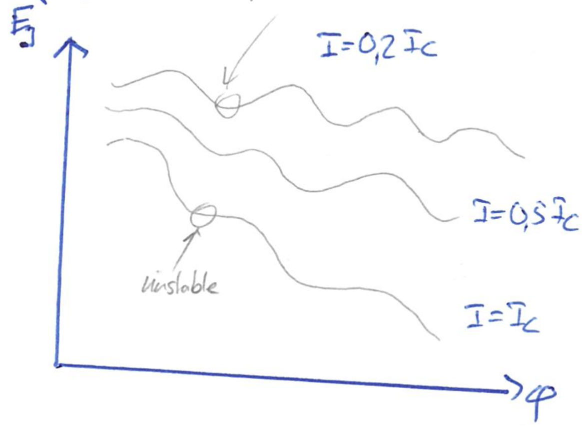
$$E_J = \int VI\,dt = \underbrace{\frac{\hbar}{2e}\frac{\partial\varphi}{\partial t}}_{V}\int \underbrace{I_c \sin\varphi}_{I}\,dt = \frac{\hbar}{2e}\int_0^{\varphi} I_c \sin\varphi'\,d\varphi' = \frac{\Phi_0 I_c}{2\pi c}\{1-\cos\varphi)$$

$\Rightarrow$ $\varphi = 0$ lowest energy state, $\varphi = \pi$ maximal energy.

Generally, $I(\varphi) = \frac{2\pi c}{\Phi_0}\frac{\partial}{\partial\varphi}E_J(\varphi)$ very general expression

If a bias current I is applied

$$E_J' = \frac{\Phi_0 I_c}{2\pi c}\{1-\cos\varphi\} - \frac{\Phi_0}{2\pi c}I\varphi$$

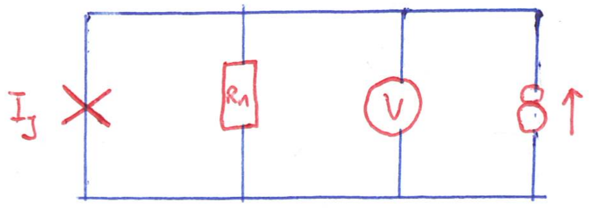


Equilibrium state is given by minimization of E_J' $\Rightarrow$ $0 \overset{!}{=} \frac{\partial E_J'}{\partial\varphi} \Rightarrow I = I_c \sin\varphi$

Current-voltage characteristics

See AC Josephson

$$I = \frac{V}{R_n} + I_c \sin\varphi = \frac{\hbar}{2eR_n}\frac{\partial\varphi}{\partial t} + I_c \sin\varphi$$



Josephson circuit

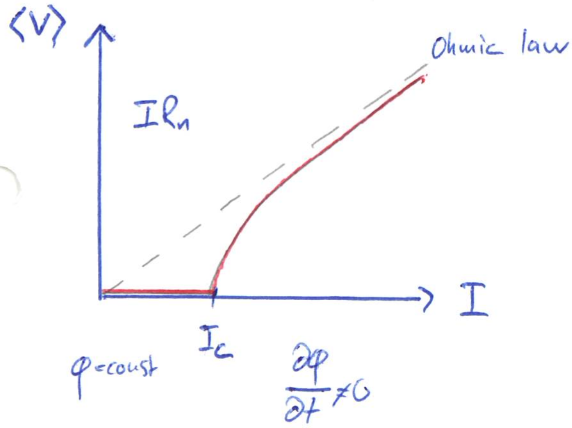
For $|I| \leq I_c \Rightarrow \quad \varphi = \arcsin\left(\frac{I}{I_c}\right)$

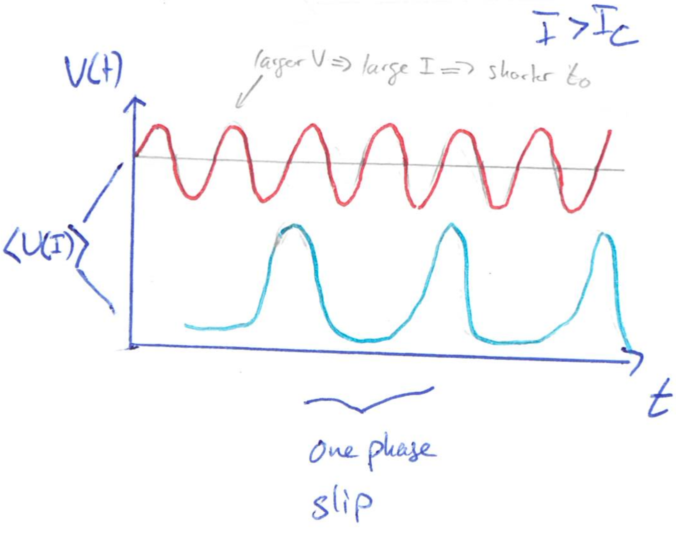
$\Rightarrow$ V over $\boxed{\times}$ exists $\Rightarrow \frac{\partial\varphi}{\partial t} \neq 0$

For $|I| > I_c$: supercurrent cannot pick up all current and the phase $\varphi(t)$ grows in time, $t_0 = \frac{\hbar}{2eR_n}\frac{2\pi}{\sqrt{I^2 - I_c^2}}$, time for one cycle

$$\Rightarrow \quad \langle V \rangle = R_n \sqrt{I^2 - I_c^2}$$ mean voltage over one cycle

$$\begin{cases} \text{For } I < I_c : & \varphi \text{ constant} \\ \text{For } I > I_c : & \varphi \text{ starts to run, phase slips, the faster, the higher the voltage.} \end{cases}$$





Shapiro steps

p.561 Abrikosov

Apply oscillatory voltage to the junction: $V(t) = V_0 + V_1 \cos(\omega t)$

$\uparrow$ DC

$\frac{\partial\varphi}{\partial t} = \frac{2e}{\hbar} V \Rightarrow$

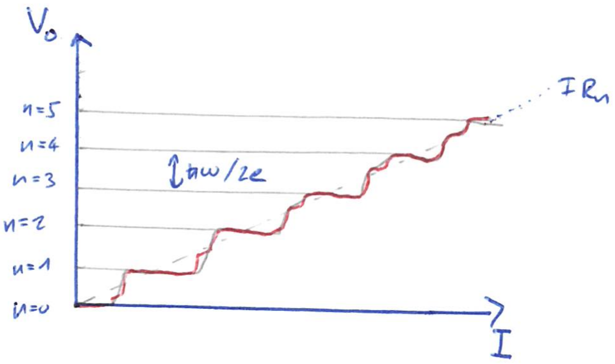
$$\frac{\partial \varphi}{\partial t} = \frac{2e}{\hbar} \underbrace{\{V_0 + V_1 \cos(\omega t)\}}_{V(t)} \Rightarrow \varphi(t) = \frac{2e}{\hbar}\left\{V_0 t + \frac{V_1}{\omega} \sin(\omega t)\right\}$$

$$I(t) = I_c \sin\Big[\underbrace{\frac{2e}{\hbar}\left\{V_0 t + \frac{V_1}{\omega} \sin \omega t\right\}}_{\varphi(t)} + \varphi_0\Big] + \frac{V_0}{R_n}$$

$$= I_c \sum_n J_n \left(\frac{2eV_1}{\hbar\omega}\right) \sin\left[\left(\frac{2e}{\hbar} V_0 + n\omega\right)t + \varphi_0\right] + \frac{V_0}{R_n}$$

$\uparrow$ Bessel

$\Rightarrow$ DC current whenever $n\hbar\omega = 2eV_0(n)$



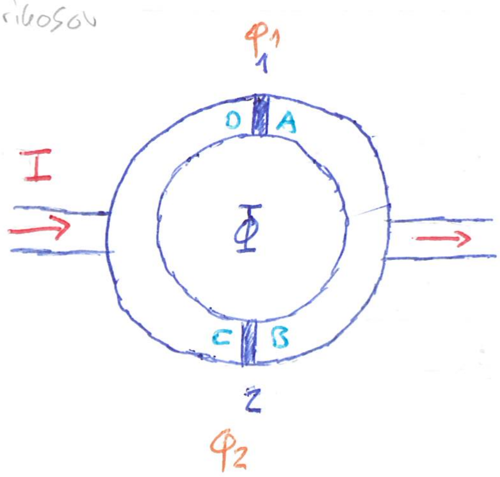
step heights independent of junction details

AC driven josephson junction

The SQUID concept

$\psi = \sqrt{n_s}\, e^{i\phi}$?? p 201 Tinkham
p. 568 Abrikosov

$\frac{e\hbar}{2mi}\{\psi^*\nabla\psi - \psi(\nabla\psi)^*\} - \frac{e^2}{mc}|\psi|^2\vec{A}$



$$\vec{J} = \frac{\hbar e n_s}{2m}\left(\vec{\nabla}\phi - \frac{2e}{\hbar c}\vec{A}\right)$$

$$\frac{2\pi\Phi}{\Phi_0} = \frac{2e}{\hbar c}\oint \vec{A}\cdot d\vec{s} = \int_A^B \vec{\nabla}\phi \cdot d\vec{s} + \int_D^C \vec{\nabla}\phi \cdot ds$$

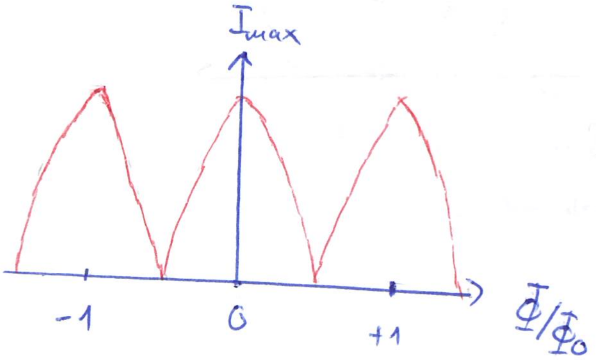
$$= \{\phi(A) - \phi(D)\} - \{\phi(B) - \phi(C)\} = \varphi_1 - \varphi_2$$

$$\Rightarrow I = I_{c1}\sin\varphi_1 + I_{c2}\sin\varphi_2 = I_{c1}\sin\left(\alpha + \frac{\pi\Phi}{\Phi_0}\right) + I_{c2}\sin\left(\alpha - \frac{\pi\Phi}{\Phi_0}\right)$$

expression skipped in Tinkham, no α introduced

for identical junctions $\Rightarrow$ $I = 2I_c\cos\left(\frac{\pi\Phi}{\Phi_0}\right)\sin\alpha$ $\Rightarrow$ $I_{max}(\Phi) = 2I_c\left|\cos\left(\frac{\pi\Phi}{\Phi_0}\right)\right|$

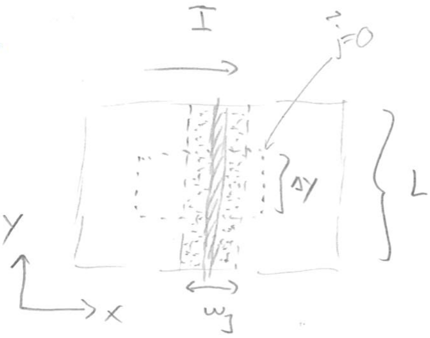
$I_{max}(\Phi)$ is periodic in Φ with a period Φ_0
$\Rightarrow$ analogon Aharanov-Bohm effect
For non identical junctions, I_{max} does not go to zero.



Interference pattern in a short Josephson junction

p. 552 Abrikosov

Junction has a finite width $L \ll \lambda_j$



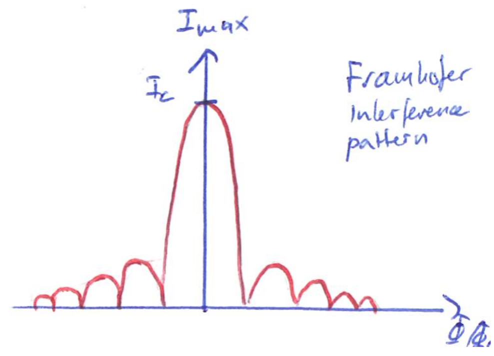
$$\frac{2\pi\Phi(y)}{\Phi_0} = \varphi(y+\Delta y) - \varphi(y)$$

$$\Phi(y) = B_z(y)(2\lambda_L + d)\Delta y = B_z(y)\, w_J\, \Delta y$$

width of tunneling barrier

$L \ll \lambda_J$ $\Leftrightarrow$ $B_z = \text{const.}$

$$\Rightarrow \varphi(y) = \frac{2\pi}{\Phi_0} H_z w_J y + \varphi_0 \Rightarrow I_{max} = I_c\left|\frac{\sin(\pi\Phi/\Phi_0)}{\pi\Phi/\Phi_0}\right|$$



p. 553 Abrikosov

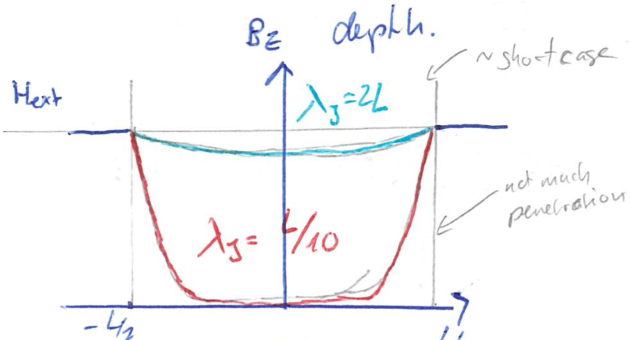
Long extended Josephson junction $L > \lambda_j$

$$\frac{\partial \varphi(y)}{\partial y} = \frac{2\pi}{\Phi_0} w_J B_z(y) \quad , \text{maxwell} \Rightarrow \quad \frac{\partial^2 \varphi}{\partial y^2} = \lambda_J^{-2} \sin\varphi$$

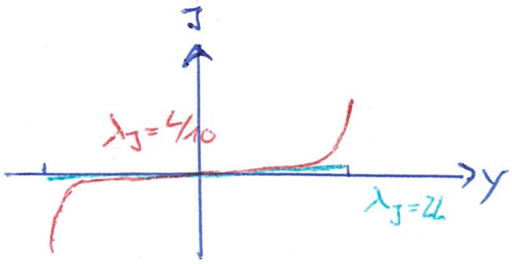
$$\lambda_J^{-2} = \frac{8\pi^2 w_J}{\Phi_0 c} j_c$$

Screening of magnetic fields: small $H_z \Rightarrow \varphi$ close to zero

$\Rightarrow \lambda_J$ screening length analogous to the London penetration depth.



Screening of field in extended Josephson junction



Short junctions: current uniformly through the extended contact

long junctions: Concentrated at the two edges $\sim \lambda_j$

Josephson vortices

p. 553 Abrikosov

long junction

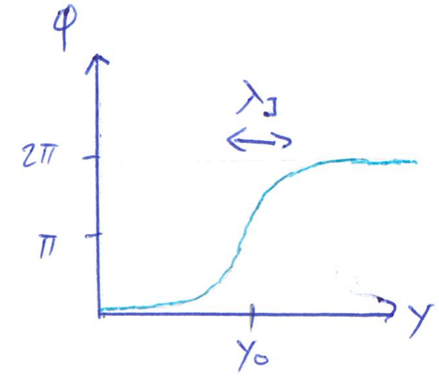
$$\frac{\partial^2 \varphi(y)}{\partial y^2} = \lambda_J^{-2} \sin \varphi(y)$$ is a variational equation of

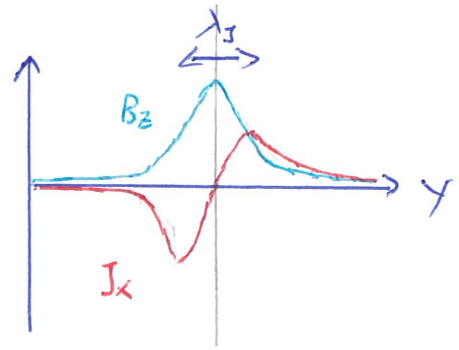
$$F_J(\varphi) = \frac{\Phi_0 j_c L'}{2\pi c} \int_{-L/2}^{L/2} dy \left\{ \underbrace{\frac{\lambda_J^2}{2} \left(\frac{\partial \varphi}{\partial y}\right)^2}_{\text{magnetic field energy density}} + \underbrace{(1-\cos\varphi)}_{\text{junction energy density}} \right\}$$

solved by

$$\Longrightarrow \quad \varphi_\pm(y) = 4 \arctan\left(e^{\pm(y-y_0)/\lambda_J}\right) \qquad \text{(soliton solution)}$$

$\Rightarrow$ Josephson vortex, one length scale λ_J





Bean Livingston

$$I_{c1} = 2{,}55\, j_c \lambda_J \quad , \quad I_{max} = 4 j_c \lambda_J \gg I_{c1}$$

The maximal current through such a long junction will be limited through the presence of such vortices. Currents generate force on vortices $\Rightarrow$ displacement along junction.
Beware time dependence of φ leads to voltage and thus dissipation.
Further, there is an analogy to a ~~Livingston~~ Bean-Livingston Barrier which prohibits nucleation at the boundary.