Disclaimer:

- Ich kann weder Korrektheit, noch Vollständigkeit garantieren.
- Bei Fehlern oder Verbesserungsvorschlägen kannst du mir eine Mail schreiben an msteinkel@ethz.ch

ZSMF Ana 3 - Marvin Steinkellner

Prof. M. Iacobelli - HS2021

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Operators and Classification

Gradient $\nabla u = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$

Laplacian $\Delta u = \nabla^2 u = u_{xx} + u_{yy} + u_{zz}$

Problem is well-posed $\Leftrightarrow$

- solution exists, is unique and stable

stable: small change in I.C.
$\Rightarrow$ small change in solution

not well-posed = ill posed

Strong solution

all derivatives in the PDE exist and are continous
(otherwise: weak solution)

Order of PDE = Order of highest derivative

Linear/Quasilinear/fully non-linear

- Linear $\Leftrightarrow$ all coefficients in front of u and it's derivatives do not depend on u and it's derivatives
- Quasi-linear $\Leftrightarrow$ PDE is linear with respect to the highest derivative
- fully non-linear $\Leftrightarrow$ neither linear nor quasi-linear

Method of Characteristics (M.O.C.)

solves (quasi-)linear 1st Order PDEs

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \; ; \; \Gamma(s) = \begin{pmatrix} x(s,0) \\ y(s,0) \\ \tilde u(s,0) \end{pmatrix}$$

initial curve

1. Finding $\Gamma(s)$ $\begin{cases} u(d,y) = f(y) \Rightarrow \Gamma(s) = \begin{pmatrix} d \\ s \\ f(s) \end{pmatrix} \\ u(x,e) = f(x) \Rightarrow \Gamma(s) = \begin{pmatrix} s \\ e \\ f(s) \end{pmatrix} \end{cases}$

2. Solve System of ODEs

$$\begin{cases} x_t(s,t) = a(x,y,u) \\ y_t(s,t) = b(x,y,u) \\ \tilde u_t(s,t) = c(x,y,u) \end{cases}$$ with initial conditions from initial curve $\Gamma(s) = \begin{pmatrix} x(s,0) \\ y(s,0) \\ \tilde u(s,0) \end{pmatrix}$

3. Check Transversality condition

$$\det\begin{pmatrix} x_t(s,0) & x_s(s,0) \\ y_t(s,0) & y_s(s,0) \end{pmatrix} = \det\begin{pmatrix} a(x(s,0),y(s,0),\tilde u(s,0)) & x_s(s,0) \\ b(x(s,0),y(s,0),\tilde u(s,0)) & y_s(s,0) \end{pmatrix} \neq 0$$

$\Leftrightarrow \begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} x_s(s,0) \\ y_s(s,0) \end{pmatrix}$ are transverse (not parallel)

$\Leftrightarrow$ we can solve for $s(x,y)$ and $t(x,y)$ in the neighborhood of $(s,0)$

$\Leftrightarrow$ there exists a unique solution in the neighborhood of $(s,0)$ if a,b,c are smooth

4. find $s(x,y)$ and $t(x,y)$

5. Plug $s(x,y)$, $t(x,y)$ in $\tilde u(s,t) \Rightarrow \tilde u(s(x,y),t(x,y)) = \underline{\underline{u(x,y)}}$

Conservation laws

$\rightarrow$ conserved quantity (e.g. mass, energy)

Strong formulation

- $u_y + f(u)_x = 0 \; ; \; u: \mathbb{R} \times [0,\infty) \to \mathbb{R} \; ; \; f: \mathbb{R} \to \mathbb{R}$ (x (spatial), flux, y (time))
- $u_y + c(u)u_x = 0 \; ; \; c(u) = f'(u)$

implicit solution

$$u(x,y) = u_0(x - c(u(x,y))y) \quad \forall (x,y) \in \mathbb{R} \times [0, y_c]$$

implicit; initial curve

Critical time

no smooth solution for $y > y_c$

$$y_c = \inf_{\substack{s \in \mathbb{R}, \\ (u_0)_s < 0}} \left\{ -\frac{1}{c(u_0)_s} \right\}$$

Integral formulation

$\forall \; x_0, x_1, y_0, y_1 \in \mathbb{R}$ with $x_0 < x_1 \; ; \; 0 < y_0 < y_1$:

$$\int_{x_0}^{x_1} u(x,y_1) - u(x,y_0)\,dx \overset{!}{=} -\int_{y_0}^{y_1} f(u(x_1,y)) - f(u(x_0,y))\,dy \quad ⊛$$

Weak solution („shock waves")

Let $D = \bigcup_{i=n}^{n} D_i$ be the original domain

$u(x,y)$ is a weak solution if

- $u(x,y)$ is continously differentiable in each D_i
- $u(x,y)$ satisfies the original PDE "
- $u(x,y)$ satisfies the integral formulation ⊛

Boundaries between D_i: shocks

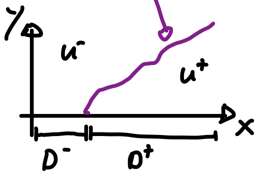
shocks need to satisfy the

Rankine-Hugoniot-Condition

Rankine - Hugoniot - Condition

shocks are curves $\begin{pmatrix} \sigma(y) \\ y \end{pmatrix}$ where

$$\sigma'(y) \overset{!}{=} \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$



Entropy - Condition

Weak solutions are not unique! $\Rightarrow$ entropy-cond helps us in choosing the best solution.

$$c(u^+) < \sigma'(y) < c(u^-) \Leftrightarrow f'(u^+) < \sigma'(y) < f'(u^-)$$

↑ flux

$\Rightarrow$ characteristics only enter shocks but do not emanate from them („information can't be generated $\rightarrow$ only lost")

2nd Order PDEs

Classification

$$L(u) = \underline{a} u_{xx} + \underline{2b} u_{xy} + \underline{c} u_{yy} + d u_x + e u_y + f u = g$$

with $a, b, \dots, g$ functions of x and y

$$\delta(L)(x_0, y_0) = b^2(x_0, y_0) - a(x_0, y_0) c(x_0, y_0)$$

- $\delta(L)(x_0, y_0) > 0$ hyperbolic
- $\delta(L)(x_0, y_0) = 0$ parabolic
- $\delta(L)(x_0, y_0) < 0$ elliptic

if a, b, c const: global property otherwise local

wave-eq.: $u_{tt} - c^2 u_{xx} = 0$ hyperbolic

Heat-eq.: $u_t - u_{xx} = 0$ parabolic

Laplace-eq: $\underbrace{u_{xx} + u_{yy}}_{\Delta u} = 0$ elliptic

Homogeneous Wave equation (hyperbolic)

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & , (x,t) \in \mathbb{R} \times [0, \infty) \\ u(x,0) = f(x) & , x \in \mathbb{R} \\ u_t(x,0) = g(x) & , x \in \mathbb{R} \end{cases}$$

$$u(x,y) = \underbrace{F(x+ct)}_{\text{goes left}} + \underbrace{G(x-ct)}_{\text{goes to the right}}$$

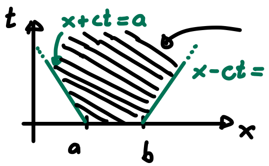
(„if $t \uparrow$: x has to grow for const. input")

solve with D'Alembert's formula

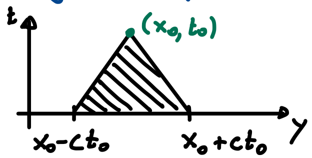
$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\, dy$$

Solution is unique! f, g odd/even/periodic $\Rightarrow u$ also odd/...

Domain of Influence



Region of Dependence



nonhomogeneous wave-equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & , (x,t) \in \mathbb{R} \times [0, \infty) \\ u(x,0) = f(x) & , x \in \mathbb{R} \\ u_t(x,0) = g(x) & , x \in \mathbb{R} \end{cases}$$

Option A D'Alembert for nonhomogeneous wave-eq:

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\, dy + \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi, \tau)\, d\xi\, d\tau$$

Option B (if F is simple)

1. find part. solution $v(x,y)$ that solves

$$v_{tt} - c^2 v_{xx} = F(x,t)$$

2. use D'Alembert (homogeneous case) to find w

where w solves

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0 & , (x,t) \in \mathbb{R} \times [0, \infty) \\ w(x,0) = f(x) - v(x,0) & , x \in \mathbb{R} \\ w_t(x,0) = g(x) - v_t(x,0) & , x \in \mathbb{R} \end{cases}$$

3. $u(x,y) = v(x,y) + w(x,y)$

Note: Solution is unique

F, f, g odd $\Rightarrow u$ odd

" even $\Rightarrow u$ even

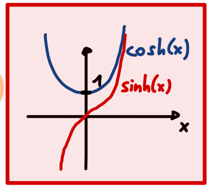
" periodic $\Rightarrow u$ periodic

$\Rightarrow$ if x not in $(-\infty, \infty)$ but $x \in [0, \infty)$ with $u(0,t) = 0$

$\tilde{f}(x), \tilde{g}(x), \tilde{F}(x,t)$: odd extensions of f, g, F

$\Rightarrow$ solve Problem with $\tilde{f}, \tilde{g}, \tilde{F}$

Seperation of Variables



homog. Heat equation

$$\begin{cases} u_t - k u_{xx} = 0 & , (x,t) \in [0, L] \times [0, \infty) \\ u(0,t) = u(L,t) = 0 & , t \in [0, \infty) \text{ Dirichl. B.C.} \\ u(x,0) = f(x) & , x \in [0, L] \text{ Initial cond} \end{cases}$$

1. Ansatz $u(x,t) = X(x) T(t)$
2. Plug into PDE $X(x) T'(t) - k X''(x) T(t) = 0$

$$\Leftrightarrow \frac{T'(t)}{k T(t)} = \frac{X''(x)}{X(x)} = -\lambda \in \mathbb{R}$$

3. ODEs

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(L) = 0 \end{cases} ; \quad T'(t) + \lambda k T(t) = 0$$

↑ here we have homog. ODE with B.C. $\Rightarrow$ solve $X(x)$ first

4. solve $X(x)$

$\lambda > 0$: $X(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$

$\lambda = 0$: $X(x) = Ax + B$

$\lambda < 0$: $X(x) = A\sinh(\sqrt{-\lambda}x) + B\cosh(\sqrt{-\lambda}x)$

from BC: non trivial solution for $\lambda > 0$

$$\Rightarrow X(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) \quad \text{(Dirichlet)}$$

5. solve $T(t)$

$$T(t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n k t}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

6. put together

$C_n = A_n B_n$

$$u(x,t) = X(x)T(t) = \sum_{n=1}^{\infty} C_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

7. find C_n with Initial condition Fourier *

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) = f(x) \overset{!}{=} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

⊛

$\Rightarrow C_n = b_n$ ← n-th. Fourier coeff. of $f(x)$

* we want Fourierseries of f with only sin

⇒ if f isn't odd: do an odd extension on $[0,L]$

⊛ Dirichlet B.C. ⇒ sin, $n = 1, 2, \dots$

Neumann B.C. ⇒ cos, $n = 0, 1, \dots$

Boundary conditions

- Dirichlet: $u(0,t) = u(L,t) = 0$ $\quad X(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$
- Neumann: $u_x(0,t) = u_x(L,t) = 0$ $\quad X(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$
- Mixed/Robin: R.I.P.

inhomogeneous heat equation

$$\begin{cases} u_t - k u_{xx} = F(x,t), & (x,t) \in [0,L] \times [0,\infty) \\ u(0,t) = u(L,t) = 0, & t \in [0,\infty) \text{ Dirichl. B.C} \\ u(x,0) = f(x), & x \in [0,L] \text{ Initial cond} \end{cases}$$

1. seperation of Variables (neglect $F(x,t)$) → $X(t)$
2. Write $u(x,t) = \sum_{n=1}^{\infty} T_n(t) \underbrace{X_n(t)}_{\text{known}}$
3. Plug this into PDE with RHS $= F(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$ (b_n: Fourier-coeff. of F)
4. match Coefficients before sinus
   ⇒ System of ODEs for $T_n(t)$
5. find $T_n(0)$ with I.C.

$$u(x,0) = \sum_{n=1}^{\infty} T_n(0)\, X_n(x) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)$$

(c_n: Fourier-coeff. of $f(x)$)

↳ this has $\sin\left(\frac{n\pi}{L}x\right)$

$\Rightarrow T_n(0) = c_n$

6. solve $T_n(t)$ and plug into $u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$

Energy method & uniqueness

1. Let $w := u_1 - u_2$
2. Energy function $E(t) = \int_0^L w_t^2(x,t) + c^2 w_x^2(x,t)\, dx$
3. Show that $\frac{d}{dt}E(t) = 0$ and $E(0) = 0$

$\Rightarrow E(t) = 0 \quad \forall t$

$\Rightarrow w_x = w_t = 0 \quad \forall t$ (because $c^2 \neq 0$)

$\Rightarrow w(x,t) = 0 \quad \forall x,t$

$\Rightarrow u_1 \equiv u_2$ □

elliptic equations

harmonic functions

solutions to the Laplace-eq. $\Delta u = 0$ are called harmonic

Poisson-equation (≙ Laplace-eq. if $\rho(x,y) \equiv 0$)

Dirichlet Problem
$$\begin{cases} \Delta u(x,y) = \rho(x,y), & (x,y) \in D \subseteq \mathbb{R}^2 \\ u(x,y) = q(x,y), & (x,y) \in \partial D \end{cases}$$

Neumann Problem
$$\begin{cases} \Delta u(x,y) = \rho(x,y), & (x,y) \in D \subseteq \mathbb{R}^2 \\ \partial_\nu u(x,y) = q(x,y), & (x,y) \in \partial D \end{cases}$$

$\partial_\nu u = \vec{\nu} \cdot \nabla u$, $\vec{\nu}$ is ⊥ on ∂D and pointing outwards

Problem of 3rd-kind
$$\begin{cases} \Delta u(x,y) = \rho(x,y), & (x,y) \in D \subseteq \mathbb{R}^2 \\ u(x,y) + \alpha\, \partial_\nu u(x,y) = q(x,y), & (x,y) \in \partial D \end{cases}$$

Existence of solution to the Neumann problem ⊛

$$\int_{\partial D} q\, dl \overset{!}{=} \iint_D \rho(x,y)\, dx\, dy \quad (dx\,dy \to r\, d\theta\, dr)$$

Weak maximum principle

if D is bounded and $u \in C^2(D) \wedge u \in C(\bar{D})$ („and") is harmonic in D ($\Delta u = 0$ in D) then:

$\max_{\bar{D}}\{u\} = \max_{\partial D}\{u\}$ and the same for min.

Strong maximum principle

u harmonic in D, D connected. Then:

u attains max/min in $\mathring{D}$ ⇒ $u = \text{const}$

Mean value principle

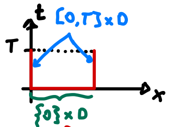
u harmonic in D, $B_R(x_0,y_0) \subseteq D$. Then:

$$u(x_0,y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos(\theta), y_0 + R\sin(\theta))\, d\theta$$

if $f(x,y)$ is smooth and satisfies m.v.p. ⇒ f harmonic

maximum principle for parabolic equations

D, bounded Domain, u solves the heat-equation in $Q_T = \overset{\text{time}}{[0,T]} \times \overset{\text{space}}{D}$ Then:



u achieves max/min on the parabolic boundary $\partial_p Q_T = \{\{0\}\times D\} \cup \{[0,T]\times\partial D\}$

Rectangular Domains

Laplace-eq. with Dirichlet-B.C.

$$\begin{cases} \Delta u = 0 & ,(x,y)\in[a,b]\times[c,d] \\ u(x,c) = u(x,d) = 0 & ,x\in[a,b] \\ u(a,y) = f(y) & ,y\in(c,d) \\ u(b,y) = q(y) & ,y\in(c,d) \end{cases}$$

minus before $Y(y)$

(homog. B.C for $X(x)$ ⇒ minus before $X(x)$)

no minus

1. $u(x,y) = X(x)Y(y) \Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda \in \mathbb{R}$
2. solve $Y(y)$ first because we have homog. B.C. here
   $\rightarrow Y(y) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi}{d-c}(y-c)\right)$, $\lambda_n = \left(\frac{n\pi}{d-c}\right)^2$
3. solve $X(x) = \sum_{n=1}^{\infty} \beta_n \sinh\left(\sqrt{\lambda_n}(x-a)\right) + \gamma_n \sinh\left(\sqrt{\lambda_n}(x-b)\right)$
4. find $\alpha_n, \beta_n, \gamma_n$ with B.C. & Fourier-coeff. of f and q

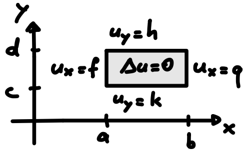
Laplace-eq. with Neumann-B.C.

$$\begin{cases} \Delta u = 0 & ,(x,y)\in[a,b]\times[c,d] \\ u_x(a,y) = u_x(b,y) = 0 & ,y\in[c,d] \\ u_x(x,c) = h(x) & ,x\in(a,b) \\ u_x(x,d) = k(x) & ,x\in(a,b) \end{cases}$$

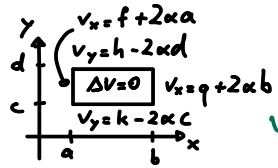
1. check condition for existence of solution (⊛)
2. $u(x,y) = X(x)Y(y) \Rightarrow -\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = \lambda \in \mathbb{R}$ — no minus
3. solve $X(x)$ first because we have homog. B.C. here
   $\rightarrow X(x) = \sum_{n=0}^{\infty} \alpha_n \cos\left(\frac{n\pi}{b-a}(x-a)\right)$, $\lambda_n = \left(\frac{n\pi}{b-a}\right)^2$
4. solve $Y(y) = \sum_{n=0}^{\infty} \beta_n \sinh\left(\sqrt{\lambda_n}(y-c)\right) + \gamma_n \sinh\left(\sqrt{\lambda_n}(y-d)\right)$
5. find coefficients with B.C (and Fourier-coeff. of h,k)

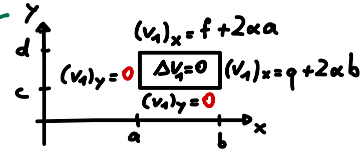
> ⊛ When writing $u(x,y) = X(x)Y(y)$:
> Let $A_n := \alpha_n\beta_n$ and $B_n := \alpha_n\gamma_n$
> ⇒ less unknowns ü

Boundary splitting with Neumann B.C.



1. add harmonic polynomial:
   $V := u + \alpha(x^2 - y^2)$
2. Write the Problem for V
3. split → $V = V_1 + V_2$





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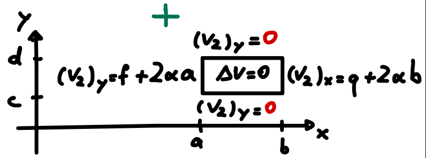


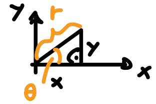
4. find α separately for $V_{1,2}$ with necessary-cond. (often: $\alpha \equiv 0$)
5. solve V_1 and V_2
6. write $u = V - \alpha(x^2 - y^2)$

Circular Domains

Laplace-eq. on circular domains

$u(x,y) = w(r,\theta) \Rightarrow$ Laplacian $\Delta w = w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\theta\theta}$

Polar-coordinates $\begin{cases} x = r\cos(\theta) ; y = r\sin(\theta) \\ r = \sqrt{x^2+y^2} ; \theta = \arctan(y/x) \end{cases}$



1. $w(r,\theta) = R(r)\cdot\Theta(\theta)$
2. $\Delta w = 0 \Leftrightarrow \frac{r^2R''(r) + rR(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda \in \mathbb{R}$ — minus — no minus
3. solve $\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0 \\ \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases} \Rightarrow \begin{cases} \Theta_n(\theta) = A_n\sin(n\theta) + B_n\cos(n\theta) \\ \lambda_n = n^2 \end{cases}$
4. solve $r^2R''(r) + rR'(r) - \lambda R(r) = 0$
   $\Rightarrow R_n(r) = \begin{cases} C_0 + D_0\ln(r) & ,n=0 \\ C_nr^n + D_nr^{-n} & ,n\neq 0 \end{cases}$

if zero is inside domain: $D_n \overset{!}{=} 0 \ \forall n\in\mathbb{N}_0$

Circular Boundary Conditions

Full circle: $\begin{cases} \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases} \Rightarrow \begin{cases} \Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta) \\ \lambda_n = n^2 \end{cases}$

Sector: $\Theta(0) = \Theta(\alpha) = 0 \Rightarrow \begin{cases} \Theta_n(\theta) = A_n\sin\left(\frac{n\pi}{\alpha}\theta\right) \\ \lambda_n = \left(\frac{n\pi}{\alpha}\right)^2 \rightarrow r^{\frac{n\pi}{\alpha}} \text{ and } r^{-\frac{n\pi}{\alpha}} \end{cases}$

Nice Stuff ü

Fourier-coefficients

if we want sinus: do odd extension of f on $[-L,L]$

$\rightarrow f^{odd}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$; $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$

if we want cos. : do even extension of f on $[-L,L]$

$\rightarrow f^{even}(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{L}x\right)$

where $a_n = \begin{cases} \frac{1}{L}\int_0^L f(x)dx & ,n=0 \\ \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx & ,n\neq 0 \end{cases}$

solving ODE with $Ch_p(\lambda)$

- $\lambda = \pm a \in \mathbb{R}$: $y_h(x) = A\sinh(ax) + B\cosh(ax)$
- $\lambda = a \pm bi$: $y_h(x) = e^{ax}(A\sin(bx) + B\cos(bx))$
- in general $\sum_i A_i e^{\lambda_i x}$ (for $\lambda_i \neq \lambda_j$, $i\neq j$)

Trigonometry/Tricks

$\sin(x\pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$

$\cos(x\pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$

$\sin(x)\sin(y) = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$

$\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right)$

$\sin(x)\cos(y) = \frac{1}{2}\left(\sin(x-y) + \sin(x+y)\right)$

$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

$\sin^3(x) = \frac{1}{4}(3\sin(x) - \sin(3x))$

$\cos^3(x) = \frac{1}{4}(3\cos(x) + \cos(3x))$

$\sin^4(x) = \frac{1}{8}(3 - 4\cos(2x) + \cos(4x))$

$\cos^4(x) = \frac{1}{8}(3 + 4\cos(2x) + \cos(4x))$

$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$; $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$

$\sinh(x) = \frac{e^x - e^{-x}}{2}$; $\cosh(x) = \frac{e^x + e^{-x}}{2}$