Linear Algebra & ODEs

$\mathbb{R}^{n\times m}$

General Matrix Properties: Let $A\in\mathbb{R}^{n\times m}$

range $(A)=\text{span}\{a_1,\dots,a_m\}$ is a subspace of $\mathbb{R}^n$

↳ Perform Gausselimin: For all pivots take the corresponding original column in A.

rank $(A)=\dim(\text{range}(A))$

null $(A)=\{x\in\mathbb{R}^m: Ax=0\}$ is a subspace of $\mathbb{R}^m$

= Set of vectors orthogonal to the rows of A

↳ Perform Gausselimin. for homogeneous Linear system: Introduce a parameter (s,t,..) for every linearly dependent set of columns. Span of solution with parameters in $\mathbb{R}$ is the nullspace.

A invertible ⇔ $\det(A)\neq 0$ ⇔ $Ax=y$ has a unique sol. ⇔ null$(A)=\{0\}$ ⇔ range$(A)=\mathbb{R}^n$ ⇔ All e-values are nonzero

Matrix Inverse ← Always unique if it exists

2x2 Matrix: $\begin{bmatrix} a & b\\ c & d\end{bmatrix}^{-1}=\frac{1}{\det(A)}\begin{bmatrix} d & -b\\ -c & a\end{bmatrix}$, $\det(A)=ad-bc$

3x3 Matrix: $\begin{bmatrix} a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}^{-1}=\frac{1}{\det(A)}\begin{bmatrix} ei-fh & ch-bi & bf-ce\\ fg-di & ai-cg & cd-af\\ dh-eg & bg-ah & ae-bd\end{bmatrix}$

$\det(A)=a\cdot\det\begin{bmatrix} e & f\\ h & i\end{bmatrix}-b\cdot\det\begin{bmatrix} d & f\\ g & i\end{bmatrix}+c\cdot\det\begin{bmatrix} d & e\\ g & h\end{bmatrix}$

Determinant with Laplace expansion ← signs

Block Diagonal Matrix: $A^{-1}=\begin{bmatrix} A_1 & 0\\ 0 & A_k\end{bmatrix}^{-1}=\begin{bmatrix} A_1^{-1} & 0\\ 0 & A_k^{-1}\end{bmatrix}$

E-Values & -Vectors: $Ax=\lambda x$, $\lambda\in\mathbb{C}$

→ $\det(A-\lambda I)=\text{chp}(\lambda)=0$ → solve for λ_i

→ insert found λ_i in $Av_i=\lambda v_i$ → solve for v_i

Hurwitz-Criterion: (only for 2x2 matrix)

$\text{chp}(\lambda)=\alpha\lambda^2+\beta\lambda+\gamma\overset{!}{=}0$

If α,β,γ have same sign ⇔ $\text{Re}\{\lambda_i\}<0$ $\forall i$

If α,β,γ not same sign ⇔ $\exists i: \text{Re}\{\lambda_i\}>0$

Diagonalizability: (AM = Algebraic, GM = Geometric Multiplicity)

If $AM_i=GM_i$ $\forall\lambda_i$ ⇒ A is diagonalizable

Thm: $AM\geq GM\geq 1$ always ⇒ if no λ_i repeated ⇒ diagonalizable

Cayley-Hamilton Thm:

Every Matrix $A\in\mathbb{R}^{n\times n}$ satisfies its characteristic polynomial.

$A^n+a_{n-1}A^{n-1}+\dots+a_1A+a_0I=0$ ⇔ $A^n=-(a_{n-1}A^{n-1}+a_{n-2}A^{n-2}+\dots+a_0I)$

⇒ If all powers of A are 0 until A^{n-1}, all higher terms $A^n, A^{n+1}\to 0$

Symmetric Matrices

have real e-values & orthogonal e-vectors ⇒ are always diagonalizable through orthogonal transformation.

- $x^TAx>0$ $\forall x\neq 0$ ⇔ A positive definite ⇔ $\lambda_i>0$ $\forall i$
- $x^TAx\geq 0$ $\forall x\neq 0$ ⇔ A positive semidefinite ⇔ $\lambda_i\geq 0$ $\forall i$

Canonical Jordan Forms:

$\begin{bmatrix} \lambda_1 & & \emptyset\\ & \ddots & \\ \emptyset & & \lambda_i\end{bmatrix}$ Jordan matrices are diagonalizable iff all Jordan blocks (blocks with same diagonal entries) are 1×1 matrices.

$GM_i=\#$ Blocks for λ_i, $AM_i=\#\lambda_i$

Lipschitz, Existence & Uniqueness of solutions:

Def.: $\|f(x,u)-f(\hat{x},u)\|^2=\|Ax-A\hat{x}\|^2\leq\sigma_i\|x-\hat{x}\|^2$ ⇔ f lipschitz

- $\frac{df}{dx}$ exists & is bounded ⇒ f lipschitz
- f linear ⇒ f lipschitz

Thm.: If either:

1) $f(x(t))$ (autonomous) is lipschitz in $x(t)$
2) $f(x(t),u(t),t)$ is lipschitz in $x(t)$, continuous in $u(t)$ & in t and $u(t)$ continuous for almost all t

⇒ State solution exists & is unique

Complex Analysis:

$\sin(z)=\frac{1}{2i}(e^{iz}-e^{-iz})$, $\cos(z)=\frac{1}{2}(e^{iz}+e^{-iz})$

	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
sin	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
cos	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1
tan	0	$\sqrt{3}/3$	1	$\sqrt{3}$	-	$-\sqrt{3}$	-1	$-\sqrt{3}/3$	0

Partial fraction decompos.:

double poles: $\frac{A_1}{x-p}+\frac{A_2}{(x-p)^2}$

Complex poles: $\frac{Ax+B}{x^2+px+q}$

Modeling

General System: $u(t)\to\boxed{\dot{x}(t)=f(t,x(t),u(t))}\to y(t)$ (Input, State, Output)

System Classification

- Time-Invariant: no explicit dependence on time
- Linear: Additivity & Homogeneity; $A(t),B(t),C(t),D(t)$ can depend on t
- Autonomous: time- & input-invariant: $\dot{x}(t)=f(x(t))$

To make a system time invariant, introduce t as a new state with $\dot{t}=1$. The new system has dimension $n+1$

LTI Systems (State Space Representation)

$$\begin{cases}\dot{x}(t)=Ax(t)+Bu(t) & \leftarrow \text{System Dynamics}\\ y(t)=Cx(t)+Du(t) & \leftarrow \text{Output measurement}\end{cases}$$

states ⇔ energy in the System

$A\in\mathbb{R}^{n\times n}$ $B\in\mathbb{R}^{n\times m}$ $C\in\mathbb{R}^{p\times n}$ $D\in\mathbb{R}^{p\times m}$

Coordinate Transforms

invertible ⇒ System properties remain unchanged. $\hat{A}$ has same ev as A as they are similar.

$\dot{x}(t)=Ax(t)+Bu(t)$, $y(t)=Cx(t)+Du(t)$ $\xrightarrow{z(t)=Tx(t)}_{x(t)=T^{-1}z(t)}$ $\dot{z}(t)=\hat{A}z(t)+\hat{B}u(t)$, $y(t)=\hat{C}z(t)+\hat{D}u(t)$

$\hat{A}=TAT^{-1}$, $\hat{B}=TB$, $\hat{C}=CT^{-1}$, $\hat{D}=D$

Modeling equations: Physical background

Electrical circuit

	Equation:	State:	Energy:
Inductor:	$v_L(t)=L\frac{\partial i_L(t)}{\partial t}$	$x(t)=i_L(t)$	$E_L(t)=\frac{1}{2}Li_L^2(t)$
Capacitor:	$i_c(t)=C\frac{\partial v_c(t)}{\partial t}$	$x(t)=v_c(t)$	$E_c(t)=\frac{1}{2}Cv_c^2(t)$

→ Kirchhoff eq → State Space representation

Mechanical system

	Equation:	State:	Energy:
Spring:	$F_s(t)=kp(t)$	$x(t)=p(t)$	(elastic) Energy: $E_s(t)=\frac{1}{2}kp^2(t)$
Damper:	$F_d(t)=d\dot{p}(t)$	$x(t)=\dot{p}(t)$	(kinetic) Energy: $E_d(t)=\frac{1}{2}m\dot{p}^2(t)$

Continuous LTI in Time Domain

$$\begin{cases}\dot{x}(t)=Ax(t)+Bu(t)=f(x(t),u(t),t)\\ y(t)=Cx(t)+Du(t)=h(x(t),u(t),t)\\ x(0)=x_0\end{cases}$$

(all LTI Systems have a unique solution)

State & Output Solution

$$x(t)=\Phi(t)x_0+\int_0^t\Phi(t-\tau)Bu(\tau)d\tau$$

ZIT ($u(t)=0$), ZIR; ($x_0=0$) ZST → $x(t)=(H*u)(t)$, ZSR → $y(t)=(K*u)(t)$

$$y(t)=C\Phi(t)x_0+\int_0^tC\Phi(t-\tau)Bu(\tau)d\tau+Du(t)$$

State Transition Matrix

$$\Phi(t)=e^{At}=\sum_{k=0}^{\infty}\frac{(At)^k}{k!}=I+At+\frac{A^2t^2}{2!}+\dots+\frac{A^kt^k}{k!}+\dots$$

If A diagonalizable: $A=W\Lambda W^{-1}$ ⇒ $e^{At}=We^{\Lambda t}W^{-1}$ $\Lambda,W\in\mathbb{C}$

If $A=N+D$ and N,D commute: $e^{At}=e^{Dt}\cdot e^{Nt}$ (N nilpotent, D diagonal)

If A nilpotent, calculate $e^{At}=\sum_{k=0}^{n}\frac{(At)^k}{k!}$, $A^{n+1}=0$, all lower powers $\neq 0$

Properties of State Transition Matrix

$\Phi(0)=I$, $\Phi(-t)=\Phi(t)^{-1}$ ⇒ $\Phi(t)\Phi(-t)=I$

$\frac{d\Phi(t)}{dt}=A\Phi(t)$, $\Phi(t_1+t_2)=\Phi(t_1)\Phi(t_2)$

Impulse Transition & Impulse Response

Let $u(t)=\delta(t)$ and $x_0=0$

Impulse Transition: $x(t)|_{u(t)=\delta(t)}:=H(t)=\Phi(t)B$ ZST: $x(t)=(H*u)(t)$

Impulse Response: $y(t)|_{u(t)=\delta(t)}:=K(t)=C\Phi(t)B+D\delta(t)$ ZSR: $y(t)=(K*u)(t)$

Stability

Definitions: A System is called

→ stable if $\forall\varepsilon>0$ $\exists\delta>0$: $\|x_0\|<\delta$ ⇒ $\|x(t)\|<\varepsilon$

→ asymptotically stable if the system is stable and $\lim_{t\to\infty}\|x(t)\|=0$

ZIT Stability

We have $x(t)=\Phi(t)x_0=e^{At}x_0$

A diagonalizable (or A non-diagonalizable but no e-value with $\text{Re}\{\lambda_i\}=0$ is repeated)

The system is...

→ stable iff $\text{Re}\{\lambda_i\}\leq 0$ $\forall i$

→ asym. stable iff $\text{Re}\{\lambda_i\}<0$ $\forall i$ } Trick: Use Hurwitz-Criterion

→ unstable iff $\exists i: \text{Re}\{\lambda_i\}>0$

A non-diagonalizable → Repeated e-values $\lambda_1=\dots=\lambda_r=\sigma\pm j\omega$

⇒ $\Phi(t)$ contains terms of form $e^{\lambda_it}, te^{\lambda_it},\dots,t^{r-1}e^{\lambda_it}$

$\sigma>0$ → $|t^ke^{\lambda_it}|\xrightarrow{t\to\infty}\infty$ ⇒ unstable

$\sigma=0$ → $r=1$ → $|e^{\lambda_it}|=$const. ⇒ stable; $r>1$ → $|te^{\lambda_it}|,\dots,|t^{r-1}e^{\lambda_it}|\xrightarrow{t\to\infty}\infty$ ⇒ unstable

$\sigma<0$ → $|t^ke^{\lambda_it}|\xrightarrow{t\to\infty}0$ ⇒ asymptotically stable.

⇒ system with non-diagonalizable A (& repeated λ_i with $\text{Re}\{\lambda_i\}=0$)

→ asym. stable iff $\text{Re}\{\lambda_i\}<0$ $\forall i$ } Trick: Use Hurwitz-Criterion

→ unstable if $\exists i: \text{Re}\{\lambda_i\}>0$

Stability with Inputs (ZST)

BIBS-stability: $\|u(t)\|\leq B_u<\infty$ ⇒ $\|x(t)\|\leq B_x<\infty$

BIBO-stability: $\|u(t)\|\leq B_u<\infty$ ⇒ $\|y(t)\|\leq B_y<\infty$

ZIT asymptotically stable ⇒ ZSR is BIBS & BIBO

Energy, Controllability, Observability

Energy: $E(t)=\frac{1}{2}x^TQx$, $Q=Q^T>0$ (symmetric & positive definite)

Power: $P(t)=\frac{dE(t)}{dt}=-\frac{1}{2}x^TRx+\frac{1}{2}[u(t)^TB^TQx(t)+x(t)^TQBu(t)]$ ($=0$ if ZIT ⇔ $u(t)=0$)

$R=R^T=-(A^TQ+QA)$

Lyapunov Stability (ZIT)

System is asym. stable (e-values of A have $\text{Re}\{\lambda_i\}<0$ $\forall i$) iff for any $R=R^T>0$ there exists a unique $Q=Q^T>0$ such that $A^TQ+QA=-R$ (Lyapunov eq.) holds

E.G.: A,R given → Find Q

→ $\exists!Q$ with $Q=Q^T>0$ ⇔ asym. stability

→ $\nexists Q$ or Q not unique or Q not sym. pos. def. ⇔ no asym. stability

Controllability

Can I steer $x(t)$ with $u(t)$?

Def: $\forall x(0)=x_0$ $\forall x_1$ $\exists u(\cdot):[0,t]\to\mathbb{R}^m$ s.t. $x(t)=x_1$

⇔ $\forall x(0)=x_0$ $\exists u(\cdot):[0,t]\to\mathbb{R}^m$ s.t. $x(t)=0$

⇔ $\forall x_1\in\mathbb{R}$ $\exists u(\cdot):[0,t]\to\mathbb{R}^m$ s.t. $x(t)=x_1$

⇔ controllable over $[0,T]$ ⇔ controllable over $[0,T']$ $\forall T'>0$

Controllability Gramian

symmetric, positive semi definite

$\mathbb{R}^{n\times n}\ni W_c(t)=\int_0^te^{A\tau}BB^Te^{A^T\tau}d\tau=W_c^T(t)\geq 0$

controllable over $[0,t]$ ⇔ $W_c(t)$ invertible ⇔ $\det(W_c(t))\neq 0$

Controllability Matrix

$P=[B\ \ AB\ \ A^2B\ \dots\ A^{n-1}B]\in\mathbb{R}^{n\times(n\cdot m)}$

Controllable over $[0,t]$ ⇔ rank$(P)=n$ $\overset{\text{For } m=1}{⇔}$ $\det(P)\neq 0$

Reachable States:

The set of reachable resp. controllable states is range (P)

The set of uncontrollable states is Null (P)

PHB test: $\lambda_1,\dots,\lambda_n$ e-values of A

rank $(\lambda_iI-A\ \ B)=n$ ⇒ λ_i is a controllable mode

→ if λ_i controllable $\forall i$ ⇒ System controllable

→ if all unstable λ_i controllable ⇒ system stabilizable

$\forall\lambda_i$ with $\text{Re}\{\lambda_i\}\geq 0$: System controllable ⇒ system stabilizable; system stable ⇒ system stabilizable

Minimum Energy input:

Thm: Assume system is controllable. Input that drives the system from $x(0)=0$ to $x(t)=x_1\in\mathbb{R}^n$, $t>0$ and has minimum energy is given by: $u_m(\tau)=B^Te^{A^T(t-\tau)}W_c(t)^{-1}x_1$ for $\tau\in[0,t]$

Observability

Can I determine $x(t)$ knowing $u(t)$ & $y(t)$?

Def.: System is called observable over $[0,t]$ if given $u(\cdot):[0,t]\to\mathbb{R}^m$ and $y(\cdot):[0,t]\to\mathbb{R}^p$, we can uniquely determine $x(\cdot):[0,t]\to\mathbb{R}^n$

⇔ $x\equiv 0$ is the only unobservable state

⇔ observable over $[0,T]$ ⇔ observable over $[0,T']$

$x'\in\mathbb{R}^n$ is unobservable ⇔ $CA^kx'=0$ $\forall k\in\{0,\dots,n-1\}$

Observability Gramian

symmetric, positive semi definite

$\mathbb{R}^{n\times n}\ni W_o(t)=\int_0^te^{A^T\tau}C^TCe^{A\tau}d\tau=W_o^T(t)\geq 0$

observable over $[0,t]$ ⇔ $W_o(t)$ invertible ⇔ $\det(W_o(t))\neq 0$

Observability Matrix

$Q=\begin{bmatrix} C\\ CA\\ \vdots\\ CA^{n-1}\end{bmatrix}\in\mathbb{R}^{(p\cdot n)\times n}$ (C $\in\mathbb{R}^{p\times n}$, CA $\in\mathbb{R}^{p\times n}$)

observable over $[0,t]$ ⇔ rank$[Q]=n$ $\overset{\text{for } p=1}{⇔}$ $\det(Q)\neq 0$

Observable States:

The set of observable states is range(Q)

The set of unobservable states is Null(Q)

PHB test: $\lambda_1,\dots,\lambda_n$ e-values of A

rank $\begin{bmatrix}\lambda_iI-A\\ C\end{bmatrix}=n$ ⇒ λ_i is an observable mode

→ if λ_i observable $\forall i$ ⇒ System observable

→ if all unstable λ_i observable ⇒ system detectable

$\forall\lambda_i$ with $\text{Re}\{\lambda_i\}\geq 0$: System observable ⇒ system detectable; system stable ⇒ system detectable

Observers & Error dynamics

$\dot{e}(t)=(A-LC)e(t)$, $e(t)=x(t)-\tilde{x}(t)$ (observer, estimate)

System observable ⇔ $\exists L$ s.t. $(A-LC)$ is asym. stable

⇔ $e(t)\xrightarrow{t\to\infty}0$ ⇔ state estimate $\tilde{x}(t)$ converges to true $x(t)$

LTI-Systems in Frequency Domain

Laplace Transform: $F(s) = \mathcal{L}[f(t)](s) = \int_0^\infty f(t)e^{-st}dt$

$af(t)+bg(t) \circ\!\!-\!\!\bullet\ aF(s)+bG(s)$ | $tf(t) \circ\!\!-\!\!\bullet -\frac{d}{ds}F(s)$ | $f(at) \circ\!\!-\!\!\bullet \frac{1}{a}F(\frac{s}{a})$

$e^{-\alpha t}f(t) \circ\!\!-\!\!\bullet\ F(s+\alpha)$ | $t^n f(t) \circ\!\!-\!\!\bullet (-1)^n \frac{d^n}{ds^n}F(s)$ | $f(t-a) \circ\!\!-\!\!\bullet e^{-as}F(s)$

$f'(t) \circ\!\!-\!\!\bullet\ sF(s)-f(0)$ | $(f*g)(t) \circ\!\!-\!\!\bullet F(s)\cdot G(s)$

$f^{(n)}(t) \circ\!\!-\!\!\bullet\ s^nF(s)-s^{n-1}f(0)-\dots-s^0f^{(n-1)}(0)$ | $\int_0^t f(\tau)d\tau \circ\!\!-\!\!\bullet \frac{1}{s}F(s)$

Initial value thm: $f(0) = \lim_{t\to 0} f(t) = \lim_{s\to\infty} s\cdot F(s)$

Final value thm: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\cdot F(s)$

Important Transformations:

$1 \circ\!\!-\!\!\bullet \frac{1}{s},\ s>0$ | $e^{at} \circ\!\!-\!\!\bullet \frac{1}{s-a},\ s>a$

$t^n \circ\!\!-\!\!\bullet \frac{n!}{s^{n+1}},\ s>0$ | $e^{at}\sin(bt) \circ\!\!-\!\!\bullet \frac{b}{(s-a)^2+b^2},\ s>a$

$\delta(t) \circ\!\!-\!\!\bullet 1$ | $\sin(at) \circ\!\!-\!\!\bullet \frac{a}{s^2+a^2},\ s>0$ | $e^{at}\cos(bt) \circ\!\!-\!\!\bullet \frac{s-a}{(s-a)^2+b^2},\ s>a$

$\cos(at) \circ\!\!-\!\!\bullet \frac{s}{s^2+a^2},\ s>0$ | $t^n e^{at} \circ\!\!-\!\!\bullet \frac{n!}{(s-a)^{n+1}},\ s>a$

LTI-Systems:

$\mathcal{L}[e^{At}](s) = \mathcal{L}[\Phi(t)](s) = (sI-A)^{-1} \in \mathbb{C}^{n\times n}$

$$\begin{cases} X(s) = (sI-A)^{-1}x_0 + (sI-A)^{-1}BU(s) \\ Y(s) = CX(s)+DU(s) = C(sI-A)^{-1}x_0 + G(s)U(s) \end{cases}$$

For $x_0=0$ (ZIT): $Y(s) = \boxed{[C(sI-A)^{-1}B+D]}U(s) = G(s)U(s)$

$G(s)$ transfer function

$$G(s) = \frac{(s-z_1)(s-z_2)\dots(s-z_k)}{(s-p_1)(s-p_2)\dots(s-p_n)}$$

$k\le n \Leftrightarrow G(s)$ proper $\Rightarrow \lim_{\omega\to\infty}|G(\pm j\omega)|<\infty$

$k<n \Leftrightarrow G(s)$ strictly proper $\Rightarrow \lim_{\omega\to\infty}|G(\pm j\omega)|=0$

No pole-zero-cancellation $\Leftrightarrow$ Denominator $=\text{chp}(A)$ $\Rightarrow$ e-values of A = poles of G(s)

Thm:
- SISO $G(s)$ from LTI-Systems are always proper
- SISO $G(s)$ from LTI-Systems are strictly proper iff $D=0$

(Single) Frequency Response:

$u(t)=\sin(\omega t)$, $y(t)=K\sin(\omega t+\varphi)$ | $G(s) = |G(s)|\cdot e^{i\angle G(s)} = K\cdot e^{i\varphi}$

Stability: (Requirement: no pole-zero cancellations!)

distinct poles:
- asym. stable iff $Re\{p_i\}<0\ \forall i$
- stable iff $Re\{p_i\}\le 0\ \forall i$
- unstable iff $\exists i: Re\{p_i\}>0$

repeated poles:
- asym. stable iff $Re\{p_i\}<0\ \forall i$
- unstable if $\exists i: Re\{p_i\}>0$
- else: further investigation necessary

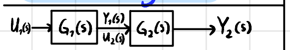
Marginally stable $\Leftarrow Re\{p_i\}\le 0$, $Re\{p_i\}=0 \ \&\ Jm\{p_i\}\ne 0$

→ If we have pole-zero cancellations, only BIBO-stability is determinable. (No info on marginal stability etc.) Pole-zero cancellation corresponds to a loss of controllability and/or observability

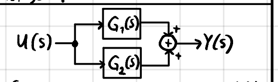
→ If $G(s)$ has less poles than the dimension of A, we know that pole-zero cancellations happened

→ Find number of pole-zero cancellations: Calculate which modes λ are uncontrollable and/or unobservable using PHB-test, If observability and/or controllability PHB-Matrix for a λ_i do not have full rank ⇒ λ_i gets pole-zero cancelled. Count these λ_i

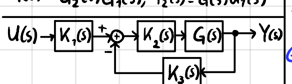
Block Diagrams:



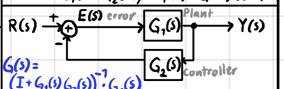
$G(s) = G_2(s)G_1(s)$, $Y_2(s) = G(s)U_1(s)$



$G(s) = G_1(s)+G_2(s)$, $Y(s) = G(s)U(s)$



$Y(s) = (I+G(s)K_2(s)K_3(s))^{-1}G(s)K_2(s)K_1(s)U(s)$

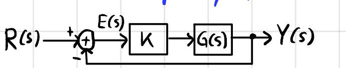


$G(s) = (I+G_1(s)G_2(s))^{-1}G_1(s)$

$E = R - G_2Y = R - G_2G_1E \Rightarrow E[1+G_2G_1]=R$

$\Rightarrow \frac{E}{R} = \frac{1}{1+G_2G_1} \Rightarrow \frac{Y}{R} = \frac{G_1}{1+G_2G_1} = G$

Closed Loop Systems:



$Y(s) = \frac{KG(s)}{1+KG(s)}\cdot R(s)$

$E(s) = \frac{1}{1+KG(s)}\cdot R(s)$

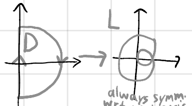
Deduction: See Block diagrams with $G_1(s)=KG(s)$, $G_2(s)=I$

We want $e(t) \xrightarrow[t\to\infty]{} 0$ (asymptotic stability)

⇒ Closed loop stability determined by roots of $1+KG(s)$

Fact: open loop poles = closed loop zeros

Principle of the Argument:



Assume clockwise D-curve that does not pass through poles/zeros of $G(s)$

always symm. wrt. real axis

$N = Z - P$

N: # clockwise encirclements of (0,0) by L-curve

Z: # of (pos.) zeros in D

P: # of (pos.) poles in D

Nyquist Stability Criterion for $K(s)=K\in\mathbb{R}$

Consider Nyquist plot of $1+KG(s)$ / $KG(s)$ / $G(s)$

→ N = # times L encircles $(0,0)$ / $(-1,0)$ / $(-\frac{1}{K},0)$ clockwise

→ Z = # closed loop unstable (RHP) poles (roots of $1+KG(s)=0$)

→ P = # open loop unstable (RHP) poles (poles of $G(s)$)

Principle of the Argument ⇒ $N=Z-P$

For closed loop stability we aim for $Z=0$

Closed loop system is asym. stable iff $N=-P$

Corollary: If open loop system is stable, the closed loop system is stable iff Nyquist plot makes no encirclements of $(-1,0)$ / $(-\frac{1}{K},0)$

Bode Plots: $G_{dB} = 20\cdot\log_{10}(G)$

$G(j\omega) = k_0\cdot(j\omega)^N\cdot G_1(j\omega)\cdot G_2(j\omega)\dots$

If $G(j0)$ exists calculate $|G(j0)|$ and start magn. plot there

If $G(j0)$ is negative start with $\pm180°$ phase otherwise with 0° phase

Else: start with smallest nonzero ω, $F_{dB}(\omega) = 20\cdot\log_{10}(|G(j\omega)|)$

		Magnitude		Phase	
$1/s^N$	Integrator	$-20\cdot N$ dB/dec	everywhere	$-90°\cdot N$	everywhere
s^N	Differentiator	$+20\cdot N$ dB/dec	everywhere	$+90°\cdot N$	everywhere
$1/(\alpha+s)^N$	LHP pole	$-20\cdot N$ dB/dec		$-90°\cdot N$	$0.1\omega - 10\omega$
$1/(\alpha-s)^N$	RHP pole	$-20\cdot N$ dB/dec		$+90°\cdot N$	$0.1\omega - 10\omega$
$(\alpha+s)^N$	LHP zero	$+20\cdot N$ dB/dec		$+90°\cdot N$	$0.1\omega - 10\omega$
$(\alpha-s)^N$	RHP zero	$+20\cdot N$ dB/dec		$-90°\cdot N$	$0.1\omega - 10\omega$

Resonance: (2nd order sys.) $G(s) = \frac{k\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

ex. spring mass damper

Natural freq. $\omega_n = \sqrt{k/M}$

Damping ratio $\zeta = \frac{d}{2\sqrt{kM}}$

1. For stability need $\zeta\ge 0$
2. For $\zeta\ge 1$ overdamped (real poles)
3. For $\zeta=1$ critically damped (real & equal p)
4. For $0\le\zeta<1$ underdamped (complex p)
5. For $\zeta=0$ undamped (imaginary p)
6. For $\zeta\ge\frac{1}{\sqrt2}$ magnitude plot decreasing in ω
7. For $0\le\zeta\le\frac{1}{\sqrt2}$ magnitude plot has maximum (Resonance) at $\omega=\omega_n\sqrt{1-2\zeta^2}$, $|G(\omega)| = \frac{k}{2\zeta\sqrt{1-\zeta^2}}$

Bode Stability Criterion:

Assume $G(s)$ asym. stable and magnitude & phase bode plot are monotonically decreasing. Then the Closed loop system (CLS) is asymptotically stable iff

$|KG(j\omega)|<1$ at the frequency where $\angle G(j\omega) = -180°$

Small Gain Thm: Under above conditions, if $|KG(j\omega)|<1$ $\forall\omega$ ⇒ CLS is asym. stable.

For GM, PM: Assume OLS G is asym. stable.

Gain Margin: = maximum gain K_m s.t. CLS still stable

$GM = \frac{1}{|G(j\omega_u)|}$ where $\angle G(j\omega_u) = -180°$ (Phase crossover frequency)

Phase Margin: = maximum phase Φ_m s.t. CLS still stable

$PM = \angle G(j\omega_c)+180°$ where $|G(j\omega_c)|=1=0dB$ (gain crossover frequency)

If ω_u does not exist ⇒ $GM\to\infty$, If ω_c does not exist ⇒ $PM\to\infty$

Discrete LTI-Systems in Time Domain

$$\begin{cases} x_{k+1} = Ax_k + Bu_k \\ y_k = Cx_k + Du_k \\ x(0)=x_0 \end{cases}$$

State solution: $x_k = \underbrace{A^k x_0}_{(ZIT)} + \underbrace{\sum_{i=0}^{k-1} A^{k-i-1}Bu_i}_{(ZST)}$

- A diagonalizable: $A = W\Lambda W^{-1} \Rightarrow A^k = W\Lambda^k W^{-1}$
- at equilibrium: $x_{k+1}=x_k$

Stability:

A diagonalizable, The System is...
- → stable iff $|\lambda_i|\le 1\ \forall i$
- → asym. stable iff $|\lambda_i|<1\ \forall i$
- → unstable iff $\exists i: |\lambda_i|>1$

A non-diagonalizable, The System is...
- → asym. stable iff $|\lambda_i|<1\ \forall i$
- → unstable if $\exists i: |\lambda_i|>1$
- → else: No statement. It depends on e-value with $|\lambda|=1$

Lyapunov Stability: (in the discrete case)

$|\lambda_i|<1\ \forall i$ iff $\forall R=R^T>0$ the eq. $A^TQA-Q=-R$ has a unique solution with $Q=Q^T$.

Deadbeat Response: (not possible in continuous) e-values of A: $\lambda_i=0\ \forall i$

$\Rightarrow A^N=0\quad N\le n$ (nilpotent Matrix)

$\Rightarrow$ ZIT $x_k = A^kx_0 = 0\quad \forall k\ge N$

Controllability & Observability (Power is the same as in cont.)

check the same way as in continuous case. Difference: $m,m'\in\mathbb{N}$ controllability for $[0,m]$ $\cancel{\Leftrightarrow}$ controllability for $[0,m']$ as we require at least $k\ge n$ (system dim.) steps to steer system from initial to final state

Sampling: $x((k+1)T) = \underbrace{e^{AT}}_{A\ \text{discrete}}x(kT) + \underbrace{\int_0^T e^{A(T-\tau)}Bd\tau}_{B\ \text{discrete}} * u(kT)$ $\quad \tilde{C}=C\quad \tilde{D}=D$

sampling autonomous cont. time systems never leads to nilpotent discrete time sys.

Forward euler approx.: $x_{k+1} = (I+\delta A)x_k + \delta Bu_k$

If autonomous: $x((k+1)\delta) = e^{A\delta}x(k\delta) \approx (I+A\delta)x_k$, $\delta := T/N$ = step

Minimum Energy input: $U = \tilde{P}(\tilde{P}\tilde{P}^T)^{-1}(\hat{x}_N - A^N\hat{x}_0)$, $\tilde{P} = [B\ AB\ \cdots\ A^{n-1}B]$

Z-Transform

$F(z) = \mathcal{Z}\{f_k\} = \sum_{k=0}^\infty f_k z^{-k}$

Linearity: $\mathcal{Z}\{af_k+bg_k\} = aF(z)+bG(z)$ | $1_k \circ\!\!-\!\!\bullet \frac{z}{z-1}$ ($1_k=1$ $k\ge0$ sonst 0)

Time shift: $\mathcal{Z}\{f_{k-k_0}\} = z^{-k_0}F(z)$ | $\delta_k \circ\!\!-\!\!\bullet 1$ ($\delta_0=1$, $\delta_k=0$ $\forall k\ne0$)

Convolution: $\mathcal{Z}\{(f*g)_k\} = F(z)\cdot G(z)$ | $a^k \circ\!\!-\!\!\bullet \frac{z}{z-a}$

Final value thm: $\lim_{n\to\infty} x_n = \lim_{z\to1}(1-z^{-1})X(z)$

Initial value thm: $\lim_{n\to0} x_n = \lim_{z\to\infty} X(z)$

Transfer function: $G(z) = \frac{Y(z)}{U(z)} = C(zI-A)^{-1}B+D$ (poles of G(z))

no pole-zero canc. ⇒ denominator of $G(z) = \det(A-zI)$ ⇒ (asym. stability ⇔ $|p_i|<1\ \forall i$)

⇒ System is both observable AND controllable

Geometric Series: $\sum_{k=0}^\infty q^k = \frac{1}{1-q}$, $\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$

Nonlinear Systems

We only look at autonomous, time-invariant systems

$\dot{x}(t) = f(x(t))$ $x(t)\in\mathbb{R}^n$, f lipschitz ⇔ sol. exists & is unique

Invariant Sets

A set of states $S\subseteq\mathbb{R}^n$ is called invariant if $\forall x_0\in S\ \forall t\ge0: x(t)\in S$

Equilibrium Let $\hat{x}$ be an equilibrium

Conditions for $\hat{x}$: continuous $\dot{\hat{x}}=0$ | discrete $\hat{x}_{k+1}=\hat{x}_k=\hat{x}$

If you start at $\hat{x}$, you will stay there forever

all equilibrium points of a system form an invariant set

Linear sys.: A regular ⇒ 1 eq. pt. | A singular ⇒ ∞ eq. pt.

nonlinear sys.: $f(\hat{x})=0$ with $\dot{x}(t)=$ | 1 | $-x(t)$ | $p(t)$ (poly.) | integer cardinality | subspace uncount.

has # solutions | 0 | 1 | finitely many | count. ∞ | ∞

Shifting Equilibria to the origin: $\omega(t) := x(t)-\hat{x}$

Periodic Orbits: Sol. $x(t)$ s.t. $\exists T>0$, $\forall t\ge0$ $x(t+T)=x(t)$ (equil. are trivial periodic orbits (T=0))

Van der Pol Oscillator: $\ddot{\theta}(t) - \varepsilon(1-\theta^2(t))\dot{\theta}(t)+\theta(t)=0$

Stability of Equilibria:

Let $\hat{x}$ be an equilibrium

Def.: $\hat{x}$ is stable ⇔ $\forall\varepsilon>0\ \exists\delta>0: \|x_0-\hat{x}\|<\delta \Rightarrow \|x(t)-\hat{x}\|<\varepsilon\ \forall t\ge0$

Otherwise the equilibrium is unstable

Local asymptotic stability (LAS): The equilibrium $\hat{x}$ is called LAS if (i) $\hat{x}$ is stable and (ii) $\exists M>0: \|x_0-\hat{x}\|<M \Rightarrow \lim_{t\to\infty}x(t)=\hat{x}$

Global asymptotic stability (GAS): Same as LAS but must hold $\forall M\ge0$

System has more than one equilibrium ⇒ ∄ a GAS equilibrium

Lyapunov first/indirect Method (by Linearization)

$$\dot{x}(t) = f(x(t)) = \begin{bmatrix} f_1(x_1,\dots,x_n) \\ \vdots \\ f_n(x_1,\dots,x_n) \end{bmatrix} = \underbrace{f(\hat{x})}_{=0} + \underbrace{\frac{d}{dx}f(x)\Big|_{\hat{x}}}_{A}\cdot\underbrace{(x(t)-\hat{x})}_{:=\delta x(t)} + \underbrace{O(\|x(t)-\hat{x}\|^2)}_{\to0}$$

$$A := J[f(x)]\Big|_{\hat{x}} = \begin{bmatrix} \frac{\partial}{\partial x_1}f_1(\hat{x}) & \cdots & \frac{\partial}{\partial x_n}f_1(\hat{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1}f_n(\hat{x}) & \cdots & \frac{\partial}{\partial x_n}f_n(\hat{x}) \end{bmatrix} \Rightarrow \dot{x}(t)\approx A\delta x(t)\quad \text{linear}$$

If $Re\{\lambda_i\}<0\ \forall i$ ⇒ linearized system asym. stable ⇒ nonlinear System is locally asym. stable around $\hat{x}$

If $\exists i: Re\{\lambda_i\}>0$ ⇒ unstable

Inconclusive if $\exists i: Re\{\lambda_i\}=0$ & others have negative real part. (no information about domain of attraction)

Lyapunov second/direct Method

Assume ∃open set $S\subseteq\mathbb{R}^n$ with $\hat{x}\in S$, $V(\cdot):\mathbb{R}^n\to\mathbb{R}$ differentiable

(i) $V(\hat{x})=0$ (ii) $V(x)>0\ \forall x\in S\setminus\{\hat{x}\}$ (iii) $\frac{d}{dt}V(x(t))\le0\ \forall x\in S$

⇒ the equilibrium $\hat{x}$ is stable

If additionally (iv) $\frac{d}{dt}V(x(t))<0\ \forall x\in S\setminus\{\hat{x}\}$ ⇒ $\hat{x}$ is locally asym. stable

If additionally (v) $\|x\|\to\infty \Rightarrow \|V(x)\|\to\infty \Rightarrow \hat{x}$ is globally asym. stable

$\frac{d}{dt}V(x(t)) = (\nabla V(x))^T\cdot f(x)$

La Salle's Theorem

mostly given

Assume ∃ compact (bounded & closed) invariant set $S\subseteq\mathbb{R}^n$ and ∃ $V(\cdot): S\to\mathbb{R}$ differentiable such that

1) $\frac{d}{dt}V(x(t)) = (\nabla V(x))^T\cdot f(x)\le0\quad \forall x\in S$

2) Let $\bar{S} = \{x\in S: \frac{d}{dt}V(x(t))=0\}\subseteq S$

3) Let M be the largest invariant set in $\bar{S}$

Then: All trajectories $x(t)\in S \xrightarrow[t\to\infty]{} M$ ⇒ LAS

If $S\to\mathbb{R}^n$ and all conditions above still hold ⇒ GAS

We either get: $S_k = \{x(t)\in\mathbb{R}^2: V(x)<k\}$

$\frac{\partial V(x(t))}{\partial t}\le0\ \forall x\in S_k \Rightarrow S_k$ is invariant

→ Find $\bar{S}\subseteq S_k$, → Find $M\subseteq\bar{S}$ (mostly (0,0))

→ Show that M is the maximal invariant set in $\bar{S}$ i.e. show that all $x\in\bar{S}\setminus M$ would leave the invariant set

La Salle ⇒ $\forall x(t)\in S_k \xrightarrow[t\to\infty]{} M$, If $k>0$ can be chosen arbitrarily when $\|x(t)\|\to\infty \Rightarrow V(x)\to\infty$ ⇒ GAS

Or we get: $S = \{x: x_1\in[a,b], x_2\in[c,d]\}$

Prove S invariant by calculating derivatives on boundary ⇒ Trajectories starting in S never leave S ⇒ S is invariant.

Find $\bar{S}$, Find M. If $\|x(t)\|\to\infty \Rightarrow V(x(t))\to\infty$ ⇒ GAS

Nyquist from Bode:

Set $\omega=0$ and read Magn. & Phase from Bode

Set $\omega\to\infty$ and read Magn. (0) & Phase from Bode

For a couple of angles e.g.: $\Phi = 180°, 90°, 45°, \dots$ read Magn. & Phase from Bode & plot in Nyquist: $|z|e^{i\Phi} \to a+ib$

Nyquist from Transfer Function:

1. Plug in $s=j\omega \to G(j\omega)$
2. Seperate $G(j\omega) = Re\{G(j\omega)\}+j\,Im\{G(j\omega)\}$
3. Find intersections with real & imaginary axis
4. Analyse phase as $\omega\to\infty$

$\measuredangle a+ib = \begin{cases} -\pi+\tan^{-1}(\frac{b}{a}) & \text{if } a<0\ \&\ b<0 \\ \tan^{-1}(\frac{b}{a}) & \text{if } a>0 \\ \pi+\tan^{-1}(\frac{b}{a}) & \text{if } a<0\ \&\ b\ge0 \end{cases}$