# Floer homology for global quotient orbifolds

Extended abstract

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## 1 Introduction

The goal of this thesis is to give a first approach to Hamiltonian Floer theory for orbifolds. We will construct (under some conditions) Floer homology $HF(\mathcal{X})$ when $\mathcal{X}$ is a compact global quotient orbifold, that is, $\mathcal{X}$ is obtained as the quotient $X/G$ of a manifold $X$ by a finite group $G$ acting on $X$. Restricting ourselves to global quotients allows us to use well established Floer theory in our construction and not have to do everything from scratch.

There are several ways why this is reasonably interesting. The first obvious reasons are that orbifolds arise naturally in symplectic geometry – for example as a result of symplectic reduction – and that there are a lot of interesting examples of symplectic orbifolds.

The idea to define Floer homology of orbifolds was motivated by prior work of the author with Abreu and Macarini in which we showed a relation between the Chen-Ruan cohomology of an orbifold filling of a contact manifolds and its contact homology. A similar phenomenon is the generalized McKay correspondence and in particular the recent approach to it via symplectic homology in [MR18]. A definition of symplectic homology for orbifolds and a proof of its main properties would give as applications new proofs of these mentioned facts. However, we won't pursue the goal of defining symplectic homology (and its versions) in the orbifold setting in this thesis; we'll restrict ourselves to the compact case.

In the smooth compact setting Floer homology is well known to be isomorphic to the singular homology of the manifold (up to a correction of the grading). The above mentioned motivating observations suggested that in the orbifold case we should replace singular homology by Chen-Ruan cohomology. This leads to the main theorem of this thesis 6.1.

The appearance of Chen-Ruan cohomology seems quite interesting – for instance the fact that the the degree shifting numbers of Chen-Ruan cohomology appear naturally in the Floer construction is quite remarkable for us. The fact that both Chen-Ruan cohomology and Floer homology (more precisely, the Fukaya category) play a role in mirror symmetry may also spark interest in this result.

As far as we know, the main content of this thesis is essentially new. However, there is some work in the direction of an orbifold version of Lagrangian Floer theory and of the Fukaya category. Cho, Hong and others seem to be pursuing this goal for instance in [CH17, CP14]. Our construction was motivated by the orbifold Morse homology defined by Cho-Hong in [CH14], which had precisely this goal in mind. Another work worth mentioning is the definition of a $\mathbb{Z}/2$-invariant Lagrangian Floer homology by Seidel and Smith in [SS10]; contrary to us, they used a Borel type construction.

## 2 Orbifolds

Orbifolds are a generalization of smooth manifolds in which we allow some "not too bad" singularities: more precisely, we allow singularities that locally look like quotients of a smooth manifold by a finite group.

In the main text we will consider general orbifolds and their general theory in chapter 2. We will focus on the approach to orbifolds via étale Lie groupoids. We discuss orbifold morphisms, orbibundles, cohomology of orbifolds, forms and de Rham cohomology, Morse homology and Chen-Ruan cohomology. Global quotient orbifolds are orbifolds obtained as the quotient $\mathcal{X} = [X/G]$ of a smooth manifold $X$ by a finite group $G$ acting on it; the underlying topological space of $\mathcal{X}$ is the quotient topological space $X/G$. These are the orbifolds that we'll consider when constructing Floer homology. This chapter contains nothing new and most of it is based on [ALR07].

### 2.1 Chen-Ruan cohomology

Chen-Ruan cohomology is one of the most interesting aspects of orbifolds. Chen and Ruan were inspired by physics and by models for string theory which were being constructed over orbifolds, namely in [DHVW85]. Their cohomology theory was seen as the classical part of a quantum cohomology for orbifolds, constructed using the space of morphisms from (orbifold) Riemann surfaces to our orbifold. Chen-Ruan cohomology has been playing a very important role in the development of mathematics (and physics) in the last 20 years. We describe now the (rationally) graded vector space structure of the Chen-Ruan cohomology of a global quotient.

Let $\mathcal{X} = [X/G]$ be a global quotient orbifold with an almost complex structure $J$. As a vector space, the Chen-Ruan cohomology is the cohomology of the so called inertia orbifold $\Lambda\mathcal{X}$, which consists of the orbifold morphisms $S^1 \to \mathcal{X}$ which are constant as topological maps. This space has a simple description:

$$\Lambda\mathcal{X} = \bigsqcup_{(g)} X^g/C(g).$$

We assume that $X^g$ are connected. When $g = 1$ we get a copy of $X$ and for $g \neq 1$ the components $\mathcal{X}^{(g)} = X^g/C(g)$ are called the twisted sectors.

To define a grading in Chen-Ruan cohomology we need to shift the usual cohomology grading by some numbers. If $x \in X^g$ we get an automorphism $(dg)_x : T_xX \to T_xX$. Let $e^{2\pi i\lambda_1}, \ldots, e^{2\pi i\lambda_n}$ be the eigenvalues of $(dg)_x$ as a complex linear transformation and define

$$\iota_{(g)} = \sum_{j=1}^{n} \{\lambda_j\} \in \mathbb{Q}.$$

Then Chen-Ruan cohomology is defined as

$$H^*_{CR}(\mathcal{X}; \mathbb{Q}) = \bigoplus_{(g)} H^{*-2\iota_{(g)}}(X^g/C(g); \mathbb{Q}).$$

Chen-Ruan cohomology satisfies a form of Poincaré duality. It also admits a product structure respecting the grading. One remarkable fact that certainly explains the interest in Chen-Ruan cohomology is that given a crepant resolution $\mathcal{X} \to Y$ we have an isomorphism of graded vector spaces $H^*(Y; \mathbb{Q}) \cong H^*_{CR}(\mathcal{X}; \mathbb{Q})$.

# 3 Floer homology with $g$-periodic boundary conditions

When we try to define Floer homology for a global quotient $[X/G]$ we will consider Hamiltonian 1-periodic orbits in $[X/G]$. Such Hamiltonian loops will lift to Hamiltonian orbits $\gamma : [0,1] \to X$ that "close" in the quotient, that is, $\gamma(1) = g\gamma(0)$ for some $g \in G$. Moreover the Floer cilinders will also lift to some maps $u : [0,1] \times \mathbb{R} \to X$ with a boundary condition $u(1,s) = gu(0,s)$ for $s \in \mathbb{R}$. So a key step in constructing the Floer complex of $\mathcal{X}$ is to fix a symplectomorphism $g$ and define a Floer homology generated by these $g$-periodic orbits. Such generalization of the usual Floer homology (that corresponds to $g = \mathrm{id}_X$) is already considered in the literature, although details are usually not given since almost everything works as in the case $g = \mathrm{id}_X$.

Let $(X, \omega)$ be a compact symplectic manifold and $g : X \to X$ a symplectomorphism. Let $H : \mathbb{R} \times X \to \mathbb{R}$ be a time dependent Hamiltonian $H_t(x) = H(t,x)$ satisfying $H_t = H_{t+1} \circ g$ and let $J = (J_t)_{t \in \mathbb{R}}$ be a time dependent almost-complex structure compatible with $\omega$ satisfying $J_t = g^* J_{t+1}$. Denote by $X_t = X_t^H$ the Hamiltonian vector field of $H_t$ and by $\varphi_t = \varphi_t^H$ its Hamiltonian flow.

Our Floer complex will be generated by Hamiltonian orbits

$$\mathcal{P}_g(H) = \{\gamma \in C^\infty([0,1], X) : \dot\gamma(t) = X_t^H(\gamma(t)) \text{ and } \gamma(1) = g(\gamma(0))\}.$$

**Definition 3.1.** *Given a Hamiltonian $H$ as above and $\gamma \in \mathcal{P}_g(H)$ we say that $\gamma$ is non-degenerate if the linearized return map*

$$d(\varphi_1^{-1} \circ g)_{x_0} : T_{x_0}X \to T_{x_0}X$$

*does not admit 1 as an eigenvalue, where $x_0 = \gamma(0)$.*

*We say that $H$ satisfies the non-degeneracy condition if every $\gamma \in \mathcal{P}_g(H)$ is non-degenerate.*

If $H$ satisfies the non-degeneracy condition then the fixed points of $\varphi_1^{-1} \circ g$ form a discrete set, so $\mathcal{P}_g(X)$ is finite.

The definition of the differential in this complex will be given by counting solutions of Floer equation

$$\partial_s u + J_t(u)\left(\partial_t u - X_t^H(u)\right) = 0 \tag{1}$$

with certain boundary conditions. We define the relevant moduli spaces of such solutions:

**Definition 3.2.** *We define the moduli spaces*

$$\begin{aligned}\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+; H, J) = \{u \in & C^\infty([0,1] \times \mathbb{R}, X) |\, u \text{ is a solution of (1)}, \\ & u(t, \pm\infty) = \gamma^\pm(t),\, u(1,s) = g(u(0,s))\}.\end{aligned} \tag{2}$$

*If $\gamma^- \neq \gamma^+$ then $\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+; H, J)$ admits an $\mathbb{R}$-action by translation in the $s$ variable and we define*

$$\mathcal{M}_g(\gamma^-, \gamma^+; H, J) = \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+; H, J)/\mathbb{R}. \tag{3}$$

## 3.1 Fredholm property and the relative index

A fundamental fact in the construction of Floer homology is that in generic transversality conditions the moduli spaces $\mathcal{M}_g(\gamma^-, \gamma^+)$ should be finite dimensional manifolds. This is controlled by the linearisation of the Floer operator, which is an operator

$$D_u : W_g^{1,p}(u^*TX) \to L_g^p(u^*TX)$$

where $W^{1,p}_g(u^*TX)$, $L^p_g(u^*TX)$ are Sobolev completions of the space of sections $\xi$ of the bundle $u^*TX \to [0,1]\times\mathbb{R}$ satisfying $\xi(1,s) = (dg)_{u(0,s)}\xi(0,s)$. To study this linearisation we can find an appropriate trivialization

$$\Psi(t,s) : \mathbb{R}^{2n} \to T_{u(t,s)}X \text{ for each } (t,s) \in [0,1]\times\mathbb{R}$$

preserving the symplectic form and the almost complex structures and satisfying

$$\Psi(1,s) = (dg)_{u(0,s)}\Psi(0,s).$$

The existence of such a trivialization essentially follows from the fact that $U(n)$-bundles over $S^1\times\mathbb{R}$ are trivial. This trivialization induces isomorphisms

$$W^{1,p}(u^*TX) \cong W^{1,p}(S^1\times\mathbb{R},\mathbb{R}^{2n}) \text{ and } L^p(u^*TX) \cong L^p(S^1\times\mathbb{R},\mathbb{R}^{2n}).$$

Under this isomorphisms, $D_u$ takes the form of a perturbed Cauchy-Riemann operator, that is, an operator $L_S : W^{1,p}(S^1\times\mathbb{R},\mathbb{R}^{2n}) \to L^p(S^1\times\mathbb{R},\mathbb{R}^{2n})$ of the form

$$L_S = \partial_s + J_0\partial_t + S$$

where $S \in C^0(S^1\times\mathbb{R}, M_{2n\times 2n}(\mathbb{R}))$. These are well understood and we can prove the following:

**Theorem 3.3** (Fredholm property). *Suppose that $\gamma^-,\gamma^+ \in \mathcal{P}_g(H)$ are non-degenerate (see definition 3.1) and $u \in \widehat{\mathcal{M}}_g(\gamma^-,\gamma^+)$. Then the operator $D_u$ is Fredholm and has index*

$$\operatorname{ind} D_u = \mu_{CZ}(\Phi^+) - \mu_{CZ}(\Phi^-) \equiv \mu(u)$$

*where $\Phi^\pm$ are defined by*

$$\Phi^\pm(t) = \Psi(t,\pm\infty)^{-1}(d\varphi^H_t)_{\gamma^\pm(0)}\Psi(0,\pm\infty) \in Sp(\mathbb{R}^{2n}) \tag{4}$$

An infinite dimensional pre-image transversality theorem shows that the condition needed to assure that the moduli spaces $\widehat{\mathcal{M}}_g(\gamma^-,\gamma^+)$ are smooth manifolds is the surjectivity of the operators $D_u$.

**Theorem 3.4.** *Suppose $\gamma^-,\gamma^+ \in \mathcal{P}_g(H)$ are non-degenerate. If $D_u$ is a surjective operator for every $u \in \widehat{\mathcal{M}}_g(\gamma^-,\gamma^+)$ then $\widehat{\mathcal{M}}(\gamma^-,\gamma^+)$, $\mathcal{M}(\gamma^-,\gamma^+)$ are smooth manifolds; moreover, their local dimensions at $u \in \widehat{\mathcal{M}}_g(\gamma^-,\gamma^+)$ are*

$$\dim_u \widehat{\mathcal{M}}(\gamma^-,\gamma^+) = \mu_{CZ}(\Phi^+) - \mu_{CZ}(\Phi^-)$$

*and*

$$\dim_u \mathcal{M}(\gamma^-,\gamma^+) = \mu_{CZ}(\Phi^+) - \mu_{CZ}(\Phi^-) - 1$$

*where $\Phi^\pm$ are defined by (4).*

When this is the case for every $u \in \widehat{\mathcal{M}}_g(H,J)$ we say that the pair $(H,J)$ is regular.

## 3.2 Floer complex

The Floer complex of $X$ with $g$-boundary conditions will be generated by $\mathcal{P}_g(H)$ and its differential will count solutions in $\mathcal{M}_g(\gamma^-,\gamma^+)$ which are rigid, i.e., which have relative index $\mu(u) = 1$. To guarantee that we're counting a finite number of objects and to prove that the differential that we'll define satisfies $\partial^2 = 0$ we'll need some compactness results on our moduli spaces. The key tool for dealing with these problems is Gromov-Floer compactness.

**Theorem 3.5** (Gromov-Floer compactness). *Let $\gamma^-, \gamma^+ \in \mathcal{P}_g(H)$ and let $u_\nu \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$ be a sequence of solutions of Floer equation with bounded energy $E(u_\nu)$ and constant index $\mu(u_\nu) = \mu$.*

*Then there is a subsequence of $u_\nu$ that converges modulo bubbling to $(u^1, \ldots, u^m)$. Moreover if $Z = \{z_1, \ldots, z_\ell\}$ is the set of singularities of the subsequence, then there are $J$-holomorphic spheres $v_1, \ldots, v_\ell : S^2 \to X$ such*

$$\lim_{\nu\to\infty} E(u_\nu) = \sum_{j=1}^m E(u^j) + \sum_{k=1}^m E(v^j) \qquad \mu = \sum_{j=1}^m \mu(u^j) + 2\sum_{k=1}^m \langle c_1(TX), v_j\rangle \tag{5}$$

*and $u^1, \ldots, u^m, v^1, \ldots, v^\ell$ is a connected family, that is,*

$$\bigcup_{j=1}^m u^j([0,1]\times\overline{\mathbb{R}}) \cup \bigcup_{k=1}^\ell v^j(S^2)$$

*is connected.*

We see from Gromov-Floer compactness that there are essentially two obstructions to the 0-dimensional manifold $\{u \in \mathcal{M}_g(\gamma^-, \gamma^+); H, J) : \mu(u) = 1\}$ being compact/finite. The first is that the energy of such solutions may be unbounded, and so we cannot apply the theorem to get convergent subsequences. The second is the existence of bubbles. The first problem is solved by using the algebraic formalism of Novikov rings, while the second is much harder to surpass, so throughout the dissertation we'll assume we have no-bubbling in sequences with constant $\mu(u_\nu) = 1$ or $\mu(u_\nu) = 2$. This is always the case for monotone manifolds and is the generic situation for Calabi-Yau manifolds; in the dissertation we discuss this.

To define the differential in characteristic different from 2 we also need coherent orientations on our moduli spaces. In particular we have an assignment of a sign $\nu(u) \in \mathcal{M}_g(\gamma^-, \gamma^+)$ to each rigid Floer trajectory.

**Definition 3.6.** *Assume that $(H, J)$ is a regular pair and has no-bubbling . Then we define the Floer complex $CF(X, g, H; \Lambda)$ with coefficients in the Novikov ring $\Lambda = \Lambda^{\text{univ}}(R)$ to be the $\Lambda$-modulo generated by $\mathcal{P}_g(H)$:*

$$CF(X, g, H; \Lambda) = \bigoplus_{\gamma\in\mathcal{P}_g(H)} \Lambda \cdot \gamma.$$

*We define a differential $\partial = \partial_{H,J} : CF(X, g, H; \Lambda) \to CF(X, g, H; \Lambda)$ by counting (with signs) isolated Floer trajectories between Hamiltonian orbits corresponding to generators. More precisely let $\partial$ be the $\Lambda$-linear map defined on generators $\gamma^+ \in \mathcal{P}_g(H)$ by*

$$\partial\gamma^+ = \sum_{\gamma^-\in\mathcal{P}_g(H)} \left( \sum_{\substack{u\in\mathcal{M}_g(\gamma^-,\gamma^+;H,J)\\ \mu(u)=1}} \nu(u) T^{\omega(u)} \right) \gamma^-. \tag{6}$$

*We define the Floer homology $HF(X, g, H, J; \Lambda)$ to be the homology of the Floer complex $(CF(X, g, H; \Lambda), \partial_{H,J})$, that is,*

$$HF(X, g, H, J; \Lambda) = \frac{\ker\partial}{\operatorname{im}\partial}.$$

# 4 Floer homology of global quotient orbifold

Chapter 4 gives the definition of Floer homology for a global quotient orbifold $\mathcal{X} = [X/G]$, the main goal of this thesis. We let $(X, \omega)$ be a compact symplectic manifold of dimension $2n$ and let $G$ be a group acting on $(X,\omega)$ by symplectomorphisms. Then the quotient orbifold $\mathcal{X}=[X/G]$ is a symplectic orbifold $(\mathcal{X},\omega)$. Let $H,J$ be a time-dependent Hamiltonian and a time dependent almost complex structure satisfying $H_{t+1}=H_t$, $H_t=H_t\circ g$, $J_{t+1}=J_t$ and $J_t=g^*J_t$ for every $g\in G$. Let

$$\widetilde{\mathcal{P}}_G(H)=\bigsqcup_g \mathcal{P}_G(H)$$

be the set of possible lifts to $X$ of closed Hamiltonian orbits of $\mathcal{X}$. An element $h$ of the group $G$ acts on $\widetilde{\mathcal{P}}_G(H)$ by sending $\gamma\in\mathcal{P}_g(H)\subseteq\widetilde{\mathcal{P}}_G(H)$ to $h\gamma\in\mathcal{P}_{hgh^{-1}}(H)\subseteq\widetilde{\mathcal{P}}_G(H)$.

The obvious idea to define the complex would be to let

$$CF(\mathcal{X},H;\Lambda)=\bigoplus_{[\gamma_g]\in\mathcal{P}_G(H)}\Lambda\cdot[\gamma_g]$$

or, equivalently, $CF(\mathcal{X},H;\Lambda)=\widetilde{CF}(\mathcal{X},H;\Lambda)^G$ where

$$\widetilde{CF}(\mathcal{X},H;\Lambda)=\bigoplus_{g\in G}CF(X,g,H;\Lambda)=\bigoplus_{\gamma_g\in\widetilde{\mathcal{P}}_G(H)}\Lambda\cdot\gamma_g$$

and $\widetilde{CF}(\mathcal{X},H;\Lambda)^G$ denotes the $G$-invariant part of $\widetilde{CF}(\mathcal{X},H;\Lambda)$ with respect to the action of $G$ that extends $\Lambda$-linearly the action of $G$ on $\widetilde{\mathcal{P}}_G(H)$. We can assemble the differentials of 3.6 to get a differential $\partial:\widetilde{CF}(\mathcal{X},H;\Lambda)\to\widetilde{CF}(\mathcal{X},H;\Lambda)$. Unfortunately in characteristic different from 2 a problem with the orientations signs $\nu(u)$ makes $\partial$ not $G$-equivariant. This leads to the need to exclude some of the Hamiltonian orbits from the complex.

## 4.1 Absolute index using trivialization of $\Lambda^n_{\mathbb{C}}TX$

In section 3.1 we showed how to assign a relative index $\mu(u)=\operatorname{ind}D_u$ to any Floer trajectory $u\in\widehat{\mathcal{M}}_g(\gamma^-,\gamma^+)$. To define a grading on the Floer homology we would like to be able to write this relative index in terms of an absolute index assigned to Hamiltonian orbits $\gamma_g\in\widetilde{\mathcal{P}}_G(H)$. This is not possible in general, so we impose the condition that $\mathcal{X}$ is Calabi-Yau.

**Definition 4.1.** *We say that a global quotient orbifold $\mathcal{X}=[X/G]$ is Calabi-Yau if the equivariant first Chern class $c_1^G(X)=c_1^G(TX)$ vanishes.*

The orbifold $\mathcal{X}$ is Calabi-Yau if and only if the $G$-equivariant line bundle $\Lambda^n_{\mathbb{C}}TX\to X$ is a trivial $G$-bundle. We fix a non-vanishing $G$-equivariant section $\mathfrak{s}:X\to\Lambda^n_{\mathbb{C}}TX$ or, equivalently, a trivialization of $\Lambda^n_{\mathbb{C}}TX\to X$.

Given $\gamma\in\mathcal{P}_g(H)$ the section $\mathfrak{s}$ also gives a trivialization of the bundle $\Lambda^n_{\mathbb{C}}(\gamma^*TX/\sim)\to S^1$ where $\sim$ identifies $T_{\gamma(0)}X$ and $T_{\gamma(1)}X$ via $(dg)_{\gamma(0)}$. Then there is an unique up to homotopy trivialization $\Psi$ of $\gamma^*TX/\sim$ that induces the above one. We write the trivialization as $\Psi(t):\mathbb{R}^{2n}\to T_{\gamma(t)}X$ with the condition that

$$\Psi(1)=(dg)_{\gamma(0)}\circ\Psi(0).$$

We say that such a trivialization is compatible with $\mathfrak{s}$.

**Definition 4.2.** *Let $\gamma\in\mathcal{P}_g(H)\subseteq\widetilde{\mathcal{P}}_G(H)$ be a non-degenerate (see 3.1) Hamiltonian orbit and consider a trivialization $\Psi$ as discussed before. Let $\Phi:[0,1]\to Sp(2n;\mathbb{R})$ be defined by*

$$\Phi(t)=\Phi_\gamma(t)=\Psi(t)^{-1}(d\varphi_t)_{\gamma(0)}\Psi(0)\ \text{for}\ t\in[0,1].$$

*Then we define a grading on the set $\widetilde{\mathcal{P}}_G(H)$ by*

$$|\gamma| = \mu_{CZ}(\Phi_\gamma).$$

With this canonical choice of trivialization we get the relative index from the absolute indices on the limits.

**Proposition 4.3.** *Let $\gamma^-, \gamma^+ \in \mathcal{P}_g(H)$ be non-degenerate orbits and let $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$. Then*

$$\mu(u) = \operatorname{ind} D_u = |\gamma^+| - |\gamma^-|.$$

## 4.2 Coherent orientations and orientable orbits

We can understand orientations and the need to exclude some orbits from our complex better in the Calabi-Yau case. The idea is to define coherent orientations by assigning orientations to certain determinant line bundles $\delta_\gamma$, for each $\gamma \in \widetilde{\mathcal{P}}_G(H)$; this is analogous to orienting the unstable manifolds in Morse homology.

Suppose that $\mathcal{X}$ is Calabi-Yau and fix a $G$-equivariant section $\mathfrak{s}$ as in 4.1. We saw that in this case there is a trivialization $\Psi(t) : \mathbb{R}^{2n} \to T_{\gamma(t)}X$ compatible with $\mathfrak{s}$ which is canonical up to homotopy. Given such trivialization we have a path of symplectic matrices $\Phi = \Phi_\gamma : [0,1] \to Sp(2n;\mathbb{R})$ defined in 4.2. We let $S : S^1 \to M_{2n\times 2n}(\mathbb{R})$ be the path of symmetric matrices defined by

$$\dot{\Phi}(t) = J_0 S(t) \Phi(t).$$

We associate to $\Phi$ a Fredholm operator $D_\Phi$ defined over $\mathbb{C}$ (instead of the cylinder $[0,1]\times\mathbb{R}$) that near the cylindrical end of $\mathbb{C}$ looks like $\partial_s + J_0\partial_t + S$; here $(t,s)$ are the cylindrical coordinates defined by $x + iy = e^{-2\pi(s+it)}$. More precisely let $B : \mathbb{C} \to M_{2n\times 2n}(\mathbb{R})$ be a matrix valued continuous function defined on $\mathbb{C}$ that is constant and equal to $S$ on the cylindrical end of $\mathbb{C}$, that is, we ask that

$$B\left(e^{-2\pi(s+it)}\right) = S(t) \text{ for s} \ll 0.$$

Moreover we pick $\alpha : \mathbb{C} \to \mathbb{R} \oplus J_0\mathbb{R} \cong \mathbb{C}$ such that

$$\alpha(x,y) = \alpha(x+iy) = \begin{cases} 1 & \text{if } s \gg 0 \\ 2\pi(-x + J_0 y) & \text{if } s \ll 0 \end{cases}$$

and $\alpha$ never vanishes. Finally we take the operator $D_\Phi : W^{1,p}_\mu(\mathbb{C},\mathbb{R}^{2n}) \to L^p_\mu(\mathbb{C},\mathbb{R}^{2n})$ defined by

$$D_\Phi Z = \alpha\left(\partial_x Z + J_0 \partial_y Z\right) + BZ. \tag{7}$$

In the cylindrical coordinates $(t,s)$ we have $D_\Phi = \partial_s + J_0\partial_t + S$ for $s \ll 0$.

**Theorem 4.4.** *Assume that the symplectic path $\Phi$ is admissible. Then the operator $D_\Phi : W^{1,p}_\mu(\mathbb{C},\mathbb{R}^{2n}) \to L^p_\mu(\mathbb{C},\mathbb{R}^{2n})$ defined in (7) is a Fredholm operator of Fredholm index $n - \mu_{CZ}(\Phi)$ for every $p \geq 2$. Moreover, its kernel does not depend on $p \geq 2$.*

With this in mind, we can now define the determinant line bundle associated to $\gamma$.

**Definition 4.5.** *Given a non-degenerate orbit $\gamma \in \widetilde{\mathcal{P}}_G(H)$ let $\Phi = \Phi_\gamma$ and $D_\Phi$ be an operator as before. We let $\delta_\gamma$ be the determinant line bundle of $D_\Phi$, that is,*

$$\delta_\gamma = \det(D_\Phi) = \left(\Lambda^{\text{top}} \ker(D_\Phi)\right) \otimes \left(\Lambda^{\text{top}} \operatorname{coker}(D_\Phi)\right)^\vee.$$

**Theorem 4.6.** *Let $\gamma^-, \gamma^+ \in \mathcal{P}_g(H) \subseteq \widetilde{\mathcal{P}}_G(H)$ for some $g \in G$ be non-degenerate Hamiltonian orbits and $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$ be a connecting orbit. Assume that $\mathcal{X}$ is Calabi-Yau. Then we have an isomorphism*

$$\delta_{\gamma^-} \cong \det(D_u) \otimes \delta_{\gamma^+} \tag{8}$$

*which is canonical up to multiplication by a positive constant.*

Now in order to get coherent orientations on the moduli spaces we fix for each $\gamma \in \widetilde{\mathcal{P}}_G(H)$ an orientation of $\delta_\gamma$. Equation (8) then induces an orientation in the determinant line bundle $\det(D_u)$. Suppose now that $(H, J)$ is regular, and thus $D_u$ is surjective. Then

$$\det(D_u) = \Lambda^{\mathrm{top}} \ker(D_u) = \Lambda^{\mathrm{top}} \left( T_u \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+) \right).$$

So we get an orientation on the manifold $\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$. Recall that $\mathcal{M}_g(\gamma^-, \gamma^+)$ is defined as the quotient $\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)/\mathbb{R}$ where $\mathbb{R}$ acts on $\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$ by $(\sigma \cdot u)(t, s) = u(t, s + \sigma)$ for $\sigma \in \mathbb{R}$. In particular we get signs $\nu(u) \in \{-1, +1\}$ for the rigid Floer trajectories.

Moreover it can be shown that we have maps

$$h_* : \delta_\gamma \to \delta_{h\gamma} \text{ and } h_* : \det(D_u) \to \det(D_{hu})$$

which are natural with respect to the isomorphism of theorem 4.6. With this we define orientable orbits:

**Definition 4.7.** *Let $\gamma \in \mathcal{P}_g(H) \subseteq \widetilde{\mathcal{P}}_G(H)$ and assume that $\mathcal{X}$ is Calabi-Yau. We say that $\gamma$ is an orientable Hamiltonian orbit if for every $h \in C(g)$ such that $h\gamma = \gamma$ the isomorphism $h_* : \delta_\gamma \to \delta_\gamma$ is orientation preserving. Otherwise we say that $\gamma$ is non-orientable. We denote by $\widetilde{\mathcal{P}}_G(H)^+$ the set of orientable Hamiltonian orbits.*

## 4.3 Orbifold Floer complex

At this point we can define the Floer complex of a Calabi-Yau orbifold $\mathcal{X}$. For this we define the complex generated only by orientable orbits, i.e.,

$$\widetilde{CF}_k(\mathcal{X}, H; \Lambda)^+ = \bigoplus_{\gamma \in \widetilde{\mathcal{P}}_G(H)^+} \Lambda \cdot \gamma.$$

Since the action of $G$ preserves orientability, we have an action on $\widetilde{CF}(\mathcal{X}, H; \Lambda)^+$ and thus we can look at the $G$-invariant part

$$CF(\mathcal{X}, H; \Lambda) = \left( \widetilde{CF}(\mathcal{X}, H; \Lambda)^+ \right)^G.$$

**Lemma 4.8.** *The differential*

$$\partial : \bigoplus_{\gamma \in \widetilde{\mathcal{P}}_G(H)} \Lambda \cdot \gamma \to \bigoplus_{\gamma \in \widetilde{\mathcal{P}}_G(H)} \Lambda \cdot \gamma$$

*given by assembling the differentials of $CF(X, g, H, J; \Lambda)$ maps $CF_k(\mathcal{X}, H; \Lambda)$ to $CF_{k-1}(\mathcal{X}, H; \Lambda)$.*

The Floer homology of $\mathcal{X}$ is then defined as

$$HF_k(\mathcal{X}, H, J; \Lambda) = \frac{\ker(\partial : CF_k \to CF_{k-1})}{\mathrm{im}(\partial : CF_{k+1} \to CF_k)}.$$

In the main text we also give a more abstract definition of the Floer complex that we think gives a more conceptual explanation for why we need to exclude non-orientable orbits.

# 5 Equivariant transversality

Transversality in orbifold Floer homology turns out to be a more subtle issue than in the smooth case. In general it's not true that the set of regular pairs $(H, J)$ is dense. This situation arises even in Morse homology, for example in the classical picture of the torus with a $\mathbb{Z}/2$ symmetry given by reflection in a plane containing the two gradient flow lines between the critical points of Morse index 1.

However, we're able to prove that given a fixed $H$ with the non-degeneracy condition we can perturb $J$ in a way that $D_u$ is surjective for every $u$ that's not contained in the singular set of $\mathcal{X}$. More precisely we proved the following:

**Theorem 5.1** (Weak equivariant transversality)**.** *Let $(X, \omega)$ be a symplectic manifold, $G$ a finite group acting on $(X, \omega)$ and $H : X \to \mathbb{R}$ a non-degenerate Hamiltonian. For $\ell \geq 1$ there is a $C^\ell$ dense subset $\mathcal{J}_G^{reg}(X, \omega)$ of $\mathcal{J}_G(X, \omega)$ such that if $J \in \mathcal{J}_G^{reg}(X, \omega)$ then, for every Floer trajectory $u \in \widehat{\mathcal{M}}_g(H, J)$ whose image is not contained in $X^h$ for any $h \in G \setminus \{1\}$, $D_u$ is surjective.*

The proof goes by showing an abundance of injective points, in an appropriate sense.

**Definition 5.2.** *Given a Floer trajectory $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+; H, J)$ we say that a point $(t_0, s_0) \in [0, 1] \times \mathbb{R}$ is $G$-injective if*

$$\partial_s u(t_0, s_0) \neq 0 \text{ and } u(t_0, s_0) \notin u(t_0, \overline{\mathbb{R}} \setminus \{s_0\}) \cup \bigcup_{g \in G \setminus \{1\}} g\, u(t_0, \overline{\mathbb{R}}).$$

*We denote by $R(u)$ the set of $G$-injective points of $u$.*

We use the following lemma in the proof of weak equivariant transversality.

**Lemma 5.3.** *Let $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+; H, J)$ be a non-constant Floer trajectory (i.e., with $\partial_s u \neq 0$) and assume that $\operatorname{im}(u)$ is not contained in $X^g$ for any $g \in G \setminus \{1\}$. Then the set of $G$-injective points $R(u)$ is dense in $[0, 1] \times \mathbb{R}$.*

Although we can't get transversality in general, this is enough to deal with the case of isolated singularities, that is, when $X^g$ is discrete for every $g \neq 1$.

We also propose a way to redefine Floer homology of orbifolds in a way that avoids equivariant transversality and only needs transversality for Floer homology with $g$-periodic boundary conditions.

# 6 Floer homology for "small" autonomous Hamiltonians

The main result in this dissertation, besides the own definition, is an isomorphism with Chen-Ruan cohomology. This isomorphism replaces the usual isomorphism between Floer homology of a smooth manifold and its singular homology. To be precise, we proved the following:

**Theorem 6.1.** *Let $\mathcal{X} = [X/G]$ be a global quotient compact Calabi-Yau orbifold with symplectic form $\omega$. Let $H \in C_G(\mathbb{R} \times X)$ and $J \in \mathcal{J}_G(X, \omega)$ be autonomous Hamiltonian and almost complex structure, respectively, and denote $H_\tau = \tau H$ for $\tau > 0$. Assume that for sufficiently small $\tau$ the pair $(H_\tau, J)$ is regular. Let $\Lambda = \Lambda^{\text{univ}}(\mathbb{Q})$ be the rational universal Novikov ring. Then for sufficiently small $\tau > 0$ we have*

$$HF_*(\mathcal{X}, H_\tau, J; \Lambda) \cong H_{CR}^{n-*}(\mathcal{X}; \Lambda).$$

The assumption that for sufficiently small $\tau$ the pair $(H_\tau, J)$ is regular can be related to a Morse-Smale hypothesis as follows:

**Proposition 6.2.** *Let $H \in C_G(\mathbb{R} \times X)$ and $J \in \mathcal{J}_G(X, \omega)$ be autonomous Hamiltonian and almost complex structure, respectively, and assume that $H$ is $C^2$-small. Then $D_u$ is surjective for every Floer trajectory $u$ which is $t$-independent if and only if $(H_{|X^g}, \mathfrak{g})$ is a Morse-Smale pair for every $g \in G$ where $\mathfrak{g}(u,v) = \omega(u, Jv)$. In particular if $(H_{|X^g}, \mathfrak{g})$ is Morse-Smale for every $g \in G$ then $(H_\tau, J)$ is regular for every sufficiently small $\tau$.*

In this summary we will just explain the ingredients of the proof. First, we can decompose $\widetilde{\mathcal{P}}_G(H)$ according to the conjugacy class $(g)$ relative to an orbit $\gamma_g \in \widetilde{\mathcal{P}}_G(H)$. Since the action of $G$ and the differential preserve this conjugacy class we get a decomposition of Floer homology

$$HF_k(\mathcal{X}, H, J; \Lambda) = \bigoplus_{(g)} HF_k^{(g)}(\mathcal{X}, H, J; \Lambda).$$

The idea is that for Hamiltonians in the conditions stated we can identify the chain complex giving the homology groups $HF_k^{(g)}(\mathcal{X}, H, J; \Lambda)$ with the orbifold Morse chain complex of the twisted sector $X^g/C(g)$; the Morse theory of orbifolds was introduced in [CH14]. The first step is to identify the generators of the complexes.

**Proposition 6.3.** *Assume that $H$ is a sufficiently $C^2$-small autonomous Hamiltonian. Then every Hamiltonian orbit $\gamma \in \widetilde{\mathcal{P}}_G(H)$ is constant.*

Thus the Hamiltonian orbits are constant maps $c_g^x$ with $gx = x$. This means that there is a bijection between $\widetilde{\mathcal{P}}_G(H)$ and

$$\bigsqcup_{g \in G} \operatorname{Crit}\big(H_{|X^g}\big).$$

The next step in establishing the connection between Floer homology and Morse homology is to prove that all the Floer trajectories are constant in the $t$-variable. This will identify the Floer trajectories with gradient flow trajectories of $H$.

**Proposition 6.4.** *Assume that $H$ is an autonomous Hamiltonian and let $H_\tau = \tau H$ for $\tau > 0$. Then for sufficiently small $\tau > 0$ every Floer trajectory $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+; H_\tau, J)$ does not depend on $t$, that is, $u(t,s) = u(s)$ is a Morse trajectory.*

This almost identifies the complexes and the differentials but there are two more ingredients missing. First, we need to compare the index of $c_g^x$ as defined in 4.2 with the Morse index of $x$ as a critical point of $H_{|X^g}$. This is done by choosing a trivialization of the form $\Psi(t) = G_t \circ \Psi(0)$ where $G_t$ is a path in $SU(n)$ from Id to $(dg)_x \in SU(T_xX) \cong SU(n)$ and computing things explicitly in appropriate local coordinates. In the end we arrive at the following expression:

$$|c_g^x| = \operatorname{ind}_x \left(H_{|X^g}\right) + 2\iota_{(g)} - n \tag{9}$$

where we can see the degree shifting numbers of Chen-Ruan cohomology appearing naturally.

The second ingredient needed concerns orientations. Essentially we have to see that orientations match in two aspects: orientable critical points correspond to orientable Hamiltonian orbits and the signs $\nu(u)$ appearing in Morse and Floer homologies agree. The key step to do this is to identify the determinant line bundle $\delta_{c_g^x}$ with the top alternating power of the unstable manifold $\Lambda^{\mathrm{top}} W^u_{H_{|X^g}}(x)$. Once again, this is done by fixing some of the choices needed to define $\delta_\gamma$ and computing things explicitly.

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